# General Toeplitz matrices and the Takenaka-Malmquist basis

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# Takenaka-Malmquist basis

Orthogonal rational functions w.r.t. Lebesgue measure on $\mathbb{T}$

$$\varphi_n(z) = \frac{\sqrt{1-|\alpha_n|^2}}{1-\overline{\alpha}_n z} \underbrace{\prod_{k=0}^{n-1} \frac{z-\alpha_k}{1-\overline{\alpha}_k z}}_{B_n(z)}, \quad n \geq 0,$$

$$\alpha_k \in \mathbb{D}, \quad \alpha_0 = 0, \quad \varphi_{-n} = \overline{\varphi_n(1/\overline{z})}$$

$\{\varphi_n\}_{k\in\mathbb{Z}}$ complete in $L^2(\mathbb{T})$ iff $\sum(1-|\alpha_k|) = \infty \Rightarrow B_n(z) \to 0$

Assumption $\boxed{|\alpha_k| \leq c < 1,\ \alpha_0 = 0}$.

# Convergence in $C^q_{2\pi}$, $q > 2$

$f(\omega) := F(e^{i\omega}) \in C^q_{2\pi}$, $q > 2$

Classical theory = all $\alpha_k = 0$: denote with a hat.

$\hat{\varphi}_k(e^{i\omega}) = e^{ik\omega}$, $\hat{c}_k = \langle f, \hat{\varphi}_k \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(\omega) \overline{\hat{\varphi}_k(e^{i\omega})} d\omega$

Thm: $f(\omega) = \sum_k \hat{c}_k \hat{\varphi}_k(e^{i\omega})$ absolutely convergent

Thm: $f \in C^q_{2\pi}$, $q > 2$ then $\sum_{|k|<p} c_k \varphi_k(e^{i\omega}) \to f(\omega), \quad c_k = \langle f, \varphi_k \rangle$
cvg uniform in $\omega$; rate of cvg is at least $1/p^{q-1}$

Proof is technical, based on

$$|c_0| \leq \|f\|_1 \text{ and } |c_{\pm k}| \leq C\|f^{(q)}\|_1/(\epsilon k)^q,\ k \geq 1$$
$$C \text{ and } \epsilon \text{ depend on } q,\ c,\ k$$

# Example

The function is $f(\omega) = (\cos 2\omega + 2)/(4\cos 2\omega + 5)$.
$f(\omega) = \mathrm{Re}F(e^{i\omega})$, $F(z) = (z^2+2)^{-1}$. Poles at $\pm i\sqrt{2}$.

