Model reduction for dynamical systems with quadratic output

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Abstract

Finite element models for structures and vibrations often lead to second order dynamical systems with large sparse matrices. For large-scale finite element models, the computation of the frequency response function and the structural response to dynamic loads may present a considerable computational cost. Padé via Krylov methods are widely used and appreciated projection-based model reduction techniques for linear dynamical systems with linear output. This paper extends the framework of Krylov methods to systems with a quadratic output arising in linear quadratic optimal control or random vibration problems. Three different two-sided model reduction approaches are formulated based on Krylov methods. For all methods, the control (or right) Krylov space is the same. The difference between the approaches lies thus in the choice of the observation (or left) Krylov space. The algorithms and theory are developed for the important particular case of Rayleigh damping. We also give a numerical example for a large-scale system corresponding to the forced vibration of a simply supported plate. In this case a block form of the Padé via Krylov method is used.

Keywords : Model reduction, moment matching, dynamical systems, structures and vibrations.
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Model reduction for dynamical systems with quadratic output

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SUMMARY

Finite element models for structures and vibrations often lead to second order dynamical systems with large sparse matrices. For large-scale finite element models, the computation of the frequency response function and the structural response to dynamic loads may present a considerable computational cost. Padé via Krylov methods are widely used and appreciated projection-based model reduction techniques for linear dynamical systems with linear output. This paper extends the framework of Krylov methods to systems with a quadratic output arising in linear quadratic optimal control or random vibration problems. Three different two-sided model reduction approaches are formulated based on Krylov methods. For all methods, the control (or right) Krylov space is the same. The difference between the approaches lies thus in the choice of the observation (or left) Krylov space. The algorithms and theory are developed for the important particular case of Rayleigh damping. We also give a numerical example for a large-scale system corresponding to the forced vibration of a simply supported plate. In this case a block form of the Padé via Krylov method is used. Copyright © 2011 John Wiley & Sons, Ltd.

KEY WORDS: model reduction; quadratic output; Arnoldi method; modal superposition; recycling

1. INTRODUCTION

In structural dynamics and vibro-acoustics, frequent use is made of finite element (FE) models to design and analyze structures subjected to dynamic loading. For large-scale FE models, the computation of the frequency response function and the structural response to dynamic loads may present a considerable computational cost. Efficient evaluation of the frequency response function is extremely important for reducing the computation time of parametric studies, see e.g. [1]. Several methods have been developed for efficient evaluation. Among the most widely used methods is model order reduction (MOR) through modal superposition. In this case, a Galerkin projection is performed on a reduced basis consisting of a subset of the eigenvectors of the generalized eigenvalue problem associated with the stiffness and mass matrix of the FE model of the structure. This method is particularly efficient in the low frequency range where the modal density is small and the structural response is dominated by a few global modes.

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The reduction does not come without a price, however, as a high number of modes may be required to obtain a good approximation and the computation of these modes may be quite expensive, particularly for a large FE model. Furthermore, it is not always easy to select the modes that should be included in the reduced basis.

Other methods, which have shown to be extremely useful, are related to the Ritz vector technique [2, 3]. This is a Krylov subspace method, which is also known as Padé via Krylov [4, 5, 6, 7]. The system's output is approximated by a low degree rational polynomial such that the first derivatives at the origin (also called moments) correspond with the exact output function. Padé via Krylov methods are therefore also called moment matching methods. Krylov methods have been used successfully for the computation of frequency response functions, with proportional [8] and non-proportional damping [5, 9, 10, 11, 12, 13, 14]. An extensive overview of model reduction methods with a large number of references are found in the recent books by Antoulas [15] and Benner et al. [16].

The aforementioned methods allow for an efficient model reduction when a linear output is considered, i.e. a linear combination of the components of the state vector or, in a finite element context, the vector that collects the degrees of freedom of the model. In many cases, however, the response quantities of interest correspond to a quadratic system output, i.e. the product in time or frequency domain of components of the state vector or the vector that collects the degrees of freedom of the system. Examples include linear quadratic optimal control problems, problems of random vibration where the second order response statistics due to stochastic excitation [17] are computed, as well as all problems where response quantities related to energy or power are considered.

In the present paper, model reduction is considered for linear dynamical systems with Rayleigh damping with quadratic output. The classical approach is to rewrite the governing equations of a system with quadratic output as an equivalent linear system with multiple outputs [18]. In this way, any method for model reduction of linear systems for multiple outputs can be used. In the present paper, we present two improvements to this idea. First, we reuse (or recycle [19, 20, 21]) the eigenmodes computed from the control Krylov space derived from the excitation. We thereby use the fact that a system with proportional Rayleigh damping has the same eigenmodes as the undamped system. The excited modes are reused to improve the construction of the left (or observation) Krylov spaces. This is a mix between moment matching and modal superposition. Second, we propose an alternative selection of the output vectors in order to obtain more matching moments for the quadratic output. These methods can be used for models in the frequency domain and in the time domain.

The outline of this paper is as follows. In Section 2, we give the problem formulation and discuss three different situations where quadratic outputs arise. We also review how reduced models for Rayleigh damping can be obtained. In Section 3, we review the notion of Padé via Krylov methods for linear systems and the concepts of moments and moment matching. We revisit the idea of recycling one-sided reduced models and introduce them for building two-sided Krylov models with Rayleigh damping. In Section 4, we propose new methods for model reduction by Padé via Krylov for systems with a quadratic output. First, we discuss the use of recycling eigenmodes in the model reduction method of an equivalent linear system with multiple outputs [18], which we call ELMOR in this paper. We then present an improvement of the ELMOR method, which we denote by DF-ELMOR. Finally, we present the QMM method, which tries to match as many moments of the quadratic output function as possible, for a given order of the reduced model. Section 5 illustrates the numerical methods for a

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structural dynamical problem with a quadratic output. We illustrate the power of recycling for model order reduction and compare the three new two-sided methods with each other and with the one-sided method. We formulate the main conclusions in Section 6.

Throughout the paper, we denote by A^T the matrix transpose and by A^* its Hermitian transpose. The complex conjugate of a is denoted by $\overline{a}$. The Euclidean inner product of two vectors, x and y, is denoted by y^*x and the induced 2-norm is denoted by $\|x\|$. The M norm $\|x\|_M$ is defined as the norm corresponding to the M inner product: $\sqrt{x^*Mx}$. The dependence on the variable s is always explicitly denoted, e.g. $x(s)$.

2. PROBLEM FORMULATION

Let us first consider a dynamical system with a quadratic output in the frequency domain:

$$\mathbf{Q} = \left\{ \begin{array}{rcl} ((1+i\gamma)K - \omega^2 M)x(\omega) & = & fu(\omega) \\ y(\omega) & = & x^*(\omega)Sx(\omega) \end{array} \right. , \tag{1}$$

where K and $M \in \mathbb{R}^{n\times n}$ are the stiffness matrix and the mass matrix respectively, $f \in \mathbb{R}^n$, γ is the structural (proportional) damping factor, $S \in \mathbb{R}^{n\times n}$ is a rank r matrix with $r \ll n$, possibly non-symmetric, $x(\omega) \in \mathbb{C}^n$ is the displacement vector, $u(\omega) \in \mathbb{C}$ is the system's input and $y(\omega) \in \mathbb{C}$ is the output. Note that $\omega \in \Omega$ is the frequency, where Ω is the frequency range or frequency interval of interest. Throughout the paper, we assume the mass matrix M is symmetric positive definite and the stiffness matrix K is symmetric and non-singular. If the latter condition would not hold, we assume there is a $\sigma \in \mathbb{R}$ such that $K - \sigma^2 M$ is nonsingular. Therefore, we can suppose, without loss of generality, that $\sigma = 0$.

