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Liftable Point-Line Configurations: Defining Equations and Irreducibility of Associated Matroid and Circuit Varieties

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Abstract: We study point-line configurations through the lens of projective geometry and matroid theory. Our focus is on their realization spaces, where we introduce the concepts of liftable and quasi-liftable configurations, exploring cases in which an n-tuple of collinear points can be lifted to a nondegenerate realization of a point-line configuration. We show that forest configurations are liftable and characterize the realization space of liftable configurations as the solution set of certain linear systems of equations. Moreover, we study the Zariski closure of the realization spaces of liftable and quasi-liftable configurations, known as matroid varieties, and establish their irreducibility. Additionally, we compute an irreducible decomposition for their corresponding circuit varieties. Applying these liftability properties, we present a procedure to generate some of the defining equations of the associated matroid varieties. As corollaries, we provide a geometric representation for the defining equations of two specific examples: the quadrilateral set and the 3×4 grid. While the polynomials for the latter were previously computed using specialized algorithms tailored for this configuration, the geometric interpretation of these generators was missing. We compute a minimal generating set for the corresponding ideals.

Keywords: matroids; point-line configurations; matroid varieties; Grassmann–Cayley algebra

MSC: 05B35; 05E40; 13P25; 51A20

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Citation: Clarke, O.; Masiero, G.; Mohammadi, F. Liftable Point-Line Configurations: Defining Equations and Irreducibility of Associated Matroid and Circuit Varieties. *Mathematics* **2024**, *12*, 3041. http://doi.org/10.3390/math12193041

Academic Editor: Qing-Wen Wang

Received: 22 August 2024
Revised: 21 September 2024
Accepted: 25 September 2024
Published: 28 September 2024

1. Introduction

The axiomatic definition of matroids was established in 1935 by Whitney [1], with MacLane highlighting their intimate connection with projective geometry soon after [2]. One prominent tool in this context is the Grassmann–Cayley algebra, which constructs polynomial equations from a given set of synthetic projective geometric statements, see e.g., [3,4]. Here, we provide other geometric tools for constructing such polynomials and apply them in specific examples to demonstrate that the constructed polynomials minimally generate the corresponding ideal.

A matroid, denoted by M, is a combinatorial object that extends the notion of linear independence from vector spaces. The matroid records all possible combinations of linearly independent vectors within a given set of vectors in a vector space; see [1,5]. If this process is reversible, meaning that given a matroid M, we can identify such a vector collection, these vectors are called a realization of M. The space of all realizations of M is denoted as Γ_M. The matroid variety V_M of M is defined as the Zariski closure of this realization space. This notion, introduced in [6], gives rise to a deep combinatorial structure called the matroid stratification of the Grassmannian. However, understanding the geometric properties of these strata, such as their irreducibility and defining equations [7], is a challenging problem. So, it is natural to consider specific classes of matroids. For instance, the matroid varieties of uniform matroids have been extensively studied in commutative algebra, in the context of determinantal varieties; see e.g., [8–12].

In this work, we focus on matroids of rank three, represented by point-line configurations, and use tools from incidence and projective geometry to study their associated varieties and define polynomial equations. Specifically, we develop methods to translate the incidence structure of the underlying configuration into a geometric representation for their polynomials.

The matroid varieties arising from point-line configurations are a rich and diverse family. For example, the Mnëv-Sturmfels Universality Theorem [13–15] shows that matroid varieties satisfy Murphy's law in algebraic geometry, i.e., given any singularity of a semialgebraic set, there is a matroid variety with the same singularity, up to a mild equivalence on singularities. See also [16]. Here, the matroid varieties associated with a point-line configuration can achieve all such singularities. Additionally, the extra structure induced by point-line configurations is mirrored in other contexts such as conditional independence constraints in algebraic statistics [17–21].

While the relationship between matroids and projective geometry is now well established [2], the utilization of projective incidence geometry in investigating matroid varieties is a relatively recent development. For example, the Grassmann–Cayley algebra can be employed to construct some polynomial equations in the matroid ideal I_M; see the example below from [7,22].

Example 1. *The associated ideal of the matroid in Figure 1a contains three degree-3 polynomials reflecting collinearities, alongside a degree-6 polynomial derived via the Grassmann–Cayley method.*

Among the generators of the ideal I_M, associated with a given matroid M, some polynomials are determinantal conditions that can be read from the dependence relations of M. We store this information in the circuit ideal $I_{\mathscr{C}(M)}$, which is generated by the circuits of the matroid M, and contains all those polynomials that must vanish on a collection of vectors satisfying the dependencies (but not necessarily the independencies) of M.

Our work centers on the challenging task of describing the generators of I_M that do not lie in $I_{\mathscr{C}(M)}$. Given a matroid M, classical tools such as the Grassmann–Cayley algebra can be employed to construct some polynomial equations in I_M that are not determined by the circuits of the matroid, i.e., they lie in $I_M \backslash I_{\mathscr{C}(M)}$. However, the description of all such polynomials remains incomplete because the construction of the ideal involves a saturation step that encodes the independence relations of the matroid, potentially introducing additional polynomials not necessarily derived from this method [7,18]. The current algorithms to compute saturation of ideals have high complexity and, consequently, provide results only for small matroids. When results are obtained in this way, the outputs often consist of non-human-parsable long polynomials that give little geometric intuition; see the following two examples (the first one computed in *Macaulay2* [23]).

Example 2. *The ideal of the quadrilateral set in Figure 1b is generated by 14 polynomials:*

a. *Four of degree 3, deduced directly from the collinearities in the point-line configuration.*
b. *Ten of degree 6, in 18 variables, each consisting of 14 or 16 terms.*

Example 3. *Consider the matroid of the* 3×4 *grid from Figure 1c. To compute the associated ideal of this matroid, the standard Gröbner basis computation algorithms do not terminate. Pfister and Steenpass, in [24], developed and optimized a specific algorithm for this case. Through numerical analysis, they demonstrated that the corresponding ideal has 44 generators:*

c. *16 of degree 3, each deduced from collinearities in the point-line configuration.*
d. *28 of degree 12 in 36 variables, each consisting of around* 250 *terms.*

However, there is no geometric description of these polynomials in terms of Grassmann–Cayley algebra.

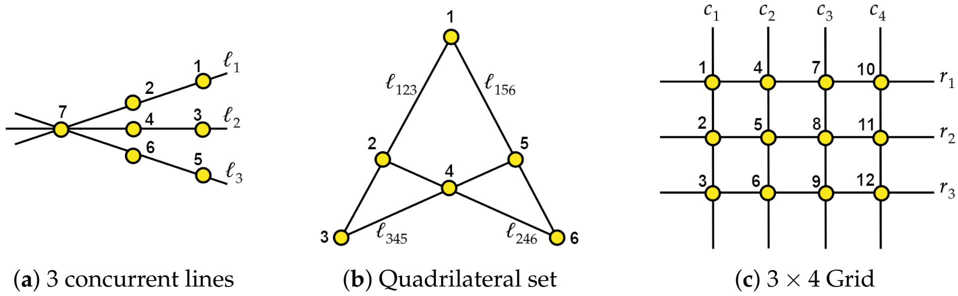

Figure 1. (**a**) Three concurrent lines; (**b**) quadrilateral set $\mathcal{L}_{QS}$; (**c**) 3×4 grid $\mathcal{L}_{G_4^3}$

Motivated by the above example, we investigate a new process to construct polynomials in the matroid ideal. Adopting an incidence-geometry viewpoint, we explore the conditions under which a tuple of collinear points in $\mathbb{P}^2_{\mathbb{C}}$ can be lifted to a nontrivial realization of a given point-line configuration $\mathcal{C}$.

Question 1. *Let $\mathcal{C}$ be a point-line configuration with n points, ℓ a line in $\mathbb{P}^2_{\mathbb{C}}$ and P a point in $\mathbb{P}^2_{\mathbb{C}} \setminus \ell$. Under what conditions is any generic n-tuple of distinct points on ℓ the image, under the projection from P to ℓ, of a nontrivial realization of $\mathcal{C}$?*

When this is always the case, we refer to $\mathcal{C}$ as *liftable*; and when this is always the case, up to removing a line, we then call $\mathcal{C}$ *quasi-liftable*. See Definitions 13 and 14. In Section 3, we analyze such configurations and their associated matroid and circuit varieties and ideals.

The following theorem summarizes our main results from Section 3. Below, we denote by $V_M = V(I_M)$ and $V_{\mathscr{C}(M)} = V(I_{\mathscr{C}(M)}$ the varieties of the matroid ideal and circuit ideal, respectively.

Theorem 1. *Let M be a rank-3 matroid whose associated point-line configuration $\mathcal{C}_M$ has no triplets of concurrent lines. Then, the following hold:*

- *The matroid variety V_M is irreducible.* (Theorem 4)
- *If $\mathcal{C}_M$ is liftable, then $V_{\mathscr{C}(M)} = V_M$ and $\sqrt{I_{\mathscr{C}(M)}} = I_M$.* (Theorem 5)
- *If $\mathcal{C}_M$ is connected quasi-liftable, then $V_{\mathscr{C}(M)} = V_0 \cup V_M$ and $\sqrt{I_{\mathscr{C}(M)}} = I_0 \cap I_M$* (Theorem 1) *where V_0 is the matroid variety whose associated configuration is a line with n marked points. Furthermore, the decompositions are, respectively, irreducible and prime.*

Specifically, Theorem 1 offers a geometric approach to generating certain polynomials in the ideal I_M, as demonstrated in Proposition 4. In Sections 4 and 5, we illustrate how these polynomials suffice to generate the matroid ideal for the quadrilateral set and the 3×4 grid. Moreover, we address Question 1 for these two matroids. More precisely, we provide a characterization of 6-tuples of collinear points that can be lifted to a quadrilateral set, and of 12-tuples that can be lifted to a 3×4 grid. For the quadrilateral set, this provides a new characterization, equivalent to the one in [25,26].

Additionally, in the context of utilizing incidence geometry to explore realization spaces of matroids, we demonstrate that a subset of the introduced polynomials generates the ideal of the matroid varieties for both configurations. Notably, these polynomials rewrite the numerically obtained high-degree polynomials as sums of determinants involving the coordinates of the points.

Theorem 2 (Theorems 8 and 11)**.** *Let $\{R_1, R_2, R_3, U\} = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix} \right\}$ be the canonical frame of reference in $\mathbb{P}^2_{\mathbb{C}}$, and let ℓ_{123} and c_1 be as in Figure 1b,c.*

- *The 10 generators in Example 2b can be replaced by the following polynomials defined in (3):*

$$QS(\ell_{123}; R_i, R_j, R_k) \qquad \text{for any } i \leq j \leq k, \text{ with } i, j, k \in \{1, 2, 3\}.$$

- *The 28 generators in Example 3d can be replaced by the following polynomials from (5):*

$$G_4^3(c_1; R_i, R_j, R_k, R_l, R_m, R_n) \qquad \text{for any } i \leq j \leq k \leq l \leq m \leq n, \text{with } i, j, k, l, m, n \in \{1, 2, 3\}.$$

To the best of our knowledge, there is currently no established method for computing the equations defining the matroid varieties, aside from the Grassmann–Cayley method [3,4,27]. Furthermore, current computer algebra programs cannot handle the computations. The above theorems provide a geometric representation of the generators. Furthermore, we prove that the generated polynomial forms a minimal generating set for the corresponding matroid ideals.

