A scalable design method for stabilising decentralised controllers for networks of delay-coupled systems

D. Dileep * **F. Borgioli** * **L. Hetel** ** **J.-P. Richard** *** **W. Michiels** *

* *Department of computer science, KU Leuven, Belgium (e-mail: {deesh.dileep, franceso.borgioli, wim.michiels}@ cs.kuleuven.be),*
** *University of Lille, UMR 9189 - CRIStAL, CNRS, Centrale Lille, France (e-mail: laurentiu.hetel@centralelille.fr),*
*** *University of Lille, UMR 9189 - CRIStAL, CNRS, Inria, Centrale Lille, France (e-mail: jean-pierre.richard@centralelille.fr).*

Abstract: A methodology is proposed to design stabilising fixed-order decentralised controllers for interconnected systems with identical nodal dynamics and identical local controllers. The systems considered are linear time invariant (LTI) multiple-input multiple-output retarded type time-delay systems. The closed-loop systems of local controllers and subsystems (nodes) are modelled by a system of delay differential algebraic equations. The proposed approach combines the non-conservative direct optimisation approach towards stabilisation of delay systems with a decoupling transformation. The latter reduces the overall design problem to a robust/simultaneous controller design problem for one parameterised subsystem, where the allowable values of the parameter correspond to eigenvalues of the adjacency matrix of the network graph. By treating the eigenvalues as perturbations, contained in specific intervals or regions in the complex plane determined by the topology of the network, and by optimising the corresponding pseudospectral abscissa using a novel structure exploiting algorithm, we can ensure that the achieved stability property, and the computational complexity of the controller design, are independent of the number of subsystems. A numerical example is presented to validate the proposed approach in MATLAB software.

Keywords: Time-delay systems, Decentralised control, Interconnected systems, Decoupling problems.

1. INTRODUCTION

In this paper we present a scalable design method for stabilising fixed-order decentralised controllers for large scale linear time invariant systems with time-delays. Time-delays are present in these systems due to delayed transfer of energy, material or information between subsystems, computation and communication, heredity, and they appear as feedback delays in control loops (see Michiels, W. et. al. (2017), Erol and İftar (2016)). The application of centralised controllers for large scale systems is often impractical or costly (see Lunze (1992), Siljak (1991), and McMillan (2012)). Generally, decentralised, distributed or other structurally constrained controllers are preferred for application to large scale systems. Traditional design methods for stabilising and $\mathcal{H}_\infty$ optimal controllers for linear time-invariant (LTI) Multiple Input Multiple Output (MIMO) systems are grounded in the Riccati equation and linear matrix inequality (LMI) framework (see Gahinet and Apkarian (1994), Fridman (2014), and references therein). In most cases, controllers designed by these methods are not structured and their dimension is equal or larger than the order of the plant.

We consider the decentralised fixed-order stabilisation problem of a class of interconnected time-delay systems with a network structure, whose nodes host identical dynamic systems, using identical local controllers. We show that it is possible to exploit the systems' network structure in the controller synthesis procedure, in order to improve computational efficiency and scalabililty with respect to the number of subsystems. More precisely, we show how to perform a decoupling transformation of the network of interconnected systems, resulting in a simultaneous controller design problem for *one parameterised (sub)system*, where the allowable values of the parameter correspond to the spectrum of the network adjacency matrix. This approach in frequency domain is analogous to the one introduced by Massioni and Verhaegen (2009), in which the LMI framework is used to design a scalable full-order controller for delay-free systems.

For some network topologies the values of the (eigenvalue) parameter related to the network structure are confined to a specific subset Ω of the complex plane (an interval on the real axis, a circle,...), which can be chosen independently of the number of nodes. In such a case, sufficient conditions for stability, achieving a desired level of performance of the network etc., can be obtained by taking a robust control

point of view and handling the network related parameter as an uncertain parameter confined to the full set Ω. Note that if the (Hausdorff) distance between the spectrum of the adjacency matrix and Ω goes to zero when the number of systems goes to infinity, the conservatism of the sufficient conditions disappears in the limit. In this paper we illustrate this approach by means of the decentralised stabilisation problem for network topologies for which the set Ω is an interval on the real axis.