In the following, we give three different situations where quadratic outputs arise. First, consider the inner product

$$\langle x_i, x_j \rangle = \frac{1}{|\Omega|} \int_\Omega x_j^*(\omega) x_i(\omega) d\omega,$$

which can be used to estimate the correlation in the frequency range Ω between two points i and j as follows:

$$\rho_{ij} = \frac{\langle x_i, x_j \rangle}{\|x_i\| \|x_j\|},$$

where $\|x\| = \sqrt{\langle x, x \rangle}$. If x_i and x_j represent the response for two different degrees of freedom in the FE model of a structure, $x_j^*(\omega)x_i(\omega)$ can be written as $x^*(\omega)Sx(\omega)$ where $S = e_j e_i^*$ is the matrix consisting of zeroes except on the position corresponding to row j and column i, where S has a unit value. Clearly, S is a non-symmetric rank one matrix. In order to compute ρ_{ij} efficiently, we require a fast routine for the evaluation of $y(\omega)$. If we are interested in the correlation matrix ρ for a selection of degrees of freedom, then the entry ρ_{ij} is computed by a model with a specific S matrix. We will therefore build a reduced model that can evaluate the output for p^2 quadratic outputs at once:

$$\mathbf{Q} = \left\{ \begin{array}{rcll} (K - \omega^2 M)x(\omega) & = & fu(\omega) & \\ y_{ij}(\omega) & = & x^*(\omega)S_{ij}x(\omega) & i,j = 1,\ldots,p. \end{array} \right.$$

Second, the auto power spectral density (PSD) function of a random excitation [17] can be written as

$$\mathrm{PSD}_j^{(y)} = e_j^* \bar{Z}(\omega)^{-1} S_x Z(\omega)^{-1} e_j$$

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with $Z(\omega) = ((1+i\gamma)K - \omega^2 M)$ and S_x a positive definite cross PSD matrix for the excitation. The vector e_j is a vector of zeroes except at position j where it has a unit value. The $\text{PSD}_j^{(y)}$ is therefore the auto PSD of the response corresponding to dof j. This problem can be reformulated as in equation (1) by choosing $f = e_j$, $S = S_x$ and $u(\omega) \equiv 1$.

Third, let us consider a system with a quadratic output in the time domain:

$$\mathbf{Q} = \left\{ \begin{array}{rcl} M\ddot{x}(t) + Kx(t) & = & Fu(t) \\ y(t) & = & x^T(t)Sx(t) \end{array} \right. . \tag{2}$$

Note, however, that the systems (1) and (2) represent different physical problems and are not simply connected by the Fourier transform as is the case for a linear system with a linear output.

For an interpretation of the quadratic output, consider first the RMS value of the signal $x_j(t)$

$$\text{RMS}_j = \frac{1}{T}\int_0^T x_j^2(t)dt.$$

Now, $x_j^2(t)$ can be written as $y(t) = x^T(t)Sx(t)$ where S is a matrix consisting of zeroes except on the jth diagonal position, where S has a unit value. Thus, the quadratic output of the problem defined in equation (2) corresponds to analyzing the RMS-value of the time dependent system output. In order to compute the RMS efficiently, we require a fast routine for the evaluation of $y(t)$.

In optimal control of a linear quadratic regulator, an objective function of the following form is minimized:

$$\frac{1}{2}\int_{t_0}^{t_1} x^T(t)Sx(t) + u^T(t)Ru(t)dt$$

which again requires a reduced model for quadratic output in order to compute the integral efficiently.

In the case of Rayleigh damping we rewrite the system (1) as

$$\mathbf{Q} = \left\{ \begin{array}{rcl} (K - sM)x(s) & = & f\tilde{u}(s) \\ y(\omega) & = & x^*(s)Sx(s) \end{array} \right. , \tag{3}$$

where $s = \omega^2/(1+i\gamma)$ and $\tilde{u} = 1/(1+i\gamma)u$. The variable s will be used throughout the paper instead of ω.

All methods, we describe next, build a reduced model of the form

$$\widehat{\mathbf{Q}} = \left\{ \begin{array}{rcl} (\widehat{K} - s\widehat{M})\widehat{x}(s) & = & \widehat{f}u(s) \\ \widehat{y}(s) & = & \widehat{x}^*(s)\widehat{S}\widehat{x}(s) \end{array} \right. \tag{4}$$

where $\widehat{K} = W^*KV$, $\widehat{M} = W^*MV$, $\widehat{S} = V^*SV$, and $\widehat{f} = W^*f$, and $V, W \in \mathbb{R}^{n\times k}$. Note that when $W \neq V$, $\widehat{K}$ and $\widehat{M}$ are in general non-symmetric. The matrix V is determined from a moment matching Krylov space using f and is independent of S. The choice of V guarantees that $y(s)$ and $\widehat{y}(s)$ match k moments, regardless the choice of W. In the paper, we discuss various alternatives for W such that $y(s)$ and $\widehat{y}(s)$ match more than k moments. We compare these methods with the case where $W = V$, leading to symmetric $\widehat{K}$ and $\widehat{M}$. We call this a one-sided Krylov method, since W needs not to be computed.

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3. LINEAR MODELS WITH MULTIPLE OUTPUTS

In this section, we review the Padé via Krylov method for linear systems with multiple outputs. We exploit the symmetry of K and M, and the positive-definite character of M. We also discuss the idea of recycling in the case of multiple outputs.

3.1. Moment matching via Lanczos' method

Consider the linear system with single input $u(s)$ and multiple outputs $y(s)$ (SIMO)

$$\mathbf{L} = \left\{ \begin{array}{rcl} (K - sM)x(s) & = & fu(s) \\ y(s) & = & C^* x(s) \end{array} \right. , \tag{5}$$

where $K, M \in \mathbb{R}^{n\times n}$, $f \in \mathbb{R}^n$, $C \in \mathbb{R}^{n\times p}$, $x(s) \in \mathbb{C}^n$ is the state, $u(s) \in \mathbb{C}$ is the input and $y(s) \in \mathbb{C}^p$ is the output. The number of states n is called the dimension or order of the system $\mathbf{L}$.

We first introduce the notion of a Krylov space. Let $A = K^{-1}M$ and $b = K^{-1}f$. Then we define a Krylov space of dimension k by

$$\mathcal{K}_k(A, b) = \text{span}\left\{b, Ab, A^2b, \cdots, A^{k-1}b\right\}. \tag{6}$$

When K and M are symmetric and M is positive definite, the Krylov space in equation (6) is usually built by the Lanczos method [22]. Note therefore that $K^{-1}M$ is self-adjoint with respect to the M inner product. Algorithm 1 describes how an M orthogonal basis $V_k = [v_1, v_2, \cdots, v_k]$ can be constructed.

Algorithm 1: Lanczos method

```
Let v_1 = b/‖b‖_M.
for j = 1, 2, ..., k − 1 do
    Compute v_{j+1} = A v_j.
    Use Gram-Schmidt orthogonalization to make v_{j+1} M-orthogonal against v_1, ..., v_j,
    i.e. v_i^* M v_j = 0 and ‖v_{j+1}‖_M = 1.
end
Let V_k = [v_1, ··· , v_k].
```

Next to V_k, the method also computes $T_k = V_k^* M K^{-1} M V_k$ as a by-product of the orthogonalization process. The matrix T_k is symmetric and tridiagonal. We will not use T_k for model reduction, but for computing the eigenvalues of

$$Ku = \lambda Mu \tag{7}$$

which is the undamped generalized eigenvalue problem [23]. We will use this eigen decomposition for recycling eigenvectors in §3.2. The Lanczos method can also be applied to a block of vectors $B \in \mathbb{R}^{n\times r}$ in a similar way, which corresponds to the case where multiple right hand sides are considered. See [23] for the implementation details. The initial block B should first be M-orthogonalized by the Gram-Schmidt method. Important to mention is that the Gram-Schmidt orthogonalization should be improved by reorthogonalization for reasons of numerical stability. This often produces more accurate results [24].

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In the context of model order reduction, we build a model that uses the input vector f as well as the output matrix C. In order to do so, we build two Krylov spaces, one with f and one with C:

$$V_k \;:\; \mathcal{K}_k(K^{-1}M, K^{-1}f) = \operatorname{span}\left\{K^{-1}f, (K^{-1}M)K^{-1}f, \cdots, (K^{-1}M)^{k-1}K^{-1}f\right\} \quad (8)$$

$$W_k \;:\; \mathcal{K}_k(K^{-1}M, K^{-1}C) = \operatorname{span}\left\{K^{-1}C, (K^{-1}M)K^{-1}C, \cdots, (K^{-1}M)^{\ell-1}K^{-1}C\right\} \quad (9)$$

where ℓ and k are chosen such that $k = \ell p$ and p is the number of columns of C. A reduced model is defined as

$$\widehat{\mathbf{L}} = \left\{ \begin{array}{rcl} (\widehat{K} - s\widehat{M})\widehat{x}(s) & = & \widehat{f}u(s) \\ \widehat{y}(s) & = & \widehat{C}^*\widehat{x}(s) \end{array} \right. , \quad (10)$$

with $\widehat{K} = W_k^*KV_k$, $\widehat{M} = W_k^*MV_k$, $\widehat{f} = W_k^*f$, and $\widehat{C} = V_k^*C$. The model is 'good' when $\|y(s) - \widehat{y}(s)\|$ is small in some sense. The key property of Krylov methods is that they lead to moment matching. The moments of $y(s)$, are obtained by developing $y(s)$ in a Taylor series as

$$y(s) = \sum_{j=0}^{\infty} \mathcal{Y}_j s^j$$

where $\mathcal{Y}_j$ is called the jth moment of $y(s)$. We assume that the power series converges. It can be shown that the moments are $\mathcal{Y}_j = C^*(K^{-1}M)^jK^{-1}f$ for $j = 0, 1, 2, \ldots$. The relation with the Lanczos algorithm is given by Theorem 1, which relies on the following lemmas.