We conclude the introduction by giving an outline of the paper. In Section 2, we fix our notation for matroids, matroid varieties, circuit varieties, and point-line configurations. In addition, we describe an explicit way of associating a matroid variety and a circuit variety with a point-line configuration. In Section 3, we present our main results, providing an irreducible decomposition of the circuit variety associated with point-line configurations having certain liftability properties. In Sections 4 and 5, we apply and complete the results of Section 3 for the quadrilateral set and the 3×4 grid.

2. Preliminaries

In this section, we provide background on the theory of matroids arising from point-line configurations and fix our notation. We recall some known results about matroid varieties and their defining equations. For further details, we refer the reader to [5,18,22], and specifically for commutative algebra, we refer to [28].

Notation 1. *Throughout, we fix natural numbers n, d with $1 \leq d \leq n$. We write $[n] = \{1, 2, \ldots, n\}$ and $\binom{[n]}{d} = \{A \subseteq [n] \mid \#A = d\}$ for the collection of d-subsets of $[n]$. Given a collection of subsets $\mathscr{D} \subseteq 2^{[n]}$, we define $\min(\mathscr{D}) = \{D \in \mathscr{D} \mid \text{if } D' \in \mathscr{D} \text{ and } D' \subseteq D \text{ then } D' = D\}$.*

Let $\mathbf{k}$ be a field and let $X = (x_{i,j})_{j=1,\ldots,n}^{i=1,\ldots,d}$ be a $d \times n$ matrix of variables. We denote by $R = \mathbf{k}[X]$ the polynomial ring in the variables $x_{i,j}$. In addition, if $A \subseteq [d]$, $B \subseteq [n]$, and $\#A = \#B$, then we denote by $[A|B]_X \in R$ the minor of the submatrix of X with rows indexed by A and columns indexed by B. If $\#A = \#B = d$, then denote this minor by $[B]_X \in R$.

2.1. Matroids and Matroid Varieties

Definition 1 (Matroid)**.** *A matroid M is the datum of a finite ground set E, which will typically be $[n]$, together with a nonempty collection $\mathscr{B}(M) \subseteq 2^E$ of bases, satisfying the basis exchange axiom:*

$$\text{if } B, B' \in \mathscr{B}(M) \text{ and } \beta \in B \setminus B', \text{ then there exists } \beta' \in B' \setminus B \text{ such that } (B \setminus \{\beta\}) \cup \{\beta'\} \in \mathscr{B}(M).$$

Remark 1. *There are other equivalent definitions of a matroid in terms of other data. The bases of a matroid determine these other data and we list the ones of relevance to us below:*

- $\mathscr{I}(M) := \{I \subseteq E \mid I \subseteq B \text{ for some } B \in \mathscr{B}(M)\}$ *the independent sets are subsets of the bases;*
- $\mathscr{D}(M) := \{D \subseteq E \mid D \notin \mathscr{I}(M)\}$ *the dependent sets of M are the nonindependent sets;*
- $\mathscr{C}(M) := \min(\mathscr{D}(M))$ *the circuits of M are the minimal dependent sets;*
- $\mathrm{rk}_M : 2^E \to \mathbb{Z}$ *defined by* $\mathrm{rk}_M(S) = \max\{\#I \mid I \subseteq S, I \in \mathscr{I}(M)\}$ *is the rank function of M. By the basis exchange axiom, each basis has the same cardinality, which is equal to $\mathrm{rk}_M(E)$ and we call this value the rank of the matroid M denoted $\mathrm{rk}(M)$;*
- $\mathscr{F}(M) := \{F \subseteq E \mid \mathrm{rk}(F \cup \{x\}) = \mathrm{rk}(F) + 1 \text{ for all } x \in E \backslash F\}$ *the flats of M;*

- *The closure operator* $\mathrm{cl}_M\colon 2^E \to \mathscr{F}(M)$ *of* M *is defined as* $\mathrm{cl}_M(S) = \{x \in E \mid \mathrm{rk}_M(S \cup x) = \mathrm{rk}_M(S)\}$. *The closure of a subset of* E *is the smallest flat containing the subset.*

Definition 2 (Realization space of a matroid)**.** *Given a matroid* M *on* E *and a field* $\mathbf{k}$, *a realization of* M *over* $\mathbf{k}$ *is a collection of vectors* $\{v_i\}_{i\in E} \subseteq \mathbf{k}^r$ *such that for any subset* $I \subseteq E$ *we have that* $\{v_i\}_{i\in I}$ *is linearly independent if and only if* $I \in \mathscr{I}(M)$ *is an independent set of* M. *Typically,* $E = [n]$, *and we arrange the vectors* v_i *as the columns of a matrix* $V \in \mathbf{k}^{r\times n}$. *Note, for any realization, the value* r *is at least the rank of* M. *The realization space of a matroid in* $\mathbf{k}^{r\times n}$ *is*

$$\Gamma_{M,r} = \{V \in \mathbf{k}^{r\times n} \mid V \textit{ is a realization of } M\}.$$

Remark 2. *The columns of a matrix always give rise to a matroid, but the converse is not true. If a matroid has (resp. does not have) a realization over a field* $\mathbf{k}$ *we call it realizable (resp. nonrealizable) over* $\mathbf{k}$. *Throughout the paper, we typically work over* $\mathbb{C}$ *so, unless otherwise specified, we will say that a matroid is realizable if it is realizable over* $\mathbb{C}$.

Definition 3 (Matroid variety)**.** *Let* M *be a matroid on* $[n]$ *and* $r \geq \mathrm{rk}(M)$ *be a positive integer. The matroid variety* $V_{M,r} = \overline{\Gamma_{M,r}}$ *is the Zariski closure in* $\mathbf{k}^{r\times n}$ *of the realization space. We denote by* $I_{M,r} = I(V_{M,r}) \subseteq \mathbf{k}[X]$ *the ideal of the matroid variety where* $X = (x_{i,j})_{j=1,\dots,n}^{i=1,\dots,r}$ *is a* $r \times n$ *matrix of variables. If* r *is fixed, then we write* Γ_M, V_M *and* I_M *for* $\Gamma_{M,r}$, $V_{M,r}$ *and* $I_{M,r}$, *respectively.*

If a matroid is nonrealizable over $\mathbf{k}$, then its realization space, and its associated variety is empty.

Definition 4 (Circuit ideal and basis ideal)**.** *Let* M *be a matroid on* $[n]$ *and fix some positive integer* $r \geq \mathrm{rk}(M)$. *Recall that* $\mathscr{C}(M)$ *are the circuits of* M. *Consider the* $r \times n$ *matrix of variables* $X = (x_{i,j})_{j=1,\dots,n}^{i=1,\dots,r}$. *We define the circuit ideal* $I_{\mathscr{C}(M)}$ *and the basis ideal* J_M *in* R *as follows:*

$$I_{\mathscr{C}(M)} = \big\langle [A|B]_X \mid B \in \mathscr{C}(M) \textit{ and } A \subseteq [r] \textit{ such that } \#A = \#B \big\rangle \quad \textit{and} \quad J_M = \prod_{B\in\mathscr{B}(M)} J_B,$$

where $J_B = \langle [A|B]_X \mid A \subseteq [r], \#A = \#B\rangle$ *for each* $B \in \mathscr{B}(M)$. *In addition, we define the circuit variety of* M *as* $V_{\mathscr{C}(M)} = V(I_{\mathscr{C}(M)}) \subseteq \mathbf{k}^{r\times n}$.

Remark 3. *For each realization* V *of* M, *observe that each generator of* $I_{\mathscr{C}(M)}$ *vanishes on* V *by the linear dependence of the circuits. Hence* $I_{\mathscr{C}(M)} \subseteq I_M$. *The points of* $V(J_B)$ *are matrices* $V = (v_{i,j})$ *where the submatrix* $V_B = (v_{i,j})_{i\in[r],j\in B}$ *are not full rank. We note that* $V(J_M) = \bigcup_{B\in\mathscr{B}(M)} V(J_B)$ *is the union of these varieties. Therefore, a point* V *of* $V_{\mathscr{C}(M)}$ *is a realization of* M *if and only if* $V \notin V(J_B)$. *Note, when* $r = \mathrm{rk}(M)$ *each ideal* J_B *is principal,* J_M *is principal, and its generator is nowhere vanishing on* Γ_M.

Notation 2. *Given two matroids* M *and* N *on the same ground set* E, *we say that* $M \leq N$ *if* $\mathscr{D}(M) \subseteq \mathscr{D}(N)$, *i.e., the dependent sets of* M *are a subset of the dependent sets of* N. *This defines a partial order on the set of matroids with ground set* E, *which we call the dependency order. We note that, in the literature, this order is the opposite of the order known as the weak order.*

Definition 5 (Combinatorial closure of a matroid variety)**.** *Let* M *be a matroid and fix* $r \geq \mathrm{rk}(M)$, *we define the combinatorial closure of the matroid variety as:*

$$V_{M,r}^{\mathrm{comb}} = \bigcup_{M\leq N} V_{N,r}.$$

We denote by $I_{M,r}^{comb}$ *the ideal* $I(V_{M,r}^{comb})$. *Whenever* r *is fixed, then we omit it from the notation.*

We recall the following result from ([18] Lemma 3.5, and Proposition 3.9) that the combinatorial closure is Zariski closed and its defining ideal is the radical of the circuit ideal $\sqrt{I_{\mathscr{C}(M)}}$.

Proposition 1. *Let M be a matroid, let $\mathscr{C}(M)$ be the set of its circuits, and fix $r \geq \mathrm{rk}(M)$. The ideal I_M of the matroid variety is the radical of the saturation $I_{\mathscr{C}(M)}$ by J_M:*

$$I_M = \sqrt{\left(I_{\mathscr{C}(M)} : J_M^{\infty}\right)},$$

where $\left(I_{\mathscr{C}(M)} : J_M^{\infty}\right) = \{f \in R \mid \text{for all } g \in J_M \text{ there exists } k \geq 1 \text{ such that } fg^k \in I_{\mathscr{C}(M)}\}$. The circuit variety of M coincides with the combinatorial closure of the matroid variety:

$$V_{\mathscr{C}(M)} = V_M^{\mathrm{comb}} \quad \text{and equivalently} \quad I_M^{\mathrm{comb}} = \sqrt{I_{\mathscr{C}(M)}}.$$

2.2. Point-Line Configurations

Throughout this paper, we consider matroids of rank at most three as incidence structures referred to as point-line configurations. This approach enables us to analyze the realization space of these matroids using techniques from incidence geometry.

Definition 6 (Abstract and linear point-line configuration)**.** *An abstract point-line configuration is a triple $(\mathcal{P}, \mathcal{L}, \mathcal{I})$, where the elements of $\mathcal{P}$ and $\mathcal{L}$ are called points and lines, respectively. The elements $(p, \ell) \in \mathcal{I} \subseteq \mathcal{P} \times \mathcal{L}$ are the incidences. In this case, we say that p lies on ℓ or, equivalently, that ℓ is incident to p. An abstract point-line configuration is linear if there is at most one line incident to a pair of given points, and every line is incident with at least two points.*

Typically, we think of linear point-line configurations that arise from the Euclidean or projective plane. In these cases, the lines are 1-dimensional affine or linear subspaces, respectively.

Notation 3. *For each line $\ell \in \mathcal{L}$, we identify ℓ with $\{p \in \mathcal{P} \mid (p, \ell) \in \mathcal{I}\}$ the set of points lying on the line. So, we write $p \in \ell$ whenever p lies on ℓ and $\#\ell$ for the number of points on ℓ. For each $p \in \mathcal{P}$, we denote by $\mathcal{L}_p = \{\ell \in \mathcal{L} \mid p \in \ell\}$ the set of lines passing through p.*

Given a realizable rank-three matroid, we define its induced point-line configuration as follows.