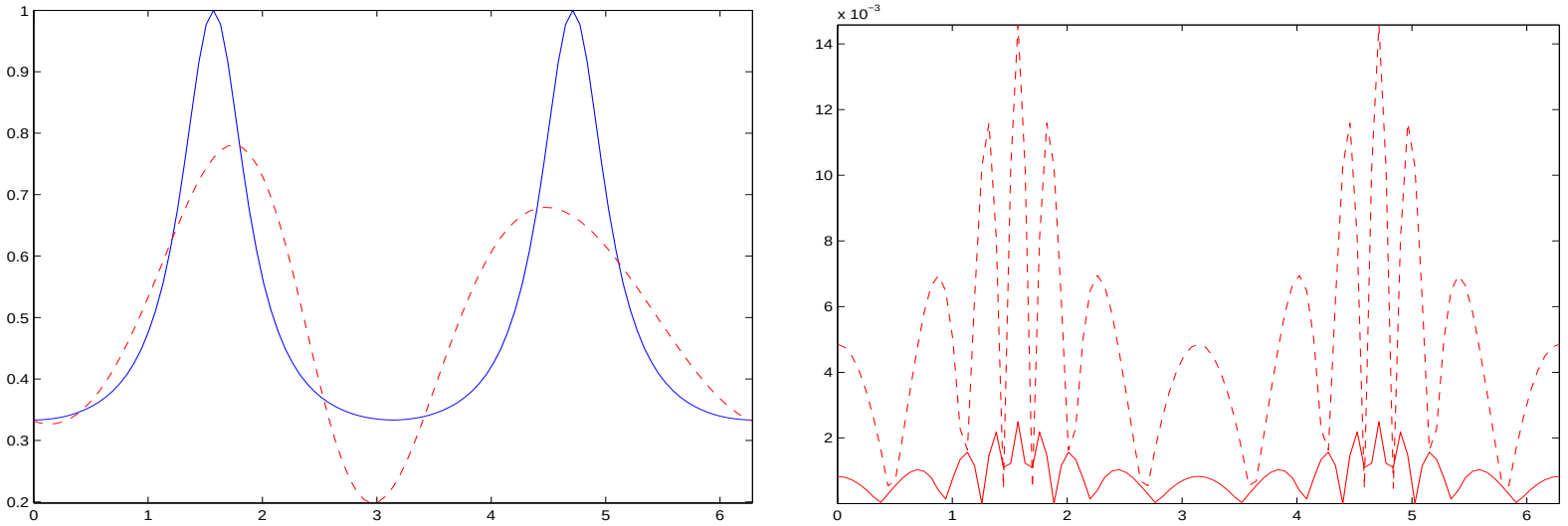



Figure: left: $f$ and $f_2$, random zeros; right: $f - f_4$ and $f - f_6$ for poles $\alpha_k = (-1)^k 0.6$.

# Example convergence

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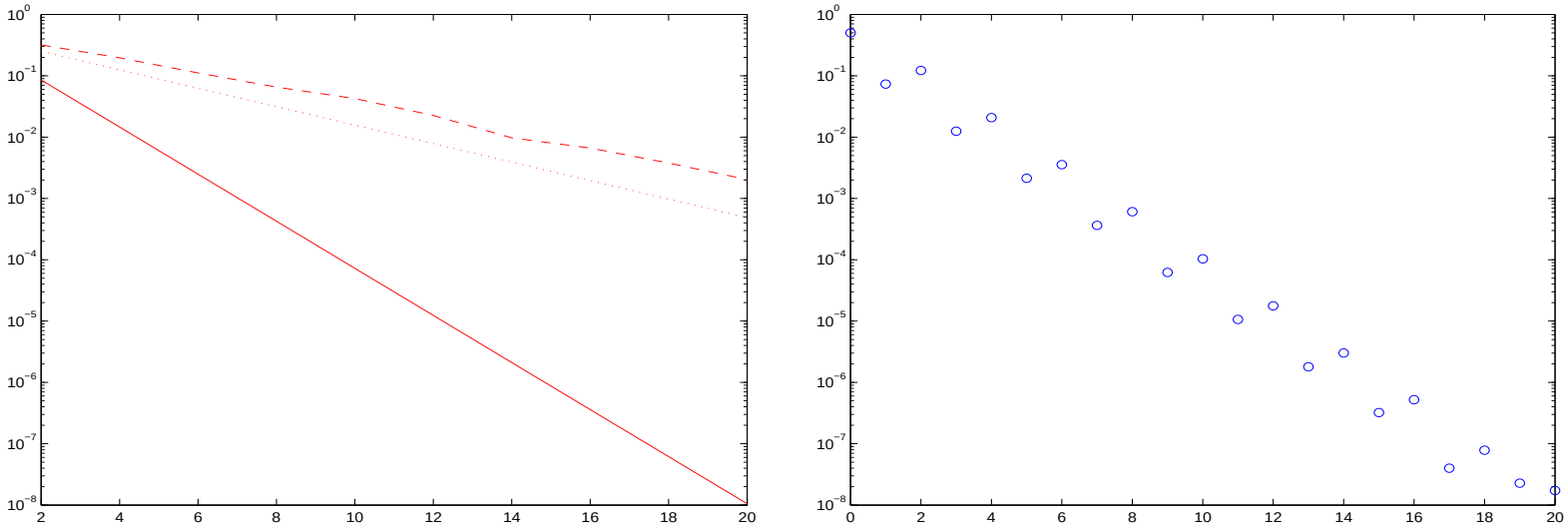



Figure: left: $\|f - f_p\|_\infty$ for (1) $\alpha_k$ random, (2) $\alpha_k = 0$, (3) $\alpha_k = (-1)^k 0.6$, right: Fourier coefs for case (3)

# Convergence in $C_{2\pi}$

Here convergence need not be uniform $\Rightarrow$ Cesàro means.