Lemma 1. *Let the columns of V_k span the Krylov space $\mathcal{K}_k(K^{-1}M, K^{-1}f)$. Let W_k be such that $\widehat{K} = W_k^*KV_k$ is invertible. Let $\mathcal{X}_j = (K^{-1}M)^jK^{-1}f$ be the jth moment of $x(s)$ for system* (5)*. Similarly, define $\widehat{\mathcal{X}}_j = (\widehat{K}^{-1}\widehat{M})^j\widehat{K}^{-1}\widehat{f}$ for the reduced system* (10)*. Then $\mathcal{X}_j = V_k\widehat{\mathcal{X}}_j$ for $j = 0, \ldots, k-1$.*

Proof. We prove the lemma using the property that all $\mathcal{X}_j$ for $j = 0, \ldots, k-1$ lie in the Krylov space $\mathcal{K}_k(K^{-1}M, K^{-1}f)$. See also [7, 24].

Let $j = 0$. Since $\mathcal{X}_0$ lies in the Krylov space, there exists a z_0 such that $\mathcal{X}_0 = K^{-1}f = V_kz_0$. Then, by multiplying on the left with K and W_k^*, and using the definitions of $\widehat{M}$ and $\widehat{K}$, we have:

$$\begin{array}{rcl} K^{-1}f & = & V_kz_0 \\ W_k^*f & = & W_k^*KV_kz_0 \\ \widehat{f} & = & \widehat{K}z_0, \end{array}$$

from which follows that $z_0 = \widehat{\mathcal{X}}_0$ and thus $\mathcal{X}_0 = V_k\widehat{\mathcal{X}}_0$.

Assume $j > 0$ and that $\mathcal{X}_{j-1} = V_k\widehat{\mathcal{X}}_{j-1}$. Then, since $\mathcal{X}_j$ lies in the Krylov space, there exists a z_j such that $\mathcal{X}_j = V_kz_j$. Hence, we find, similarly to the case where $j = 0$:

$$\begin{array}{rcl} K^{-1}M\mathcal{X}_{j-1} & = & V_kz_j \\ W_k^*M(V_k\widehat{\mathcal{X}}_{j-1}) & = & W_k^*KV_kz_j \\ \widehat{M}\widehat{\mathcal{X}}_{j-1} & = & \widehat{K}z_j, \end{array}$$

from which $z_j = \widehat{\mathcal{X}}_j$ and thus $\mathcal{X}_j = V_k\widehat{\mathcal{X}}_j$. This proves the lemma. □

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Lemma 2. *Let the columns of W_k span the Krylov space $\mathcal{K}_k(K^{-1}M, K^{-1}C)$. Let V_k and W_k be such that $\widehat{K} = W_k^* K V_k$ is invertible. Let $Z_j = (K^{-1}M)^j K^{-1}C$. Similarly, define $\widehat{Z}_j = (\widehat{K}^{-*}\widehat{M}^*)^j \widehat{K}^{-*}\widehat{C}$. Then $Z_j = W_k \widehat{Z}_j$ for $j = 0, \ldots, \ell - 1$.*

Proof. The proof follows from Lemma 1, where the role of V_k and W_k is interchanged. We therefore have transposes for $\widehat{K}$ and $\widehat{M}$. □

Theorem 1. *Given the reduced model defined by the system of equations* (10)*. Let $\widehat{\mathcal{Y}}_j$ be the jth moment of $\widehat{y}(s)$. Then $\widehat{\mathcal{Y}}_j = \mathcal{Y}_j$, for $j = 0, \ldots, k + \ell - 1$.*

Proof. See [7, 15, 25, 26] for moment matching in Krylov methods. We prove the theorem for the present case where K and M are symmetric and M is positive definite.

The jth moments $\mathcal{Y}_j$ and $\widehat{\mathcal{Y}}_j$ are respectively

$$\begin{aligned} \mathcal{Y}_j &= C^*(K^{-1}M)^j K^{-1} f \\ \widehat{\mathcal{Y}}_j &= \widehat{C}^*(\widehat{K}^{-1}\widehat{M})^j \widehat{K}^{-1}\widehat{f}. \end{aligned}$$

Following Lemma 1, we have

$$(K^{-1}M)^j K^{-1} f = V_k \widehat{\mathcal{X}}_j \qquad \text{for} \qquad j = 0, \ldots, k-1.$$

Similarly, by Lemma 2, we have

$$C^*(K^{-1}M)^i K^{-1} = \widehat{C}^*(\widehat{K}^{-1}\widehat{M})^i \widehat{K}^{-1} W_k^* \qquad \text{for} \qquad i = 0, \ldots, \ell - 1.$$

Hence, we find for $j = 0, \ldots, k-1$,

$$\begin{aligned} \mathcal{Y}_j &= C^*(K^{-1}M)^j K^{-1} f \\ &= C^* V_k \widehat{\mathcal{X}}_j \\ &= \widehat{C}^* \widehat{\mathcal{X}}_j \;=\; \widehat{\mathcal{Y}}_j. \end{aligned}$$

For $i = 0, \ldots, \ell - 1$, we have

$$\begin{aligned} \mathcal{Y}_{k+i} &= C^*(K^{-1}M)^i (K^{-1}M)(K^{-1}M)^{k-1} K^{-1} f \\ &= \left(\widehat{C}^*(\widehat{K}^{-1}\widehat{M})^i \widehat{K}^{-1} W_k^*\right) M \left(V_k (\widehat{K}^{-1}\widehat{M})^{k-1} \widehat{K}^{-1}\widehat{f}\right) \\ &= \widehat{C}^*(\widehat{K}^{-1}\widehat{M})^{k+i} \widehat{K}^{-1}\widehat{f} \;=\; \widehat{\mathcal{Y}}_{k+i}. \end{aligned}$$

This proves the theorem. □

By Theorem 1, the output of the reduced order model (10) is a Padé approximation of the output of the model (5).

3.2. Connection with modal superposition and recycling

Modal superposition is often the desired method for model reduction, provided it is cheap to compute the modes. Let (λ_j, φ_j) for $j = 1, \ldots, n$ be the eigenvalues and eigenmodes of the generalized eigenvalue problem (7). The system's output can be written as

$$y(s) = \sum_{j=1}^{n} \frac{(C^*\varphi_j)(\varphi_j^* f)}{\lambda_j - s} u(s).$$

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Now, the k lowest modes and eigenfrequencies (λ_j, φ_j) for $j = 1, \ldots, k$ can be computed by the Lanczos method as follows. Let $T_k = V_k^* M K^{-1} M V_k$ be computed by the Lanczos method for building V_k and

$$T_k z_j = \theta_j z_j \qquad j = 1, \ldots, k \tag{11}$$

be the eigenpairs of T_k. Then, we call $\widehat{\varphi}_j = V_k z_j$ a Ritz vector and $\widehat{\lambda}_j = \theta_j^{-1}$ a Ritz value of the generalized eigenvalue problem (7). Typically, the lowest modes are well approximated by the method. The computed eigenpairs can be used to approximate the output

$$\widehat{y}(s) = \sum_{j=1}^{k} \frac{(C^* \widehat{\varphi}_j)(\widehat{\varphi}_j^* f)}{\widehat{\lambda}_j - s} u(s). \tag{12}$$

Recycling is studied in the literature for one-sided methods, i.e. methods for fast computation of the state vector $x(s)$ [20, 21]. This technique was previously proposed to speed up Krylov based model reduction methods [19, 21]. In this paper, we use recycling also for two-sided methods. The output of (5) is decomposed as

$$y(s) = y_R(s) + y_L(s),$$

where $y_R(s)$ is computed by modal superposition (12). This can be done very cheaply, because the Ritz values and the Ritz vectors are computed from T_k, which is a by-product of the orthogonalization process of the Lanczos method for constructing the Krylov space (8). Let us now assume that $\lambda_1, \cdots, \lambda_q$ are the eigenfrequencies in the frequency range of interest Ω and are used in the modal superposition for the computation of $y_R(s)$. It this case, it has been observed [20, 21] that the computation of $y_L(s)$ requires only very few Lanczos iterations. This is due to the fact that vertical asymptotes of $y(s)$ are all contained by $y_R(s)$, whereas $y_L(s)$ is a smooth function in the frequency range of the recycled modes. Outside this frequency range, $y_L(s)$ has vertical asymptotes corresponding to the poles that are not recycled. Decomposing the computation of the output is called frequency sweeping in [20] and recycling of eigenmodes in [21].