The robust stabilisation problem of a system, affected by "uncertainty" confined to an interval on the real axis, fits within the framework of the (real structured) pseudospectral abscissa optimisation. The novel method proposed in Borgioli and Michiels (2018)) for the computation of the pseudospectral abscissa of a delay system (see Section 3 for the definition) is able to fully exploit the structure of the uncertainty. Hence, the related robust stability conditions are necessary and sufficient. The method in Borgioli and Michiels (2018)) is based on the discretisation of a gradient flow maximising the real part of the right most eigenvalue in the space of perturbations. The optimisation of the pseudospectral abscissa as a function of the controller parameters, which may be a non-convex and non-smooth function, is performed with the HANSO algorithm (Hybrid Algorithm for Non-Smooth Optimisation, see Overton (2009)).

The main contribution of the paper consists of combining the two ingredients spelled out above, the decomposition approach of networks and the pseudospectral abscissa optimisation, which allows us to design stabilising decentralised fixed-order controllers for network of coupled systems, with computational complexity independent of the number of nodes.

The remainder of this paper is organised as follows. Section 2 introduces the models used to describe the considered class of interconnected systems with delays (we refer to Dileep et al. (2018b) for an extended version). Section 3 reviews the direct optimisation based approach for designing robust fixed-order controllers (using the work of Borgioli and Michiels (2018) as a starting point). Section 4 presents a simple numerical example to illustrate the proposed methodology and corresponding MATLAB software [1] . Finally, section 5 contains the concluding remarks and the direction of future work.

2. INTERCONNECTED SYSTEMS

In this section, we describe the decomposition approach for designing a decentralised controller for an interconnected system which has by itself a network structure, consisting of identical subsystems to be controlled by identical local controllers.

2.1 Mathematical model

We consider a network described by a directed graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, A_M\}$ with a set of nodes $\mathcal{V} = \{1, 2, 3, ..., n\}$ and a set of edges $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. The edge $(i, j) \in \mathcal{E}$ connects from node $j \in \mathcal{V}$ to node $i \in \mathcal{V}$. The graph $\mathcal{G}$ need not be strongly connected. However, we assume that the graph is simple, which means that there is no self-coupling. We denote by

$$A_M = [a_{Mij}]_{i,j=1}^{n}$$

the weighted adjacency matrix with zero diagonal entries and non-negative off-diagonal entries such that $a_{Mij} > 0$ if and only if the corresponding edge $(i, j) \in \mathcal{E}$.

Each of the n nodes hosts a dynamical system with time-delays described by the following equations,

$$\begin{cases} \dot{x}_{pi}(t) &= A_{p0}x_{pi}(t) + A_{p1}x_{pi}(t-\tau_1) + B_{p1}w_i(t) \\ &\quad + B_{p2}u_i(t-\tau_2) + B_{p3}u_{ci}(t-\tau_3), \\ y_i(t) &= C_{p1}x_{pi}(t) + D_{p1}w_i(t) + D_{p2}u_i(t-\tau_2), \\ z_i(t) &= C_{p2}x_{pi}(t) + D_{p4}w_i(t) + D_{p5}u_i(t), \\ y_{ci}(t) &= C_{p3}x_{pi}(t) \qquad\qquad \forall\ i = 1, \ldots, n. \end{cases} \tag{1}$$

The instantaneous state vector is denoted by $x_{pi}(t) \in \mathbb{R}^{n_p}$ at time t. Similarly, $u_i(t) \in \mathbb{R}^{n_u}$ and $y_i(t) \in \mathbb{R}^{n_y}$ are instantaneous controlled input and measured output vectors at time t, whereas the instantaneous exogenous input and the instantaneous exogenous (or controlled) output are represented as $w_i(t)$ and $z_i(t)$ respectively. The additional input u_{ci} and output y_{ci} are related to the coupling with other nodes/subsystems in the network, which can be described by

$$u_{ci}(t) = \sum_{j=1}^{n} a_{Mij}y_{cj}(t), \quad i = 1, \ldots, n. \tag{2}$$

Hence, we consider the case where the interconnected system is composed of subsystems (1) and has a structure imposed by (2). We also consider each subsystem to be controlled by a fixed-order LTI feedback controller (order $= n_c$) of the form

$$\begin{cases} \dot{x}_{ci}(t) = A_c x_{ci}(t) + B_c y_i(t), \\ u_i(t) = C_c x_{ci}(t) + D_c y_i(t). \end{cases} \tag{3}$$

Here a static controller ($n_c = 0$) would have only the D_c component. We assume either $D_c = 0$ or $D_{p2} = 0$ to ensure that the closed-loop system is a delay system of retarded type.