Definition 7 (Point-line configuration of a matroid)**.** *Let M be a matroid of rank 3. We define the point-line configuration $\mathcal{C}_M := ([n], \mathcal{L}_M, \mathcal{I}_M)$ as follows. The point set of $\mathcal{C}_M$ is the ground set $[n]$ of M. The set of lines $\mathcal{L}_M$ is defined as follows: $\mathcal{L}_M := \{F \in \mathscr{F}(M) \mid \mathrm{rk}(F) = 2, |F| \geq 3\}$. Here, $\mathscr{F}(M)$ denotes the collection of all flats of the matroid M. The elements F in $\mathcal{L}_M$ are specifically the rank-two flats, meaning that they span a two-dimensional subspace. Furthermore, we require that the size of each flat F is at least three, ensuring that these lines consist of a sufficient number of points. The incidences $\mathcal{I}_M$ are given by inclusion, i.e., $(p, \ell) \in \mathcal{I}_A$ if and only if $p \in \ell$. Furthermore, if M has a realization, then we write $\mathcal{C}_A$ for the point-line configuration $\mathcal{C}_M$.*

As examples, we introduce here the point-line configurations that have inspired this work, namely, quadrilateral sets and 3×4 grids.

Example 4 (Quadrilateral set)**.** *A quadrilateral set in a projective plane $\mathbb{P}^2_{\mathbf{k}}$ is the datum of 4 lines and their 6 intersection points. By Definition 6, the linear point-line configuration corresponding to a quadrilateral set consists of 6 points and 4 lines (see Figure 1b), denoted by $\mathcal{C}_{QS} = \{\mathcal{P}_{QS}, \mathcal{L}_{QS}, \mathcal{I}_{QS}\}$, where $\mathcal{P}_{QS} = \{P_1, \ldots, P_6\}$, $\mathcal{L}_{QS} = \{\ell_{123}, \ell_{156}, \ell_{246}, \ell_{345}\}$ and $\mathcal{I}_{QS}$ is the incidence relation shown in the picture.*

The configuration $\mathcal{C}_{QS}$ arises, as in Definition 7, from the simple matroid QS over $[6]$ whose set of circuits are $\min\left(\Delta \cup \binom{[6]}{4}\right)$, where $\Delta = \{123, 156, 246, 345\}$.

Example 5 (3 × 4 grid)**.** *An $n \times m$ grid configuration is a plane point-line configuration consisting of the $n \cdot m$ points of intersection of two pencils of n and m parallel lines. When we take realizations of such a grid, the realizations of two lines of the same pencil will intersect in the projective plane. However, these intersection points will not be part of the realization itself.*

Our main example is the 3×4 grid G_4^3. Mirroring our construction for the quadrilateral set, we begin with the linear point-line configuration with 7 lines and 12 points $\mathcal{C}_{G_4^3} = \left(\mathcal{P}_{G_4^3}, \mathcal{L}_{G_4^3}, \mathcal{I}_{G_4^3}\right)$, see Figure 1c. Following the diagram, the lines are referred to as rows and columns in $\mathcal{L}_{G_4^3}$. The underlying simple matroid G_4^3 on $[12]$, has circuits $\min\left(\Delta \cup \binom{[12]}{4}\right)$ where $\Delta = \{123, \ldots, 10\,11\,12\}$.

In general, the point-line configuration of a realizable matroid is not linear. As the next example shows, this is due to the *loops* and nontrivial *parallel classes* of the matroid.

Definition 8 (Loops and parallel classes of a matroid)**.** *Let M be a matroid on ground set E. An element $e \in E$ is a loop if, for every basis B of M, we have $e \notin B$. Equivalently, an element $e \in E$ is a loop if, for every flat F of M, we have $e \in F$. The parallel classes of M are the rank-one flats. A parallel class is trivial if it contains one nonloop element. Equivalently, a parallel class is an inclusion-wise maximal subset of E such that any pair of elements of the subset are dependent.*

Example 6. *Consider the matroid M on ground set $[6]$, realized by the columns of the matrix:*

$$A = \begin{bmatrix} 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The first column of A is the zero vector, hence it belongs to no bases of M, and so 1 is a loop of M. The parallel classes of M are $\{123, 14, 15, 16\}$. So 123 is a nontrivial parallel class of M. The point-line configuration $\mathcal{C}_A$ has lines

$$\mathcal{L}_A = \{12345, 1236, 146, 156\}.$$

This configuration is not linear, since the lines 12345 and 1236 are incident to 3 common points.

We now recall the notion of *simplification* of a matroid, leading to linear point-line configurations.

Definition 9 (Simple matroid and simplification of a matroid)**.** *We say that a matroid M is simple if M has no loops and every parallel class is trivial. The simplification of M is a matroid M' on ground set $E' = \{F \in \mathscr{F}(M) \mid \mathrm{rk}(F) = 1\}$ of rank-one flats of M. A set $\{F_1, F_2, \ldots, F_k\} \subseteq E'$ is a basis of M' if for each $i \in [k]$ there exists $f_i \in F_i$ such that $\{f_1, f_2, \ldots, f_k\}$ is a basis of M.*

It is straightforward to check that the simplification of a matroid is indeed a matroid and moreover that it is simple. The simplification of a simple matroid M is itself, and so simplification is a closure operator on the class of matroids. In the language of matroid deletion, see [5], the simplification of M can be viewed as the matroid obtained by deleting the loops of M and, for each parallel class of M, deleting all but one nonloop element.

Proposition 2. *Let M be a simple matroid over $[n]$. Then, the point-line configuration $\mathcal{C}_M$ is linear. In particular, if M has a realization $A \in \mathbb{C}^{3n}$, then the columns of A are distinct points in $\mathbb{P}^2_{\mathbb{C}}$ and the lines of $\mathcal{C}_M$ naturally correspond to 1-dimensional linear subspaces in $\mathbb{P}^2_{\mathbb{C}}$.*

Proof. We start the proof by pointing out the following:

- Singletons in $[n]$ are all the rank-one flats of M and correspond to points of $\mathbb{P}^2_{\mathbb{C}}$;
- Rank-two flats of M of cardinality at least 3 correspond to lines in $\mathbb{P}^2_{\mathbb{C}}$.

Thus, assuming that M is simple, immediately yields that its rank-two flats of cardinality ≥ 3 contain at least three nontrivial parallel classes. Thus, every line of $\mathcal{C}_M$ is incident with at least $3 \geq 2$ points. We additionally have to check that any pair of points in $\mathcal{C}_M$ lies in at most one line. We prove equivalently that, if $\{pqrs\} \subseteq [n]$ is such that $\text{rk}_M(pqr) = \text{rk}_M(pqs) = 2$, then $\text{rk}_M(pqrs) = 2$. As M is a simple matroid, all 2-subsets of $[n]$ are elements of $\mathscr{I}(M)$. Thus pqr and pqs are circuits of M and the statement follows by the Circuit Elimination Axiom (see e.g., ([5] Lemma 1.1.3)). □

For the rest of this paper, we fix the following conventions for working with realizable matroids.

Notation 4. *Let M be a realizable matroid on ground set E of rank at most three, and M' its simplification. The point-line configuration $\mathcal{C}$ of M is taken to mean $\mathcal{C}_{M'}$ the point-line configuration of M'. So the points of $\mathcal{C}$ are the subsets of E given by the parallel classes of M. By a slight abuse of terminology, we say loops of $\mathcal{C}$ for the loops of M, and two points coincide in $\mathcal{C}$ when two nonloop elements of E belong to the same parallel class of M.*

Suppose that M and N are matroids of rank at most three on the same ground set E. Recall from Notation 2 the dependency order on matroids. Assume that $M \leq N$, i.e., that every dependent set of M is dependent in N. In general, the point-line configurations $\mathcal{C}_M$ and $\mathcal{C}_N$ are different as they have different numbers of points and lines. We introduce the following notation for the purpose of comparing the configurations $\mathcal{C}_M$ and $\mathcal{C}_N$.

Notation 5. *Assume the above setup. We say that a pair of lines ℓ_1 and ℓ_2 in $\mathcal{C}_M$ coincide in $\mathcal{C}_N$ if the rank-two flats F_1 and F_2 of M, which give rise to ℓ_1 and ℓ_2 respectively, are contained in a common rank-two flat of N. Suppose that $\ell \in \mathcal{C}_M$ is a line that arises from a rank-two flat F of M. The closure $\text{cl}_N(F)$ gives rise to a collection of points of $\mathcal{C}_N$. If these points do not lie on a line of $\mathcal{C}_N$, then we say that these points lie on the ghost line of ℓ in $\mathcal{C}_N$.*

We conclude the section by stating some results that correlate the irreducibility of certain matroid varieties with properties of the corresponding point-line configuration.

Proposition 3 ([18], Theorem 4.2)**.** *If M is a simple matroid of rank 3, whose point-line configuration $\mathcal{C}_M$ has at most six lines, then V_M is irreducible with respect to the Zariski topology.*

Notation 6. *Let $\mathcal{C} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a point-line configuration and ℓ a line in $\mathcal{L}$. We denote as $\mathcal{C} \setminus \ell$ the point-line configuration $(\mathcal{P}', \mathcal{L}', \mathcal{I}')$ where:*

- $\mathcal{P}' = \mathcal{P} \setminus \{p \in \mathcal{P} \mid (p, \ell) \in \mathcal{I} \text{ and } \# \mathcal{L}_p = 1\}$,
- $\mathcal{L}' = \mathcal{L} \setminus \{\ell\}$, *and*
- $\mathcal{I}' = \mathcal{I} \setminus \{(q, \ell) \mid q \in \mathcal{P}\}$.

Theorem 3 ([18], Theorem 4.5)**.** *Let M be a simple rank-3 matroid, and let $\mathcal{C}_M$ be its point-line configuration. Suppose that ℓ is a line of $\mathcal{C}_M$ such that $\#\{p \in \ell \mid \# \mathcal{L}_p \geq 3\} \leq 2$. Let M_ℓ be the simple matroid such that $\mathcal{C}_{M_\ell} = \mathcal{C}_M \setminus \ell$. If Γ_{M_ℓ} is irreducible, then so is Γ_M.*

3. (Quasi-)Liftable Configurations

In this section, we introduce the notions of liftable and quasi-liftable for point-line configurations and prove the associated varieties of such configurations are irreducible. Moreover, we present an irreducible decomposition for their circuit varieties. In particular, given a point-line configuration $\mathcal{C}$ with n points, we explore the property that an n-tuple of collinear points can be lifted to a nondegenerate realization of $\mathcal{C}$. Typically, this task is highly nontrivial and requires additional conditions on the coordinates of the collinear points.

We first introduce the notion of liftability property for point-line configurations, which plays a central role in the irreducibility of the corresponding circuit variety.

Definition 10 (Strictly liftable configuration)**.** *A linear point-line configuration $\mathcal{C}$ with n points is called strictly liftable over a field* $\mathbf{k}$ *if any n-tuple of distinct collinear points in $\mathbb{P}^2_{\mathbf{k}}$ is the image under a projection of a nondegenerate realization of $\mathcal{C}$. Here, by nondegenerate realization, we mean the datum of an n-tuple of points in $\mathbb{P}^2_{\mathbf{k}}$, and a bijection with the points of $\mathcal{C}$ such that (non-)collinear points in $\mathcal{C}$ correspond to (non)collinear points in $\mathbb{P}^2_{\mathbf{k}}$.*

For example, the next result shows that *forest configurations* are strictly liftable over $\mathbb{C}$. We first recall their definition. Consider a point-line configuration $\mathcal{C} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ whose points $\mathcal{P}$ are endowed with a total order $p_1 < \cdots < p_n$. We associate to $\mathcal{C}$ the graph $G_{\mathcal{C}} = (\mathcal{P}, E)$, where:

$$E = \{p_i p_j \mid p_i, p_j \in \ell \text{ for some } \ell \in \mathcal{L},\ p_i < p_j, \text{ and for all } p_k \in \ell \text{ we do not have } p_i < p_k < p_j\}.$$

We define the *connected components* of $\mathcal{C}$ to be set of subconfigurations corresponding to the connected components of $G_{\mathcal{C}}$. We write ω for the number of connected components of $\mathcal{C}$. The configuration $\mathcal{C}$ is called a forest if its graph is a forest. This definition is well posed by ([18] Lemma 5.2).