The following example of a dynamical system with linear output $\mathbf{L}$ (5) is considered

$$K = \begin{bmatrix} 1^2 & & & & \\ & 1^2 & & & \\ & & \ddots & & \\ & & & m^2 & \\ & & & & m^2 \end{bmatrix}, \quad M = I_{n\times n}\ , \quad f = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \quad C = \left[\begin{array}{c|c} \mathbf{1}_{m\times 1} & \mathbf{0}_{m\times 1} \\ \hline \mathbf{0}_{m\times 1} & \mathbf{1}_{m\times 1} \end{array}\right]$$

with $n = 200$ and $m = n/2$. Assume the eigenpairs (λ_j, φ_j) for $j = 1, \ldots, 10$ are given. Figure 1 shows that the modulus of the recycled part $y_R(s)$ contains all peaks of the modulus of the total response in the frequency range of the recycled modes. The modulus of the remaining part, $y_L(s)$, is very smooth in this frequency range. This is a typical result of recycling. For approximating $y_L(s)$, only a few iterations of the Lanczos method are required. This example illustrates that recycling also works well for dynamical systems with multiple eigenvalues.

The fact that the Lanczos method can be used to compute eigenvalues and eigenvectors allows us to compute the reduced model for multiple outputs more efficiently. This is called *recycling*, as we now explain.

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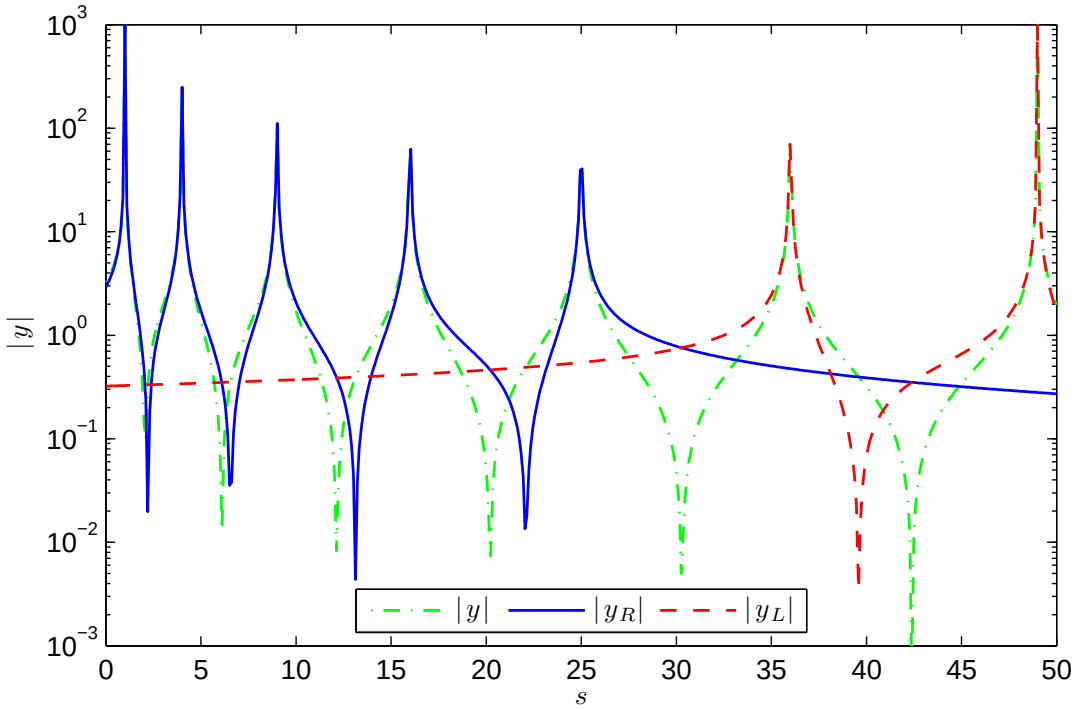


Figure 1: Modulus of the total response y (dashed-dotted line), modulus of y_R (solid line) and modulus of y_L (dashed line) in the frequency range of interest Ω.

Theorem 2. *Let* $U_q = [\varphi_1, \cdots, \varphi_q]$ *and* $\Lambda_q = \mathrm{diag}([\lambda_1, \cdots, \lambda_q])$. *We can decompose the output of* (5) *in two independent terms*

$$\begin{aligned} y_R(s) &= C^* U_q (\Lambda_q - sI)^{-1} U_q^* f u(s) \\ y_L(s) &= C^* (I - U_q U_q^* M)(K - sM)^{-1}(I - M U_q U_q^*) f u(s) \end{aligned}$$

where $y(s) = y_R(s) + y_L(s)$.

Proof. The external force f can be decomposed in two components: $f_R = M U_q U_q^* f$ and $f_L = (I - M U_q U_q^*) f$. The corresponding state vectors are the solutions of the following problems:

$$(K - sM) x_R(s) = f_R u(s), \tag{13}$$
$$(K - sM) x_L(s) = f_L u(s). \tag{14}$$

Since the columns of U_q are eigenvectors of the generalized eigenvalue problem (7), we have that $x_R(s) = U_q z_R(s)$ and equation (13) can be rewritten as

$$\begin{aligned} (K - sM) U_q z_R(s) &= f_R u(s), \\ (M U_q \Lambda_q - sM U_q) z_R(s) &= M U_q U_q^* f u(s), \\ (\Lambda_q - sI) z_R(s) &= U_q^* f u(s), \end{aligned} \tag{15}$$

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where we used $KU_q = MU_q\Lambda_q$ and $U_q^*MU_q = I$, respectively. Multiplying equation (14) on the left and right by U_q^*, we have

$$\begin{aligned}(U_q^*K - sU_q^*M)x_L(s) &= (U_q^* - U_q^*MU_qU_q^*)fu(s),\\ (\Lambda_qU_q^*M - sU_q^*M)x_L(s) &= 0,\\ (\Lambda_q - sI)U_q^*Mx_L(s) &= 0,\end{aligned}$$

or

$$U_q^*Mx_L = 0. \tag{16}$$

Based on equations (15) and (16), the output can be decomposed in two components:

$$y_R(s) = C^*x_R(s) = C^*U_q(\Lambda_q - sI)^{-1}U_q^*fu(s),$$
$$y_L(s) = C^*x_L(s) = C^*(I - U_qU_q^*M)x_L(s) = C^*(I - U_qU_q^*M)(K - sM)^{-1}(I - MU_qU_q^*)fu(s).$$

This proves the theorem. □

By Theorem 2, the reduced order model $\widehat{\mathbf{L}}$ (10) is now constructed as follows. We compute V_k from the Krylov space (8). For the construction of W_k we work in two steps. First, we compute $\widehat{\Lambda}_q$ and $\widehat{U}_q$ from V_k. Second, we define define $W_k = \left[\widehat{U}_q, \widehat{W}_{k-q}\right]$, where $\widehat{W}_{k-q}$ is the basis of Lanczos vectors of the space

$$\mathcal{K}_{k-q}(K^{-1}M, K^{-1}C_L) = \text{span}\left\{K^{-1}C_L, (K^{-1}M)K^{-1}C_L, \cdots, (K^{-1}M)^{\ell-1}K^{-1}C_L\right\}$$

with $C_L = (I - MU_qU_q^*)C$ and $k - q = p\ell$.

The output $y_L(s)$ matches $k - q + \ell$ moments. Note that in this case, less moments for $y(s)$ and $\widehat{y}(s)$ are matched then in the case without recycling. However, by recycling, the approximation of $y(s)$ may be more accurate because $y_L(s)$ is smooth and therefore needs less moments to be matched.

4. PADÉ VIA KRYLOV METHODS FOR QUADRATIC OUTPUT

In this section, we explain how model reduction for linear SIMO systems can be used to construct a reduced model for the case where the output is quadratic.