Example: The function is $f(\omega) = |\omega - \pi|$.

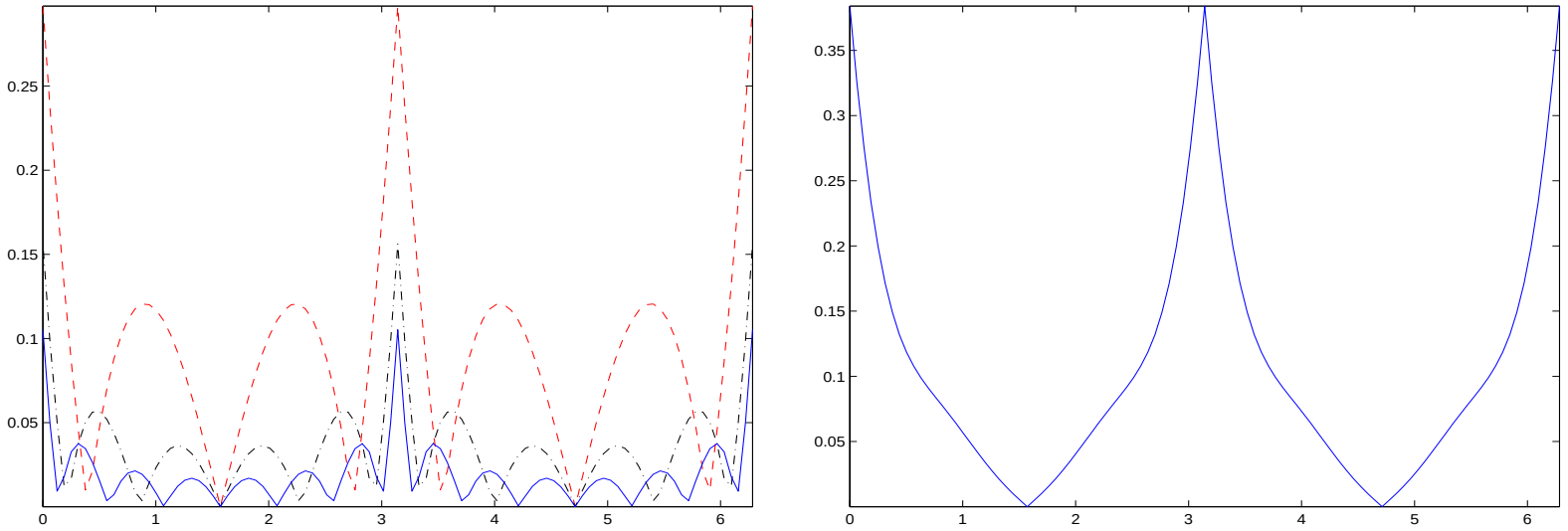



Figure: left: $|f - f_n|$, $n = 2, 4, 6$ for poles $\alpha_k = 0$. right: Cesàro sum for $n = 6$.

# Cesàro sums and Toeplitz matrix

$$\hat{M}_p(f) = \left[\frac{1}{2\pi}\int_0^{2\pi} e^{ik\omega} f(\omega)\overline{e^{il\omega}}d\omega\right]_{k,l=0}^{p-1} = \frac{1}{2\pi}\int_0^{2\pi} \hat{\Gamma}_p(\omega) f(\omega)\hat{\Gamma}_p^*(\omega)d\omega$$

$$\hat{\Gamma}_p(\omega) = [1, e^{-i\omega}, \ldots, e^{-i(p-1)\omega}]^* = [\overline{\hat{\varphi}_0(e^{i\omega})}, \ldots, \overline{\hat{\varphi}_{p-1}(e^{i\omega})}]^*$$

Note

$$\hat{\Gamma}_p^*(\mu)\hat{\Gamma}_p(\sigma) = \sum_{k=0}^{p-1} \hat{\varphi}_k(e^{i\sigma})\overline{\hat{\varphi}_k(e^{i\mu})} = \hat{K}_p(\mu,\sigma)$$

$$\hat{\gamma}_p = \hat{\Gamma}_p^*(\omega)\hat{\Gamma}_p(\omega) = \hat{K}_p(\omega,\omega) = p$$