Lemma 1. *Forest configurations are strictly liftable over* $\mathbb{C}$*.*

Proof. Let $\mathcal{C} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a forest configuration with n points. Starting from n collinear points and a projection center P in $\mathbb{P}^2_{\mathbb{C}}$, one can concretely construct a realization of $\mathcal{C}$ projecting from P to the n-tuple of given points. We pick a point $P' \in \mathcal{P}$ such that $\mathcal{L}_{P'} = 1$ (the existence of such a point is ensured by the forest assumption), and lying on a line $\ell \in \mathcal{L}$. We associate P' to an arbitrary point in the n-tuple, which is the projection image of any point in the line joining itself with the center. We fix a point on this line and we take a line ℓ' through it. By taking the intersections of ℓ' with $\#\ell$ fibers of points in the p-tuple, ℓ' becomes a realization of ℓ.

The points in ℓ' projecting to points Q, such that $\mathcal{L}_Q \geq 2$, give rise to branches of the configuration $\mathcal{C}$ which do not contain any other point of ℓ, because of the forest assumption. So, we take arbitrary lines through these points and iterate the argument until the configuration $\mathcal{C}$ is fully chased. Figure 2 shows how this is performed in a sample configuration. □

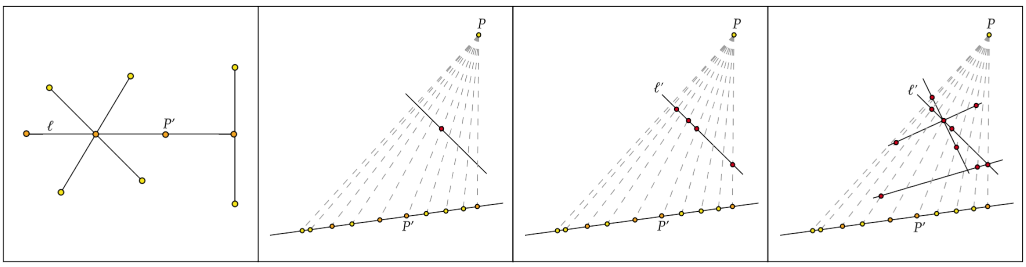

Figure 2. From left to right the figure shows a forest planar configuration with ten points and how it is realized starting from ten collinear points, following the proof of Lemma 1.

3.1. Realization Space of Liftable Configurations

The liftability problem involves uniquely realizable configurations. Nevertheless, the general setting can be formulated for all kind of configurations. Now, for any point-line configuration $\mathcal{C}$ with n points, we introduce an algebraic tool that plays a crucial role in the liftability problem. In particular, it gives insights about whether $\mathcal{C}$ is (quasi-)liftable and the conditions that n collinear points must satisfy so that they may be lifted to a nondegenerate realization of $\mathcal{C}$. For the following construction, we keep in mind the purpose of checking the liftability of a point-line configuration over $\mathbb{C}$.

Construction 1 (Collinearity matrix)**.** *Let $\mathcal{C} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a point-line configuration with $n = \#\mathcal{P}$ points and let $\mathbf{k}$ be a field. Consider an n-tuple of collinear points $P_1, \ldots, P_n$ in the projective plane $\mathbb{P}^2_{\mathbf{k}}$. Let P be a point that is not collinear with $P_1, \ldots, P_n$. After a change in coordinates, we may assume the points lie on the line $z = 0$ and we may write*

$$P = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ and } P_1 = \begin{pmatrix} x_1 \\ 1 \\ 0 \end{pmatrix}, \ldots, P_n = \begin{pmatrix} x_n \\ 1 \\ 0 \end{pmatrix} \text{ for some } x_i \in \mathbf{k}\,.$$

The problem is to find $z_1, \ldots, z_n \in \mathbf{k}$ such that the points $\begin{pmatrix} x_i \\ 1 \\ z_i \end{pmatrix}$ form a realization of $\mathcal{C}$.

We denote by $[P_iP_j]$ the 2×2 minor $x_i - x_j$. The collinearity matrix Λ of $\mathcal{C}$ encodes the collinearity conditions imposed by the configuration. The columns of Λ are indexed by the points P_j of $\mathcal{C}$ for $j \in [n]$, and the rows of Λ are indexed by each triple $i = (i_1, i_2, i_3)$, where $P_{i_1}, P_{i_2}, P_{i_3}$ are collinear points. The entries of Λ are given by:

$$(\Lambda)_{i,j} = \begin{cases} [P_{i_2}P_{i_3}] & \text{if } j = i_1, \\ -[P_{i_1}P_{i_3}] & \text{if } j = i_2, \\ [P_{i_1}P_{i_2}] & \text{if } j = i_3, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $\ell \subseteq \mathcal{P}$ is a set of collinear points with $\#\ell > 3$. For each 3-subset of ℓ, the above construction gives a row of Λ. These rows are linearly dependent. Moreover, it is straightforward to show that the set of such rows has rank $\#\ell - 2$.

Definition 11 (Space of lifts, trivial and degenerate liftings)**.** *Let $\mathcal{C}$ be a linear point-line configuration with n points. We follow the notation from Construction 1 for the collinearity matrix Λ. The matrix Λ defines the linear system:*

$$\Lambda \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{1}$$

The solution space of (1), denoted as $\mathscr{L}_{\mathcal{C}}$, is the space of lifts for $\mathcal{C}$.

For each $z \in \mathscr{L}_{\mathcal{C}}$, we obtain a lifted configuration of points P_i with coordinates $\begin{pmatrix} x_i \\ 1 \\ z_i \end{pmatrix}$.

By construction, these points realize all the collinearity conditions of C. We call z trivial if these lifted points are collinear. We say z is a degenerate lift if these lifted points are not a realization of C but do not lie on a single line.

Lemma 2. *The linear system (1) has nontrivial solutions if and only if $\dim(\mathscr{L}_{\mathcal{C}}) \geq 3$.*

Proof. Trivial liftings of the n points contribute 2 dimensions to the dimension of $\mathscr{L}_{\mathcal{C}}$. Therefore, a nontrivial lift exists if and only if the solution space has a dimension of at least three. □

This leads to an efficient method to check if an arrangement is not strictly liftable.

Example 7. *By applying Lemma 2 to the collinearity matrices of the quadrilateral set and the 3×4 grid, in the proofs of Theorems 7 and 10, we observe that they are not strictly liftable.*

For the following family of point-line configurations, the fact that System (1) has a solution space of dimension $\geq 3 \cdot \omega$ completely solves the liftability problem.

Definition 12 (Maximal matroid)**.** *Let M be a simple matroid of rank 3, whose configuration $\mathcal{C}_M$ has no triplets of concurrent lines. We say that M is maximal if it is maximal, with respect to the dependency order (see Notation 2), among the realizable simple matroids of rank 3 whose point-line configuration has no triplets of concurrent lines.*

Example 8. *The underlying simple matroids of the quadrilateral set are maximal. This can be deduced by how the property of being maximal reflects on the point-line configuration associated to the matroid. In particular, the fact that there are no simple rank-three matroids $N > M$ over $[n]$, with $\mathcal{C}_N$ having no triplets of concurrent lines, is equivalent to the fact that there is no linear point-line configuration $\mathcal{C}' \neq \mathcal{C}_M$, with n different points, satisfying the following conditions:*

- *All the collinearities (i.e., triplets of collinear points) of $\mathcal{C}_M$ are collinearities of $\mathcal{C}'$;*
- *The points of $\mathcal{C}'$ do not lie on a single line.*

In other words, the underlying simple matroids of the quadrilateral set is maximal because any nontrivial lifting of them in $\mathbb{P}^2_{\mathbb{C}}$ is a realization of the matroid.

Differently, the simple matroid M over $[12]$, whose circuits are $\mathscr{C}(M) = \min\left(\Delta \cup \binom{[12]}{4}\right)$, where $\Delta = \binom{[6]}{3} \cup \{178, 289, 379, 4\,10\,11, 5\,10\,12, 6\,11\,12\}$, is not maximal. Indeed, one can consider, for instance, the simple matroid M_1 over $[12]$, whose circuits are $\mathscr{C}(M_1) = \min\left(\Delta_1 \cup \binom{[12]}{4}\right)$, where $\Delta_1 = \binom{[9]}{3} \cup \{4\,10\,11, 5\,10\,12, 6\,11\,12\}$. As $\mathscr{C}(M) \subsetneq \mathscr{C}(M_1)$, we have that $M_1 > M$. In particular, M is contained in the two maximal matroids M_1 and M_2, whose configurations are depicted in Figure 3. Their maximality can be verified by the method mentioned at the beginning of the Example. In particular, both of them have triplets of non-collinear points but they are not a realization of M. Adding further dependencies on M_1 and M_2 would lead to a nonsimple matroid or to a rank drop (see matroid M_3, in Figure 3).

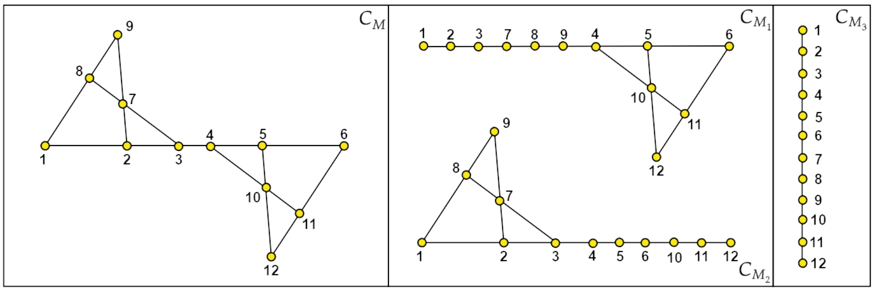

Figure 3. In the notation of Example 8, from left to right there are the linear point-line configurations of the matroids M, M_1, M_2, and M_3.

Example 9. *Similarly, one sees that 3×3 grids (and $m \times n$ grids in general) are not maximal. Let G_3^3 be the simple matroid $([9], \mathscr{C}(G_3^3))$, with $\mathscr{C}(G_3^3) = \min\left(\{123, 147, 258, 369, 456, 789\} \cup \binom{[9]}{4}\right)$. Consider the matroid $G = ([9], \mathscr{C}(G))$, with associated point-line configuration depicted in Figure 4, with $\mathscr{C}(G) = \min\left(\{123, 147, 148, 149, 178, 179, 189, 258, 369, 456, 478, 479, 489, 789\} \cup \binom{[9]}{4}\right)$. The matroid G is strictly bigger than G_3^3, depicted in Figure 5a, with respect to the dependency order. Furthermore, the matroid G is maximal. Indeed, consider the associated point-line configuration $\mathcal{C}_G$, and impose any additional collinearity (avoiding triplets of concurrent lines). This forces the collinearity of points laying on different lines of a same pencil of a 3×3 grid, leading to a rank drop.*

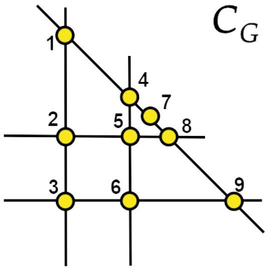

Figure 4. The configuration $\mathcal{C}_G$ realizes all the dependencies of the 3×3 grid matroid, but not all independencies.