Consider the linear system (3) with quadratic output where K, M and f are defined as before, and $S \in \mathbb{C}^{n\times n}$ is a rank r matrix with $r \ll n$. The quadratic output function uses the state vector $x(s)$, which is a large vector of dimension n. The goal is to replace this system by the lower order system (4) where $\widehat{K} = W_k^*KV_k$, $\widehat{M} = W_k^*MV_k$, $\widehat{S} = V_k^*SV_k$, and $\widehat{f} = W_k^*f$. Similar to the linear case, we obtain this reduced system by replacing $x(s) = V_k\widehat{x}(s)$ in (3) and by multiplying the state equation of (3) on the left by W_k^*. Here, we assume that $V_k, W_k \in \mathbb{R}^{n\times k}$ with $k \ll n$. The matrix V_k is determined in the same way as in §3. Following Lemma 1, $x(s)$ and $V_k\widehat{x}(s)$ match the first k moments. Therefore, $y(s)$ and $\widehat{y}(s)$ match k moments as well for any W_k with $W_k^*KV_k$ nonsingular. The goal is now to choose W_k such that more moments of $y(s)$ are matched.

We assume S is Hermitian. If this is not the case, we replace S by $(S+S^*)/2$. This produces the same $y(s)$, since

$$y(s) = \frac{y(s) + y^*(s)}{2} = x^*(s)\frac{S + S^*}{2}x(s).$$

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Let us first consider the following situation as an introduction to this problem. Let S be a rank one symmetric positive definite matrix, written as $S = cc^*$ with $c \in \mathbb{R}^n$. Then $y(s) = (x^*(s)c)(c^*x(s)) = |c^*x(s)|^2$. This suggests choosing W_k following equation (9) with $C = c$ such that $2k$ moments of $c^*x(s)$ are matched. This guarantees that $2k$ moments of $y(s)$ and $\widehat{y}(s)$ are matched as well. This idea is now further explored.

4.1. An equivalent linear system with multiple outputs (ELMO)

We first rewrite the system with quadratic output (3) as a linear system with r outputs. The quadratic output function is then rewritten as a quadratic function of r variables.

Consider the eigendecomposition $S = LDL^*$ with $L \in \mathbb{R}^{n\times r}$ and D an $r \times r$ diagonal matrix. Then the output can be written as $y(s) = z^*(s)Dz(s)$ where $z(s) \in \mathbb{C}^r$ is the output of

$$\mathbf{L_Q} = \left\{ \begin{array}{rcl} (K - sM)x(s) & = & fu(s) \\ z(s) & = & L^*x(s) \end{array} \right. . \tag{17}$$

This is a linear SIMO system and a reduced model for this system with output $\widehat{z}(s)$ is obtained as explained before. The approximation of the quadratic output is then computed as $\widehat{y}(s) = \widehat{z}^*(s)D\widehat{z}(s)$. We call this method the Equivalent Linear Multiple Output method, which we denote by ELMO. The different steps are summarized in Algorithm 2. Assuming that k is a multiple of r, V_k and W_k are build with respectively k and k/r Krylov iterations. Since (17) is a linear system, we can use the concepts of recycling discussed before.

Algorithm 2: Equivalent linear multiple output method (ELMO)

1 Decompose $S = LDL^*$ with D an $r \times r$ matrix
2 Choose k such that k is a multiple of r
3 Let V_k span the Krylov space $\mathcal{K}_k(K^{-1}M, K^{-1}f)$
4 Let W_k span the Krylov space $\mathcal{K}_k(K^{-1}M, K^{-1}L)$
5 Compute $\widehat{K} = W_k^*KV_k$, $\widehat{M} = W_k^*MV_k$, $\widehat{S} = V_k^*SV_k$, $\widehat{f} = W_k^*f$

Let $\mathcal{Z}_j$ denote the moments of $z(s)$. Now, by identifying the powers of s in the Taylor series of $y(s) = z^*(s)Dz(s)$, the moments of $y(s)$ are defined as

$$\mathcal{Y}_j = \sum_{i=0}^{j} \mathcal{Y}_{i,j-i}, \tag{18}$$

where $\mathcal{Y}_{i,j} = \mathcal{Z}_i^* D \mathcal{Z}_j$ are called the *partial moments* of $y(s)$. Note also that $\mathcal{Y}_{j,i} = \overline{\mathcal{Y}_{i,j}}$. Following Theorem 1, $k + k/r$ moments of $z(s)$ and $\widehat{z}(s)$ are matched: $\mathcal{Z}_j = \widehat{\mathcal{Z}}_j$, for $j = 0, \ldots, k + k/r - 1$ and so do the partial moments $\mathcal{Y}_{i,j}$ for $i, j = 0, \ldots, k + k/r - 1$. This also corresponds to a matching of $k + k/r$ moments between $y(s)$ and $\widehat{y}(s)$.

As discussed in §3.2, recycling can also be combined with the ELMO method. We denote this Equivalent Linear Multiple Output method with Recycling as ELMOR. Algorithm 3 gives the different steps.

4.2. The decomposition free ELMO method (DF-ELMO)

A drawback of the ELMO method is that a decomposition of S in the form LDL^* is required. Often, S has a special structure which allows the fast computation of this decomposition. In

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Algorithm 3: Equivalent linear multiple output method with recycling (ELMOR)

1 Decompose $S = LDL^*$ with D an $r \times r$ matrix
2 Choose k and q such that $k-q$ is a multiple of r
3 Let V_k span the Krylov space $\mathcal{K}_k(K^{-1}M, K^{-1}f)$
4 Recycle q eigenvectors of T_k from the Lanczos method for V_k: $\widehat{U}_q$
5 Let $\widehat{W}_{k-q}$ span the Krylov space $\mathcal{K}_{k-q}(K^{-1}M, K^{-1}(I - M\widehat{U}_q\widehat{U}_q^*)L)$
6 Let $W_k = \left[\widehat{U}_q, \widehat{W}_{k-q}\right]$
7 Compute $\widehat{K} = W_k^* K V_k$, $\widehat{M} = W_k^* M V_k$, $\widehat{S} = V_k^* S V_k$, $\widehat{f} = W_k^* f$

some cases, however, such a decomposition may not be easy to obtain. The following method is therefore proposed as the Decomposition Free ELMO, denoted by DF-ELMO.

First we write the partial moments of $y(s)$ as $\mathcal{Y}_{i,j} = \mathcal{X}_i^* S \mathcal{X}_j$. The following lemma forms the key theory for the DF-ELMO method.

Lemma 3. *Let the columns of V_k span $\mathcal{K}_k(K^{-1}M, K^{-1}f)$ and let $\mathcal{K}_k(K^{-1}M, K^{-1}S\mathcal{X})$, with $\mathcal{X} = [\mathcal{X}_0, \mathcal{X}_1, \cdots, \mathcal{X}_{r-1}]$ and $k = \ell r$, be spanned by the columns of W_k. Then*

$$\mathcal{Y}_{i,j} = \widehat{\mathcal{Y}}_{i,j} \qquad \text{for} \qquad \begin{cases} i,j = 0,\ldots,k-1 \\ i = k,\ldots,k+\ell-1,\ j = 0,\ldots,\ell-1 \\ i = 0,\ldots,\ell-1,\ j = k,\ldots,k+\ell-1 \end{cases}.$$

Proof. From Lemma 1 we have $\mathcal{X}_j = V_k \widehat{\mathcal{X}}_j$, such that

$$\mathcal{Y}_{i,j} = \mathcal{X}_i^* S \mathcal{X}_j = \widehat{\mathcal{X}}_i^* V_k^* S V_k \widehat{\mathcal{X}}_j = \widehat{\mathcal{Y}}_{i,j}, \quad \text{for} \quad i,j = 0,\ldots,k-1.$$

The Krylov spaces $\mathcal{K}_k(K^{-1}M, K^{-1}f)$ and $\mathcal{K}_k(K^{-1}M, K^{-1}S\mathcal{X})$ are respectively the right and left Krylov spaces for the linear system

$$\mathbf{L} = \begin{cases} (K - sM)x(s) &= fu(s) \\ y_\star(s) &= (S\mathcal{X})^* x(s) \end{cases}.$$

