Two-point Padé approximation to Herglotz-Riesz transforms

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Abstract

Padé approximation is the rational generalization of Hermite interpolating polynomial. On its own merits, it has earned a relevant place in the theory of constructive approximation. In this article we will develop an exhaustive analysis of two-point Padé approximations to Herglotz-Riesz transforms. We study the convergence problem when the poles are partially preassigned. In this analysis the Stieltjes polynomials on the unit circle naturally arise. Finally, some illustrative numerical examples are discussed.

Keywords: Padé approximation, Herglotz-Riesz transform, Carathéodory function, Stieltjes polynomial, quasi-paraorthogonal, unit circle

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1 Introduction

Padé approximants are rational functions that approximate a function f by letting their Taylor expansion in some point coincide with the expansion of f as far as possible, that is, up to the sum of the degrees of the numerator and denominator. Then, both numerator and denominator of a Padé approximant

are completely determined by this condition. If some poles of f are known or if we have information on where their singularities are located, it can be interesting to use this information. Padé-type approximants are interpolating rational functions with an arbitrary denominator, whose numerator is determined by the condition that the expansions of the Padé-type approximant and of f at some point coincide as far as possible, that is, up to the degree of the numerator, see e.g. [1]. Rational interpolation with all poles fixed was discussed in depth by Walsh, see [2], less oriented to applications as Brezinski did. It is also possible to choose some of the zeros of the denominator of the Padé approximants and determine its other zeros and the numerator in order to let their expansions coincide as far as possible. Such approximants, intermediating between Padé and Padé-type approximants, are called partial Padé approximants [3],[4]. From this approach, we can say that Padé approximants and Padé-type approximants are extreme cases of partial Padé approximants. In the first case the numerator and the denominator are employed to match the maximal number of coefficients in the expansion and in the second case only the numerator is available for that purpose. Thus we could have called the partial Padé approximants also lower-order Padé approximants or higher-order Padé-type approximants because by removing freedom in the denominator the order of approximation decreases with respect to Padé approximation, while by adding more freedom to the Padé-type approximants, the order of approximation is increased with respect to Padé-type approximants. Because the name partial Padé has been coined before in the literature, we shall stick to that name. In general, Padé-type approximants provide an approximation to a broader class of functions than the classical Padé approximants because the knowledge of the poles can be incorporated into their construction, see e.g. [5]. For certain functions, the speed of convergence for Padé-type approximants is the same as for Padé approximants (see [6]).

When interpolation conditions are imposed at two points in the Padé approximation process, they are called Two-point Padé approximants and similarly for Padé-type and partial Padé approximants that will also get the prefix 'Two-point' attached.

Let us first introduce some notation. We write $\mathbb{T} = \{z \in \mathbb{C} \ : \ |z| = 1\}$, while $\mathbb{D}$ and $\mathbb{E}$ denotes the interior and exterior of the closed unite disk, respectively. We denote the extended complex plane by $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and the set of all holomorphic (analytic) functions in $S \subset \hat{\mathbb{C}}$ is denoted by $\mathcal{H}(S)$. $\mathbb{P}$ is the vector space of all polynomials and $\mathbb{P}_n = \operatorname{span}\{z^k : k = 0, \ldots, n\}$ is the $(n+1)$-dimensional space of polynomials of degree at most n. We define the para-Hermitian conjugate of a function f as $f_*(z) = \overline{f}(1/z) = \overline{f(1/\overline{z})}$ and when $P \in \mathbb{P}_n \backslash \mathbb{P}_{n-1}$ then $P^*(z) = z^n P_*(z)$ is the reciprocal (or reversed) polynomial of P. It is invariant if $P(z) = cP^*(z)$, with $c \in \mathbb{T}$.

In this paper μ is a finite positive Borel measure that is supported on $\mathbb{T}$ with infinitely many points and L^2_μ is the corresponding classical Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. The monic orthogonal polynomials in L^2_μ are called Szegő polynomials and will be denoted by $\rho_n \in \mathbb{P}_n \backslash \mathbb{P}_{n-1}$.

They satisfy the recurrence relation, $\rho_n(z) = z\rho_{n-1}(z) + \delta_n\rho^*_{n-1}(z)$ where $\delta_n \in \mathbb{D}$ is the nth Szegő parameter. An invariant polynomial $B_n \in \mathbb{P}_n\backslash\mathbb{P}_{n-1}$ is paraorthogonal iff $\langle B_n, z^k\rangle = 0$, $k = 1, \ldots, n-1$ and if it is monic then $B_n(0) \in \mathbb{T}$ and $B_n = B_n(0)B^*_n$, and it is given by $B_n(z) = z\rho_{n-1}(z) + B_n(0)\rho^*_{n-1}(z) = \nu_n^{-1}(\rho_n(z) + \tau\rho^*_n(z))$, with $\tau \in \mathbb{T}$ and $\nu_n = 1 + \tau\overline{\delta_n}$, see [7]. This notion is generalized to quasi-paraorthogonal polynomials (QPOP) when the number of orthogonality conditions is reduced. The polynomial B_n will be called a QPOP of degree n and order $2\ell+1$ or briefly an $(n, 2\ell+1)$-QPOP iff it satisfies $\langle B_n, z^k\rangle = 0$, $k = \ell+1, \ldots, n-\ell-1$ where ℓ can take integer values from 0 to $\lfloor\frac{n}{2}\rfloor - 1$. Note that an nth degree paraorthogonal polynomial is a $(n, 1)$-QPOP.

Finally we denote the Herglotz-Riesz kernel as $D(t, z) = \frac{t+z}{t-z}$, $t \neq z$ and define the Herglotz-Riesz transform of a finite positive Borel measure μ on $\mathbb{T}$ as

$$\Omega_\mu(z) = \int_{\mathbb{T}} D(t, z)d\mu(t), \quad z \notin \mathbb{T}.$$

Such functions were introduced by Carathéodory (see [8]) and therefore they are also called Carathéodory functions or positive real functions (because $\mathrm{Re}(\Omega_\mu(z)) > 0$ for $z \in \mathbb{D}$). It is well known that $\Omega_\mu \in \mathcal{H}(\hat{\mathbb{C}}\backslash\mathbb{T})$.

In this paper we study the convergence of two-point partial Padé Approximants (2PPA) to a Carathéodory function (and two-point Padé-type Approximants (2PTA) as a particular case).

In several papers (see [9], [10], [11], [5], [12], [13]), important contributions were made. Before we give a precise formulation of the problem, we recall some relevant properties of the Herglotz-Riesz transform. It will be required that these properties are also verified by their rational approximants in order to improve the approximation.

a) Ω_μ is inversive, i.e.,
$$\Omega_\mu(z) = -\Omega_{\mu*}(z), \quad z \notin \mathbb{T}. \tag{1}$$
b) Both its singularities and its zeros are on $\mathbb{T}$.
c) The following expansions hold
$$\Omega_\mu(z) = \mu_0 + 2\sum_{\nu\geq 1}\mu_\nu z^\nu, \qquad z \in \mathbb{D}, \tag{2}$$
$$\Omega_\mu(z) = -\mu_0 - 2\sum_{\nu\geq 1}\mu_{-\nu} z^{-\nu}, \quad z \in \mathbb{E}. \tag{3}$$
where $\mu_\nu = \int_{\mathbb{T}} t^{-\nu}d\mu(t)$, $\nu \in \mathbb{Z}$ are the moments of μ. Note that $\overline{\mu_\nu} = \mu_{-\nu}$.
d) There exists a finite positive Borel measure $\tilde{\mu}$ such that [7, Section 1.3.9]
$$\frac{1}{\Omega_\mu(z)} = \Omega_{\tilde{\mu}}(z), \quad z \in \hat{\mathbb{C}}\backslash\mathbb{T}.$$

Note that property a) is very important for approximation, since if we estimate Ω_μ in $\mathbb{D}$ by a rational approximant R_n, and if it also satisfies $R_n(z) = -R_{n*}(z)$,

then this para-Hermitian conjugate will automatically give the corresponding rational approximant in $\mathbb{E}$.

Let us now state the problem considered in this paper in more detail. Recall the definition of a *Two-point Padé-type approximant* (2PTA): given the two power series (2)-(3), two nonnegative integers p and n $(0 < p < n+1)$ and an arbitrary monic polynomial B_n of degree n, with $B_n(0) \neq 0$, preferably with all its zeros on $\mathbb{T}$ (where also Ω_μ has its singularities), then there exists a unique polynomial A_n of degree n at most, such that with $q = p - 1$

$$B_n(z)\Omega_\mu(z) - A_n(z) = d_p z^p + \mathcal{O}(z^{p+1}), \quad z \to 0, \tag{4}$$

$$B_n(z)\Omega_\mu(z) - A_n(z) = d_q^* z^q + \mathcal{O}\left(z^{q-1}\right), \quad z \to \infty. \tag{5}$$

The unique rational function $\frac{A_n}{B_n}$ is called a 2PTA associated to Ω_μ, and it is denoted by $(p/n)_\mu$. Note that A_n has degree n and $A_n(0) \neq 0$ because $\frac{A_n}{B_n}$ interpolates Ω_μ at the origin and at infinity. As mentioned before rational interpolation with prescribed poles were studied in detail by Walsh (see e.g. [2]). In the mid 1970's the specific two-point case was introduced in some physical applications and the theoretical analysis was picked up for example by J. McCabe (see [14]) and in contributions by W.B. Jones and W. Thron in [15]. The name 2PTA for the two-point case with prescribed denominator was coined by Draux in [16], where he followed the formal procedure used by Brezinski in [1] for the one-point Padé-type Approximants (PTAs). For more details about this topic see [5] and the references found there. We point out that, in addition to B_n having all its zeros simple and on $\mathbb{T}$, it is interesting to choose B_n such that A_n also has all its zeros on $\mathbb{T}$, (where also the possible zeros of Ω_μ will be). Moreover, if possible, choose B_n such that the zeros of B_n and A_n are interlacing so that the approximant A_n/B_n generates better qualitative properties as approximation of Ω_μ.

When considering rational approximation with all its poles freely available to maximize the interpolation order we have $2n+1$ degrees of freedom to match as many coefficients in the two series. In this case, we are given two nonnegative integers p and n with $0 < p < 2n+1$. Then there exist two polynomials A_n and B_n, of degree n at most, such that (4)-(5) holds with $q = p - 1 - n$.

Any pair (A_n, B_n) satisfying these conditions defines a unique rational function $\frac{A_n}{B_n}$ to which we shall refer as the *Two-point Padé approximant* (2PA) associated to Ω_μ, and we will denote it by $[p/n]_\mu$.

When some, but not all the poles are free, we obtain *Two-point Partial Padé approximants* (2PPA). Now there are $n+m+1$ degrees of freedom in the couple (A_n, B_n) and if B_n has degree n and $B_n(0) \neq 0$, then $\frac{A_n}{B_n}$ matches Ω_μ up to order p at the origin and order $n+m+1-p$ at infinity, and m is the number of free poles. Hereafter the 2PPA will be denoted by $R_{n,p}^m$. To the best of our knowledge, these approximants will be analyzed for the first time in depth in section 3. Thus in this case (4)-(5) holds with $q = p - 1 - m$. Clearly for $m = 0$ we get a 2PPA, with prescribed B_n of degree n and with

$B_n(0) \neq 0$, while $m = n$ gives 2PA. If $1 \leq m < n$, this corresponds to 2PPA in a strict sense.

The case $R_{n,n}^{n-1}$ with an invariant denominator was analyzed for the first time in [17], a paper of special relevance in solving the trigonometric moments problem. In this seminal paper, the authors demonstrate that B_n is a paraorthogonal polynomial of degree n with respect to μ. All its zeros are simple and they are on $\mathbb{T}$ and A_n is the associated polynomial of the second kind, i.e. (this is essentially in [17, Eq. 5.6], see also [10, Remark 4]).

$$A_n(z) = z^n \left(\int_{\mathbb{T}} D(t,z) \left(\frac{B_n(z)}{z^n} - \frac{B_n(t)}{t^n} \right) d\mu(t) - \int_{\mathbb{T}} B_n(t) \frac{d\mu(t)}{t^n} \right). \tag{6}$$

Moreover, as a consequence of the invariance of B_n, it also holds that A_n is invariant and $A_n = -B_n(0)A_n^*$, so that

$$\frac{A_n(z)}{B_n(z)} = -\frac{A_{n*}}{B_{n*}} = -\frac{A_n^*(z)}{B_n^*(z)}, \tag{7}$$

just like the function it approximates satisfies $\Omega_\mu(z) = -\Omega_{\mu*}(z)$, $z \notin \mathbb{T}$. It enables, among other advantages, to restrict the approximation problem to $\mathbb{D}$, as we have pointed out. It follows from the results in [17] that the zeros of A_n are simple and are on $\mathbb{T}$, and they interlace with the zeros of B_n. Assume B_n is monic, and that we fix one of its zeros on $\mathbb{T}$ (or equivalently $B_n(0) \in \mathbb{T}$), then B_n is uniquely defined (see e.g. [18]), and since A_n is defined by (6), it follows that the approximant $R_{n,n}^{n-1} = \frac{A_n}{B_n}$ is unique.

