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Convexity in Newton's method. (English) Zbl 08010232
Frontiers in Mathematics. Cham: Birkhäuser (ISBN 978-3-031-85753-9/pbk; 978-3-031-85754-6/ebook). xii, 242 p. (2025).

This is the third book by these authors on the topic of Newton's method, and some of its variants, for solving nonlinear equations and optimization problems in Banach spaces. The titles of the first and second volume [*J. A. Ezquerro Fernández* and *M. A. Hernández Verón*, Newton's method: an updated approach of Kantorovich's theory. Cham: Birkhäuser/Springer (2017; Zbl 1376.65088)] and [*J. A. Ezquerro Fernández* and *M. Á. Hernández Verón*, Mild differentiability conditions for Newton's method in Banach spaces. Cham: Birkhäuser (2020; Zbl 07214192)] make it clear what their main focus is. This concluding volume completes the analysis by highlighting how convexity influences convergence.

The first two chapters introduce log-convexity (i.e., convexity of $\log f(x)$ implying usual convexity but not conversely), first in the scalar and then in the Banach setting. For Newton's method with iteration function $N_f(x) = x - \frac{f(x)}{f'(x)}$ the second derivative of $\log f$ is given by $L_f(x) = N_f'(x) = \frac{f(x)f''(x)}{f'(x)^2}$. Therefore log-convexity is influential to define convergence conditions and the speed of convergence that are analysed throughout the book.

Under certain conditions, L_f can be easily generalized to a Banach space setting using Fréchet derivatives. Then, by the Kantorovich approach that constructs majorizing operators for the Newton iterates, it is possible to formulate conditions that will guarantee convergence. However, a simpler approach is using a majorizing technique due to *J. M. Ortega* [Am. Math. Mon. 75, 658–660 (1968; Zbl 0183.43004)], which derives scalar bounds for the norms of operators, and therefore essentially reduces the convergence analysis to the convergence of a scalar sequence.

In this way conditions are defined for operators f, f' and L_f, to define regions in which convergence is assured from any starting point (global convergence) or where convergence is certain under some conditions on the starting point (semilocal convergence) or convergence is guaranteed if the starting point is close enough to the solution (local convergence). There is often a proved speed of convergence.

Chapter 3 deals with convex acceleration methods to speed up this convergence. Particular examples lead to iteration formulas of the form $G(x) = x - Q(x)\frac{f(x)}{f'(x)}$ including methods of Chebyshev ($Q = 1 + \frac{1}{2}L_f$), super-Halley ($Q = 1 + \frac{L_f}{2(1-L_f)}$) and Halley ($Q = \frac{2}{2-L_f}$). Also for these methods the convergence conditions and their speed is analysed, first in the scalar case and then in a Banach setting.

Chapter 4 studies other Newton-like methods that increase the order of convergence to 3 or even 4 by using more complex expressions for Q. The norms of the operators are bounded and analysis of the scalar sequence of bounds for the iterations proves the convergence under the appropriate conditions

In Chapter 5, modifications are applied to the Chebyshev method that avoid second derivaties, yet are as efficient as possible, usually involving multipoint methods or hybrid methods mixing different iterations in predictor-corrector steps with semilocal convergence. Since this is dealing with efficiency, it includes an analysis of the number of linear systems to be solved or matrices to be inverted and the computational complexity as the methods are applied. The semilocal convergence analysis reveals fractal regions for Julia sets.

The three volumes can be read independent from each other. The text is clearly written by reducing much of the analysis to the scalar case and giving many examples that illustrate the concepts.

Reviewer: Adhemar Bultheel (Leuven)

MSC:

65-02 Research exposition (monographs, survey articles) pertaining to numerical analysis
65K10 Numerical optimization and variational techniques
90C25 Convex programming

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