We define the new state as

$$x_i(t) = [x_{pi}^{\mathrm{T}}(t)\ u_i^{\mathrm{T}}(t)\ \gamma_{w_i}^{\mathrm{T}}\ x_{ci}^{\mathrm{T}}(t)\ y_i^{\mathrm{T}}(t)\ u_{ci}^{\mathrm{T}}(t)]^{\mathrm{T}},$$

where γ_{w_i} is a dummy variable. The closed-loop system including subsystems, controllers and interconnections can be written using the new state $x_i \in \mathbb{R}^{n_{cl}}$ in the form of differential algebraic equations (DDAEs)

$$\begin{cases} E\dot{x}_i(t) &= A_0 x_i(t) + \sum_{k=1}^{3} A_k x_i(t-\tau_k) \\ &\quad + \sum_{j=1}^{n} a_{Mij} BC x_j(t) + B_2 w_i(t) \\ z_i(t) &= C_2 x_i(t), \qquad\qquad i = 1, \ldots, n, \end{cases} \tag{4}$$

where

$$E = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, A_0 = \begin{bmatrix} A_{p0} & 0 & B_{p1} & 0 & 0 & 0 \\ C_{p1} & 0 & D_{p1} & 0 & -I & 0 \\ 0 & 0 & -I & 0 & 0 & 0 \\ 0 & 0 & 0 & \begin{matrix} A_c & B_c \end{matrix} & 0 \\ 0 & -I & 0 & \begin{matrix} C_c & D_c \end{matrix} & 0 \\ 0 & 0 & 0 & \begin{matrix} 0 & 0 \end{matrix} & -I \end{bmatrix} \tag{5}$$

and

[1] The software and the numerical example are available from http://twr.cs.kuleuven.be/research/software/delay-control/dec-cont-pseudospectral.zip

$$A_1 = \begin{bmatrix} A_{p1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & B_{p2} & 0 & 0 & 0 & 0 \\ 0 & D_{p2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ I \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & B_{p3} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, C^{\mathrm{T}} = \begin{bmatrix} C_{p3} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$C_2 = [C_{p2} \;\; D_{p5} \;\; D_{p4} \;\; 0 \;\; 0 \;\; 0]$.

Note that DDAEs are amendable for modelling interconnected systems with time-delays at any part of the system, and neutral equations can be rephrased as DDAEs (see Dileep et al. (2018b)). Therefore the methodology proposed in this paper is not restricted to retarded time-delay system (1) with three time-delays, which we have chosen for the sake of simplicity.

The state-space representation for the overall structured system takes the form

$$\begin{aligned} I \otimes E\dot{x}(t) &= I \otimes A_0 x(t) + \sum_{k=1}^{3} I \otimes A_k x(t-\tau_k) \\ &\quad + A_M \otimes BCx(t) + I \otimes B_2 w(t), \\ z(t) &= I \otimes C_2 x(t). \end{aligned} \tag{6}$$

Here $x = [x_1 \;\ldots\; x_n]^{\mathrm{T}}$, $w = [w_1 \;\ldots\; w_n]^{\mathrm{T}}$, $z = [z_1 \;\ldots\; z_n]^{\mathrm{T}}$ and the Kronecker product is denoted by "$\otimes$".

It is important to point out that all controller parameters are contained in matrix A_0 in (5), as indicated with the dashed box. In the direct optimisation approach of Dileep et al. (2018a), stability and performance measures, expressed in terms of the spectral abscissa (the real part of the rightmost eigenvalue) and $\mathcal{H}_\infty$ norms, are optimised as a function of the elements of controller matrices A_c, B_c, C_c and D_c. In what follows we denote the controller parameters by vector $\boldsymbol{p}$,

$$\boldsymbol{p} = \mathrm{vec} \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix}, \tag{7}$$

and, whenever appropriate, stress the dependence of A_0 on these parameters with the notation $A_0(\boldsymbol{p})$.