2 Two-point Padé-type approximation

As we have noted before, it is advisable to require that the rational approximants verify certain properties to enlarge the domain of convergence or the class of functions that can be approximated, see [5].

Particularly, for the Herglotz-Riesz transform, important contributions have been made already see [19]. Especially for our interests, for example the following result holds (see equations (2.16) and (2.24) in [20]):

Theorem 1 *Let Ω_μ be a Herglotz-Riesz transform and $(p/n)_\mu = \frac{A_n}{B_n}$ a 2PTA to Ω_μ. Then, the following expressions hold true.*

1. *Its numerator A_n satisfies*

$$A_n(z) = z^p \left(\int_{\mathbb{T}} D(t,z) \left(\frac{B_n(z)}{z^p} - \frac{B_n(t)}{t^p} \right) d\mu(t) - \int_{\mathbb{T}} B_n(t) \frac{d\mu(t)}{t^p} \right).$$

2. Its residual is given by

$$\Omega_\mu(z) - (p/n)_\mu(z) = \frac{2z^p}{B_n(z)} \int_{\mathbb{T}} \frac{B_n(t)}{t-z} \frac{d\mu(t)}{t^{p-1}}, \quad z \notin \mathbb{T}. \tag{8}$$

Now, our aim is to find suitable sequences of denominators B_n which enable us to approximate Ω_μ everywhere in $\hat{\mathbb{C}}\backslash\mathbb{T}$. Therefore, we have to make use of sequences of extremal polynomials, whose properties are summarized in the following definition.

Definition 1 The sequence of monic polynomials $\{P_n\}_{n\in\mathbb{N}}$ is a sequence of extremal polynomials for $\mathbb{T}$ if it satisfies

$$\lim_{n\to\infty} |P_n(z)|^{1/n} = |z|, \quad \text{uniformly in compact subsets of } \mathbb{E}$$

and

$$\lim_{n\to\infty} |P_n(z)|^{1/n} = 1 \quad \text{uniformly in compact subsets of } \mathbb{D}.$$

We have the following characterization of the behaviour in $\mathbb{E}$ in terms of the uniform norm, see [2, Section 7.3 & Thm. 3.4].

Lemma 1 *Consider a sequence of monic polynomials $\{P_n\}_{n\in\mathbb{N}}$, then $\lim_{n\to\infty}|P_n(z)|^{1/n} = |z|$ uniformly in compact subsets on $\mathbb{E}$ if and only if $\lim_{n\to\infty} \|P_n\|_\infty^{1/n} = 1$, where $\|P_n\|_\infty = \max_{t\in\mathbb{T}}|P_n(t)|$.*

Now, the following result is immediate.

Theorem 2 *Let $\{p_n\}_{n\in\mathbb{N}}$ be a sequence of positive integers such that $0 < p_n < n+1$ and $\lim\limits_{n\to\infty} \dfrac{p_n}{n} = r \in [0,1]$ and let $\{B_n\}_{n\in\mathbb{N}}$ be a sequence of extremal polynomials for $\mathbb{T}$. Then the sequence $\{(p_n/n)_\mu\}_{n\in\mathbb{N}}$ of 2PTA with denominator B_n converges uniformly to Ω_μ on $\hat{\mathbb{C}}\backslash\mathbb{T}$. Moreover, a geometric rate of convergence is achieved.*

$$\limsup_{n\to\infty} |\Omega_\mu(z) - (p_n/n)_\mu(z)|^{\frac{1}{n}} \le |z|^r, \quad \forall z \in \mathbb{D},$$

and

$$\limsup_{n\to\infty} |\Omega_\mu(z) - (p_n/n)_\mu(z)|^{\frac{1}{n}} \le |z|^{r-1}, \quad \forall z \in \mathbb{E}.$$

Proof From (8), with $d(z,\mathbb{T}) = \min\{|z-t| : t \in \mathbb{T}\}$

$$|\Omega_\mu(z) - (p_n/n)_\mu(z)| \le \frac{2\mu_0|z|^{p_n}}{|B_n(z)|} \frac{\|B_n\|_\infty}{d(z,\mathbb{T})}$$

so that, because $\{B_n\}$ is an extremal sequence,

$$\limsup_{n\to\infty} |\Omega_\mu(z) - (p_n/n)_\mu(z)|^{\frac{1}{n}} \le \frac{|z|^r}{\lim_{n\to\infty}|B_n(z)|^{\frac{1}{n}}} = \min\left\{|z|^r, \frac{1}{|z|^{1-r}}\right\},$$

which proves the result. □

Remark 1 If $\{B_n\}_{n\in\mathbb{N}}$ is a sequence of paraorthogonal polynomials associated to finite positive Borel measure η with $\eta' > 0$ then $\{B_n\}_{n\in\mathbb{N}}$ is extremal for $\mathbb{T}$, which is the essential condition that is used in the proof. The previous theorem is also true for a complex measure μ (see [20]).

In this paper we are interested in the class of real regular measures η which is the largest class of measures whose associated paraorthogonal polynomials are extremal.

Definition 2 A finite positive Borel measure with support $\mathbb{T}$ is said to be regular if the corresponding sequence of monic orthogonal polynomials $\{\varrho_n\}_{n\geq 0}$ satisfies $\lim_{n\to\infty} \|\varrho_n\|^{\frac{1}{n}} = 1$.

There are several characterizations of a regular measure.

Theorem 3 *Let η be a finite positive Borel measure with support $\mathbb{T}$ with monic orthogonal polynomial sequence $\{\varrho_n\}_{n\in\mathbb{N}}$. Then the following are equivalent:*

1. *η is a regular measure on $\mathbb{T}$.*
2. *$\lim_{n\to\infty}|\varrho_n(z)|^{\frac{1}{n}} = |z|$, uniformly on compact subsets of $\mathbb{E}$.*
3. *For any sequence of polynomials $\{P_n\}_{n\in\mathbb{N}}$ with P_n monic and of degree n we have*
$$\lim_{n\to\infty} \frac{\|P_n\|^{\frac{1}{n}}}{\|P_n\|_\infty^{\frac{1}{n}}} = 1.$$

As a consequence (see [21, Lemma 5]) we have

Lemma 2 *Let η be a regular measure on $\mathbb{T}$, then the sequence of monic paraorthogonal polynomials associated to η is extremal for $\mathbb{T}$.*

As we have noted before, it is advisable that the approximant R_n to Ω_μ should satisfy $R_n(z) = -R_{n*}(z)$ for all $z \in \hat{\mathbb{C}}\backslash\mathbb{T}$. That this is true for a 2PTA under certain symmetry conditions can be verified as follows. Take minus the para-Hermitian conjugate of (4)-(5) and use $\Omega_\mu = -\Omega_{\mu*}$, and

$$R_{n*} = \left[\frac{A_n}{B_n}\right]_* = \frac{A_{n*}}{B_{n*}} = \frac{A_n^*}{B_n^*}$$

to arrive at

$$\begin{aligned} B_n^*(z)\Omega_\mu(z) + A_n^*(z) &= \mathcal{O}(z^{n+1-p}), \quad z \to 0, \\ B_n^*(z)\Omega_\mu(z) + A_n^*(z) &= \mathcal{O}(z^{n-p}), \quad z \to \infty. \end{aligned}$$

Note that $-\frac{A_n^*}{B_n^*} = (n+1-p/n)_\mu$. Comparing this with (4)-(5), it follows if the interpolation conditions are exact, i.e. $d_p d_q^* \neq 0$ in (4)-(5), we can obtain the following result.

Theorem 4 *Let B_n be a monic polynomial of degree n with $B_n(0) \neq 0$ and let (A_n, B_n) satisfy (4)-(5) with $d_p d_q^* \neq 0$, $q = p-1$ and $1 \leq p \leq n$. Then, the following are equivalent*

1. *n is odd, $p = \frac{n+1}{2}$, and B_n is invariant.*
2. *n is odd, $p = \frac{n+1}{2}$, and A_n is invariant, $A_n = -B_n(0)A_n^*$*
3. *A_n/B_n is an inversive 2PTA for Ω_μ.*

The proof is postponed till the next section where it will appear as a special case of Theorem 9.

Note that the $R_n = A_n/B_n$ of this theorem is actually an inversive 2PTA $(p/n)_\mu$ to Ω_μ whose denominator (and numerator) is invariant. It is balanced in the sense that there are as many interpolation condition at 0 as there are at ∞ and these conditions are strict (i.e. $d_p d_q^* \neq 0$).

As a consequence we have:

Corollary 1 *Let $\{(n+1/2n+1)_\mu\}_{n\in\mathbb{N}}$ be the sequence of inversive 2PTA whose denominators form a sequence $\{B_{2n+1}\}_{n\in\mathbb{N}}$ of paraorthogonal polynomials associated to a regular measure η on $\mathbb{T}$. Then, $\{(n+1/2n+1)_\mu\}_{n\in\mathbb{N}}$ converges uniformly to Ω_μ in compact subsets of $\hat{\mathbb{C}}\backslash\mathbb{T}$. Moreover, a geometric rate of convergence is achieved.*

$$\limsup_{n\to\infty} |\Omega_\mu(z) - (n+1/2n+1)_\mu(z)|^{\frac{1}{2n}} \leq \sqrt{|z|}, \quad \forall z \in \mathbb{D},$$

and

$$\limsup_{n\to\infty} |\Omega_\mu(z) - (n+1/2n+1)_\mu(z)|^{\frac{1}{2n}} \leq \frac{1}{\sqrt{|z|}}, \quad \forall z \in \mathbb{E}.$$

Proof The paraorthogonal denominators imply an inversive approximant by the previous theorem. The convergence is a consequence of Theorem 2, since the respective denominators are paraorthogonal polynomials for a regular measure η on $\mathbb{T}$ and thus are extremal polynomials by Lemma 2. □

Remark 2 Note that from the previous result about inversive 2PTA, it is enough to restrict ourselves to the approximation on $\mathbb{D}$. The para-Hermitian conjugate of the approximation error $\Omega_\mu(z) - \frac{A_n(z)}{B_n(z)}$ for $z \in \mathbb{D}$ and the invariance of A_n and B_n gives the corresponding approximation error in $\mathbb{E}$. On the other hand, in the next section we will analyze the case where μ is a regular measure on $\mathbb{T}$, since the rate of convergence for such measures is substantially improved.

The 2PTA $(n/p)_\mu$ being inversive in addition to the quantitative properties in convergence, also has qualitative properties if B_n is chosen appropriately.

Thus if all the zeros x_j, $j = 1, \ldots, n$ of B_n are simple, then $(p/n)_\mu$, admits the following unique decomposition

$$(p/n)_\mu(z) = D_0 + \sum_{j=1}^{n} D_j \frac{x_j + z}{x_j - z},$$

where $D_j = -\frac{A_n(x_j)}{2x_j B_n'(x_j)}$, $j = 1, \ldots, n$, see [10], and since it interpolates at least once at the origin and at infinity,

$$(p/n)_\mu(0) = D_0 + \sum_{j=1}^{n} D_j = \Omega_\mu(0) = \mu_0$$

and

$$(p/n)_\mu(\infty) = D_0 - \sum_{j=1}^{n} D_j = \Omega_\mu(\infty) = -\mu_0,$$

and hence $D_0 = 0$.

About the nature of the D_j we can prove the following two theorems. Their proofs will again be postponed to the more general case of 2PPA, see Theorems 10 and 11 below.

Theorem 5 *Let $B_n \in \mathbb{P}_n \backslash \mathbb{P}_{n-1}$, be an invariant monic polynomial with only simple zeros $\{x_i\}_{j=1}^n$ and let $(p/n)_\mu = \frac{A_n}{B_n} = \sum_{j=1}^n D_j \frac{x_j+z}{x_j-z}$ be a 2PTA, then the following are equivalent*

1. *n is odd and $p = \frac{n+1}{2}$.*
2. *The coefficients D_j associated to poles on $\mathbb{T}$ are real while those associated to poles symmetric with respect to $\mathbb{T}$ are complex conjugates.*

As a consequence of the analytic properties of Ω_μ, we can be more precise about the nature of the D_j in the decomposition.

Theorem 6 *Let n be odd and $p = \frac{n+1}{2}$ and $(p/n)_\mu$ a 2PTA whose denominator is monic, invariant and has only simple zeros. Suppose that in the decomposition of $(p/n)_\mu(z) = \sum_{j=1}^n D_j \frac{x_j+z}{x_j-z}$ there are s' poles on $\mathbb{T}$ and r' poles in $\mathbb{D}$ for which the corresponding D_j has positive real part, then $s' + r' \geq p$.*

As we will see, it is fundamental that all coefficients D_k are positive since it guarantees convergence. Moreover, if the poles are simple and lie on $\mathbb{T}$, then the following result guarantees that the approximants accumulate interlacing zeros and poles on $\mathbb{T}$, improving both quantitatively and qualitatively convergence, see [22, Lemma 4.1].