The reduced model using V_k and W_k becomes

$$\widehat{\mathbf{L}} = \begin{cases} (\widehat{K} - s\widehat{M})\widehat{x}(s) &= \widehat{f}u(s) \\ \widehat{y}_\star(s) &= (\widehat{\mathcal{X}}^* \widehat{S})\widehat{x}(s) \end{cases},$$

where $\widehat{\mathcal{X}} = [\widehat{\mathcal{X}}_0, \widehat{\mathcal{X}}_1, \cdots, \widehat{\mathcal{X}}_{r-1}]$. We note that $S\mathcal{X}_j = SV_k\widehat{\mathcal{X}}_j$ and so, $V_k^* S \mathcal{X}_j = \widehat{S}\widehat{\mathcal{X}}_j$. Following Theorem 1, the first $k+\ell$ moments of $y_\star(s)$ and $\widehat{y}_\star(s)$ match:

$$\mathcal{Y}_{i,j} = \mathcal{X}_i^* S \mathcal{X}_j = \widehat{\mathcal{X}}_i^* \widehat{S} \widehat{\mathcal{X}}_j = \widehat{\mathcal{Y}}_{i,j}, \quad \text{for} \quad i = 0,\ldots,k+\ell-1,\ j = 0,\ldots,\ell-1$$

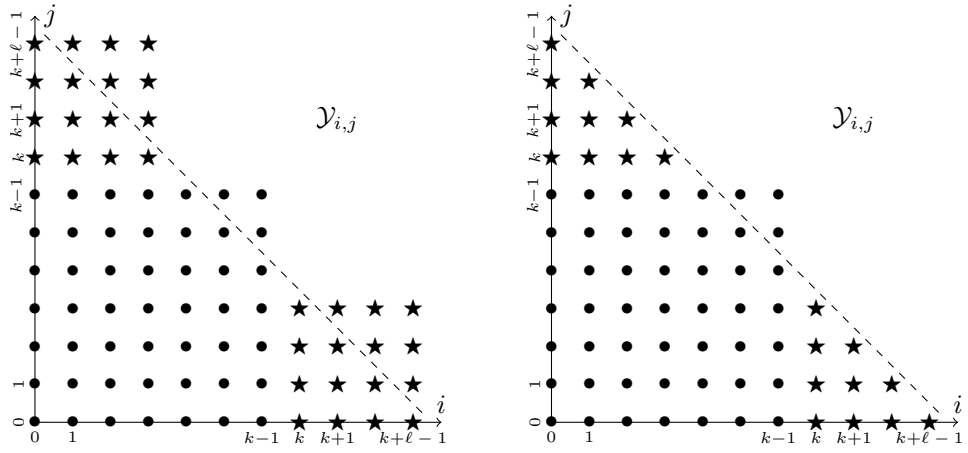
which, together with $\mathcal{Y}_{i,j} = \overline{\mathcal{Y}_{i,j}}$, proves the lemma. □

In the DF-ELMO method, we do not decompose S, but match the partial moments $\mathcal{Y}_{i,j}$, $i,j = 0,\ldots,k-1$ and $i = k,\ldots,k+\ell-1$, $j = 0,\ldots,\ell-1$. A graphical representation is given in Figure 2(a). From Lemma 3 it follows that we have to construct W_k from $\mathcal{K}_k(K^{-1}M, K^{-1}S\mathcal{X})$

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to match those partial moments. Note that this Krylov space operates on the vectors $K^{-1}S\mathcal{X}_j$, where we would like to avoid the computation of the moments $\mathcal{X}_j$. Since $\mathcal{X}_j$ is a linear combination of the first $j+1$ columns of V_k, we therefore build W_k from $\mathcal{K}_k(K^{-1}M, K^{-1}SV_k)$. Note also that the rank of SV_k is bounded by the rank r of S and k and, therefore, this method should not be more expensive than the ELMO method.



(a) DF-ELMO (b) QMM

Figure 2: The partial moments $\mathcal{Y}_{i,j}$ matched by (a) the DF-ELMO method and (b) the QMM method: moments matched due to V_k (bullets) and moments matched due to W_k (stars).

The different steps of the DF-ELMO method are summarized in Algorithm 4. Following Lemma 3, this method matches $k+\ell$ moments between $y(s)$ and $\widehat{y}(s)$. So, all moments $\mathcal{Y}_j$ and $\widehat{\mathcal{Y}}_j$ for $j = 0, \ldots, k+\ell-1$ are matched.

Algorithm 4: Decomposition free ELMO method (DF-ELMO)

1 Choose k such that k is a multiple of r
2 Let V_k span the Krylov space $\mathcal{K}_k(K^{-1}M, K^{-1}f)$
3 Let W_k span the Krylov space $\mathcal{K}_k(K^{-1}M, K^{-1}SV_k)$
4 Compute $\widehat{K} = W_k^*KV_k$, $\widehat{M} = W_k^*MV_k$, $\widehat{S} = V_k^*SV_k$, $\widehat{f} = W_k^*f$

A note is in order about recycling. Since the output is quadratic, we cannot simply apply a modal decomposition of the output function. However, we can use the alternative linear system, as in §4.1. For the DF-ELMOR method, the equivalent linear system is

$$\mathbf{L_Q} = \left\{ \begin{array}{rcl} (K - sM)x(s) & = & fu(s) \\ z(s) & = & V_k^*Sx(s) \end{array} \right. ,$$

where the output matrix $C = SV_k$ and the output is then computed as $y(s) = \widehat{x}^*(s)z(s)$. Algorithm 5 gives the different steps of the DF-ELMOR method.

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Algorithm 5: Decomposition free ELMO method with recycling (DF-ELMOR)

1 Choose k and q such that $k-q$ is a multiple of r
2 Let V_k span the Krylov space $\mathcal{K}_k(K^{-1}M, K^{-1}f)$
3 Recycle q eigenvectors of T_k from the Lanczos method for V_k: $\widehat{U}_q$
4 Let $\widehat{W}_{k-q}$ span the Krylov space $\mathcal{K}_{k-q}(K^{-1}M, K^{-1}(I - M\widehat{U}_q\widehat{U}_q^*)SV_k)$
5 Let $W_k = \left[\widehat{U}_q, \widehat{W}_{k-q}\right]$
6 Compute $\widehat{K} = W_k^*KV_k$, $\widehat{M} = W_k^*MV_k$, $\widehat{S} = V_k^*SV_k$, $\widehat{f} = W_k^*f$

4.3. Quadratic moment matching method (QMM)

An important conclusion from the analysis of the previous method is that too many partial moments are matched for matching $k+\ell$ moments of $y(s)$. It is sufficient to match only the partial moments below the dashed antidiagonal in Figure 2. Moreover, it is possible that we could increase ℓ slightly by not matching unnecessary partial moments. This is the idea of the Quadratic Moment Matching method, which we denote by QMM.

Following equation (18), we must only match the partial moments $\mathcal{Y}_{k+i,j}$, $i = 0, \ldots, \ell-j-1$, $j = 0, \ldots, \ell-1$ instead of $i, j = 0, \ldots, \ell-1$, i.e. W_k must span the Krylov spaces

$$\mathcal{K}_\ell(K^{-1}M, K^{-1}S\mathcal{X}_0), \mathcal{K}_{\ell-1}(K^{-1}M, K^{-1}S\mathcal{X}_1), \cdots, \mathcal{K}_1(K^{-1}M, K^{-1}S\mathcal{X}_{\ell-1}).$$

The matched partial moments are shown in Figure 2(b). Since $\mathcal{X}_j$ is a linear combination of $v_1, \ldots, v_{j+1}$, we can replace the list of required Krylov spaces by

$$\mathcal{K}_\ell(K^{-1}M, K^{-1}Sv_1), \mathcal{K}_{\ell-1}(K^{-1}M, K^{-1}Sv_2), \cdots, \mathcal{K}_1(K^{-1}M, K^{-1}Sv_\ell).$$

If $r < \ell$, the starting vectors $K^{-1}Sv_1, \ldots, K^{-1}Sv_\ell$ are linearly dependent. This means that only r Krylov spaces will have to be built. We illustrate this for the case $r = 2$, $k = 19$, $\ell = 10$. Assume that Sv_1 and Sv_2 are not parallel. Note that $\ell + \ell - 1 = k$. In this case, we build the space $\mathcal{K}_k(K^{-1}M, K^{-1}f)$ for V_k and build the following spaces for W_k:

$$\mathcal{K}_\ell(K^{-1}M, K^{-1}Sv_1), \mathcal{K}_{\ell-1}(K^{-1}M, K^{-1}Sv_2).$$

These spaces also span the other Krylov spaces

$$\mathcal{K}_\ell(K^{-1}M, K^{-1}Sv_3), \cdots, \mathcal{K}_1(K^{-1}M, K^{-1}Sv_\ell).$$

If Sv_1 and Sv_2 are parallel, we build the spaces

$$\mathcal{K}_\ell(K^{-1}M, K^{-1}Sv_1), \mathcal{K}_{\ell-2}(K^{-1}M, K^{-1}Sv_3).$$

instead, with $k = \ell + \ell - 2$.