2.2 Decoupling for the stabilisation problem

There always exists a transformation matrix $T \in \mathbb{C}^{n\times n}$ and an upper-triangular matrix $Z \in \mathbb{C}^{n\times n}$ such that

$$TA_MT^{-1} = Z. \tag{8}$$

Note that spectrum of A_M appears on the diagonal of Z. If we apply the transformation $x(t) = \hat{T}^{-1}\bar{x}(t)$ to (6), with $\hat{T} = T \otimes I$, we obtain equation

$$\begin{aligned} I \otimes E\dot{\bar{x}}(t) &= I \otimes A_0 \bar{x}(t) + \sum_{k=1}^{3} I \otimes A_k \bar{x}(t-\tau_k) \\ &\quad + Z \otimes BC\bar{x}(t) + \underbrace{\hat{T}\; I \otimes B_2}_{(I\otimes B_2)\hat{T}} w(t), \\ z(t) &= \underbrace{I \otimes C_2 \hat{T}^{-1}}_{\hat{T}^{-1}(I\otimes C_2)}\; \bar{x}(t). \end{aligned} \tag{9}$$

As the system in (9) has a cascaded structure, the following theorems holds true.

Theorem 1. Let the spectrum of A_M be denoted by $\boldsymbol{\lambda_a} = \{\lambda_{a1}, \ldots, \lambda_{an}\}$. Let us consider the group of subsystems given by

$$\begin{aligned} E\dot{\bar{x}}_i(t) &= (A_0 + \lambda_{ai}BC)\,\bar{x}_i(t) + \sum_{k=1}^{3} A_k \bar{x}_i(t-\tau_k) \\ &\quad + B_2 \bar{w}_i(t), \\ \bar{z}_i(t) &= C_2 \bar{x}_i(t), \qquad\qquad i = 1, \ldots, n, \end{aligned} \tag{10}$$

where $\bar{w}_i = Tw_i$ and $\bar{z}_i = Tz_i$. Then the following results hold:

(1) System (9) with $w \equiv 0$ is exponentially stable if and only if the system (10) with $\bar{w}_i \equiv 0$ for all $i = 1, ..., n$ is exponentially stable. Moreover the spectral abscissa $c(\boldsymbol{p})$ of (9) satisfies

$$c(\boldsymbol{p}) = \max_{1\le i\le n} \sup_{\lambda\in\mathbb{C}} \{\Re(\lambda) : \det\Delta(\lambda, \boldsymbol{p}, \lambda_{ai}) = 0\},$$

where

$$\Delta(\lambda, \boldsymbol{p}, \lambda_{ai}) = \lambda E - A_0(\boldsymbol{p}) - \lambda_{ai}BC - \sum_{k=1}^{3} A_k \mathrm{e}^{-\lambda\tau_k}. \tag{11}$$

(2) If there exists an unitary transformation matrix T (recall that a matrix T is unitary if $T^*T = TT^* = I$) such that

$$TA_MT^{-1} = \Lambda_a, \tag{12}$$

with $\Lambda_a = \mathrm{diag}(\lambda_{a1}, \ldots, \lambda_{an})$. Then by considering $G(jw)$ and $\tilde{G}(jw, \boldsymbol{p}; \lambda_{ai})$ as the transfer functions of systems (6) from w to z and of system (10) from $\bar{w}_i$ to $\bar{z}_i$ respectively, we can express

$$||G(jw, \boldsymbol{p})||_{\mathcal{H}_\infty} = \max_{i\in\{1,..,n\}} ||\tilde{G}(jw, \boldsymbol{p}; \lambda_{ai})||_{\mathcal{H}_\infty} \tag{13}$$

and

$$||G(jw, \boldsymbol{p})||_{\mathcal{H}_2} = \sqrt{\sum_{i=1}^{n} ||\tilde{G}(jw, \boldsymbol{p}; \lambda_{ai})||^2_{\mathcal{H}_2}}. \tag{14}$$

Proof. The assertions for the first part of Theorem 1 directly follow from the block-triangular structure of (9), with (10) appearing as a diagonal block, and from the structure of the associated eigenvalue problem.

In what concerns the second assertion, since Λ_a is diagonal we can completely decouple system $G(s)$ using the unitary matrix ($\hat{T} = T \otimes I$) in (12). However, the input and output signals get mixed. Let $\bar{G}(s)$ be the transfer function of system (9) from $\bar{w} = [\bar{w}_1 \;\ldots\; \bar{w}_n]^{\mathrm{T}}$ to $\bar{z} = [\bar{z}_1 \;\ldots\; \bar{z}_n]^{\mathrm{T}}$ (with $\bar{w} = \hat{T}w$ and $\bar{z} = \hat{T}z$), then we can express