Lemma 3 *Let $R = P/Q$ be a rational function such that $\mathrm{Re}(R(\infty)) \neq 0$ and $R(\infty) = -R(0)$. Suppose that the polynomial Q has only simple zeros $z_1, \ldots, z_k$, $k \geq 2$ that are all on $\mathbb{T}$. Then, the following statements are equivalent.*

1. *The zeros of P lie on $\mathbb{T}$ and interlace with the zeros of Q.*
2. *All the numbers*
$$\frac{P(z_j)}{z_j Q'(z_j)}, \quad j = 1, \ldots, k,$$
are real with the same sign as $\mathrm{Re}(R(\infty))$

Remark 3 In our case, where $R_n = \frac{A_n}{B_n}$, we have $R_n(\infty) = -\mu_0 = -R_n(0) < 0$ and thus $D_k = -\frac{A_n(x_k)}{2x_k B_n'(x_k)} > 0$, $k = 1, \ldots, n$.

3 Two-point Partial Padé approximation

In this section, we consider Two-point Partial Padé approximants (2PPA). As we have pointed out, we should choose the prescribed poles in an appropriate way in order to get an inversive 2PPA with simple poles, most of which on $\mathbb{T}$.

3.1 Orthogonality

Concretely, let $p \in \mathbb{N}$ be such that $0 < p < n+1+m$, with $0 \leq m < n$, and let $d = n - m$ zeros of B_n be fixed in advance. Then there exist two polynomials $A_n, B_n \in \mathbb{P}_n$ such that (4)-(5) hold with $q = p - 1 - m$. Any pair (A_n, B_n) satisfying the above interpolation conditions defines a unique rational function $R_{n,p}^m = \frac{A_n}{B_n}$ that we call a Two-point Partial Padé approximant. As noted before, we shall use the notation $R_{n,p}^m$ for a 2PPA of degree n with m free poles and an order of contact p at the origin and $n - q = n + m + 1 - p$ at infinity. Note that $m = 0$ identifies 2PTA as a special case of 2PPA. The increase (or decrease) in the number of interpolation conditions with respect to 2PTA (or 2PA) corresponds to an increase (or decrease) in the number of orthogonality conditions satisfied by the denominator B_n. It is appropriate to introduce the inner product induced by μ. Let f and g be two functions of $L_\mu^2(\mathbb{T})$, then

$$\langle f, g \rangle = \int_{\mathbb{T}} f(t)\overline{g(t)} d\mu(t).$$

Theorem 7 *Let (A_n, B_n) satisfy (4)-(5) with $q = p-1-m$, $m \geq 1$. Then $R_{n,p}^m = \frac{A_n}{B_n}$ is a 2PPA and*
$$\langle B_n, z^k \rangle = 0, \quad k = p - m, \ldots, p - 1. \tag{9}$$

Proof It follows from (4) and (5) that for any $Q \in \mathbb{P}_{m-1}$:
$$Q(z)\frac{B_n(z)\Omega_\mu(z) - A_n(z)}{z^p} = \begin{cases} \mathcal{O}(z^{p'}), & p' \geq 0, \quad z \to 0 \\ \mathcal{O}(z^{-q'}), & q' \geq 2, \quad z \to \infty. \end{cases}$$

Since it is holomorphic in $\hat{\mathbb{C}}\backslash\mathbb{T}$ and $\mathcal{O}(z^{-2})$ as $z \to \infty$, there exist Jordan curves $\Gamma_0 \subset \mathbb{D}$ and $\Gamma_\infty \subset \mathbb{E}$ with the origin in the interior of both curves such that

$$\int_{\Gamma_0} Q(z)\frac{B_n(z)\Omega_\mu(z) - A_n(z)}{z^p}dz = 0 = \int_{\Gamma_\infty} Q(z)\frac{B_n(z)\Omega_\mu(z) - A_n(z)}{z^p}dz$$

and

$$\int_{\Gamma_0} \frac{Q(z)A_n(z)}{z^p}dz = \int_{\Gamma_\infty} \frac{Q(z)A_n(z)}{z^p}dz.$$

Hence with $\Gamma = \Gamma_0 - \Gamma_\infty$:

$$0 = \int_\Gamma \frac{Q(z)B_n(z)\Omega_\mu(z)}{z^p}dz = \int_\Gamma \frac{Q(z)B_n(z)}{z^p}\int_{\mathbb{T}} D(t,z)d\mu(t)dz.$$

By Fubini's theorem thus

$$\int_{\mathbb{T}}\int_\Gamma \frac{Q(z)B_n(z)(t+z)}{(t-z)z^p}dzd\mu(t) = 0,$$

and using Cauchy's theorem

$$\int_{\mathbb{T}} \frac{Q(t)B_n(t)2t}{t^p}d\mu(t) = 2\int_{\mathbb{T}} \frac{Q(t)B_n(t)}{t^{p-1}}d\mu(t) = 0, \quad \forall Q \in \mathbb{P}_{m-1}$$

or equivalently

$$0 = \int_{\mathbb{T}} t^{-\nu}B_n(t)d\mu(t) = \langle B_n, z^\nu\rangle, \quad \nu = p-m,\ldots,p-1.$$

□

For the orthogonality conditions (9) to be in effect, the interpolation conditions at the origin p must satisfy $0 \le p-m \le p-1$, i.e., $p \ge m \ge 1$.

These orthogonality conditions allows us to extend our Theorem 1 above, about 2PTA.

Theorem 8 *Let Ω_μ be a Herglotz-Riesz transform and $R^m_{n,p} = \frac{A_n}{B_n}$ a 2PPA to it with $m \ge 1$. Then*

1. Its numerator A_n satisfies

$$A_n(z) = z^p\left(\int_{\mathbb{T}} D(t,z)\left(\frac{B_n(z)}{z^p} - \frac{B_n(t)}{t^p}\right)d\mu(t) - \int_{\mathbb{T}} B_n\frac{d\mu(t)}{t^p}\right).$$

2. The interpolation error is given by

$$\Omega_\mu(z) - R^m_{n,p}(z) = \frac{2z^p}{Q(z)B_n(z)}\int_{\mathbb{T}} \frac{B_n(t)Q(t)}{t-z}\frac{d\mu(t)}{t^{p-1}}, \quad z \notin \mathbb{T}, \quad \forall Q \in \mathbb{P}_m\backslash\{0\}. \tag{10}$$

Proof Consider the following identity

$$\frac{B_n(z)}{z^p}\Omega_\mu(z) - \int_{\mathbb{T}} D(t,z)\left(\frac{B_n(z)}{z^p} - \frac{B_n(t)}{t^p}\right)d\mu(t) = \int_{\mathbb{T}} D(t,z)\frac{B_n(t)}{t^p}d\mu(t)$$

or, using $D(t,z) = -1 + \frac{2t}{t-z}$ in the right-hand side, we obtain

$$\frac{B_n(z)\Omega_\mu(z) - A_n(z)}{z^p} = 2\int_{\mathbb{T}} \frac{B_n(t)d\mu(t)}{(t-z)t^{p-1}} \tag{11}$$

where

$$A_n(z) = z^p \left(\int_{\mathbb{T}} D(t,z) \left(\frac{B_n(z)}{z^p} - \frac{B_n(t)}{t^p} \right) d\mu(t) - \int_{\mathbb{T}} \frac{B_n(t)}{t^p} d\mu(t) \right).$$

For $m \geq 1$ recall the orthogonality conditions from (9) implying that

$$\int_{\mathbb{T}} B_n(t)\, \frac{P(t)}{t^{p-1}} d\mu(t) = 0, \quad \forall P \in \mathbb{P}_{m-1}.$$

Therefore

$$\int_{\mathbb{T}} B_n(t) \frac{Q(z) - Q(t)}{z-t} \frac{d\mu(t)}{t^{p-1}} = 0, \quad \forall Q \in \mathbb{P}_m \backslash \{0\}.$$

and thus

$$\frac{B_n(z)\Omega_\mu(z) - A_n(z)}{z^p} = \frac{2}{Q(z)} \int_{\mathbb{T}} \frac{1}{t-z} \frac{Q(t)B_n(t)}{t^{p-1}} d\mu(t). \tag{12}$$

The equation (11) shows the necessary interpolation at the origin, and taking Q of degree m in (12) shows the interpolation at infinity, which implies that A_n is the numerator of $R^m_{n,p}$. □

Note that if B_n has degree n and is invariant, then from orthogonality conditions (9) we obtain

$$\langle B_n, z^k \rangle = 0, \quad \begin{cases} k = p-m, \ldots, n-(p-m), & \text{if } \ p \leq \frac{n+m+1}{2}, \\ k = n+1-p, \ldots, p-1, & \text{if } \ p > \frac{n+m+1}{2}. \end{cases} \tag{13}$$

In the first case $R^m_{n,p}$ over-interpolates at the origin and in the other case at infinity. Invariant monic polynomials of degree n verifying the orthogonality conditions (13) with $p - m \leq \frac{n}{2} \leq p-1$ were named by the authors, quasi-paraorthogonal polynomials of order $2\ell - 1$, with $\ell = p - m$ in the first case and $\ell = n+1-p$ in the second case. Their existence is guaranteed, see [18]. In order to take advantage of all the resources offered by orthogonality, we must determine those $R^m_{n,p}$ whose denominator B_n has degree n and it is invariant, a property that, as we have seen for 2PTA, is closely linked to $R^m_{n,p}$ being inversive if the interpolation conditions are exact, i.e., $d_p\, d_q^* \neq 0$ in (4)-(5). Then we have the following theorem that includes Theorem 4 as a special case when $m = 0$.

Theorem 9 *Let B_n be a monic polynomial of degree n with $B_n(0) \neq 0$. Let (A_n, B_n) solve the equations (4)-(5) with $q = p-1-m$, $0 \leq m < n$, and $1 \leq p \leq n+m$, and assume $d_p\, d_q^* \neq 0$. Then, the following are equivalent*

1. *$n+m$ is odd, $p = \frac{n+m+1}{2}$, and B_n is invariant.*
2. *$n+m$ is odd, $p = \frac{n+m+1}{2}$, and A_n is invariant, $A_n = -B_n(0)A_n^*$*
3. *$R^m_{n,p} = A_n/B_n$ is inversive.*

Proof 1. $\Leftrightarrow$ 2. $\Rightarrow$ 3. is straightforward and it holds without the condition $d_p\, d_q^* \neq 0$. 3. $\Rightarrow$ 1. Note that $R_{n,p}^m$ is indeed a 2PPA and the 2PTA $R_{n,p}^0 = (p/n)_\mu$ appears as a special case. From (8) for $m = 0$ and (10) for $m \geq 1$, we can treat both cases simultaneously by choosing $Q \in \mathbb{P}_m$ monic of degree m. We obtain for $z \notin \mathbb{T}$.

$$Q(z)B_n(z)\Omega_\mu(z) - Q(z)A_n(z) = 2z^p \int_{\mathbb{T}} \frac{B_n(t)Q(t)}{t-z} \frac{d\mu(t)}{t^{p-1}}, \tag{14}$$

and as $\frac{A_n^*}{B_n^*}$ is equal to $R_{n,n+m+1-p}^m$,

$$Q(z)B_n^*(z)\Omega_\mu(z) - Q(z)A_n^*(z) = 2z^{n+m+1-p} \int_{\mathbb{T}} \frac{B_n^*(t)Q(t)}{t-z} \frac{d\mu(t)}{t^{n+m-p}}. \tag{15}$$

Since $R_{n,p}^m$ is inversive, $B_n(z)A_n^*(z) + B_n^*(z)A(z) = 0$, so that

$$B_n^*(z) \int_{\mathbb{T}} \frac{B_n(t)}{t-z} \frac{Q(t)d\mu(t)}{t^{p-1}} - z^{n+m+1-2p} B_n(z) \int_{\mathbb{T}} \frac{B_n^*(t)Q(t)}{t-z} \frac{d\mu(t)}{t^{n+m-p}} = 0,$$

If we evaluate at $z = \infty$ and we use $d_p d_q^* \neq 0$ or equivalently $\int_{\mathbb{T}} B_n(t)\bar{t}^p d\mu(t) \cdot \int_{\mathbb{T}} B_n(t)\bar{t}^{p-m-1} d\mu(t) \neq 0$, we deduce that $2p = m + n + 1$ and thus that

$$B_n^*(z) \int_{\mathbb{T}} \frac{B_n(t)}{t-z} \frac{Q(t)d\mu(t)}{t^{p-1}} - B_n(z) \int_{\mathbb{T}} \frac{B_n^*(t)}{t-z} \frac{Q(t)d\mu(t)}{t^{p-1}} = 0. \tag{16}$$

We shall prove by contradiction that $B_n(z) = B_n(0)B_n^*(z)$. Suppose that $B_n(z) \neq B_n(0)B_n^*(z)$, then there must exist α_i, $i = 1, 2, \ldots, \ell$ such that $B_n(\alpha_i) = 0$ but $B_n^*(\alpha_i) \neq 0$. Set $r(z) = (z-\alpha_1)\cdots(z-\alpha_\ell)$. So we can write $B_n(z) = s(z)r(z)$ with $s(z) = s(0)s^*(z)$ of degree $n-\ell$ and $r^*(\alpha_i) \neq 0$, $i = 1, \ldots, \ell$. Substitute this in (16) then after adding and subtracting $r(z)r^*(z)\int_{\mathbb{T}} \frac{s(t)Q(t)}{(t-z)t^{p-1}} d\mu(t)$ we obtain

$$r(z) \int_{\mathbb{T}} \frac{r^*(t) - r^*(z)}{t-z} \frac{s(t)Q(t)}{t^{p-1}} d\mu(t) - r^*(z) \int_{\mathbb{T}} \frac{r(t) - r(z)}{t-z} \frac{s(t)Q(t)}{t^{p-1}} d\mu(t) = 0, \quad z \notin \mathbb{T}.$$