Lemma 3. *Let $\mathcal{C} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a linear point-line configuration with n points, with connected components $\mathcal{C}_1, \ldots, \mathcal{C}_\omega$. Assume that $\mathcal{C}_1, \ldots, \mathcal{C}_\omega$ derive from a maximal matroid. Then, the following statements are equivalent:*

- *The configuration $\mathcal{C}$ is strictly liftable.*
- *The linear system $\Lambda z = 0$, where Λ is the collinearity matrix of $\mathcal{C}$, has solution space of dimension at least $3 \cdot \omega$, or equivalently $n - 3 \cdot \omega \geq \operatorname{rk} \Lambda$.*

Proof. Follows from Lemma 2 and Example 8. □

Example 10. *Let $1, \ldots, 9$ be collinear points in the projective plane. On the one hand, the collinearity matrix Λ of a 3×3 grid configuration imposes a linear system with solution space of dimension 3. On the other hand, the 3×3 grid matroid is not maximal, so we cannot use Lemma 3 to conclude information about its strict liftability.*

$$\Lambda = \begin{pmatrix} [23] & -[13] & [12] & 0 & 0 & 0 & 0 & 0 & 0 \\ [47] & 0 & 0 & -[17] & 0 & 0 & [14] & 0 & 0 \\ 0 & [58] & 0 & 0 & -[28] & 0 & 0 & [25] & 0 \\ 0 & 0 & [69] & 0 & 0 & -[39] & 0 & 0 & [36] \\ 0 & 0 & 0 & [56] & -[46] & [45] & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & [89] & -[79] & [78] \end{pmatrix}$$

However, the fact that $\operatorname{rk}(\Lambda) = 6$ ensures the existence of nontrivial liftings of $1, \ldots, 9$. Furthermore, up to relabelling the points, the only simple rank-3 matroid over $[9]$ with no triplets of concurrent lines which is strictly bigger than G_3^3 in the dependency order is the maximal matroid G from Example 9. As a consequence, there are only two options for a nontrivial solution of the linear system $\Lambda(z_1 \ldots z_9)^t = (0 \ldots 0)^t$. Either it fully realizes G_3^3, or it realizes the matroid G (because of its maximality).

We conclude the example by pointing out that, an arbitrarily small translation of 2 points in any realization of G allows to fully realize a 3×3 grid.

Notation 7. *Let $\mathcal{C}$ be a linear point-line configuration with n points that is not strictly liftable over $\mathbb{C}$, and let ε be a positive real number. We say that $\mathcal{C}$ has property $*$ if for any n-tuple of collinear points in $\mathbb{P}^2_{\mathbb{C}}$ and any value of ε, there exists a lifting of the n points to a degenerate realization $C \in \mathbb{C}^{3n}$ of $\mathcal{C}$ and a full realization $C' \in \mathbb{C}^{3n}$ of $\mathcal{C}$ such that $||C - C'|| < \varepsilon$.*

Example 10 shows that 3×3 grids have the property $*$.

Definition 13 (Liftable point-line configuration)**.** *A linear point-line configuration $\mathcal{C}$ is liftable over $\mathbb{C}$ if:*

- *$\mathcal{C}$ is strictly liftable over $\mathbb{C}$, or*
- *$\mathcal{C}$ has the property $*$ in $\mathbb{P}^2_{\mathbb{C}}$.*

Remark 4. *The liftability property is preserved by certain operations on point-line configurations. If $\mathcal{C}$ is liftable, then $\mathcal{C} \setminus \ell$ (if realizable) is still liftable for any $\ell \in \mathcal{L}$. In addition, adding a point to a* *line (with no extra collinearity requirement) does not affect liftability. In this case, both the number of points and the rank of the collinearity matrix increase by one.*

Lemma 4. *Let M be a simple matroid of rank 3 over $[n]$, whose associated point-line configuration $\mathcal{C}_M$ is liftable. Then, any realization of the 2-uniform matroid over $[n]$, in $\mathbb{C}^{3n}$, is an element of the matroid variety V_M.*

Proof. It is enough to prove the result when $\mathcal{C}_M$ is strictly liftable. Let $C \in \mathbb{C}^{3n}$ be a realization of the 2-uniform matroid over $[n]$. Then the coordinates $C_{1,1}, \ldots, C_{3,n}$ of C can be seen as the (x, y, z)-coordinates of n collinear points in the projective plane. These can be represented with the $3 \times n$ matrix below, up to performing a change in coordinates in $\mathbb{P}^2_{\mathbb{C}}$—as in Construction 1—which yields, in turn, a change in coordinates on $\mathbb{C}^{3n}$.

$$C = \begin{pmatrix} C_{1,1} & C_{1,2} & \ldots & C_{1,n} \\ C_{2,1} & C_{2,2} & \ldots & C_{2,n} \\ C_{3,1} & C_{3,2} & \ldots & C_{3,n} \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & \ldots & x_n \\ 1 & 1 & \ldots & 1 \\ 0 & 0 & \ldots & 0 \end{pmatrix}.$$

The liftability of $\mathcal{C}_M$ ensures the existence of $z_1, \ldots, z_n \in \mathbb{C}$, such that the points of coordinates $(x_i \, 1 \, z_i)^t$, for $i = 1, \ldots, n$, are a nondegenerate realization of M. Now, let ε be an arbitrary positive real number and set $z := \max_{i=1,\ldots,n} |z_i|$. Let $D \in \mathbb{C}^{3n}$ be the point of the following coordinates:

$$D = \begin{pmatrix} x_1 & x_2 & \ldots & x_n \\ 1 & 1 & \ldots & 1 \\ \frac{\varepsilon}{nz} z_1 & \frac{\varepsilon}{nz} z_2 & \ldots & \frac{\varepsilon}{nz} z_n \end{pmatrix}.$$

The point D is still a realization of M because Γ_M is a semialgebraic set defined by the (non-)vanishing of determinants of 3×3 matrices whose columns are (x, y, z)-coordinates of points in the projective plane. In our case, by multilinearity of determinants:

$$\begin{bmatrix} x_i & x_j & x_k \\ 1 & 1 & 1 \\ \frac{\varepsilon}{nz} z_i & \frac{\varepsilon}{nz} z_j & \frac{\varepsilon}{nz} z_k \end{bmatrix} = \frac{\varepsilon}{nz} \begin{bmatrix} x_i & x_j & x_k \\ 1 & 1 & 1 \\ z_i & z_j & z_k \end{bmatrix}, \qquad \text{for any } 1 \leq i < j < k \leq n.$$

As the left-hand side vanishes if and only if the right-hand side vanishes; this proves that any Euclidean open subset of $\mathbb{C}^{3n}$ containing C intersects Γ_M. In turn, this implies that any Zariski open subset containing C intersects Γ_M, thus $C \in \overline{\Gamma_M} = V_M$. □

Definition 14 (Quasi-liftable configuration)**.** *A linear point-line configuration $\mathcal{C} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ is called quasi-liftable if $\mathcal{C}$ is not liftable but $\mathcal{C} \setminus \ell$ is liftable for every $\ell \in \mathcal{L}$.*

Example 11. *Our configurations of interest, namely the quadrilateral set and the* 3×4 *grid, are quasi-liftable (see Figure 5b,c). However, when a line is added to a quasi-liftable configuration, the quasi-liftability property is not preserved.*

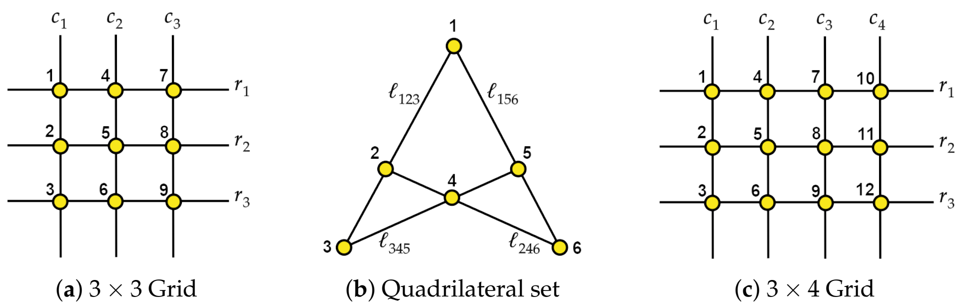

Figure 5. Examples of liftable and quasi-liftable plane arrangements.

3.2. Irreducible Decomposition of Varieties of (Quasi-)Liftable Configurations

In this section, we prove the irreducibility of the matroid varieties of liftable and quasi-liftable point-line configurations, assuming there are no triplets of concurrent lines in the configuration. Specifically, in Theorem 4, we establish the irreducibility of the matroid variety in both cases. However, for the circuit variety, we demonstrate that while it is irreducible for the liftable configuration (Theorem 5), the circuit variety of the quasi-liftable matroid has two irreducible components (Corollary 1).

In this section, unless stated otherwise, we assume that all configurations are for simple matroids of rank three and preserve their realizability when considering a subconfiguration.

We first prove the irreducibility of the matroid and circuit varieties of liftable configurations.

Theorem 4. *Let M be a matroid, whose associated point-line configuration $\mathcal{C}_M$ has no triplets of concurrent lines. Then, the matroid variety V_M is irreducible.*

Proof. If $\mathcal{C}_M$ has less than 6 lines, then V_M is irreducible by Proposition 3. Assume that $\mathcal{C}_M$ has more than 6 lines. Consider a subconfiguration $\mathcal{C}' = \mathcal{C}_M \setminus \{\ell_1, \ldots, \ell_{\#\mathcal{L}_M - 6}\}$, for $\{\ell_1, \ldots, \ell_{\#\mathcal{L}_M - 6}\} \subseteq \mathcal{L}_M$. Now, let M' be the corresponding matroid. Then, $\Gamma_{M'}$ is irreducible with respect to the Zariski topology, as $V_{M'}$ is irreducible by Proposition 3 and $\Gamma_{M'}$ is Zariski dense in $V_{M'}$. Note that $\#\mathcal{L}_p \leq 2$ for any $p \in \mathcal{P}_M$ by assumption. Let us now consider $\mathcal{C}'' = \mathcal{C}_M \setminus \{\ell_1, \ldots, \ell_{\#\mathcal{L}_M - 7}\}$. Among the intermediate configurations between $\mathcal{C}'$ and $\mathcal{C}_M$, $\mathcal{C}''$ is such that $\mathcal{C}' = \mathcal{C}'' \setminus \ell_{\#\mathcal{L}_M - 6}$ (denote as M'' the corresponding matroid). Theorem 3 implies that $\Gamma_{M''}$ is irreducible and so is its Zariski closure $V_{M''}$. If $\mathcal{C}'' = \mathcal{C}_M$, we are done. If not, we introduce a configuration $\mathcal{C}'''$ and we argue analogously. The process terminates after $\#\mathcal{L}_M - 6$ steps. □

Remark 5. *In view of Lemma 1, Theorem 4 generalizes Proposition 5.9, and Theorem 5.11 from [18]. Thereby, an analogous result was proved for forest configurations.*

Recall that for every matroid M, the circuit variety $V_{\mathscr{C}(M)}$ is equal to the combinatorial closure V_M^{comb}, but not necessarily to the matroid variety V_M; see and Definition 5 and Proposition 1. In the following theorem, we demonstrate that for liftable configurations, all these varieties coincide. We first establish the theorem and subsequently prove the technical lemma, Lemma 5, used in the proof.