$$G(s) = \hat{T}^{-1}\bar{G}(s)\hat{T}.$$

Also, we obtain

$$\left(\hat{T}^{-1}\bar{G}(s)\hat{T}\right)^* \left(\hat{T}^{-1}\bar{G}(s)\hat{T}\right) = \hat{T}^* \left(\bar{G}^*(s)\bar{G}(s)\right)\hat{T},$$

which has the same spectrum as $G^*(s)G(s)$, resulting in

$$||G(s)||_{\mathcal{H}_\infty} = ||\bar{G}(s)||_{\mathcal{H}_\infty}, \quad ||G(s)||_{\mathcal{H}_2} = ||\bar{G}(s)||_{\mathcal{H}_2}. \tag{15}$$

The assertions regarding the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms directly follows from the expression (15) due to the block-diagonal structure of $\bar{G}(s)$ (see Dileep et al. (2018b)). □

As an important observation, system (10) can be interpreted as the closed-loop system formed by (1)-(3), wherein the coupling given by (2) is replaced with

$$u_{ci}(t) = \lambda_{ai} y_{ci}(t).$$

This yields a full decoupling of the structured system. For more details and generalisations we refer to the extended version Dileep et al. (2018b).

3. PSEUDOSPECTRAL ABSCISSA

From the first assertion of Theorem 1 the design of stabilising decentralised controllers can be recast as the simultaneous stabilisation problem of finding controller parameters such that (10), with $\bar{w}_i \equiv 0$, is exponentially stable for all $i = 1, \ldots, n$.

For some typologies, the eigenvalues corresponding to their network adjacency matrices (λ_{ai}) for any number of nodes (n) are confined to a real interval, i.e $\lambda_{ai} \in \Omega := [p, q]$ where $p \in \mathbb{R}$, $q \in \mathbb{R}$ and $p < q$. Since the only difference between the systems' equations in (10) lies in parameters λ_{ai}, a sufficient condition for stability is given by the *robust stability* of the "uncertain" system

$$\dot{\eta}(t) = (A_0 + \lambda_a BC)\,\eta(t) + \sum_{k=1}^{3} A_k \eta(t - \tau_k), \tag{16}$$

subject to interval uncertainty $\lambda_a \in [p, q]$, for which necessary and sufficient conditions can be obtained within the real structured pseudospectral framework developed in Borgioli and Michiels (2018). The characteristic matrix of (16) can be written as

$$\breve{\Delta}(\lambda) = \lambda E - (A_0 + B\lambda_{av} C + B\delta A_0 C) - \sum_{k=1}^{3} A_k \mathrm{e}^{-\lambda \tau_k}, \tag{17}$$

with $\lambda_{av} = \frac{p+q}{2}$ and the uncertainty $|\delta A_0| \leq \frac{|q-p|}{2}$. We note that this is a special type of the problem mentioned in Borgioli and Michiels (2018), where the characteristic matrix takes the form.

$$\hat{\Delta}(\lambda) = \left(\lambda \hat{E} - \sum_{k=0}^{m} \left(\hat{A}_k + \hat{B}_k \delta \hat{A}_k \hat{C}_k \right) \mathrm{e}^{-\lambda \hat{\tau}_k} \right). \tag{18}$$

Here δA_k (δA_0 in our setting) is dedicated to real valued perturbations on A_k (A_0), and matrices $\hat{B}_k$ (B) and $\hat{C}_k$ (C) define the structure of the perturbation on each matrix A_k (A_0).

Considering all real valued perturbations such that

$$|\delta A_0| \leq \epsilon := \frac{|q-p|}{2},$$

we can define the real structured pseudospectrum of (17) as

$$\Lambda_\epsilon := \bigcup_{|\delta A_0| \leq \epsilon} \left\{ \lambda \in \mathbb{C} : \det \breve{\Delta}(\lambda) = 0 \right\}.$$

The pseudospectral abscissa (α_ϵ) is given by

$$\alpha_\epsilon := \sup\{\Re(\lambda) : \lambda \in \Lambda_\epsilon\}. \tag{19}$$

Hence, a *necessary and sufficient* condition for exponential stability of (16) for all $\lambda_a \in [p, q]$ is given by $\alpha_\epsilon < 0$. An algorithm to compute the pseudospectral abscissa is provided in Borgioli and Michiels (2018). It is based on generating a sequence of ϵ-bounded perturbations, such that the spectral abscissa is monotonically increasing, and the corresponding eigenvalue converges to the globally right-most point of the pseudospectrum.