Since the integrands of both integrals are polynomials in z of degree $\ell - 1$, we can write

$$r(z)V(z) - r^*(z)W(z) = 0, \quad \forall z \notin \mathbb{T}$$

and both V and W are polynomials of degree $\ell - 1$. Evaluating this in the zeros α_i of $r(z)$ and knowing that $r^*(\alpha_i) \neq 0$ implies $W(\alpha_i) = 0$, $i = 1, \ldots, \ell$, but since its degree is at most $\ell - 1$, W must be identically zero, which is a contradiction because we assumed $\ell \geq 1$. Thus $B_n(z) = B_n(0)B_n^*(z)$. □

Let's consider once again 2PPA $R_{n,p}^m = \frac{A_n}{B_n}$ with $0 \leq m < n$ and $1 \leq p \leq n + m$. Then, exactly like we have seen in the previous section for 2PTA (i.e., $m = 0$): if B_n has degree n and all its zeros x_j, $j = 1, \ldots, n$ are simple, then $R_{n,p}^m$, admits the following unique decomposition

$$R_{n,p}^m(z) = \sum_{j=1}^{n} D_j \frac{x_j + z}{x_j - z}.$$

We have the following theorems about the D_j:

Theorem 10 *Let B_n, be an invariant monic polynomial of degree n with only simple zeros $\{x_i\}_{j=1}^n$ and let $R_{n,p}^m = \frac{A_n}{B_n} = \sum_{j=1}^n D_j \frac{x_j+z}{x_j-z}$ be a 2PPA to Ω_μ and $0 \leq m < n$, then the following are equivalent*

1. *$n+m$ is odd and $p=\frac{n+m+1}{2}$.*
2. *The coefficients D_j associated to poles on $\mathbb{T}$ are real while those associated to poles symmetric with respect to $\mathbb{T}$ are complex conjugate.*

Proof We denote by $\{t_j\}_{j=1}^s$ the poles on $\mathbb{T}$ and by $\{(\omega_j,\omega_j^*)\}_{j=1}^r$ the symmetric couples not on $\mathbb{T}$. Then,

$$R_{n,p}^m(z)=\sum_{j=1}^s D_j D(t_j,z)+\sum_{j=1}^r D_{j+s}D(\omega_j,z)+\sum_{j=1}^r D_{j+s+r}D(\omega_j^*,z).$$

Note that for $z_j\in\mathbb{C}$: $D_*(z_j,z)=-D(1/\overline{z_j},z)$. Thus

$$-R_{n,p*}^m(z)=\sum_{j=1}^s \overline{D_j}D(t_j,z)+\sum_{j=1}^r \overline{D_{j+s}}D(\omega_j^*,z)+\sum_{j=1}^r \overline{D_{j+s+r}}D(\omega_j,z).$$

Because $R_{n,p}^m=-R_{n,p*}^m$ by the previous theorem and these decompositions are unique, $D_j\in\mathbb{R}$ for $j=1,\dots,s$, and $D_{s+j}=\overline{D_{s+r+j}}$ for $j=1,\dots,r$. □

As a consequence of the analytic properties of Ω_μ, we can be even more precise about the nature of the D_j in the decomposition.

Theorem 11 *Let $n+m$ be odd and $p=\frac{n+m+1}{2}$ and $R_{n,p}^m$ a 2PPA (with $0\le m<n$) to Ω_μ whose denominator is invariant monic and of degree n with only simple zeros. Suppose that in the decomposition of $R_{n,p}^m(z)=\sum_{j=1}^n D_j\frac{x_j+z}{x_j-z}$ there are s' poles on $\mathbb{T}$ and r' poles in $\mathbb{D}$ for which the corresponding D_j has positive real part, then $s'+r'\ge p$.*

Proof Let us recover the notation of the previous theorem and denote by $\{x_j=t_j\}_{j=1}^s$ the poles on $\mathbb{T}$ and by $\{x_{s+j}=\omega_j\}_{j=1}^r$ the poles in $\mathbb{D}$ and let $\{D_j\}_{j=1}^{s'}$ and $\{D_{s+j}\}_{j=1}^{r'}$ be the coefficients with positive real part. We give a proof by contradiction. Assume $\ell=s'+r'<p$ and define $Q(z)=\prod_{j=1}^{s'}(z-t_j)\prod_{j=1}^{r'}(z-\omega_j)$ where $\prod_{j=1}^0(\cdot)=1$. Set $R(z)=\frac{Q(z)Q^*(z)}{z^\ell}$. Then if $R_n=R_{n,p}^m$

$$\begin{aligned}R(z)\left(\Omega_\mu(z)-R_n(z)\right)&=\mathcal{O}\left(z^{p-\ell}\right), && z\to 0,\\ &=\mathcal{O}\left(\frac{1}{z^{n+m+1-p-\ell}}\right), && z\to\infty,\end{aligned}$$

with $p-\ell\ge 1$ and thus also $n+m+1-p-\ell\ge 1$. Note $R(t_i)=0$ for $i=1,\dots,s$ and $R(\omega_j)=R(\omega_j^*)=0$, $j=1,\dots,r$. Now using $R(t)=|Q(t)|^2$ for $t\in\mathbb{T}$ we get

$$R(z)\Omega_\mu(z)=\int_{\mathbb{T}} D(t,z)\left(R(z)-R(t)\right)d\mu(t)+\int_{\mathbb{T}} D(t,z)|Q(t)|^2 d\mu(t)$$

and

$$R(z)R_n(z)=\sum_{j=1}^n D_j D(x_j,z)(R(z)-R(x_j))+\sum_{j=1}^n D_j D(x_j,z)|Q(x_j)|^2.$$

Note that

$$L(z) = \int_{\mathbb{T}} D(t,z)(R(z) - R(t))d\mu(t) - \sum_{j=1}^{n} D_j D(x_j, z)\left(R(z) - R(x_j)\right)$$

belongs to span $\{z^k : -\ell \leq k \leq \ell\}$ and can be rewritten as

$$R(z)\left(\Omega_\mu(z) - R_n(z)\right) - \int_{\mathbb{T}} D(t,z)|Q(t)|^2 d\mu(t) + \sum_{j=1}^{n} D_j D(x_j, z)|Q(x_j)|^2 \in \mathcal{H}(\hat{\mathbb{C}}\backslash\mathbb{T}).$$

Then, since it is analytic both at the origin and at infinity, it must be a constant:

$$R(z)\left(\Omega_\mu(z) - R_n(z)\right) - \int_{\mathbb{T}} D(t,z)|Q(t)|^2 d\mu(t) + \sum_{j=1}^{n} D_j D(x_j, z)|Q(x_j)|^2 = c \in \mathbb{C}.$$

Evaluating this last equality at the origin and infinity, gives $c = 0$ and thus by Theorem 10

$$\begin{aligned} 0 < \int_{\mathbb{T}} |Q(t)|^2 d\mu(t) &= \sum_{j=s'+1}^{s} D_j |Q(t_j)|^2 + \sum_{j=r'+1}^{r} \left[D_{s+j}|Q(\omega_j)|^2 + D_{s+r+j}|Q(\omega_j^*)|^2\right] \\ &= \sum_{j=s'+1}^{r} D_j |Q(t_j)|^2 + \sum_{j=r'+1}^{r} \mathrm{Re}(D_{s+j})\left(|Q(\omega_j)|^2 + |Q(\omega_j^*)|^2\right) \end{aligned}$$

which is a contradiction, since all the D's appearing in the previous line have non-positive real parts. Thus $\ell \geq p$ and the result follows. □

3.2 One prescribed pole

For an inversive approximant the maximal number of poles that can be free is $m = n-1$, leaving a minimum of one pole to be prescribed (recall $1 \leq m < n$). This optimal situation is collected in the following result, which is a direct consequence of Theorem 8.

Corollary 2 *Let $m = n-1$ and $p = n$ and $R_{n,n}^{n-1} = \frac{A_n}{B_n}$ a 2PPA with one prescribed pole at $z_0 \in \mathbb{T}$. Then,*

$$\Omega_\mu(z) - \frac{A_n(z)}{B_n(z)} = 2\frac{z^n(z - z_0)}{B_n^2(z)} \int_{\mathbb{T}} \frac{B_n^2(t)}{(t-z)(t-z_0)} \frac{d\mu(t)}{t^{n-1}},$$

where B_n is paraorthogonal and $B_n(z_0) = 0$.

Theorem 12 *Let μ be a regular measure on $\mathbb{T}$ and $\{R_{n,n}^{n-1}\}_{n\geq 1}$ the sequence of 2PPA for Ω_μ with one prescribed pole z_0. Then the sequence converges uniformly to Ω_μ in compact subsets of $\hat{\mathbb{C}}\backslash\mathbb{T}$. Moreover, an exact geometric rate of convergence is achieved*

$$\lim_{n\to\infty} |\Omega_\mu(z) - R_{n,n}^{n-1}(z)|^{\frac{1}{n}} = |z|, \quad \forall z \in \mathbb{D},$$

and

$$\lim_{n\to\infty} |\Omega_\mu(z) - R_{n,n}^{n-1}|^{\frac{1}{n}} = \frac{1}{|z|}, \quad \forall z \in \mathbb{E},$$

Proof Define $f_z(t) = \frac{2t}{t-z}$ and $K_n(z) = \frac{z^n B_n(0)\|B_n\|^2}{B_n^2(z)}$, and denote $R_n = R_{n,n}^{n-1} = \frac{A_n}{B_n}$, then from the previous Corollary 2

$$\begin{aligned}
\Omega_\mu(z) - R_n(z) &= \frac{2z^n}{B_n^2(z)} \int_{\mathbb{T}} \frac{z - z_0}{(t-z)(t-z_0)} \frac{B_n^2(t)}{t^{n-1}} d\mu(t) \\
&= \frac{z^n}{B_n^2(z)} \left(\int_{\mathbb{T}} f_z(t) \frac{B_n^2(t)}{t^n} d\mu(t) - 2 \int_{\mathbb{T}} \frac{B_n(t)}{t-z_0} \frac{B_n(t)}{t^{n-1}} d\mu(t) \right) \\
&= \frac{z^n B_n(0)}{B_n^2(z)} \left(\int_{\mathbb{T}} f_z(t) |B_n(t)|^2 d\mu(t) - 2\overline{B_n(0)} \langle B_n, 1 \rangle \right) \\
&= K_n(z) \left(\int_{\mathbb{T}} f_z(t) |B_n(t)|^2 \frac{d\mu(t)}{\|B_n\|^2} - 2\overline{B_n(0)} \frac{\langle B_n, 1 \rangle}{\|B_n\|^2} \right).
\end{aligned}$$

On the second line use $\frac{z-z_0}{(t-z)(t-z_0)} = \frac{1}{t-z} - \frac{1}{t-z_0}$ and on the third line we use that $\frac{B_n(t)}{t^n} = B_n(0)\overline{B_n(t)}$ and that $\frac{B_n(t)}{(t-z_0)t^{n-1}} \in \mathbb{P}_{n-1}$ and the orthogonality of B_n. Now we set $d\mu_n(t) = \frac{|B_n(t)|^2}{\|B_n\|^2} d\mu(t)$ so that $\int d\mu_n(t) = 1$.