Theorem 5. *Let M be a matroid whose associated point-line configuration $\mathcal{C}_M$ is liftable and has no triplets of concurrent lines. Then,*

$$V_{\mathscr{C}(M)} = V_M \qquad \textit{and equivalently} \qquad \sqrt{I_{\mathscr{C}(M)}} = I_M.$$

Proof. We prove the result by induction on the number of lines in $\mathcal{C}_M$.

If $\mathcal{C}_M$ has no lines, then M is the 3-uniform matroid. In this case, the realization space Γ_M is a Zariski open subset, and $V_M = \overline{\Gamma_M} = \mathbb{C}^{3n} = V_{\mathscr{C}(M)}$.

Assume that the result is true for any matroid fulfilling the hypotheses with $\leq m$ lines. Consider a configuration $\mathcal{C}_M$ with $m+1$ lines and let C be a point in $V_{\mathscr{C}(M)} \setminus \Gamma_M$, which equals to $V_M^{\text{comb}} \setminus \Gamma_M$, by Proposition 1. We want to prove that $C \in V_M$. If M is a matroid over $[n]$, the point $C \in \mathbb{C}^{3n}$ realizes a matroid $N \geq M$, over $[n]$, which is dependent for M. In particular, the coordinates of C can be represented as a matrix whose columns are the (x, y, z)-coordinates of n points in the projective plane. As $\mathcal{C}_M$ is liftable (Lemma 5), we restrict it to the case where N is a simple matroid.

If the associated point-line configuration $\mathcal{C}_N$ consists of collinear points, the result holds by Lemma 4. Otherwise, for any line $\ell \in \mathcal{L}_M$, there is a well-defined projection $\pi_\ell \colon \mathbb{C}^{3n} \to \mathbb{C}^{3k}$, for some $k \leq n$, which deletes from the matrix C the columns that are coordinates of points in $\mathcal{C}_M$, but not in $\mathcal{C}_M \setminus \ell$. Now, for any line ℓ, let us denote M_ℓ for the simple matroid whose point-line configuration is $\mathcal{C}_M \setminus \ell$. Then, $\pi_\ell(C) \in V_{M_\ell}$ because $\mathcal{C}_M \setminus \ell$ is liftable. Thus, $C \in \bigcap_{\ell \in \mathcal{L}} \pi_\ell^{-1}(V_{M_\ell})$. To conclude, we prove that $\bigcap_{\ell \in \mathcal{L}} \pi_\ell^{-1}(V_{M_\ell}) \subseteq V_M$. This is equivalent to the inclusion:

$$\left(\bigcap_{\ell \in \mathcal{L}} \pi_\ell^{-1}(V_{M_\ell})\right)^c \supseteq (V_M)^c = \left(\overline{\Gamma_M}\right)^c = (\Gamma_M^c)^\circ.$$

Now, as M is not 3-uniform, $(\Gamma_M^c)^\circ$ is not empty. Let $D = (D_{1,1}, \ldots, D_{3,n})$ be in the Zariski interior of Γ_M^c. We show that there exists a line ℓ such that $\pi_\ell(D) \notin V_{M_\ell}$. Now, Γ_M is the semialgebraic set defined by $p_i(x_{1,1}, \ldots, x_{3,n}) = 0$ and $q(x_{1,1}, \ldots, x_{3,n}) \neq 0$ for certain homogeneous polynomials p_i and q imposing, respectively, the dependence and independence relations for the matroid M.

If one of the polynomials p_i does not vanish when evaluated at the coordinates of D, then there is a collinearity of $\mathcal{C}_M$, which is not satisfied by D. Let ℓ be any line in $\mathcal{L}_M$ not requiring the critical collinearity, then $\pi_\ell(D) \notin V_{M_\ell}$. Hence, $C \in \bigcap_{\ell \in \mathcal{L}} \pi_\ell^{-1}(V_{M_\ell}) \subseteq V_M$.

The only excluded case is when, for all i, $p_i(D_{1,1}, \ldots, D_{3,n}) = q(D_{1,1}, \ldots, D_{3,n}) = 0$. We show that this cannot happen; specifically, we prove that D is a point of V_M. On the one hand, the vanishing of the p_i's implies that $D \in V_{C(M)} \setminus \Gamma_M$. On the other hand, as long as D is in the interior of Γ_M^c, there exists a Zariski open subset entirely contained in Γ_M^c and containing D.

Now, Lemma 5 allows us to restrict to the case where D realizes a simple matroid, dependent for M, and Lemma 4 excludes the possibility that the n points given by the realization of D are collinear. In other words, the point-line configuration realized by D has n distinct points, among which at least three of them are not collinear, and it respects all the collinearities of M. There are some extra-collinear points which are the projective image of a subconfiguration of $\mathcal{C}_M$. As $\mathcal{C}_M$ is liftable, all its subconfigurations are liftable as well. Thus, the unwanted collinearity can be resolved by a small arbitrary lifting.

To conclude, if D satisfies $p_i(D_{1,1}, \ldots, D_{3,n}) = q(D_{1,1}, \ldots, D_{3,n}) = 0$, then any Euclidean open subset containing D intersects Γ_M, contradicting that D is in the Zariski interior of Γ_M^c. □

We now prove the technical lemma used in the proof of Theorem 5.

Lemma 5. *Let M be a matroid whose point-line configuration $\mathcal{C}_M$ is liftable and has no triplets of concurrent lines. Let $C \in \mathbb{C}^{3n}$ be a point in $V_{\mathscr{C}(M)} \setminus \Gamma_M$, realizing a nonsimple matroid $N > M$. Then, for any $\varepsilon > 0$, there exists C' in the Euclidean open ball $B(C, \varepsilon)$ realizing a simple matroid N' which is dependent for M, or equivalently, is $M \leq N'$. Further, if $C' \in V_M$, then $C \in V_M$.*

Proof. The coordinates of the point C can be represented by the $3 \times n$ matrix:

$$C = \begin{pmatrix} C_{1,1} & C_{1,2} & \ldots & C_{1,n} \\ C_{2,1} & C_{2,2} & \ldots & C_{2,n} \\ C_{3,1} & C_{3,2} & \ldots & C_{3,n} \end{pmatrix} = (c_1\ c_2\ \ldots\ c_n),$$

where $c_i \in \mathbb{C}^3$ for each i. Since the matroid N, realized by C, is nonsimple, the matrix above might contain zero vectors $c_i = 0$ or linearly dependent pairs of columns $c_i = \lambda \cdot c_j$ for some $\lambda \in \mathbb{C}^*$. To prove the result, we proceed by induction on n. Since the base case is trivial, we may assume that the result holds for any subconfiguration of $\mathcal{C}_M$, having less than n points. We recall that the point-line configuration $\mathcal{C}_N$ of N is associated with the simplification of the matroid.

Fixed a positive real number ε, the procedure to obtain the point C' will consist in perturbing the columns of C finitely many times by adding to them vectors whose norm is bounded by ε. For the choice of the base-field $\mathbb{C}$, and the continuity property of the Euclidean distance over $\mathbb{C}^{3n}$, the value of $||C - C'||$ will eventually be bounded by a continuous function of ε.

By assumption we have that $N > M$, so every flat of rank 2 in M is dependent in N. This means that for every line ℓ of $\mathcal{C}_M$, the points $c_\ell := \{c_i : i \in \ell\}$ lie in a 2-dimensional linear subspace $\hat{\ell} \subseteq \mathbb{C}^3$, which we call the *ghost line* of ℓ. There are two possibilities for the points c_ℓ:

- Either their linear span is 2-dimensional, in which case $\hat{\ell}$ is uniquely determined;
- The linear span has dimension strictly less than 2, in which case we choose $\hat{\ell}$ generically such that it contains c_ℓ.

We construct the point C' algorithmically by performing the following steps.

Loops: Suppose there is a zero column $c_i = 0$ of C. By assumption, the point i of $\mathcal{C}_M$ lies on at most two lines ℓ_1 and ℓ_2 (if this is not the case, then the same strategy can be performed with fewer constraints). Let $v \in \hat{\ell}_1 \cap \hat{\ell}_2$ be a point in the intersection of the two ghost lines. Without loss of generality, we may assume that $|v| = 1$. We perturb c_i by moving it to the point $\varepsilon' v$. This results in a configuration N' with $N' \geq M$.

From now on, we assume that all points c_i are nonzero.

Multiple points: Let $\mathscr{I} = \{I_1, \ldots, I_K\}$ be the set of multiple points of $\mathcal{C}_N$, i.e., the rank-one flats of N of cardinality strictly higher than one. For any $I_k \in \mathscr{I}$ let $\mathscr{I}_k = \{i_1, \ldots, i_{n_k}\}$ be the corresponding set of points in $\mathcal{C}_M$ such that $\mathrm{Span}_{\mathbb{C}}\{c_{i_1}, \ldots, c_{i_{n_k}}\}$ is one-dimensional. We introduce a procedure to perturb these points so that they realize the point-line configuration of a simple matroid, dependent on M. We address cases based on the i_l's. Initially, we handle points lying on no lines of $\mathcal{C}_M$ (step S1) and points lying on one line of $\mathcal{C}_M$ (step S2). At this stage, all multiple points can be assumed to consist of points belonging to two lines of $\mathcal{C}_M$. Now, either all (ghost) lines through a multiple point coincide or not. In the former case, we proceed as in step S3; in the latter case, we apply steps S4 and S5.

S1. For any index $k = 1, \ldots, K$ and for any $l = 1, \ldots, n_k$, if the point $i_l \in \mathscr{I}_k$ does not belong to any line in $\mathcal{C}_M$, then it can be translated along any direction. In other words, we perturb the column c_{i_l} by adding to it a vector εv, where v has unitary norm (see Figure 6 for an example). In turn, we can assume that for any $k = 1, \ldots, K$, all points in $\mathscr{I}_k$ lie at least on a line in $\mathcal{C}_M$.

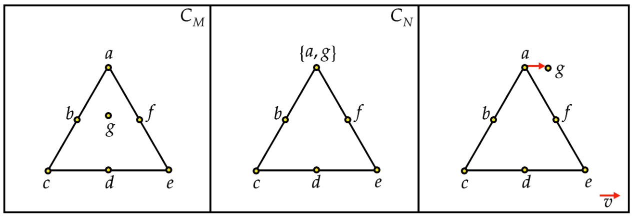

Figure 6. The figure refers to step S1 and illustrates how to solve the double point $\{a, g\}$. In view of the fact that the point g does not lie on any line of $\mathcal{C}_M$, it can be translated along the direction v.

S2. For any $k = 1, \ldots, K$ and for any $l = 1, \ldots, n_k$, we check whether the point $i_l \in \mathscr{I}_k$ belongs to a single line ℓ in $\mathcal{C}_M$. If this is the case, the point i_l can be realized by translating the corresponding column c_i along the ghost line $\hat{\ell}$, as in Figure 7. Thus, we can further assume that for any $k = 1, \ldots, K$, all points in $\mathscr{I}_k$ lie at the intersection of two lines in $\mathcal{C}_M$.

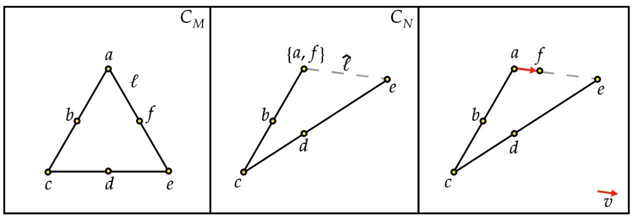

Figure 7. The figure illustrates step S2, and how to solve the double point $\{a, f\}$. In the starting configuration $\mathcal{C}_M$, the point f lies solely on the line ℓ and can be translated along the direction v.