Recall

$$\hat{M}_p(f) = \frac{1}{2\pi}\int_0^{2\pi} \hat{\Gamma}_p(\omega) f(\omega)\hat{\Gamma}_p^*(\omega)d\omega$$

Set $\hat{G}_p(\omega) = \hat{\Gamma}_p(\omega)/\sqrt{\hat{\gamma}_p}$. Then

$$\hat{G}_p^*(\omega)\hat{M}_p(f)\hat{G}_p(\omega) = \sum_{|k|<p} \left(1 - \frac{|k|}{p}\right) \hat{c}_k \hat{\varphi}_k(e^{i\omega})$$

This is Cesàro sum and

$$\lim_{p\to\infty} \hat{G}_p^*(\omega)\hat{M}_p(f)\hat{G}_p(\omega) = f(\omega)$$

$$\lim_{p\to\infty} \hat{G}_p^*(\omega)\hat{M}_p(f)\hat{M}_p(g)\hat{G}_p(\omega) = f(\omega)g(\omega)$$

# Generalized Cesàro sums

Assume $f \in C^q_{2\pi}$ with $q > 4$.

Generalized Toeplitz: $M_p(f_i) = \frac{1}{2\pi}\int_0^{2\pi} \Gamma_p(\omega) f_i(\omega) \Gamma_p^*(\omega) d\omega$

$$\Gamma_p(\omega) = [\overline{\varphi_0(e^{i\omega})}, \ldots, \overline{\varphi_{p-1}(e^{i\omega})}]^*$$

$$G_p(\omega) = \frac{\Gamma_p(\omega)}{\sqrt{\gamma_p(\omega)}}, \quad \gamma_p(\omega) = \Gamma_p^*(\omega)\Gamma_p(\omega)$$

$$G_p^*(\sigma) T(M_p(f_1), \ldots, M_p(f_n)) G_p(\mu) = \begin{cases} T(f_1, \ldots, f_n), & \sigma = \mu \\ 0, & \sigma \neq \mu \end{cases}$$

$T(x_1, \ldots, x_n)$ analytic function in $n$ variables.
Rate of convergence as fast as $1/p$ if $\sigma = \mu$ or $\frac{\ln p}{p}$ if $\sigma \neq \mu$.

Depends on analysis of the generalized Toeplitz matrix $M_p(f)$.

# Spectral properties

Classical case

$$\lim_{p\to\infty} \frac{1}{p} \sum_{i=0}^{p-1} \lambda_i(\hat{M}_p(f)) = \frac{1}{2\pi} \int_0^{2\pi} f(\omega) d\omega$$

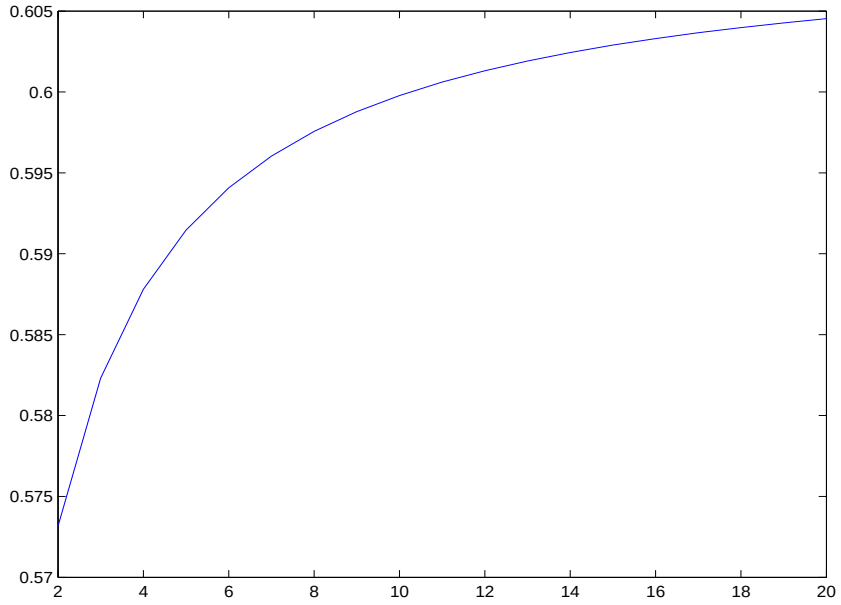
Generalized to

$$f \in C^q_{2\pi},\ q > 4$$

$$\lim_{p\to\infty} \frac{1}{p} \sum_{i=0}^{p-1} T\left(\lambda_i(M_p(f))\right) = \frac{1}{2\pi} \int_0^{2\pi} T\left(f(\chi^{-1}(\omega))\right) d\omega$$