Recall that $B_n(z) = z\rho_{n-1}(z) + B_n(0)\rho_{n-1}^*(z)$ so that by using orthogonality of the Szegő polynomials we can obtain

$$\langle B_n, 1 \rangle = B_n(0)(1 - \delta_n \overline{B_n(0)}) \|\rho_{n-1}\|^2$$

and

$$\langle B_n, B_n \rangle = \|B_n\|^2 = 2\|\rho_{n-1}\|^2 \left(1 - \mathrm{Re}(\delta_n \overline{B_n(0)})\right)$$

so that

$$2\overline{B_n(0)} \frac{\langle B_n, 1 \rangle}{\langle B_n, B_n \rangle} = \frac{1 - \delta_n \overline{B_n(0)}}{1 - \mathrm{Re}(\delta_n \overline{B_n(0)})}.$$

Since $f_z(t) = 1 + \frac{t+z}{t-z}$, the above expression for $\Omega_\mu(z) - R_n(z)$ gives

$$\frac{\Omega_\mu(z) - R_n(z)}{K_n(z)} = \int_{\mathbb{T}} \frac{t+z}{t-z} d\mu_n(t) + 1 - \frac{1 - \delta_n \overline{B_n(0)}}{1 - \mathrm{Re}(\delta_n \overline{B_n(0)})}. \tag{17}$$

Therefore, taking the real part of the right-hand side and setting $d(z) = \max_{t\in\mathbb{T}} |z - t| = 1 + |z|$, we obtain the following inequality for all $z \notin \mathbb{T}$

$$\begin{aligned}
\left| \frac{\Omega_\mu(z) - R_n(z)}{K_n(z)} \right| &\geq \left| \mathrm{Re} \left(\int_{\mathbb{T}} \frac{t+z}{t-z} d\mu_n(t) \right) \right| = |1 - |z|^2| \int_{\mathbb{T}} \frac{d\mu_n(t)}{|t-z|^2} \\
&\geq \frac{|1 - |z|^2|}{d(z)^2} = \left| \frac{1 - |z|}{1 + |z|} \right|.
\end{aligned}$$

On the other hand, (17) gives for $z \notin \mathbb{T}$

$$\left| \frac{1 - |z|}{1 + |z|} \right|^{\frac{1}{n}} \leq \left| \frac{\Omega_\mu(z) - R_n(z)}{K_n(z)} \right|^{\frac{1}{n}} \leq \left| \left| 1 - \frac{1 - \delta_n \overline{B_n(0)}}{1 - \mathrm{Re}(\delta_n \overline{B_n(0)})} \right| + \max_{t\in\mathbb{T}} \left| \frac{t+z}{t-z} \right| \right|^{\frac{1}{n}},$$

so that by applying the limit

$$\lim_{n\to\infty} \left| \frac{\Omega_\mu(z) - R_n(z)}{K_n(z)} \right|^{\frac{1}{n}} = 1, \quad z \notin \mathbb{T} \tag{18}$$

because

$$\lim_{n\to\infty} \left| \frac{1 - \delta_n \overline{B_n(0)}}{1 - \mathrm{Re}(\delta_n \overline{B_n(0)}} \right|^{\frac{1}{n}} = 1,$$

given that

$$\frac{1-|\delta_n|^2}{4} \leq \frac{1-|\delta_n|^2}{(1+|\delta_n|)^2} \leq \left|\frac{1-\delta_n\overline{B_n(0)}}{1-\mathrm{Re}(\delta_n\overline{B_n(0)}}\right| \leq \frac{(1+|\delta_n|)^2}{1-|\delta_n|^2} \leq \frac{4}{1-|\delta_n|^2},$$

and $1-|\delta_n|^2 = \frac{\|\rho_n\|^2}{\|\rho_{n-1}\|^2}$.

Furthermore, note also that

$$\lim_{n\to\infty}|K_n(z)|^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{|z|^n}{\|B_n\|^2|B_n(z)|^2}\right)^{\frac{1}{n}} = \min\left(|z|, \frac{1}{|z|}\right), \quad z \notin \mathbb{T}. \tag{19}$$

The combination of (19) and (18) concludes the proof. □

Recall that the nth root convergence for 2PTA behaved like $\sqrt{|z|}$ in $\mathbb{D}$ (Corollary 1) while, as we just proved, it behaves like $|z|$ for 2PPA. This extends the result in Padé Approximation for Markov functions, see [23].

Remark 4 Note that z_0 does not have to be fixed, it can vary with n.
In [17], the authors found another estimate for the error that leads to the same order of convergence but with an inequality.

Corollary 3 *Let μ a regular measure on $\mathbb{T}$ and $\{R_{n,n}^{n-1}\}_{n\geq 1}$ be the sequence of 2PPA to Ω_μ with one prescribed pole on $\mathbb{T}$. Then the sequence $\left\{1/R_{n,n}^{n-1}\right\}_{n\geq 1}$ converges uniformly to $1/\Omega_\mu$ in compact subsets of $\hat{\mathbb{C}}\backslash\mathbb{T}$. Moreover, an exact geometric rate of convergence is achieved:*

$$\lim_{n\to\infty}\left|\frac{1}{\Omega_\mu(z)} - \frac{1}{R_{n,n}^{n-1}(z)}\right|^{\frac{1}{n}} = |z|, \quad \forall z \in \mathbb{D},$$

and

$$\lim_{n\to\infty}\left|\frac{1}{\Omega_\mu(z)} - \frac{1}{R_{n,n}^{n-1}(z)}\right|^{\frac{1}{n}} = \frac{1}{|z|}, \quad \forall z \in \mathbb{E}.$$

Proof Note that all the coefficients $D_{k,n}$, $k = 1, \ldots, n$ of

$$R_{n,n}^{n-1}(z) = \sum_{k=1}^{n} D_{k,n}D(x_{k,n}, z),$$

are positive by Theorem 11 because $p = n$ and all the zeros of B_n are simple and on $\mathbb{T}$ because it is monic paraorthogonal polynomial associated to μ. Then the zeros of $R_{n,n}^{n-1}$ are also simple and on $\mathbb{T}$ by Lemma 3. Moreover $\{R_{n,n}^{n-1}\}_{n\geq 1}$ is a sequence of nonzero functions in $\mathcal{H}(\hat{\mathbb{C}}\backslash\mathbb{T})$ uniformly bounded from below on compact subsets of $\hat{\mathbb{C}}\backslash\mathbb{T}$. Therefore, $\{\Omega_\mu R_{n,n}^{n-1}\}_{n\geq 1}$ is a sequence of nonzero analytic functions and uniformly bounded from below too in compacts of $\hat{\mathbb{C}}\backslash\mathbb{T}$. Note that the numerator of $R_{n,n}^{n-1}$ is a paraorthogonal polynomial for the measure $\tilde{\mu}$ defined by $\Omega_{\tilde{\mu}} = 1/\Omega_\mu$.

□

3.3 More prescribed poles

Given this optimal answer for case $m = n - 1$, one can ask if a similar result holds for some $1 \leq m \leq n-2$ and if possible with $n+m$ odd. Concretely, will there exist a quasi-paraorthogonal polynomial of degree n with an appropriate order that has all its simple zeros on $\mathbb{T}$? Moreover, if such a polynomial is used as the denominator in a 2PPA, will then all the $D_{k,n}$ of its decomposition be positive? Since $d = n - m$ of its zeros can be fixed in advance on $\mathbb{T}$, the answer will obviously depend on how these are chosen.

We provided in [18] a sufficient condition for the existence of quasi-paraorthogonal polynomials, supplemented with Schur-Cohn's algorithms to check if in addition the D_k are positive. Take into account that it is possible that the quasi-paraorthogonal polynomial coincides with the paraorthogonal polynomial of an auxiliary finite positive Borel measure $\tilde{\mu}$ whose moments up to the order $2n-d$ are equal to those of the measure μ. Therefore the numerator of the 2PPA associated to Ω_μ coincides with the denominator of the 2PPA to $\Omega_{\tilde{\mu}}$ in the case $m = n - 1$ and thus, for this situation, the coefficients D_k are positive.

3.3.1 Special case: Stieltjes polynomials

The indicated procedure in [18] is very appropriate if the number of fixed poles is low, since the verification of the existence of the quasi-paraorthogonal polynomial is equivalent to a problem of interpolating unimodular values by Blaschke products, which is a hard problem. When the number of prescribed zeros in a quasi-paraorthogonal polynomial is comparable to the number of orthogonality conditions, we must look for a more practical criterion that gives us invariant polynomials, if possible with simple zeros, by properly choosing the fixed zeros. This analysis leads us to the concept of Stieltjes polynomials on the unit circle. Concretely, we consider the question: does there exist a quasi-paraorthogonal polynomial B_n of degree n with prescribed zeros at the zeros of a paraorthogonal polynomial? That is, if we factor B_n as $B_n = S_m W_d$ with W_d the paraorthogonal polynomial of degree d containing the prescribed zeros, can we find invariant polynomials S_m of degree $m = n - d$, satisfying appropriate orthogonality conditions such that B_n is a quasi-paraorthogonal polynomial? Note that if it exists, we can say that B_n has at least d zeros on $\mathbb{T}$. As we know from [18], the answer depends on the parity of d. For $d = 2\ell+1$ we need $m = n - d$ orthogonality conditions

$$\left\langle S_m W_d, z^k \right\rangle = 0, \quad k = \ell + 1, \ldots, n - (\ell + 1),$$

and if $d = 2\ell$, we need to fix $B_n(0) = \omega \in \mathbb{T}$, which gives again a total of $m = n - d$ conditions. The paraorthogonal polynomial W_d of degree d, is a quasi-paraorthogonal polynomial of order $\ell = 0$, and thus it is orthogonal to $\operatorname{span}\{z, \ldots, z^{d-1}, z^d - \tau\}$ for some $\tau \in \mathbb{T}$, and clearly both S_m and W_d depend on this τ. If B_n with the previous properties exists, we know that B_n has $m+1$ zeros of odd multiplicity on $\mathbb{T}$, and we also know that W_d has $d = n - m$

simple zeros on $\mathbb{T}$. Will it be possible to choose τ and an appropriate ω in the case that d is even in such a way that B_n has all its zeros simple and on $\mathbb{T}$? Or, failing that, can we choose it in such a way that its zeros that are not on $\mathbb{T}$ approach $\mathbb{T}$ as the degree of the polynomials goes to infinity? A positive answer can be given in the case $m = d$ when $d \in \{2\ell, 2\ell+1\}$ and in the case $m = d+1$ when $d = 2\ell$. This will be the subject of this subsection.

Remark 5 Let us recall that on the real line the situation is as follows. If q_{2n+1} is a quasi-orthogonal polynomial of degree $2n+1$ and order n with n zeros fixed at the zeros of the nth orthogonal polynomial p_n, then the Stieltjes polynomial S_{n+1} of degree $n+1$ satisfies, $q_{2n+1} = S_{n+1}p_n$. We have argued in [18], that as far as the zeros are concerned, the analogues to the (quasi-)orthogonal polynomial on the real line are the (quasi-)paraorthogonal polynomials on the unit circle. With this observation in mind, it is reasonable to refer to the polynomials S_m introduced above as *Stieltjes polynomials on the unit circle* if $m = d+1$ but also if $m = d$ (as was also done in [21]).

Theorem 13 (Stieltjes polynomials) *Let μ be a finite positive Borel measure on $\mathbb{T}$ and consider $n = m + d$, $m, d \in \mathbb{N}$. Then there exists a unique invariant monic polynomial S_m such that $B_n = S_m W_d$ is a $(n, 2\ell+1)$-quasi-paraorthogonal polynomial with prescribed zeros at the zeros of the monic paraorthogonal polynomial W_d of degree d associated to μ if either $m = d$ and $d \in \{2\ell, 2\ell+1\}$ or $m = d+1$ and $d = 2\ell$.*

Proof The existence of a quasi-paraorthogonal polynomial (QPOP) is essentially guaranteed for all n, as a consequence of [21, Lemma 1]. However, we believe it is appropriate to spell it out for clarity since we start from a different point of view. Below we shall consider 3 situations differentiating between parity combinations for d and m. In each case the intention is to show that there exists a unique invariant S_m monic and of degree m, such that $B_n = S_m W_d$ is a uniquely defined (n, d)-QPOP that has to satisfy for $\ell = \lfloor \frac{d}{2} \rfloor$ the system of $n - 2\ell - 1$ equations

$$\langle S_m W_d, z^k \rangle = 0, \quad k = \ell+1, \ldots, n-(\ell+1).$$

Using $W_d(t) = W_d(0) t^d \overline{W_d(t)}$ for $t \in \mathbb{T}$ and a change of index

$$\left\langle \frac{S_m z^j}{z^{m-\ell-1}}, W_d \right\rangle = 0, \quad j = 0, \ldots, n-2(\ell+1). \tag{20}$$

If d is odd, then these are just enough equations to define S_m uniquely (case 1 below) but if d is even, there is one equation short and an extra one needs to be added (case 2 and 3 below). This results in a homogeneous linear system of m equations in the $m+1$ unknown coefficients of S_m so that a non-zero solution for the Stieltjes polynomial S_m of degree at most m is guaranteed to exist. It will be uniquely defined if we choose the monic one of minimal degree. We shall prove by contradiction that $\deg(S_m) = m$. The last step is to prove that S_m is invariant. This is worked out in the different cases below.

It is useful to recall that the monic paraorthogonal polynomial W_d is by definition orthogonal to span $\{z, z^2, \ldots, z^{d-1}, z^d - \tau\}$ if and only if $W_d = c^{-1}(\rho_d + \tau \rho_d^*)$, $c = 1 + \tau\overline{\delta_d}$. Here the parameter $\tau \in \mathbb{T}$ that is fixed by the prescribed zeros of W_d.