S3. For any k, if all the (ghost) lines through I_k coincide on a unique (ghost) line ℓ, then we perturb all points of $\mathscr{I}_k$ along ℓ. Figure 8 illustrates that this operation might not lead to a full realization of $\mathcal{C}_M$, as it does not take the independence relations into account. However, it generates all required collinearities, leading to a dependent matroid for M. Any remaining multiple point now lies at the intersection of the realization of at least two nonoverlapping lines.

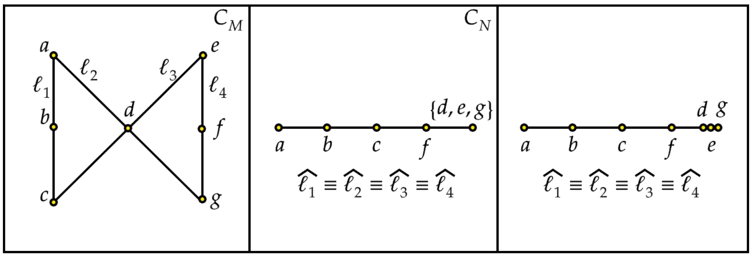

Figure 8. The realization of lines $\ell_1, \ldots, \ell_4$ of $\mathcal{C}_M$ all coincide in $\mathcal{C}_N$. Therefore, following step S3, the triple point $\{d, e, g\}$ can be split along the line $\hat{\ell}_1 \equiv \cdots \equiv \hat{\ell}_4$ to realize all collinearities of $\mathcal{C}_M$.

S4. We now remove all remaining multiple points from the configuration, and we consider the multiple lines L that are incident to them. We resolve the multiple lines using a lifting procedure, which can be followed graphically from Figure 9. We consider the following two steps:

S4.1. In view of the liftability property of $\mathcal{C}_M$ and the induction hypothesis, we can perform a lifting of the subconfiguration of N that involves only the remaining points on L and the (ghost) lines that coincide with L. Here, we pick our projection point in general position, away from all lines generated by points of $\mathcal{C}_N$. It follows from the proof of Lemma 4 that the distance between C and the lifted configuration can be bounded by ε.

S4.2. If a lifted point i was the intersection of L with another (ghost) line $\hat{\ell}$, prior to the lifting, then we need to further perturb it. In $\mathcal{C}_M$, the point i lies at the intersection of two lines ℓ and ℓ'. In $\mathcal{C}_N$, $\hat{\ell}'$ coincides with the line L. After the lifting, we have that $\hat{\ell}'$ is lifted and intersects $\hat{\ell}$ at a unique point. We redefine i to be this unique intersection point, whose distance from the previous lifted point is a continuous function of ε.

For any lifted (ghost) line resulting from this operation and containing a point in one of the multiple points through ℓ, we can assume, without loss of generality, that it is still arbitrarily close to the concerned multiple points. This can be explained considering that if the multiple points had not been removed from the configuration, the lifting could still have been performed. In particular, any such lifted line intersects all the other lines incident to the multiple point in a neighborhood of the multiple point itself.

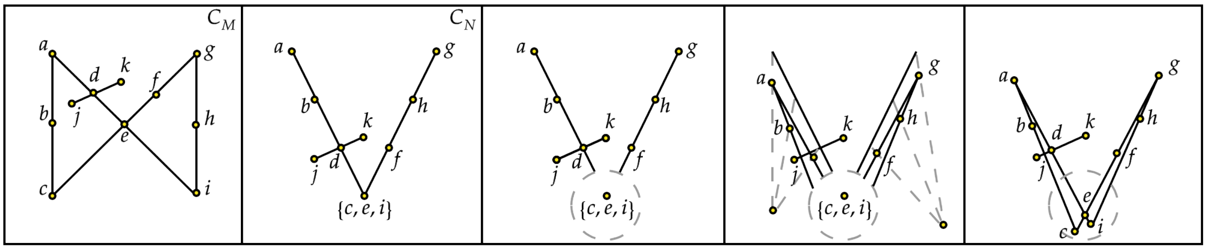

Figure 9. The figure demonstrates step S4 in action from left to right. Here, we address a triple point through liftings. Both lines incident to c, e, i in $\mathcal{C}_N$ realize two rank-two flats of M. Therefore, a lifting procedure is necessary to resolve the multiple point. In the final square, we illustrate how to intersect the lifted lines to complete the configuration.

S5. For any multiple point $I_k \in \mathscr{I}$, we consider the lines through I_k excluded from step S4. Namely, those corresponding to rank-two flats of N that do not contain more than a rank-two flat of M. Let m_k be the number of such lines. If $m_k \geq 3$, we have to perturb $m_k - 2$ of them so that they are not all concurrent on the same point in the projective plane. This can be performed considering that none of these lines is a multiple line, and for any such line ℓ, we perform the following two steps, depicted in Figure 10:

S5.1. We pick a generic direction $v \in \mathbb{C}^3$. Our goal is moving all points on ℓ in the direction of v.

S5.2. We select a basis B for the (ghost) line ℓ. Then, fixing a unitary vector v, we consider the linear subspace $\mu = \mathrm{Span}_{\mathbb{C}}\{b + \varepsilon v : b \in B\}$. For each point i lying on ℓ, we perturb i as follows. By assumption, i belongs to exactly one or two lines of $\mathcal{C}_M$. If i belongs to two lines ℓ and ℓ', then we move i to the closest point in the intersection of the ghost line $\hat{\ell}'$ and μ. Otherwise, if i belongs uniquely to the line ℓ, we move i to the closest point of μ.

The total distance that points have moved is a continuous function in ε'.

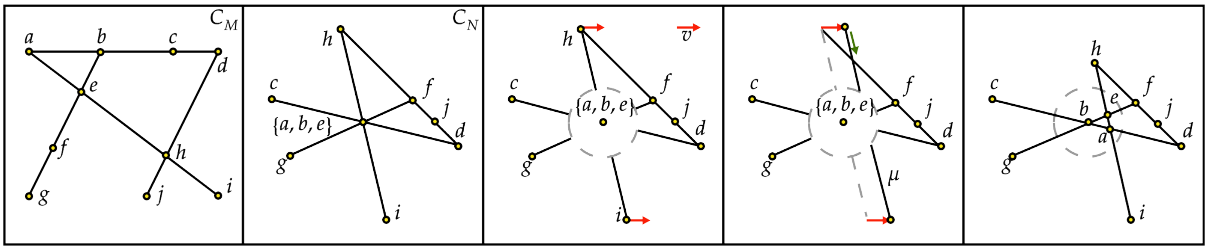

Figure 10. From left to right, the figure shows how a triple point can be solved via the line translation introduced in step S5. The point $\{a, b, e\}$ in $\mathcal{C}_N$ is such that $m_{\{a,b,e\}} = 3$. Therefore, it suffices to perform the translation of one of the three lines incident to it. As in Figure 8, the final configuration realizes a matroid which is dependent for M.

S6. For any k, after having performed steps 4 and 5, all lines incident to $I_k \in \mathscr{I}$ now intersect in points whose coordinates can be obtained by adding a vector εv to the coordinates of I_k (v is supposed to have unit norm). Therefore, we can finally resolve all multiple points by taking the intersection of the corresponding lines, as in the right-most representations of Figures 9 and 10.

By construction, the point C' and its associated matroid N' has the desired properties. $\square$

Before stating our decomposition theorem for quasi-liftable configurations, we state the following remark which allows us to restrict to connected point-line configurations.

Remark 6. *Let I and J be ideals in polynomial rings $\mathbb{C}[x_1, \ldots, x_n]$ and $\mathbb{C}[y_1, \ldots, y_m]$, respectively, with prime decompositions $I = I_1 \cap \cdots \cap I_k$ and $J = J_1 \cap \cdots \cap J_h$. Then,*

$$I + J = I \otimes \mathbb{C}[y_1, \ldots, y_m] + \mathbb{C}[x_1, \ldots, x_n] \otimes J = \bigcap_{\substack{i=1,\ldots,k \\ j=1,\ldots,h}} I_i + J_j.$$

Moreover, [29], Theorem 7.4.i ensures that, in our setting, tensor product and finite intersections commute, and [30], Proposition 5.17.b, implies that the ideals $I_i + J_j$ are prime.

The above remark applies in particular to disconnected configurations, allowing us not to lose any generality by adding the connectivity assumption to the statement below.

Corollary 1 (Decomposition theorem for quasi-liftable configurations)**.** *Let M be a matroid, whose point-line configuration $\mathcal{C}_M$ is connected, quasi-liftable and has the property that every point lies on at most two lines. Then,*

$$V_{\mathscr{C}(M)} = V_0 \cup V_M \qquad \text{and equivalently} \qquad \sqrt{I_{\mathscr{C}(M)}} = I_0 \cap I_M, \tag{2}$$

where V_0 is the matroid variety whose associated configuration is a line with n marked points. Furthermore, the decompositions in (2) are, respectively, irreducible and prime.

Proof. Let $C \in V_{\mathscr{C}(M)}$. By Proposition 1, we have that $C \in V_M^{\text{comb}}$. If $C \in \Gamma_M$, then $C \in V_M = \overline{\Gamma_M}$. Thus, from now on, we assume that $C \in V_M^{\text{comb}} \setminus \Gamma_M$ and we prove that $C \in V_0 \cup V_M$. If M is a matroid over $[n]$, the point $C \in \mathbb{C}^{3n}$ realizes a matroid $N \geq M$ over $[n]$, which is dependent for M.

Now, if the point-line configuration $\mathcal{C}_N$ consists of collinear points, then $C \in V_0$. If $\mathcal{C}_N$ has more than one line, then $\mathcal{C}_M$ has at least two lines as well. In particular, if some subconfigurations of $\mathcal{C}_M$ are flattened in $\mathcal{C}_N$, then these subconfigurations are liftable as $\mathcal{C}_M$ is quasi-liftable and $\mathcal{C}_N$ consists of more than one line. Thus, in view of Lemma 5, we can assume that N is simple. For any line $\ell \in \mathcal{L}_M$, there is a well-defined projection $\pi_\ell \colon \mathbb{C}^{3n} \to \mathbb{C}^{3m}$, for some $m \leq n$, which deletes the coordinates of the points which are in $\mathcal{C}_M$, but not in $\mathcal{C}_M \setminus \ell$. Now, for any line ℓ, let us denote M_ℓ for the simple matroid whose point-line configuration is $\mathcal{C}_M \setminus \ell$. Then, $\pi_\ell(C) \in V_{M_\ell}$ because $\mathcal{C}_M$ is quasi-liftable. Thus, $C \in \bigcap_{\ell \in \mathcal{L}} \pi_\ell^{-1}(V_{M_\ell})$ which is contained in V_M by Theorem 5. □

3.3. The Ideal of Quasi-Liftable Configurations

We apply the results to compute the generators of an ideal whose radical contains the ideal of the matroid variety; see Proposition 4. We first set up our notation.

Let M be a matroid over $[n]$ with point-line configuration $\mathcal{C}_M$ and the collinearity matrix Λ. Consider the notation of Construction 1. Then, the nonzero entries of Λ are

$$x_i - x_j = \begin{bmatrix} x_i & x_j \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} x_i & x_j & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

for certain $1 \leq i < j \leq n$, where all columns are the coordinates of points P_i, P_j, and P, in a frame of reference $\{R_1, R_2, P, U\}$ with R_1, R_2, U being in general position with P.