This is the motivation for the QMM method. Algorithm 6 summarizes the different steps of the QMM method with Recycling, which we denote by QMMR. In a practical implementation, the choice of k and ℓ is rather tricky. We do not fix ℓ beforehand, but continue the algorithm until W_k contains k vectors.

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Algorithm 6: Quadratic moment matching method with recycling (QMMR)

Let V_k span the Krylov space $\mathcal{K}_k(K^{-1}M, K^{-1}f)$
Recycle q eigenvectors of T_k from the Lanczos method for V_k: $\widehat{U}_q$
Set $\widehat{W}_{k-q} = [\,]$, $i = 1$, $\ell = 0$
while *number of columns of* $\widehat{W}_{k-q}$ *is less than* $k - q$ **do**
    Let $c_i = (I - M\widehat{U}_q\widehat{U}_q^*)Sv_i$
    if c_i *is not linearly dependent on* $\widehat{W}_{k-q}$ **then**
        Add c_i to $\widehat{W}_{k-q}$ and $\ell = \ell + 1$
    end
    Perform 1 Krylov iteration on $\widehat{W}_{k-q}$
    Increment $i = i + 1$
end
Let $W_k = \left[\widehat{U}_q, \widehat{W}_{k-q}\right]$
Compute $\widehat{K} = W_k^* K V_k$, $\widehat{M} = W_k^* M V_k$, $\widehat{S} = V_k^* S V_k$, $\widehat{f} = W_k^* f$

5. NUMERICAL EXAMPLES

This section discusses the results of two numerical experiments carried out in MATLAB® version 7.9.0 (R2009b) on a MacBook Pro with a 2.4 GHz Intel Core 2 duo processor with 4 GB 667 MHz DDR2 SDRAM memory. We start with a simple experiment to show the power of recycling for model order reduction of dynamical systems with a quadratic output. Next, we discuss the results of an experiment with a system arising from the analysis of a simply supported plate. This model is approximated by the methods developed in §4. Furthermore, the different methods are compared with the single-sided method where $W_k = V_k$. We also give relative errors of the approximated outputs and compare the computational efforts of the proposed methods.

5.1. Recycling

We first illustrate that recycling reduces the computational cost and does not reduce the accuracy of the reduced model. In fact, in many cases the accuracy is even improved by using recycling. We use the equivalent linear output method of §4.1 without and with recycling. We denote these methods by ELMO and ELMOR(q), respectively, where q is the number of recycled vectors of V_k.

The computational cost of all the methods is primarily determined by the Krylov iterations. The methods without recycling need k Krylov iterations for the computation of V_k and k/r Krylov iterations, with a block size of r, for the computation of W_k. On the other hand, the methods with recycling need k iterations for the computation of V_k, but only $(k-q)/r$ iterations, with a block size of r, for W_k. The additional computational cost due to the recycling of q vectors is negligible, because it only requires an eigenvalue decomposition of the small scale tridiagonal matrix T_k.

Recall the example in §3.2. Consider the dynamical system $\mathbf{Q}$ with quadratic output defined

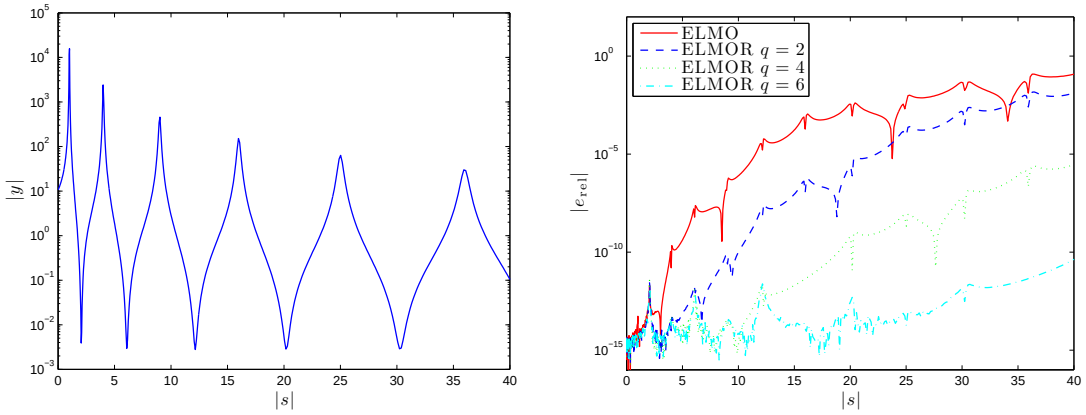
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in equation (1), where

$$K = \begin{bmatrix} 1^2 & & & & \\ & 1^2 & & & \\ & & \ddots & & \\ & & & m^2 & \\ & & & & m^2 \end{bmatrix}, \quad M = I_{n\times n}\ , \quad f = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \quad S = \left[\begin{array}{c|c} \mathbf{1}_{m\times m} & \mathbf{0}_{m\times m} \\ \hline \mathbf{0}_{m\times m} & \mathbf{1}_{m\times m} \end{array}\right]$$

with $n = 200$, $m = n/2$, $\gamma = 0.01$ and $u(\omega) \equiv 1$. The modulus of the output of the system $\mathbf{Q}$ is shown in Figure 3(a). The approximated models $\widehat{\mathbf{Q}}$ of size $k = 20$ are obtained by the ELMO method and the ELMOR method for $q = 2, 4, 6$. The moduli of the corresponding relative errors of the approximated outputs are given in Figure 3(b). This figure shows how models with recycling produce a beter approximation of the output for the same order k than the model without recycling. We see from Figure 3(b) that the relative error is close to machine precision when we recycle the six associated modes (ELMOR $q = 6$) that correspond to the vertical asymptotes in the frequency range of interest shown in Figure 3(a).



(a) Output (b) Relative error

Figure 3: Recycling experiment: (a) modulus of the quadratic output and (b) relative error of the outputs obtained by the ELMO method (solid line) and the ELMOR(q) method for $q = 2$ (dashed line), $q = 4$ (dotted line) and $q = 6$ (dashed-dotted line).

5.2. Simply supported plate

This section discusses the results and the computation times of model reduction performed on an application with a model with a quadratic output. In this application, we consider a simply supported plate representing a concrete floor supported by four walls [27]. The dimensions of the plate are 10 m × 10 m × 0.3 m (Figure 4). The Young's modulus, Poisson's ratio, proportional damping ratio and density are 30 GPa, 0.3, 0.1 and 2500 kg/m^3, respectively. For the discretization of the plate we use a regular mesh of 100×100 DKT shell elements [28]. This leads to a system with a total number of 29799 dofs. The excitation is a point load at the centre of the floor. The matrix S computes the square mean value of the displacement in four

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points selected around the point of excitation. This leads to a positive semidefinite matrix S of rank four and the model $\mathbf{Q}$ defined by equation (1) with $\gamma = 0.1$.

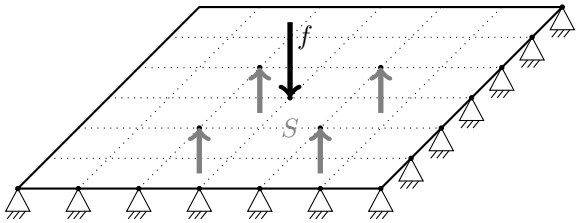


Figure 4: Supported plate with excitation f and observation points.

Figure 5 shows the modulus of the quadratic output and the relative error for the outputs of the reduced models of order $k = 32, 36, 40$ obtained by the ELMO, DF-ELMO and QMM method. We observe that the DF-ELMO method always gives a beter approximation of the output than the ELMO method. Furthermore, the QMM method still improves the approximation.

The corresponding computation times are given in Table I(a), including the construction of the reduced models as well as the evaluation in 200 frequency points ($n_\omega = 200$). However, the time required for the evaluation is negligible to the time required for the construction of these reduced models. Table I(a) shows that the computation times for the three proposed methods are similar. Note also that the computation times scale linearly with the order k of the computed reduced model. Indeed, the computational cost is dominated by the number of Krylov iterations. The times required for the evaluation of the large scale model are shown in Table I(b). We note that this computation time scales linearly with the number of evaluation points n_ω. Comparing the evaluation time of the large scale model in 200 points and the computational and evaluation time of the reduced model with the QMM method of order $k = 32$ shows that a reduction of the computation time by a factor of more than 60 has been obtained. This factor is expected to be even higher with an increasing number of evaluation points. This illustrates the importance of model reduction for systems with a quadratic output.