1. We first consider the case $d = 2\ell + 1$ and $m = d$. The orthogonality conditions (20) are

$$\left\langle \frac{S_m z^j}{z^\ell}, W_d \right\rangle = 0, \quad j = 0, \ldots, m-1. \tag{21}$$

A monic Stieltjes polynomial S_m of degree at most m will always exist. We show that the minimum degree is m, and that S_m is invariant. Clearly $s = \deg(S_m) \leq m$. To show that $\deg(S_m) = m$, we give a proof by contradiction. So assume that S_m is monic and of degree $s < m$. First, we assume that $\ell < s < m$. Then $\ell + 1 \leq m - s + \ell \leq m - 1$, and we choose $j = \ell + m - s$ in (21) which gives (recall $m = d$)

$$0 = \left\langle S_m z^{m-s}, W_m \right\rangle = \langle z^m, W_m \rangle ,$$

which is absurd since $\langle W_m, z^m \rangle \neq 0$ because W_m is a paraorthogonal polynomial of degree m. Next, suppose that $s \leq \ell$ then there exists some integer k such that $S_m(z) = s_k z^k + \cdots + z^s$ with $s_k \neq 0$, for some $0 \leq k \leq s$. Now choose $j = \ell - k$ in (21), then noting that $0 \leq j \leq \ell$,

$$0 = \left\langle S_m z^{-k}, W_m \right\rangle = s_k \langle 1, W_m \rangle,$$

we conclude that $\langle W_m, 1 \rangle = 0$, which is absurd for the same reason as before. Therefore, S_m has degree m. For B_n to be invariant, it is necessary that $S_m(0) \in \mathbb{T}$. This is true since if we choose $j = \ell$ in (21),

$$0 = \langle S_m, W_m \rangle = \langle (z^m + S_m(0)), W_m \rangle,$$

so that $S_m(0) = -\tau$, since W_m is orthogonal to span $\{z, z^2, \ldots, z^{m-1}, z^m - \tau\}$. Furthermore, it is easy to verify that $S_m^* W_m$ satisfies the same orthogonality conditions as $S_m W_m$, that is S_m^* safisties the same system as S_m, by uniqueness, S_m is invariant. We conclude that $B_n = S_m W_d$ is invariant. It will be unique since $\tau \in \mathbb{T}$ is fixed by the prescribed zeros in W_d.

2. Now consider $d = 2\ell$ and $m = d = 2\ell$. Because d is even, the orthogonality conditions for B_n are not sufficient to define S_m uniquely and we introduce as an extra condition $s_0 = \mathrm{i}\sqrt{\tau} s_\ell$, where $S_m(t) = s_0 + s_1 t + \cdots + s_m t^m$ and τ is the (fixed) parameter of W_d. This completes the set of orthogonality conditions that in this case is

$$\left\langle \frac{S_m z^j}{z^{\ell-1}}, W_d \right\rangle = 0, \quad j = 0, \ldots, m-2 \tag{22}$$

with a nontrivial solution S_m. We show by contradiction that $s = \deg(S_m) \leq m$ can not be less than m by considering three cases: $\ell < s < m$, $s = \ell$ and $s \leq \ell - 1$. If $\ell < s < m$, then $\ell \leq m - (s - (\ell - 1)) \leq m - 2$, so

that for $j = m - (s - (\ell - 1))$ in (22),

$$0 = \left\langle S_m z^{m-s}, W_d \right\rangle = \langle z^m, W_d \rangle$$

with $m = d$ which is impossible because $\langle z^d, W_d \rangle \neq 0$.
If $s = \ell$, then $S_m(t) = \mathrm{i}\sqrt{\tau} + \cdots + t^\ell$ because S_m is monic, and $s_0 = \mathrm{i}\sqrt{\tau} s_\ell$. Now choose $j = \ell - 1$ in (22) to get

$$0 = \langle S_m, W_d \rangle = \mathrm{i}\sqrt{\tau} \langle 1, W_d \rangle \neq 0,$$

again a contradiction since W_d is paraorthogonal.
Finally consider $s \leq \ell - 1$, then there exists some integer k such that $S_m(z) = s_k z^k + \cdots + z^s$ with $s_k \neq 0$, for some $1 \leq k \leq s$ since $s_0 = 0$, because $s_0 = \mathrm{i}\sqrt{\tau} s_\ell$. Now choose $j = \ell - 1 - k$ in (22), then noting that $0 \leq j \leq \ell - 2$, (recall $d = m$)

$$0 = \left\langle S_m z^{-k}, W_m \right\rangle = s_k \langle 1, W_m \rangle,$$

we conclude that $\langle W_m, 1 \rangle = 0$, which is absurd for the same reason as before. Therefore, S_m has degree m. It remains to show that S_m is invariant. First, choosing $j = \ell - 1$ in (22) gives

$$0 = \langle S_m, W_m \rangle = \langle (z^m + S_m(0)), W_m \rangle,$$

so that $S_m(0) = -\tau$ by exactly the same argument as in case 1. Secondly, as $s_0 = \mathrm{i}\sqrt{\tau} s_\ell$, we deduce $s_\ell = -\tau \overline{s_\ell}$. We conclude that S_m is invariant because $S_m(0) S_m^*$ verifies the same orthogonality conditions and the additional equation $s_\ell = -\tau \overline{s_\ell}$. Therefore $S_m = S_m(0) S_m^*$ and thus $B_n = S_m W_d$ is a monic quasi-paraorthogonal polynomial of de degree n and order d and it is uniquely defined.

3. We finally consider $d = 2\ell$ and $m = d + 1$. The parity of d, forces us again to introduce and extra condition $s_0 = \omega s_m$, $\omega \in \mathbb{T}$ in addition to the orthogonality conditions so that in this case S_m is a solution of

$$\begin{aligned} \left\langle \frac{S_m z^j}{z^\ell}, W_d \right\rangle &= 0, \quad j = 0, \ldots, m-2 \\ s_0 &= \omega s_m \end{aligned} \tag{23}$$

whose solution is a non-zero Stieltjes polynomial S_m. We show that its degree is m and that it is invariant. Since $s = \deg(S_m) \leq m$, we prove like in the previous cases by contradiction that $\deg(S_m) = m$. We consider two cases: $\ell + 1 \leq s < m$ and $s \leq \ell$.

If $\ell + 1 \leq s < m$, then $\ell \leq d - (s - \ell)) \leq m - 2$, so that for $j = d - (s - \ell))$ in (23),

$$0 = \left\langle S_m z^{d-s}, W_d \right\rangle = \langle z^d, W_d \rangle.$$

This follows from the paraorthogonality of W_d if $d - s \geq 1$ and $s_0 = 0$ for $s = d$ because $s_0 = \omega s_m$. Since $\langle z^d, W_d \rangle \neq 0$ this is a contradiction.
If $s \leq \ell$, then there exists some integer k such that $S_m(z) = s_k z^k + \cdots + z^s$ with $s_k \neq 0$, for some $1 \leq k \leq s$ since $s_0 = 0$ because $s_0 = \omega s_m$ and $s_m = 0$. Now choose $j = \ell - k$ in (23), then noting that $0 \leq j \leq \ell - 1$,

$$0 = \left\langle S_m z^{-k}, W_d \right\rangle = s_k \langle 1, W_d \rangle,$$

we conclude that $\langle W_d, 1 \rangle = 0$, which is absurd for the same reason as before. Therefore, S_m is monic, of degree m and $S_m(0) = \omega \in \mathbb{T}$. To prove that B_n is invariant note that

$$\begin{aligned} \left\langle \frac{S_m^* z^j}{z^\ell}, W_d \right\rangle &= 0, \quad j = 0, \ldots, m-2 \\ \overline{s_m} &= \omega \overline{s_0}, \end{aligned}$$

is the same system whose solution was S_m, so by uniqueness, $S_m = S_m(0) S_m^*$. Thus B_n is uniquely determined by fixing ω or equivalently $B_n(0)$ since $B_n(0) = \omega W_d(0)$.

□

Remark 6 Note that B_n has at least $m + 1$ different zeros of odd multiplicity on $\mathbb{T}$ (see [18, Theorem 3.2]). That may be zeros of S_m or zeros of W_d and zeros of S_m may even coincide with zeros of W_d. There can also be zeros of B_n that occur in symmetric pairs with respect to $\mathbb{T}$. We can use the Schur-Cohn algorithm, see [18], to know how many simple zeros there really are on $\mathbb{T}$, depending on how long it takes until a Schur parameter with modulus greater than or equal 1 occurs.

Theorem 14 *Let μ a regular measure on $\mathbb{T}$ and let $n = m_n + d_n$, $m_n, d_n \in \mathbb{N}$ and either $m_n = d_n$ or $m_n = d_n + 1$ and d_n even. Let $\{R_{n,p_n}^{m_n}\}_{n \geq 1}$ be the sequence of 2PPA with prescribed poles that are the zeros of W_{d_n}, being a paraorthogonal polynomial of degree d_n with respect to μ with $p_n = \lfloor \frac{n+m_n+1}{2} \rfloor$, fixing $\omega \in \mathbb{T}$ when $m_n = d_n + 1$ and d_n even. Then $R_{n,p_m}^{m_n}$ converges uniformly to Ω_μ in compact subsets of $\hat{\mathbb{C}} \backslash \mathbb{T}$. Moreover, a geometric rate of convergence is achieved:*

$$\limsup_{n \to \infty} |\Omega_\mu(z) - R_{n,p_n}^{m_n}(z)|^{\frac{1}{n}} \leq \sqrt[4]{|z|^3}, \quad \forall z \in \mathbb{D},$$

and

$$\limsup_{n \to \infty} |\Omega_\mu(z) - R_{n,p_n}^{m_n}(z)|^{\frac{1}{n}} \leq \frac{1}{\sqrt[4]{|z|^3}}, \quad \forall z \in \mathbb{E}.$$

Proof The existence of quasi-paraorthogonal denominator of $R_{n,p_n}^{m_n}$, is guaranteed for all n by the previous Theorem. The convergence of the case $d_n = m_n$ is obtained from [21, Theorem 5]. In the other case, the result remains true after some slight adaptations. □

Remark 7 Note that in the previous Theorem $R_{n,p_n}^{m_n}$ is inversive only when d_n is odd. Furthermore, since Ω_μ has no zeros in $\mathbb{C}\backslash\mathbb{T}$, convergence implies that both the zeros and poles of $R_{n,p_n}^{m_n}$ must accumulate on $\mathbb{T}$. Therefore, an analogue result is also available for $\Omega_{\tilde{\mu}} = 1/\Omega_\mu$.

3.3.2 General case

What can be said when $m_n \notin \{d_n, d_n + 1\}$? We know that if $R_{n,p}^m$ interpolates to Ω_μ both at least once at the origin and at infinity, $R_{n,p}^m$ admit this decomposition

$$R_{n,p}^m(z) = \sum_{k=1}^{n} D_{k,n} D(x_{k,n}, z).$$

Can we preselect poles among $\{x_{k,n}\}_{k=1}^n$ such that $R_{n,p}^m$ has only simple poles, all on $\mathbb{T}$ and such that all coefficients $D_{k,n}$ are positive, or failing that, can we guarantee that $\sum_{k=1}^n |D_{k,n}| \leq M$, $\forall n \in \mathbb{N}$? If the answer is yes, we have convergence as the following result indicates.

Theorem 15 *Let $\{m_n\}_{n\in\mathbb{N}}$ be a sequence of positive integers such that $0 \leq m_n \leq n$ such that $\lim_{n\to\infty} \frac{m_n}{n} = \theta \in [0,1]$ and $\lim_{n\to\infty} \frac{p_n}{n+m_n} = r \in [0,1]$. Let $\{R_{n,p_n}^{m_n}\}_{n\in\mathbb{N}}$ be the sequence of 2PPA associated to Ω_μ with all its poles simple and on $\mathbb{T}$ and assume that $\sum_{k=1}^n |D_{k,n}|$ is uniformly bounded. Then, the sequence $\{R_{n,p_n}^{m_n}\}_{n\in\mathbb{N}}$ converges uniformly to Ω_μ in compact subsets of $\hat{\mathbb{C}}\backslash\mathbb{T}$. Moreover, a geometric rate of convergence is achieved,*

$$\limsup_{n\to\infty} |\Omega_\mu(z) - R_{m_n,p_n}^{m_n}(z)|^{\frac{1}{n}} \leq |z|^{(1+\theta)r}, \quad \forall z \in \mathbb{D},$$

and

$$\limsup_{n\to\infty} |\Omega_\mu(z) - R_{n,p_n}^{m_n}(z)|^{\frac{1}{n}} \leq \frac{1}{|z|^{(1+\theta)(1-r)}}, \quad \forall z \in \mathbb{E}.$$

Proof As $\{R_{n,p_n}^{m_n}\}_{n\in\mathbb{N}}$ is a sequence of analytic functions, uniformly bounded on compact subsets of $\hat{\mathbb{C}}\backslash\mathbb{T}$, to ensure convergence, it is enough to show that all convergent subsequences have the same limit. That is immediate, by the interpolation conditions that $R_{n,p_n}^{m_n}$ satisfies, since the coefficients of the expansion at the origin and at infinity of all limit functions coincide with the corresponding expansion coefficients of Ω_μ. On the other hand, from (4)-(5) with $p = p_n$ and $q = p_n - 1 - m_n$, we obtain

$$\begin{aligned} \Omega_\mu(z) - R_{n,p_n}^{m_n}(z) &= \mathcal{O}(z^{p_n}), \qquad z \to 0, \\ \Omega_\mu(z) - R_{n,p_n}^{m_n}(z) &= \mathcal{O}(z^{p_n - m_n - 1 - n}), \quad z \to \infty. \end{aligned}$$