$$\chi(\omega) = \lim_{p\to\infty} \frac{\chi_p(\omega)}{p}, \quad \chi_p(\omega) = \text{phase of } B_p(\omega).$$

# Example




$$f(\omega) = \frac{\cos 2\omega + 2}{4\cos 2\omega + 5}$$

$$\lim_{p\to\infty} \frac{1}{p} \sum_{k=0}^{p-1} \lambda_k(M_p(f)) = \frac{1}{2\pi} \int_0^{2\pi} f(\tilde{\chi}^{-1}(\omega)) d\omega = \int_0^{2\pi} f(\mu)\tilde{\chi}(\mu) d\mu$$

left-hand side for $p = 20$ is 0.6045, right-hand side integral $= 0.6098$

# Quadrature formulas

The basic idea in the proofs is to approximate integrals by quadratures.

$$M_p(f) = \frac{1}{2\pi} \int_0^{2\pi} \Gamma_p(\omega) f(\omega) \Gamma_p^*(\omega) d\omega$$

Use

$$\frac{1}{2\pi} \int_0^{2\pi} g(\omega) d\omega \approx \sum_{k=0}^{p-1} H_k g(\omega_k).$$

Then

$$M_p(f) \approx W_p(f) = \Upsilon_p F_p \Upsilon_p^*$$

$$F_p = \mathrm{diag}(f(\omega_0), \ldots, f(\omega_{p-1})),$$

$$\Upsilon_p = [\tilde{\Gamma}_p(\omega_0), \ldots, \tilde{\Gamma}_p(\omega_{p-1})], \quad \tilde{\Gamma}_p(\omega_k) = \Gamma_p(\omega_k)\sqrt{H_k}$$

The quadrature formula is exact in

$$\mathcal{L}_p \cdot \mathcal{L}_{-p} = \mathrm{span}\{\varphi_k(e^{i\omega})\overline{\varphi_l(e^{i\omega})} : k,l = 0,\ldots,p-1\}$$

if $\omega_k$ are zeros of para-orthogonals

$$\varphi_p(e^{i\omega}) - \eta B_p(e^{i\omega})\overline{\varphi_p(e^{i\omega})}, \quad \eta \in \mathbb{T}$$

and $H_k = 1/\gamma_p(\omega_k)$.