Table I: Computation times for the output: (a) with MOR and (b) without MOR.

(a) with MOR

	ELMO	DF-ELMO	QMM
$k = 8$	1.55 s	1.53 s	1.67 s
$k = 12$	2.09 s	2.06 s	2.22 s
$k = 16$	2.68 s	2.65 s	2.83 s
$k = 20$	3.31 s	3.22 s	3.49 s
$k = 24$	4.00 s	3.98 s	4.20 s
$k = 28$	4.74 s	4.72 s	4.96 s
$k = 32$	5.55 s	5.52 s	5.77 s
$k = 36$	6.37 s	6.35 s	6.63 s
$k = 40$	7.26 s	7.24 s	7.54 s

(b) without MOR

full model	
$n_\omega = 10$	17.5 s
$n_\omega = 20$	35.0 s
$n_\omega = 40$	70.0 s
$n_\omega = 80$	140.0 s
$n_\omega = 100$	175.0 s
$n_\omega = 200$	350.0 s
$n_\omega = 400$	700.0 s
$n_\omega = 800$	1400.0 s
$n_\omega = 1000$	1750.0 s

It is also important to notice that the Krylov subspaces are built using real arithmetic, since

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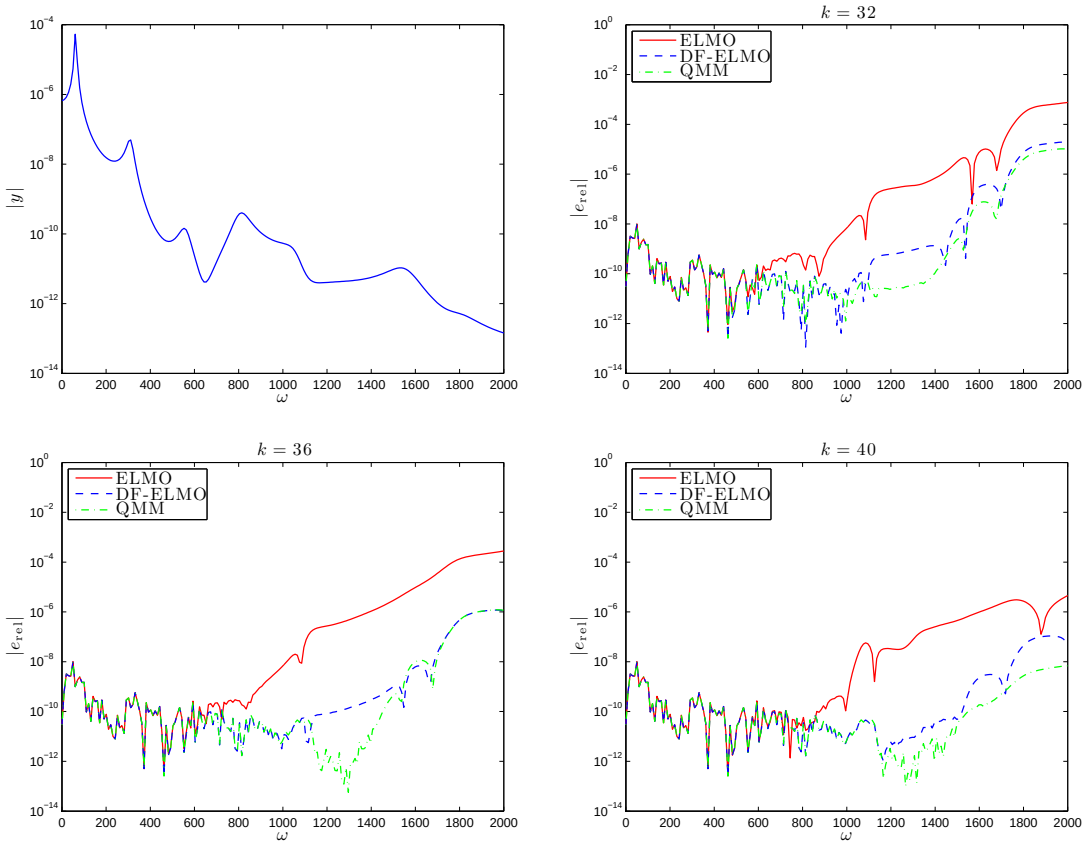


Figure 5: Modulus of the quadratic output and relative errors for the outputs of the reduced models of order $k = 32, 36, 40$ obtained by the ELMO method (solid line), the DF-ELMO method (dashed line) and the QMM method (dashed-dotted line).

all matrices and vectors involved are real. However, the direct approach has to factor a large scale complex matrix, which is expensive both in time as in storage. The reduced model is evaluated for a complex value of s, but since it is small, this cost is not high.

We will now compare the proposed methods with the single-sided ELMO method, where $W_k = V_k$ and which we denote by SELMO. The computational cost for this method is almost half of the ELMO method, because the SELMO method only requires k Krylov iterations. We compare the SELMO method with the QMMR method, since this is the best of the three proposed methods.

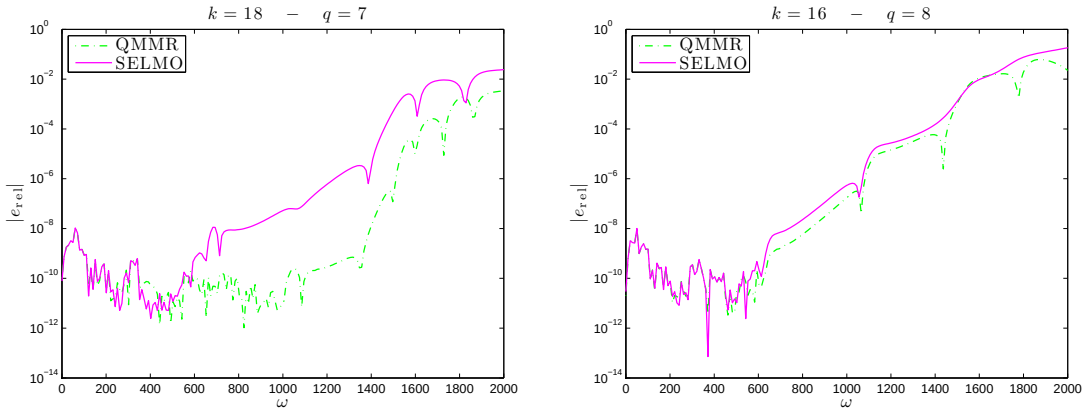
First, we suppose that the model of the supported plate only has a single observation point such that the matrix S has rank 1. The relative error for the outputs of the reduced models of order $k = 18$ is shown in Figure 6(a). In this figure, we observe that the QMMR method gives a much better approximation of the quadratic output than the SELMO method in this case.

Second, we apply the SELMO and QMMR methods for the case where 4 observation points

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are considered and thus with a matrix S of rank 4. The corresponding relative error for the outputs of the reduced models of order $k = 16$ is shown in Figure 6(b). In this case the QMMR method gives an approximation of the quadratic output of the same accuracy as the one of the reduced model obtained by the SELMO method. This is due to the fact that, in the case $\text{rank}(S) > 1$, a smaller number of Krylov iterations is performed by the QMMR method for the construction of W_k when compared to the number of Krylov iteration performed for the construction of V_k.



(a) $\text{rank}(S) = 1$ (b) $\text{rank}(S) = 4$

Figure 6: Relative error of the outputs of the reduced models obtained by the SELMO method (solid line) and the QMMR method (dashed-dotted line) for the models with (a) $\text{rank}(S) = 1$ and (b) $\text{rank}(S) = 4$.

6. CONCLUSIONS

The general framework of Krylov methods is extended to systems with a quadratic output. Three two-sided methods are proposed where the control (or right) Krylov space is the same. The difference between the methods lies in the choice of the observation (or left) Krylov space. First, the ELMO method writes the SISO system with a quadratic output as a linear SIMO system and uses this system to construct the matrices V_k and W_k. Second, the DF-ELMO method avoids the decomposition of the matrix S for the construction of the reduced model. Third, the QMM method tries to match as much as possible moments between the outputs of the large-scale and the reduced system. Using recycling, the proposed methods try to combine the strength of modal superposition and Padé via Krylov methods. The numerical experiments show that the proposed model reduction methods lead to a significant reduction of the computation time required for the evaluation of systems with a quadratic output. In the case where the model has a matrix S of rank higher than 1, a single-sided model reduction method as the SELMO method performs better than a two-sided method.

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