We consider the disk $\mathbb{D}_\rho = \{z \in \mathbb{C} : |z| < \rho\}$ with $\rho < 1$. Since $\left\{\frac{\Omega_\mu - R_{n,p_n}^{m_n}}{z^{p_n}}\right\}_{n\in\mathbb{N}}$ is a sequence of functions analytic in $\mathbb{D}$ and the sequence $\left\{\Omega_\mu - R_{n,p_n}^{m_n}\right\}_{n\in\mathbb{N}}$ is uniformly bounded on compact subsets of $\mathbb{C}\backslash\mathbb{T}$ it follows that for $\rho < 1$:

$$\frac{1}{|z|^{p_n}} \left|\Omega_\mu(z) - R_{n,p_n}^{m_n}(z)\right| \leq \max_{z\in\mathbb{D}_\rho} \frac{1}{|z|^{p_n}} \left|\Omega_\mu(z) - R_{n,p_n}^{m_n}(z)\right| = \frac{M_\rho}{\rho^{p_n}}, \quad \forall z \in \mathbb{D}_\rho.$$

Therefore

$$\limsup_{n\to\infty} \frac{1}{|z|^{\frac{p_n}{n}}} \left|\Omega_\mu(z) - R^{m_n}_{n,p_n}(z)\right|^{\frac{1}{n}} \leq \frac{1}{\rho^{(1+\theta)r}}, \quad \forall \rho < 1,$$

and we conclude

$$\limsup_{n\to\infty} \frac{1}{|z|^{\frac{p_n}{n}}} \left|\Omega_\mu(z) - R^{m_n}_{n,p_n}(z)\right|^{\frac{1}{n}} \leq 1.$$

For $z \in \mathbb{E}$, an analogous proof holds considering that the functions in the sequence $\{z^{n+m_n+1-p_n}\left(\Omega_\mu - R^{m_n}_{n,p_n}\right)\}_{n\in\mathbb{N}}$ are analytic in $\mathbb{E}$ and we and we we restrict ourselves to the exterior of a disk $\mathbb{D}_R$ with $R > 1$. □

This last result commits us to write another paper in which we should characterize how the prescribed poles must be chosen in order to achieve convergence in a general context as we have done when they have been fixed to be the zeros of the paraorthogonal polynomial associated with the corresponding measure.

4 Numerical examples

We consider the same positive measure of our previous paper [18]. Its derivative is given by the weight function

$$w(\theta) = \frac{1}{\sqrt{2\pi \log(1/q)}} \sum_{j=-\infty}^{\infty} \exp\left(-\frac{(\theta - 2\pi j)^2}{2\log(1/q)}\right), \quad 0 < q < 1.$$

Its moments are given by $\mu_k = q^{\frac{k^2}{2}}$, $k \in \mathbb{Z}$. The orthogonal polynomials associated with it are usually called the Rogers-Szegő polynomials. In our examples we took $q = 1/2$. The absolutely continuous part of the measure is given by the weight $w(t) = \mathrm{Re}(\Omega_\mu(t))$, $t \in \mathbb{T}$. All computations were performed with Matlab in double precision.

In a first example we compute $R^m_{n,p}$ with $d = 7$ poles prescribed on $\mathbb{T}$. These poles are $\alpha_k = e^{\pi t_k}$ with $\{t_k\}_{k=1}^7 = \{\pm 1/4, \pm 1/2, \pm 3/4, 1/10\}$. For $n = 14$ they are indicated on Figure 1.B by a star. The remaining $m = 14 - 7 = 7$ poles, indicated as empty red circles, are chosen to maximize the 2PPA interpolation conditions for the balanced situation where $p = \frac{n+m+1}{2} = 11$, so that we consider an approximant that we denoted as $R^7_{14,11}$. The weight $w(t) = \mathrm{Re}(\Omega_\mu(t))$, $t \in \mathbb{T}$ is shown in Figure 1.A as a black line. Since the approximation $\mathrm{Re}(R^7_{14,11}(t))$, has all poles on $\mathbb{T}$ we illustrate the approximation in the same graphic but on a smaller circle $t = 0.9e^{\mathrm{i}\theta}$, $\theta \in [0, 2\pi]$ with $\mathrm{Re}(\Omega_\mu(t))$ (red line) and $\mathrm{Re}(R^7_{14,11}(t))$ (blue line). Note that this corresponds to case 1 of Theorem 13. The relative error $(\Omega_\mu - R^7_{14,11})/\Omega_\mu$ is plotted for $z = 0.9e^{\mathrm{i}\theta}$, $\theta \in [0, 2\pi]$ in Figure 1.D with the real part (red), the imaginary part (blue), and the absolute value (black). Minus the 10-logarithm of the absolute value everywhere in the disk $\mathbb{D}$ can be found in Figure 1.C. This means that it represents an approximation for the number of correct digits.

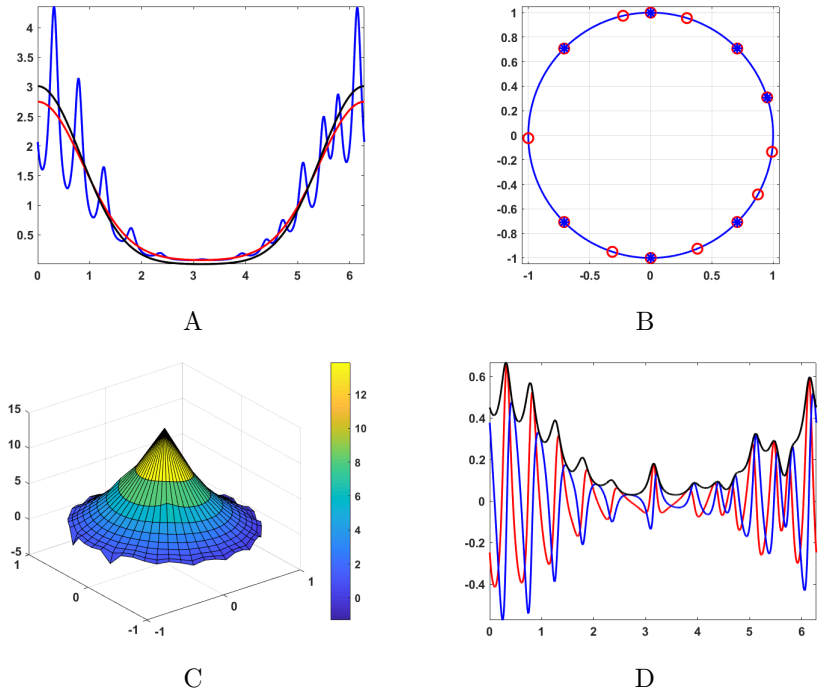
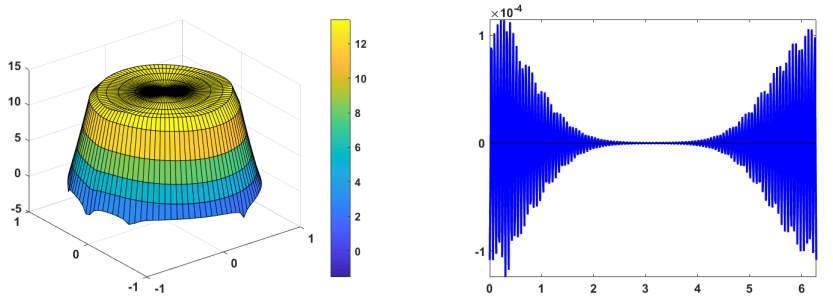


Fig. 1 Top: A: The weight $w(t) = \mathrm{Re}(\Omega_\mu(t))$, $t = e^{\mathrm{i}\theta}$, $\theta \in [0, 2\pi]$ (black), $\mathrm{Re}(\Omega_\mu(re^{\mathrm{i}\theta}))$ (red) and the approximant $\mathrm{Re}(R^7_{14,11}(re^{\mathrm{i}\theta}))$ (blue) for $r = 0.9$ and $\theta \in [0, 2\pi]$. The poles of $R^7_{14,11}$ are shown on the right. The preselected ones are indicated with a star.
Bottom: C: Minus the 10-logarithm of the absolute value of the relative error in $\mathbb{D}$. The color bar gives approximately the number of correct digits. D: This is the real part (red), imaginary part (blue), and absolute value (black) of the absolute value of the relative error on the circle $t = 0.9e^{\mathrm{i}\theta}$, $\theta \in [0, 2\pi]$.

The error is smallest near the origin. This is obvious since we matched the Taylor expansion at the origin (and infinity) as much as possible. However, as we approach the circle $\mathbb{T}$, the error becomes larger. Also that is easily explained because the approximant has its poles and zeros on $\mathbb{T}$, while the limit function Ω_μ, has a jump discontinuity on $\mathbb{T}$. As n becomes large, the approximation in $\mathbb{D}$ becomes better in the disk $\mathbb{D}$ (convergence in compact subsets of $\mathbb{D}$) but the error remains problematic close to $\mathbb{T}$.

Figure 2 shows the situation for $n = 104$. Comparing Figure 2.A with Figure 1.C it is clear that the error is small in a larger part of the disk $\mathbb{D}$, but it is still large near the boundary $\mathbb{T}$. A plot like Figure 1.A would show no visual difference between the $\mathrm{Re}(\Omega_\mu(z))$ and its approximant $\mathrm{Re}(R^{97}_{104,101}(z))$ on a circle $|z| = 0.9$, so we plotted instead the error $|\mathrm{Re}(\Omega_\mu(e^{\mathrm{i}\theta})) - \mathrm{Re}(R^{97}_{104,101}(e^{\mathrm{i}\theta}))|$ for $\theta \in [0, 2\pi]$ in Figure 2. showing that there are approximately 4 correct digits.

As we know, the 2PPA, despite being inversive, can have poles outside of $\mathbb{T}$. For example when $n = 20$, $R^{13}_{20,17}$ has 4 poles outside of $\mathbb{T}$, two of them



A B

Fig. 2 In A, we see the error $-\log_{10}|\Omega_\mu(z) - R^{97}_{104,101}(z)|$ just like in Figure 1.C but now with $n = 104$. In B, we see the error $\mathrm{Re}(\Omega_\mu(z)) - \mathrm{Re}(R^{97}_{104,101}(z))$ for $z = 0.9e^{i\theta}$ and $\theta \in [0, 2\pi]$.

in $\mathbb{D}$, as we know because it is inversive. We know that at least 17 of the 18 different coefficients $D_{k,20}$ have a positive real part, however in this example, they all have a positive real part so that the approximation is still comparable to the case where $n = 16$. This situation is shown in Figure 3 with the same kind of graphics as in Figure 1, but as can be seen in Figure 3.B, two poles are inside $\mathbb{D}$ (the symmetric ones in $\mathbb{E}$ are not shown). In this case, the further the poles are from the circle, the smaller the D_j are.

In figure 4 we plot the 10-logarithm of the mean value over $\mathbb{D}$ of $|\Omega_\mu(z) - R^m_{n,p}(z)|$ as a function of $n = 14, 15, \ldots, 110$ for $|z| = r$ where $r = 0.95, 0.85, 0.7, 0.5$. Throughout this process we keep always the same 7 prescribed poles we had before. The root convergence is clear from the linear decrease of the error curves, i.e., $|E_n|^{1/n} \sim r^\alpha$ with $\alpha > 0$ or $\log_{10}(|E_n|) \sim n\alpha \log_{10}(r)$ as $n \to \infty$. Note however that this behaviour of the error is not proved in this paper, but it predicts what can be expected when we will consider this more general convergence in a subsequent paper.

In general, we know that those poles x_k on $\mathbb{T}$ with their corresponding D_k being positive guarantee a zero in their neighborhood. Numerical evidence illustrates that this remains true for poles outside $\mathbb{T}$ if the real part of their corresponding D_k is positive, so that we still have a good approximation. What is true for poles and zeros of the approximant $R^m_{n,p}$ for Ω_μ of course also holds for the approximant $1/R^m_{n,p}$ for $1/\Omega_\mu$ with an interchange of poles and zeros. To illustrate this we have plotted in Figure 5.B for $z = 0.999e^{i\theta}$ and $\theta \in [0, 2\pi]$ the approximant $\mathrm{Re}(R^{13}_{20,17}(z))$ and $\mathrm{Re}(\Omega_\mu(z))$ on top and $\mathrm{Re}(1/R^{13}_{20,17}(z))$ and $\mathrm{Re}(1/\Omega_\mu(z))$ at the bottom. The nearby zeros and poles on $\mathbb{T}$ can be clearly observed where the spikes occur. Note also in the bottom figure the weight $\tilde{w}$ associated with $\Omega_{\tilde{\mu}} = 1/\Omega_\mu$. It is a narrow spike near $z = -1$. In Figure 5.A we copied Figure 3.B but we removed the indication of the prescribed poles, but plotted all the poles (red o) and zeros (blue x) to show the interlacing of the zeros on $\mathbb{T}$ and the how the poles not on $\mathbb{T}$ are accompanied with a corresponding zero. These deliver the complex conjugate

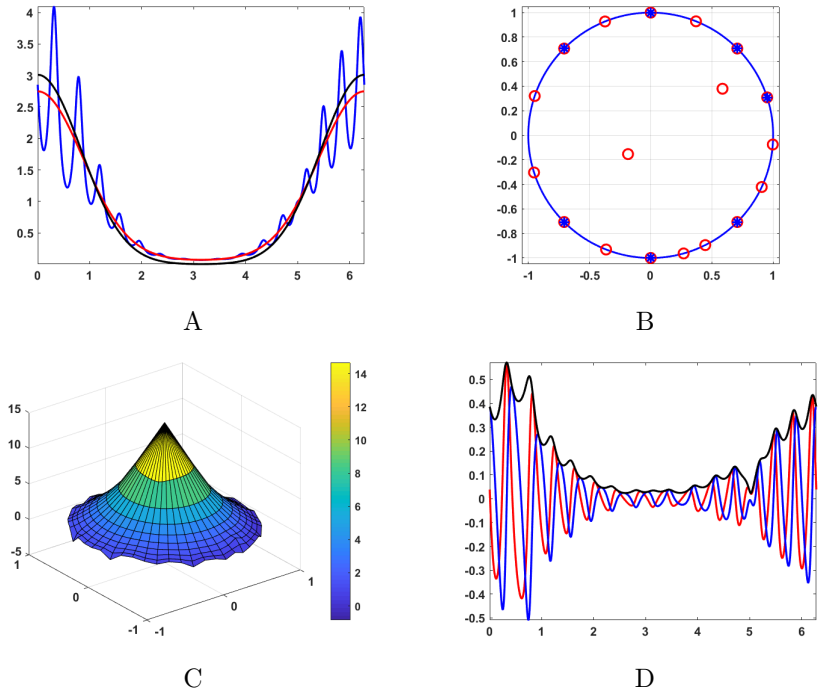


Fig. 3 The same example as in Figure 1 but now with $n = 20$, thus for $R^{13}_{20,17}$. Four of the poles moved away from $\mathbb{T}$. In B, only the two inside $\mathbb{D}$ and one in $\mathbb{E}$ but close to $\mathbb{T}$ are shown.