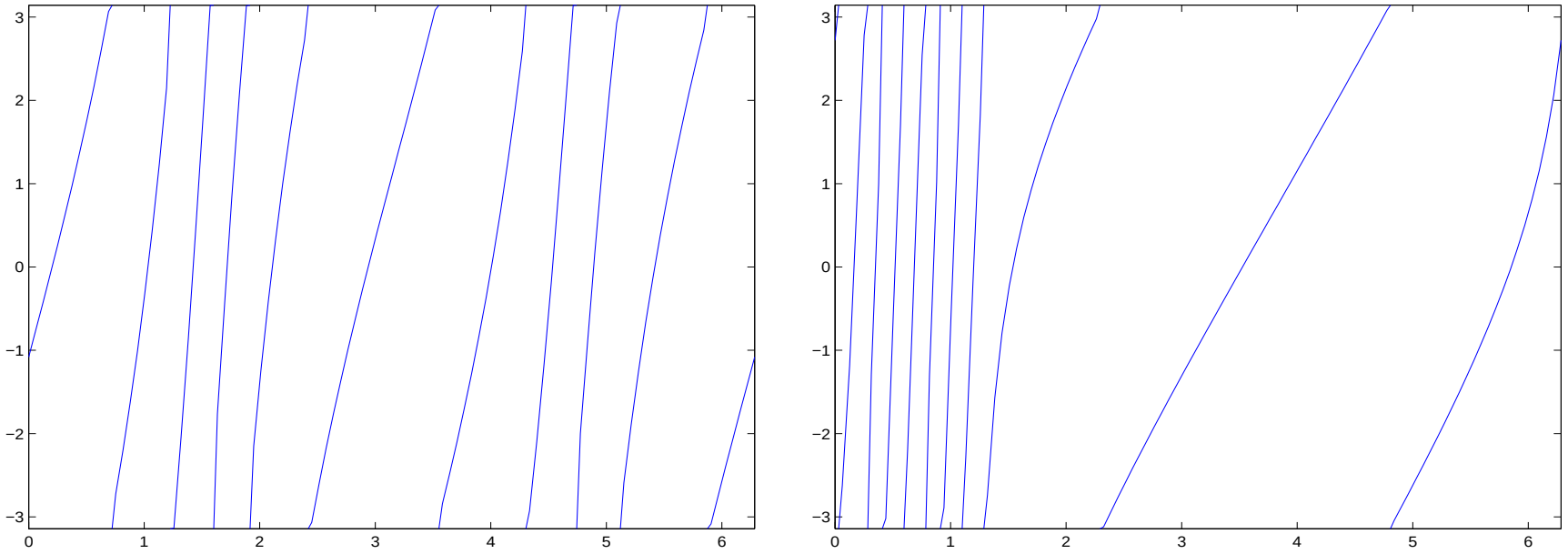
The classical case = Szegő quadrature: $\omega_k$ equidistant, $H_k = 1/p$. Then $\Upsilon_p$ is the (unitary) FFT matrix.

$$M_p(f) \approx W_p(f) = \Upsilon_p F_p \Upsilon_p^*$$

EVD of $W_p(f) \approx M_p(f)$, so $f(\omega_k) \approx f_p(\omega_k)$ approximate the eigenvalues of $M_p(f)$.

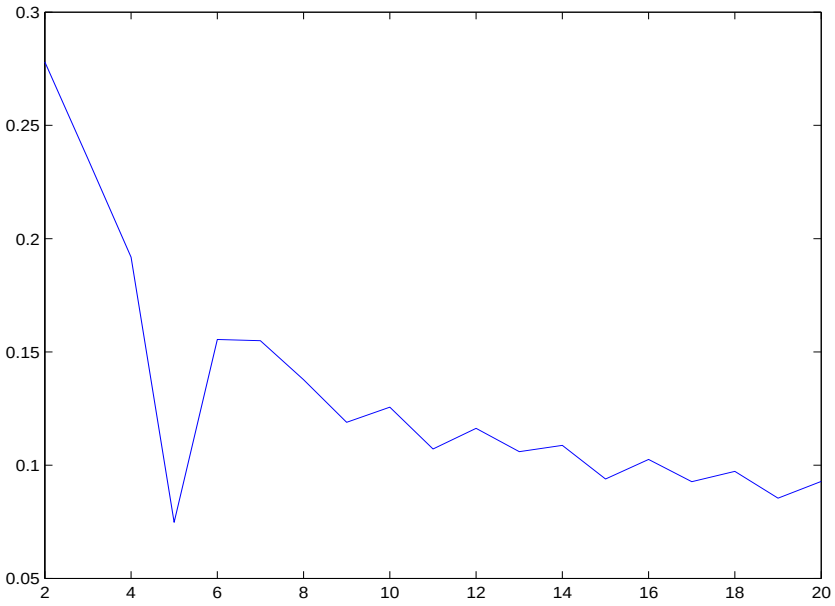
The general case = rational Szegő: $\omega_k \in \Omega_p(\theta)$, $H_k = 1/\gamma_p(\omega_k)$.