In order to find a robustly stabilising controller and minimise the worst-case asymptotic decay rate of solutions, we can minimise the pseudospectral abscissa by tuning controller parameters, i.e., we solve the optimisation problem

$$\min_{\boldsymbol{p}} \alpha_\epsilon. \tag{20}$$

Recall that vector $\boldsymbol{p}$ contains all the tunable controller parameters, which are contained in the closed-loop system matrix A_0 (see (5) and (7)). As the objective function is non-convex and not-everywhere differentiable we use a dedicated optimisation algorithm for this task, HANSO (see Overton (2009)).

In Section 4, we show an example for which $\lambda_{ai} \in \mathbb{R}$ for all $i = 1, ..., n$, however there are other network topologies for which the adjacency matrix has non-real eigenvalues (e.g., a ring topology with unidirectional coupling). This would translate to a problem with complex valued perturbations ($B\lambda_a C$) to system matrix A_0. The algorithm in Borgioli and Michiels (2018) can be extended to treat such problems.

The pseudospectral abscissa optimisation approach for designing decentralised controller has many advantages. First, the size (n) of A_M increases with the number of subsystems in the structured system, leading to more eigenvalues (λ_{ai}). Therefore, the computational complexity of optimising the spectral abscissa of the overall system depends on the number of subsystems ($\mathcal{O}(n \cdot (n_{cl})^3)$) (and even $\mathcal{O}((n \cdot n_{cl})^3)$ if the decomposition is not employed). However, the computational complexity of the pseudospectral abscissa optimisation does not depend on the number of subsystems. Second, a controller designed to stabilise "n" subsystems might not be stabilising for a different value of "n", as will be demonstrated using a numerical example in the next section.

4. A NUMERICAL EXAMPLE

Let us consider a simple numerical example for which a stabilising controller has to be designed such that it can control a group of identical subsystems arranged in a network topology (with adjacency matrix A_M whose eigenvalues satisfy $\lambda_{ai} \in [-1, 1]\ \forall\ i \in \{1, ..., n\}$). The subsystems can be described by

$$\begin{cases} \dot{x}_{pi}(t) = \begin{bmatrix} 1 & 3 \\ 2 & -0.5 \end{bmatrix} x_{pi}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_i(t - 0.1) \\ \qquad\qquad + \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \overbrace{u_{ci}(t - 0.2)}^{\lambda_{ai} y_{pi}(t-0.2)}, \\ y_i(t) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_{pi}(t), \\ y_{ci}(t) = I x_{pi}(t), \quad u_{ci}(t) = \sum_{j=1}^{n} a_{Mij} x_{pj}, \end{cases} \tag{21}$$

for all $i = 1, ..., n$. In the following subsections, we will consider systems (21), arranged in two network topologies.

4.1 Series network topology

Let us consider a series network topology with bidirectional coupling, for which matrix A_M is of the form

$$A_M = \begin{bmatrix} 0 & 0.5 & 0 & \dots & 0 & 0 \\ 0.5 & 0 & 0.5 & \dots & 0 & 0 \\ 0 & 0.5 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0.5 \\ 0 & 0 & 0 & \dots & 0.5 & 0 \end{bmatrix}_{n\times n}, \quad (22)$$

whose eigenvalues consist of

$$\lambda_{ai} = \cos(i\frac{\pi}{n+1}) : \ i = 1, ..., n. \quad (23)$$

For $n = 3$, the adjacency matrix has a spectrum given by $\hat{\boldsymbol{\lambda}}_{\boldsymbol{a}} = \{-0.7071, 0, 0.7071\}$. Choosing the controller order equal to one and minimising the spectral abscissa $c(\boldsymbol{p})$ of the coupled system lead us to the controller K_1 given by

$$\begin{cases} \dot{x}_{ci}(t) &= -9.5309x_{ci}(t) + [-15.3689 \ -3.5134]y_i(t), \\ u_i(t) &= -2.9752x_{ci}(t) + [-19.3607 \ 5.8863]y_i(t) \end{cases} \quad (24)$$

for $1 \leq i \leq 3$.