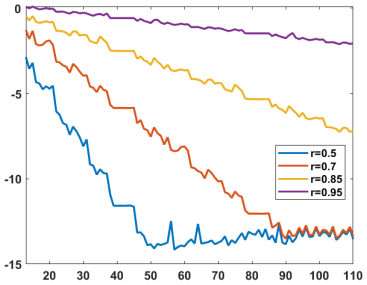
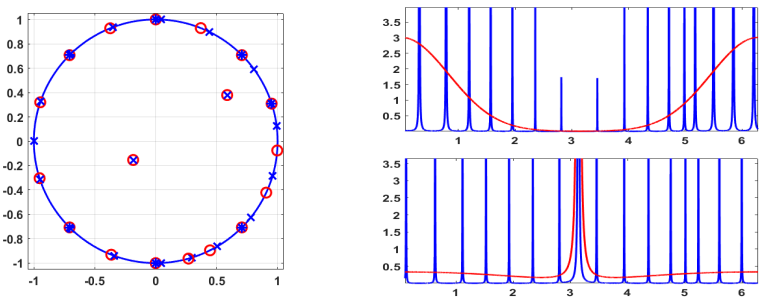


Fig. 4 This represents the 10-logarithm of the mean value of $|\Omega_\mu(z) - R^m_{n,p}(z)|$, $m = n-7$ as a function of $n = 14, 15, \ldots, 110$ for $|z| = r$ where $r = 0.95, 0.85, 0.7, 0.5$. $r = 0.95$ (in that order top to bottom). Full accuracy is obtained for $r = 0.5$ when $n \approx 48$, while for $r = 0.95$ the accuracy is only about one correct digit for $n = 110$.

D_j. The nearer pole/zero couples are, the smaller are the corresponding D_j. For example the two poles that are furthest away from $\mathbb{T}$ almost cancel with nearby zeros, which corresponds to an almost vanishing $D_j = -\frac{A_k(x_j)}{2x_j B'_n(x_j)}$.

As a second example we consider $R^m_{n,p}$ where $d = 6$ poles are prescribed for the same Rogers-Szegő example. We took another set of prescribed poles



A B

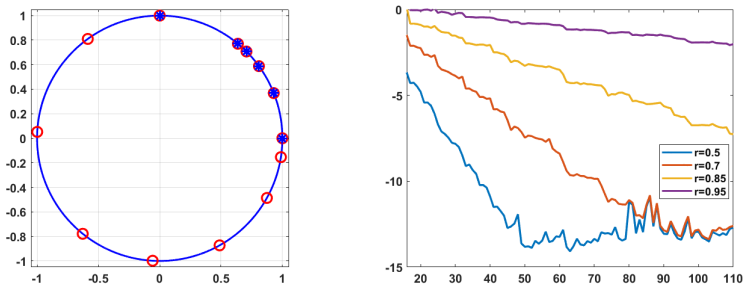
Fig. 5 We plot on the right $\mathrm{Re}(\Omega_\mu(z))$ (red) and $\mathrm{Re}(R^{13}_{20,17}(z))$ (blue) on top and $\mathrm{Re}(1/\Omega_\mu(z))$ (red) and $\mathrm{Re}(1/R^{13}_{20,17}(z))$ (blue) at the bottom for $z = 0.999e^{\mathrm{i}\theta}$ and $\theta \in [0, 2\pi]$. The spikes in B show for $R^{13}_{20,17}$ the location of the poles (top) and zeros (bottom) that are nearby on $\mathbb{T}$. Observe in the bottom figure also the weight of the measure associated with $1/\Omega_\mu$ (red line) which has a narrow spike (near dirac-delta) near $z = -1$.
On the left we see the same plot as in Figure 3.B but we did not indicate the prescribed poles to show more clearly the interlacing of poles (red o) and zeros (blue x) on $\mathbb{T}$. There are also zeros near the poles that are not on $\mathbb{T}$.

that is less symmetric. The prescribed poles can be seen on Figure 6.A as blue stars. They are located at $e^{\mathrm{i}\pi t_k}$ with $t_k \in \{0, 0.12, 0.2, 0.25, 0.28, 0.5\}$. The 12 poles in Figure 6.A for $R^7_{13,10}$ are indicated by red circles. They correspond to a choice of $B_{13}(0)$ for which they are all simple and on $\mathbb{T}$. See case 3 of Theorem 13. Also here there can be pairs of poles symmetric with respect to $\mathbb{T}$ whose D_j are complex conjugate.

The errors behave very much the same as in the previous case where $d = 7$. In Figure 6.B we plot similar to Figure 4 the 10-logarithm of the mean value of $|\Omega_\mu(z) - R^m_{n,p}(z)|$ as a function of $n = 13, 14, \ldots, 110$ for $|z| = r$ where $r = 0.95, 0.85, 0.7, 0.5$. The maximal accuracy is reached with $r = 0.5$ for $n \approx 50$, while for $r = 0.95$ the error is only approximately 10^{-2} for $n = 110$.

5 Conclusion

We have considered two-point partial Padé approximation to the Herglotz-Riesz transform Ω_μ of a finite positive Borel measure. Those are Padé approximants that interpolate the transform both at zero and infinity simultaneously but for which some of the poles are prescribed. Because of the inversive property $\Omega_{\mu^*} = -\Omega_\mu$ of the transform, rational approximants R_n of degree n that have the same inversive property are considered. This resulted in R_n whose denominator turns out to be quasi-paraorthogonal polynomials of degree n and appropriate order. This allowed to make a link with Stieltjes polynomials in the case of the unit circle if the degree equals $n = 2d$ or $n = 2d + 1$. The corresponding nth-root convergence is formulated in Theorem 14. The numerical examples illustrated that the same kind of convergence is obtained in more general situations, which will be treated in a subsequent paper.



A B

Fig. 6 In A we plot the 6 prescribed poles as blue stars and all the poles of $R^7_{13,10}$ as red circles where $B_{13}(0)$ of the denominator B_{13} is chosen such that all the poles are on $\mathbb{T}$. In B, we represent the 10-logarithm of the mean value of $|\Omega_\mu(z) - R^m_{n,p}(z)|$ as a function of $n = 13, 14, \ldots, 110$ for $|z| = r$ where $r = 0.95, 0.85, 0.7, 0.5$ (in that order top to bottom). The same 6 prescribed poles are maintained throughout. Full accuracy is obtained for $r = 0.5$ when $n \approx 50$, while for $r = 0.95$ the accuracy is only about 10^{-2} for $n = 110$.

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Data availability. The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

Conflict of interest. The authors declare that they have no conflict of interest.

References

[1] Brezinski, C.: Padé-type Approximation and General Orthogonal Polynomials. ISNM, vol. 50. Birkhäuser, Basel (1980)

[2] Walsh, J.L.: Interpolation and Approximation, 5th edn. Amer. Math. Soc. Colloq. Publ., vol. 20. Amer. Math. Soc., Providence, Rhode Island (1969). First edition 1935

[3] Brezinski, C.: Partial Padé approximantion. J. Approx. Th. **54**, 210–233 (1988). https://doi.org/10.1016/0021-9045(88)90020-2

[4] Brezinski, C.: New representations of Padé, Padé-type and partial Padé approximants. J. Comput. Appl. Math. **284**, 69–77 (2015). https://doi.org/10.1016/j.cam.2014.07.007

[5] Díaz Mendoza, C., González-Vera, P., Orive, R.: On convergence of two-point Padé-type approximants. Numer. Math. **72**, 295–312 (1996). https://doi.org/10.1007/s002110050171

[6] Ambroladze, A., Wallin, H.: Convergence rates of Padé-type approximants. J. Approx. Th. **86**(3), 310–319 (1996)

[7] Simon, B.: Orthogonal Polynomials on the Unit Circle. Colloquium Publications, vol. 54. Amer. Math. Soc., Providence, RI (2005)

[8] Carathéodory, C.: Über den Variabilitätsbereich der Koeffizienten von Potenzreihen die gegebene Werte nicht annehmen. Math. Anal. **64**, 95–115 (1907). https://doi.org/10.1007/BF01449883

[9] Bultheel, A.: Laurent Series and Their Padé Approximations. Oper. Th. Adv. Appl., vol. OT-27. Birkhäuser, Basel-Boston (1987). https://doi.org/10.1007/978-3-0348-9306-0

[10] Bultheel, A., González-Vera, P., Hendriksen, E., Njåstad, O.: Quadrature formulas on the unit circle and two-point Padé approximation. In: Cuyt, A.M. (ed.) Nonlinear Numerical Methods and Rational Approximation II, pp. 303–318. Kluwer, Dordrecht (1994)

[11] González-Vera, P., Jiménez-Paíz, M., Orive, R.: On the convergence of quadrature formulas connected to multipoint Padé-type approximantion. J. Math. Anal. Appl. **202**, 747–775 (1996). https://doi.org/10.1006/jmaa.1996.0345

[12] Díaz Mendoza, C., González-Vera, P., Orive, R.: Two-point Padé-type approximation to the Cauchy transform of certain strong distributions. J. Comput. Appl. Math. **105**, 229–243 (1999). https://doi.org/10.1016/S0377-0427(99)00043-6

[13] De Andrade, E.X., McCabe, J.H.: On the two point Padé table for a distribution. Rocky Mountain J. Math. **33**(2), 545–566 (2003). https://doi.org/10.1216/rmjm/1181069966

[14] McCabe, J.H.: A formal extension of the Padé table to include two-point Padé quotients. J. Inst. Math. Appl. **15**, 363–372 (1975). https://doi.org/10.1093/imamat/15.3.363

[15] Saff, E.B., Varga, R.S. (eds.): Padé and Rational Approximation : Theory and Applications. Academic Press, London (1977)

[16] Draux, A.: Polynômes Orthogonaux Formels. Applications. ISNM, vol. 974. Springer, Berlin (1983)

[17] Jones, W.B., Njåstad, O., Thron, W.B.: Moment theory, orthogonal polynomials, quadrature, and continued fractions associated with the unit circle. Bull. London Math. Soc. **21**, 113–152 (1989). https://doi.org/10.1112/blms/21.2.113

[18] Bultheel, A., Cruz-Barroso, R., Díaz Mendoza, C.: Zeros of quasi-paraorthogonal polynomials and positive quadrature. J. Comput. Appl. Math. **407**, 14039 (2022). https://doi.org/10.1016/j.cam.2021.114039

[19] González-Vera, P., Santos León, J.C., Njåstad, O.: Some results about numerical quadrature on the unit circle. Adv. Comput. Math. **5**, 297–328 (1996). https://doi.org/10.1007/BF02124749

[20] Bultheel, A., González-Vera, P., Hendriksen, E., Njåstad, O.: On the convergence of multipoint Padé-type approximants and quadrature formulas associated with the unit circle. Numer. Algorithms **13**, 321–344 (1996). https://doi.org/10.1007/BF02207699

[21] de la Calle Ysern, B., Lopez Lagomasino, G., Reichel, L.: Stieltjes-type polynomials on the unit circle. Math. Comput. **78**(266), 969–997 (2009). https://doi.org/10.1090/S0025-5718-08-02195-9

[22] de la Calle Ysern, B.: Optimal extension of the Szegő quadrature. IMA J. Numer. Anal. **35**, 722–748 (2015)

[23] Gonchar, A.A.: On the convergence of generalized Padé approximants of meromorphic functions. Sb. Math **27**, 503–514 (1975)