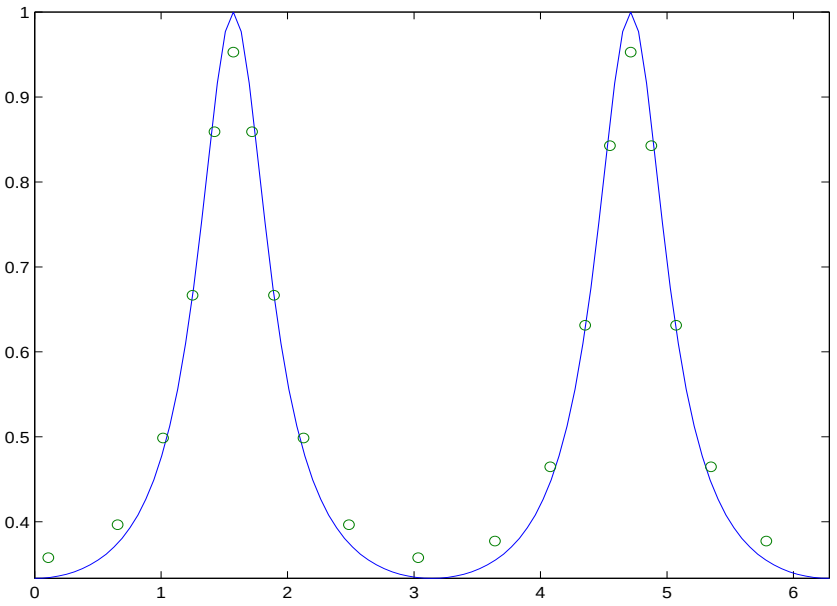
$$\Omega_p(\theta) = \{\omega \in [0, 2\pi) : \chi_p(\omega) = \theta \bmod 2\pi;\ \eta = e^{i\theta};\ B_p(e^{i\omega}) = e^{i\chi_p(\omega)}\}.$$




Figures: $\chi_{10}(\omega)$, left $\alpha_k = \pm 0.6i$, right $\alpha_k = 0.9e^{ik\pi/10}$

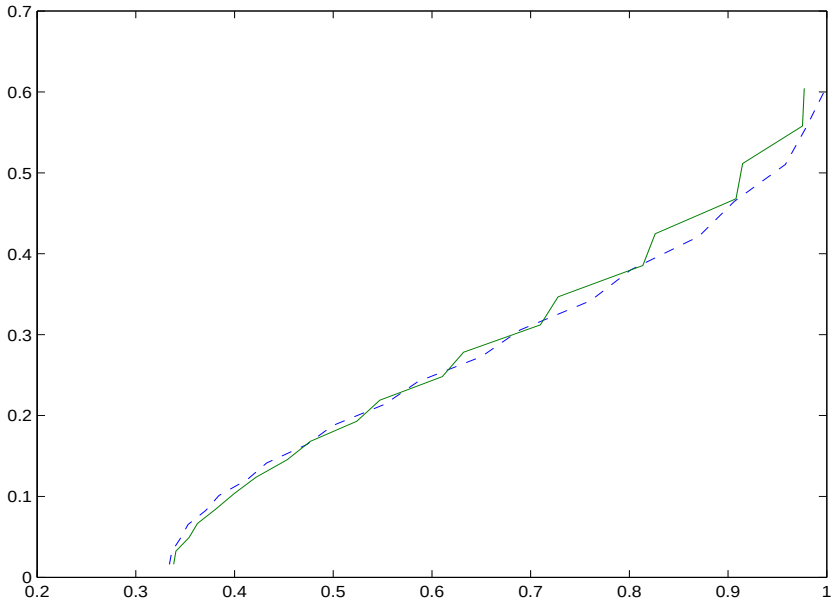
# Spectral approximation




$f(\omega) = (\cos 2\omega + 2)/(4\cos 2\omega + 5)$. Left: 2-norm of the vector of differences $\lambda_i(M_p(f)) - f(\omega_i)$ as a function of $p$. Right $\mathrm{diag}(\Upsilon_p^* M_p(f) \Upsilon_p)$ (circles) and $f(\omega)$.

# Spectral approximation




cumulative distribution of $f(\omega_k)$ for $\omega_k \in \Omega_{20}(1)$ (dashed) and cumulative distribution of $\lambda_k(M_{20}(f))$ (solid).

# Reference

[1] A. Bultheel, P. Carrette. Algebraic and spectral properties of general Toeplitz matrices. *SIAM J. Control Optim.*, 41 (2003), pp. 1413–1439.