The optimisation of the pseudospectral abscissa of (17), constraining λ_{ai} to the interval $[-1, 1]$, leads us to controller K_α given by

$$\begin{cases} \dot{x}_{ci}(t) &= -4.3275x_{ci}(t) + [-4.3547 \ 6.2463]y_i(t) \\ u_i(t) &= 6.9677x_{ci}(t) + [-13.8842 \ 0.8332]y_i(t) \end{cases} \quad (25)$$

for $1 \leq i \leq n$, which is in fact stabilising for any $n \in [1, \infty]$.

In Fig. 1 and Fig. 2 we compare the performance of both controllers. The pseudospectral abscissa (when $\lambda_{ai} \in [-1, 1]$) for the system reduced from $\alpha_\epsilon = 4.0793$ for the case of no feedback control to $\alpha_\epsilon = -0.4151$ when controlled using K_α (see Fig. 1). Therefore, the structured closed loop system is stable as long as the adjacency matrix of the network structure A_M has all its eigenvalues in the interval $[-1, 1]$. This means that the closed loop system is guaranteed to be stable (with spectral abscissa ≤ -0.4151) for any number of nodes in the series, ring, star and fully connected network topologies with bidirectional coupling. Since, under mild conditions on the coupling weights, the eigenvalues of their adjacency matrices are contained in the interval $[-1, 1]$.

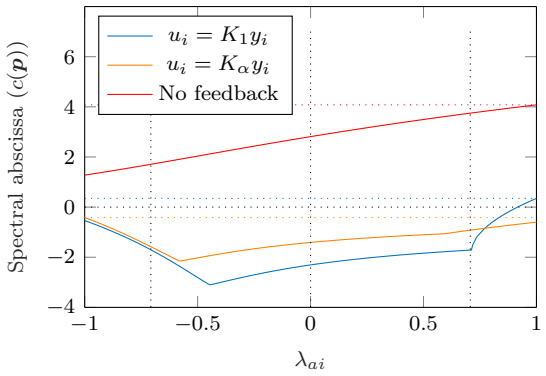

Fig. 1. The spectral abscissa of (21) with (according to the decomposition) $u_{ci}(t) = \lambda_{ai}y_{ci}(t)$, for varying $\lambda_{ai} \in [-1, 1]$. We consider i) local controller K_1 ($\alpha_\epsilon = 0.3478$) designed by minimising $c(\boldsymbol{p})$ for $n = 3$ (i.e., for $\hat{\boldsymbol{\lambda}}_{\boldsymbol{a}} = \{-0.7071, 0, 0.7071\}$), ii) local controller K_α ($\alpha_\epsilon = -0.4151$) designed by minimising the pseudospectral abscissa of (16) for $\lambda_a \in [-1, 1]$, and iii) the absence of feedback control ($\alpha_\epsilon = 4.0793$).

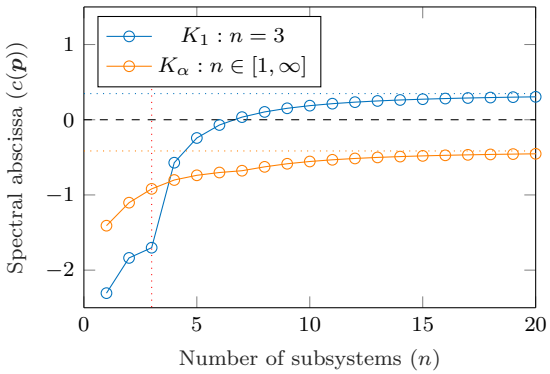

Fig. 2. The local controller K_1 obtained by optimising the spectral abscissa of three ($n = 3$) subsystems in a bidirectionally coupled series network, with the spectrum of the adjacency matrix given by $\hat{\boldsymbol{\lambda}}_{\boldsymbol{a}} = \{-0.7071, 0, 0.7071\}$, results in a unstable network system for more than seven subsystems. Therefore, for improved scalability we could design a controller K_α through pseudospectral abscissa optimisation, $\forall \ \lambda_{ai} \in [-1, 1]$ leading to a stable network system for any number of subsystems $n \in [1, \infty]$.

Although controller K_1 achieves lower spectral abscissa for $\hat{\boldsymbol{\lambda}}_{\boldsymbol{a}}$, it results in a positive pseudospectral abscissa.

