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Extremal problems of analysis and applications. (English) Zbl 08074057

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To minimize the distance between two mathematical objects, $f, g \in X$, the authors place this in a very general context by defining a distance function $h_X(f, g)$ (not a metric) on X with values in a partially ordered set R that has a minimal element θ that need not be 0. This allows them to formulate a versatile optimization problem and this book is all about estimating (sharp) bounds for that problem in very different settings and illustrating its use in applications like approximation theory, cubature, and signal processing.

The original Ostrowski paper (1938) gave a sharp bound for how much the value of a real differentiable function on an interval can deviate from its average. Leaving out the technical details, an example of an Ostrowski-like theorem in this book goes as follows: Let $f : T \to X$ be in a class $H(T, X)$ of Lipschitz-like continuous maps and $\Lambda : H(T, X) \to Y$, then, under some conditions, it is possible to bound $h_Y(\Lambda f(x), \Lambda f(t))$ in terms of $h_T(x, t)$ for all $x \in T$. If the bound is sharp, this corresponds to reaching an extremal value for the bound, hence the title of the book.

The previous description is like for a typical approximation problem if $\Lambda f(x)$ is the evaluation of f at a point x, but many other interpretations are possible. For example if f represents an integral operating on functions $x, t \in T$, and Λ is the identity, then $f(t)$ may be a quadrature formula for functions x in the class T, etc. Another interpretation of this framework gives bounds for the error when an unbounded operator is approximated by a bounded one (Stechkin problem) and several other variants are possible.

Also the Lipschitz condition in the example can be replaced by others, e.g. conditions involving derivatives for f (as in Sobolev spaces), or functions with bounded Laplacian or restricted gradient, or conditions involving the modulus of continuity, or functions that are of bounded variation, or perhaps in a stochastic context, belong to a certain class of random processes. To obtain the Ostrowski-like bounds, one has to find bounds for not only the (integrals of) functions involved but also for their (higher order, mixed partial, or Radon-Nykodym) derivatives.

All this gives rise to a plethora of inequalities that are adaptations of the general principle but that need careful analysis of all the ingredients that will lead to the (optimal) bounds. The optimalization of cubature formulas with lower and upper bounds for several function classes is an elaborated example.

All this is the subject of Chapter 1, which is the longest because all the general definitions and concepts had to be introduced. So Chapter 2 and 3 can be shorter discussing respectively the Bojanov-Naidenov problem and the Remez inequalities.

While the general Ostrowski inequalities for real functions requires bounding the norms of some intermediate derivatives over the whole real line, in the Bojanov-Naidenov problem these bounds are obtained for finite sections (relating it to a problem by Erdős), and (sharp) Remez inequalities are important to characterise best approximations. The inequalities are known for trigonometric and algebraic polynomials. In Chapters 2 and 3, these problems are generalized to include functions in a Sobolev class, trigonometric polynomials and polynomial splines. This relies on information about the ratio of the norms of their positive and negative parts.

Chapter 4 and 5 discuss applications in image processing. New definitions and tools have to be introduced to formulate a regularized objective function and its derivatives, and prove the optimality conditions and existence of a solution.

A new variational model in Sobolev-Orlicz spaces with variable exponent is proposed in Chapter 4 to simultaneously denoise and enhance contrast over the different spectral channels of colored images.

The application of Chapter 5 is the fusion of two color images with different spectral resolution, like if you want to add a picture of certain objects into a scene of another picture. Here the complicated constrained variational optimization problem involves the L^1-norm of the noise and the anisotropic diffusion tensor

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as a regularization term. The solution belongs to a Sobolev-Orlicz space with variable exponent for each spectral channel and these exponents are considered states in the optimization problem, hence computed dynamically. The optimization problem is nonlinear, non-smooth, and non-convex and hence very difficult. A proximal alternating algorithm is proposed.

The proposed algorithms for both image processing problems of these two chapters are illustrated numerically with several examples.

The book brings together and generalizes several results from the literature. The relevant references are collected after each chapter. It has innovative results of interest for researchers both in theory and applications. Only smaller, non-essential typos could be spotted. For example on page 168, it should define G_L and G_H and not G_H twice, and there are some others that are not really harmful.

Reviewer: Adhemar Bultheel (Leuven)

MSC:

41-02	Research exposition (monographs, survey articles) pertaining to approximations and expansions
41A50	Best approximation, Chebyshev systems
41A55	Approximate quadratures
41A17	Inequalities in approximation (Bernstein, Jackson, Nikol'skiĭ-type inequalities)
42A05	Trigonometric polynomials, inequalities, extremal problems
46N10	Applications of functional analysis in optimization, convex analysis, mathematical programming, economics
47N10	Applications of operator theory in optimization, convex analysis, mathematical programming, economics
49Jxx	Existence theories in calculus of variations and optimal control
65K10	Numerical optimization and variational techniques
92C55	Biomedical imaging and signal processing
65D32	Numerical quadrature and cubature formulas
94A08	Image processing (compression, reconstruction, etc.) in information and communication theory

Cited in **1** Document

Keywords:

Ostrowski-type inequality; Bojanov-Naidenov problem; Kolmogorov-type inequality; Remez-type inequality; numerical cubature; image processing; variational optimization

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