4.2 Ring network topology

Let us consider the sub-systems in (21) to be arranged in a ring topology with bidirectional coupling. We have a circulant A_M matrix of the form

$$A_M = \begin{bmatrix} 0 & 0.5 & 0 & \dots & 0 & 0.5 \\ 0.5 & 0 & 0.5 & \dots & 0 & 0 \\ 0 & 0.5 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0.5 \\ 0.5 & 0 & 0 & \dots & 0.5 & 0 \end{bmatrix}_{n\times n}, \quad (26)$$

with the spectrum given in Michiels and Nijmeijer (2009) as

$$\lambda_{ai} = \cos(\frac{2\pi}{n}(i-1)) : \ i = 1, ..., z \text{ and } n \in \mathbb{Z}^+ \setminus \{1\}, \quad (27)$$

where $z = \frac{n+2}{2}$ or $\frac{n+1}{2}$ for even or odd values of n respectively. All the eigenvalues given by (27) have a multiplicity of two, except for '± 1' which have multiplicity one.

Optimising the spectral abscissa of the closed-loop system for $n = 3$ (for which the adjacency matrix has spectrum $\tilde{\boldsymbol{\lambda}}_{\boldsymbol{a}} = \{-0.5, -0.5, 1\}$) results in the controller K_2 given by

$$\begin{cases} \dot{x}_{ci}(t) &= -10.1209x_{ci}(t) + [3.0604 \ 12.1140]y_i(t), \\ u_i(t) &= 3.5645x_{ci}(t) + [-19.3998 \ 6.0459]y_i(t), \end{cases} \quad (28)$$

for $1 \leq i \leq 3$ ($n = 3$). Note that we can control the subsystems in this network also using the controller K_α in (25), as the eigenvalues of the network adjacency matrix satisfy $\lambda_{ai} \in [-1, 1] \ \forall i \in \{1, \dots, n\}, \ \forall \ n \in [1, \infty]$. We observe in Fig. 3 that the stability of this network is affected adversely by small change in n for the closed-loop

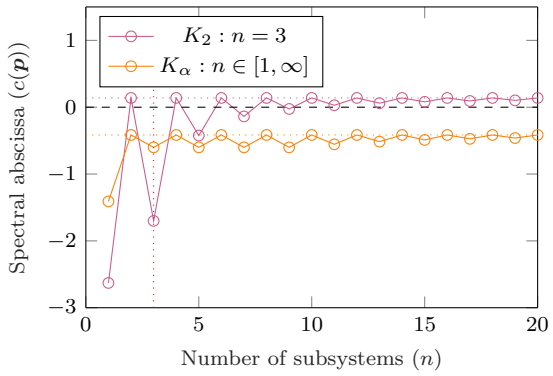

Fig. 3. The local controller K_2 obtained by optimising the spectral abscissa for three ($n = 3$) subsystems in a bidirectionally coupled ring network, with the spectrum of the adjacency matrix given by $\tilde{\boldsymbol{\lambda}}_{\boldsymbol{a}} = \{-0.5, -0.5, 1\}$, results in an unstable network system for two or four subsystems. On the contrary, the local controller K_α obtained by minimising the pseudospectral abscissa stabilises the system for any number of subsystems n.

system with the controller K_2. This is due to the presence of the eigenvalue "-1" in the spectrum of A_M for even numbers of subsystems (n).

5. CONCLUDING REMARKS

A methodology to design dynamic (LTI) fixed-order controllers for large-scale interconnected systems was presented. We conclude that using a direct optimisation approach and a decomposition, it might be possible to design a stabilising decentralised controller independent of the number of nodes, where the network related eigenvalue parameter is handled by considering the pseudospectral abscissa as the objective function. As already mentioned in Dileep et al. (2018b), the decomposition approach can be easily extended to classes of distributed or overlapping controllers, including the case of interacting controllers communicating their state with neighbours. The implementation is based on the algorithm described in Borgioli and Michiels (2018) for the pseudospectral abscissa computation. The non-convexity of the objective function is addressed by using randomly generated initial values for controller parameters, along with initial controllers specified by the user, and choosing the solution with the lowest pseudospectral abscissa in the resulting set. The original algorithm presented in Dileep et al. (2018b) uses the algorithms from Michiels (2011) and Gumussoy and Michiels (2011), which in general have a computational complexity dependent on the number of nodes n in $\mathcal{G}$ (approximately cubic). However, by framing the control design problem in terms of the pseudospectral abscissa optimisation, we have a computational complexity independent of the number of nodes. As the next steps, we will consider uncertainty on the mathematical model for the systems at the nodes and consider $\mathcal{H}_\infty$ type cost functions.

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