Let $\mathfrak{P}(\Lambda_{i,j})$ be a polynomial in the entries of Λ, which is the determinant of a square submatrix of Λ itself. In this case, $\mathfrak{P}$ is the sum of products of k nonzero entries of Λ.

In Construction 1, the choice of P as a reference point of the projective plane was intended to simplify the notation. In general, one can fix an arbitrary frame of reference $\{R_1, R_2, R_3, U\}$ in $\mathbb{P}^2_{\mathbb{C}}$, and use it to take coordinates for $P = (x_P, y_P, z_P)$. Let Λ' be the matrix constructed as Λ but considering an arbitrary frame of reference. The nonzero entries of Λ' are

$$[P_i P_j P] = \begin{bmatrix} x_i & x_j & x_P \\ y_i & y_j & y_P \\ z_i & z_j & z_P \end{bmatrix}.$$

Let $\mathfrak{P}'(\Lambda'_{i,j})$ be the polynomial defined as $\mathfrak{P}$, but with entries in Λ'. Now, the vanishing of the polynomial $\mathfrak{P}$ is a projective invariant property. Since changes in coordinates are projective transformations, $\mathfrak{P}$ vanishes if and only if the polynomial $\mathfrak{P}'$ vanishes.

By multilinearity of determinants, the polynomial $\mathfrak{P}'$ is the linear combination of 3^k copies of $\mathfrak{P}'$ itself, each of them corresponding to a k-tuple of points of the frame of reference (with repetitions). Each copy is the sum of products of k 3×3 determinants of matrices whose first two columns are as in $\mathfrak{P}$ and whose last column is a point of the frame of reference, determined by the corresponding k-tuple (see Example 12). We call each copy of $\mathfrak{P}'$ an *extension* of $\mathfrak{P}$.

Example 12. *Consider a matroid M over $[5]$ with circuits $\mathcal{C}(M) = \{123, 345\}$. The configuration $\mathcal{C}_M$ consists of two lines incident at a point. Take 5 distinct points of $\mathbb{P}^2_{\mathbb{C}}$ lying on the same line and a point P not collinear with them. With the above notation and Construction 1, we have*

$$\Lambda = \begin{pmatrix} [23] & -[13] & [12] & 0 & 0 \\ 0 & 0 & [45] & -[35] & [34] \end{pmatrix}, \quad \text{and} \quad \Lambda' = \begin{pmatrix} [23P] & -[13P] & [12P] & 0 & 0 \\ 0 & 0 & [45P] & -[35P] & [34P] \end{pmatrix}.$$

Take $\mathfrak{P}$ (resp. $\mathfrak{P}'$) to be the minor generated by the second and third column of Λ (resp. Λ'). Then,

$$\begin{aligned} \mathfrak{P} &= -[13][45], \quad \text{and} \\ -\mathfrak{P}' &= [13P][45P] \overset{P=x_PR_1+y_PR_2+z_PR_3}{=} [13\, x_PR_1 + y_PR_2 + z_PR_3][45\, x_PR_1 + y_PR_2 + z_PR_3] \\ &= x_P^2[13R_1][45R_1] + x_Py_P[13R_1][45R_2] + x_Pz_P[13R_1][45R_3] + \\ &\quad + x_Py_P[13R_2][45R_1] + y_P^2[13R_2][45R_2] + y_Pz_P[13R_2][45R_3] + \\ &\quad + x_Pz_P[13R_3][45R_1] + y_Pz_P[13R_3][45R_2] + z_P^2[13R_3][45R_3]. \end{aligned}$$

Here, each summand of $\mathfrak{P}'$ is a copy of $\mathfrak{P}$ associated with a pair (R_i, R_j), with $i, j \in \{1, 2, 3\}$, which is an extension of $\mathfrak{P}$.

Proposition 4. *Let M be a maximal matroid over $[n]$, whose point-line configuration $\mathcal{C}_M$ is quasi-liftable. Then the ideal I_M is contained in the radical of the ideal I generated by the following:*

- *The collinearity conditions of $\mathcal{C}_M$;*
- *The extensions of the $n-2$ minors of the collinearity matrix $\Lambda_{\mathcal{C}_M}$.*

Proof. From Example 8, we know that an n-tuple of different points in $\mathbb{C}^{3n}$ realizes M if and only if it contains at least 3 non-collinear points. Every point P in $V(I)$ belongs to the interior of the matroid variety V_M since, after a suitable change in coordinates, it satisfies the equality $\Lambda_{\mathcal{C}_M}(z_1 \dots z_n)^t = (0 \dots 0)^t$ and has (at least) three non-collinear points in the projective plane (we are considering $n-2$ minors). Thus, $V(I) \subset V_M$ and so $I_M \subset \sqrt{I}$. □

In the following sections, we prove that for the quadrilateral set and the 3×4 grid, the two ideals in Proposition 4 are equal. Furthermore, we see that for both examples, the ideal I is actually radical, and we provide a minimal generating set for I.

4. The Quadrilateral Set

We now apply the results of Section 3 to the quadrilateral set. Furthermore, we provide a minimal and geometrically meaningful set of generators for the corresponding ideal. Finally, we outline the method for interpreting these generators from the arrangement, highlighting the underlying symmetries.

We recall that, as point-line configurations $\mathcal{C}_{QS}$, quadrilateral sets arise from the simple matroid QS, introduced in Example 4. The matroid QS is realizable over $\mathbb{C}$, thus $\mathcal{C}_{QS}$ can be embedded in $\mathbb{P}^2_{\mathbb{C}}$. We take $(x\,y\,z)$-coordinates for $1, \dots, 6$, with respect to a fixed frame of reference $\{R_1, R_2, R_3, U\}$. Without loss of generality, we assume that the points of the frame are in general position with any couple of points of the embedded configuration, and we encode the coordinates in the matrix:

$$X = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\ z_1 & z_2 & z_3 & z_4 & z_5 & z_6 \end{pmatrix}.$$

Our aim is to understand the variety V_{QS}, whose ideal, by Proposition 1 is:

$$I_{QS} = \sqrt{\langle [123]_X, [156]_X, [246]_X, [345]_X \rangle : J_{QS}^{\infty}}$$

where J_{QS} is the principal ideal $\langle [124]_X \cdot [125]_X \cdots [456]_X \rangle$. We achieve this by identifying a geometrically meaningful set of generators for the ideal I_{QS}.

Notation 8. *Let ℓ be a line in $\mathcal{L}$, whose points are P^1_ℓ, P^2_ℓ and P^3_ℓ. Let P, P^1, P^2 and P^3 be points of the projective plane, not necessarily distinct. Denote as P^1_m, P^2_m and P^3_m the three points of $\mathcal{P}$ which do not belong to ℓ: two of them will be collinear with P^1_ℓ (wlog P^1_m and P^2_m), two with P^2_ℓ (wlog P^2_m and P^3_m), and two with P^3_ℓ (wlog P^1_m and P^3_m). We denote the following:*

$$QS(\ell; P^1, P^2, P^3) = [P^1_\ell P^1_m P^1][P^2_\ell P^2_m P^2][P^3_\ell P^3_m P^3] - [P^1_\ell P^2_m P^1][P^2_\ell P^3_m P^2][P^3_\ell P^1_m P^3]$$
$$QS(\ell; P) = [P^1_\ell P^1_m P][P^2_\ell P^2_m P][P^3_\ell P^3_m P] - [P^1_\ell P^2_m P][P^2_\ell P^3_m P][P^3_\ell P^1_m P].$$

Note that, once the line ℓ and the points P, P^1, P^2, P^3 are fixed, the polynomials above are well-defined up to the sign. However, for our purposes, we only concern ourselves with their (non)vanishing.

Example 13. *Let us compute the QS-polynomials for $\ell = \ell_{123}$, $P^1 = P^2 = R_1$ and $P^3 = R_2$.*

$$\begin{aligned} QS(\ell_{123}; R_1, R_1, R_2) &= [15R_1][26R_1][34R_2] - [16R_1][25R_1][34R_2] \\ &= \begin{bmatrix} x_1 & x_5 & 1 \\ y_1 & y_5 & 0 \\ z_1 & z_5 & 0 \end{bmatrix} \begin{bmatrix} x_2 & x_6 & 1 \\ y_2 & y_6 & 0 \\ z_2 & z_6 & 0 \end{bmatrix} \begin{bmatrix} x_3 & x_4 & 0 \\ y_3 & y_4 & 1 \\ z_3 & z_4 & 0 \end{bmatrix} - \begin{bmatrix} x_1 & x_6 & 1 \\ y_1 & y_6 & 0 \\ z_1 & z_6 & 0 \end{bmatrix} \begin{bmatrix} x_2 & x_4 & 1 \\ y_2 & y_4 & 0 \\ z_2 & z_4 & 0 \end{bmatrix} \begin{bmatrix} x_3 & x_5 & 0 \\ y_3 & y_5 & 1 \\ z_3 & z_5 & 0 \end{bmatrix} \\ &= -x_5y_4y_6z_1z_2z_3 + x_4y_5y_6z_1z_2z_3 - x_3y_5y_6z_1z_2z_4 + x_5y_2y_6z_1z_3z_4 + \\ &\quad + x_3y_4y_6z_1z_2z_5 - x_4y_1y_6z_2z_3z_5 - x_3y_2y_6z_1z_4z_5 + x_3y_1y_6z_2z_4z_5 + \\ &\quad - x_4y_2y_5z_1z_3z_6 + x_5y_1y_4z_2z_3z_6 + x_3y_2y_5z_1z_4z_6 - x_5y_1y_2z_3z_4z_6 + \\ &\quad - x_3y_1y_4z_2z_5z_6 + x_4y_1y_2z_3z_5z_6. \end{aligned}$$

Remark 7. *We introduce two different multidegrees on the monomials of the ring $R = \mathbb{C}[x_1, \dots, z_6]$.*

- *The letter multidegree $(d_x, d_y, d_z) \in (\mathbb{Z}_{\geq 0})^3$, where d_x, d_y, and d_z are, respectively, the numbers of x, y, and z variables.*
- *The point multidegree $(d_1, \dots, d_6) \in (\mathbb{Z}_{\geq 0})^6$ where d_i is the number of variables corresponding to coordinates of the point p_i for any $i = 1, \dots 6$.*

Notice that, by construction, the QS-polynomials are homogeneous of point multidegree $(1, \dots, 1)$.

We now provide a family of polynomials that vanish when evaluated on the coordinates of the points of a quadrilateral set.

Theorem 6. *Let $\mathcal{C}_{QS}$ be a quadrilateral set in $\mathbb{P}^2_{\mathbb{C}}$. Then, for any line $\ell \in \mathcal{L}_{QS}$, and any three points $P^1, P^2, P^3 \in \mathbb{P}^2_{\mathbb{C}}$:*

$$QS(\ell; P^1, P^2, P^3) = 0.$$

Proof. We have to show that $QS(\ell; P^1, P^2, P^3) = 0$ for any choice of ℓ and points P^1, P^2, P^3. As a summand of $QS(\ell; P^1, P^2, P^3)$ vanishes if and only if the other vanishes too, we can assume, without loss of generality, that the points P^1, P^2, P^3 do not belong to any of the lines in $\mathcal{L}$. Due to the multilinearity of determinants, the claim follows if $QS(\ell; R_i, R_j, R_k) = 0$, for any line ℓ and any $(i, j, k) \in \{1, 2, 3\}^3$. (Note that we fixed $\{R_1, R_2, R_3, R_1 + R_2 + R_3\}$ as the frame of reference.)

We now show the argument for a particular choice of ℓ. It can be easily adapted for other possible choices. Let us assume $\ell = \ell_{123}$. We want to show that

$$QS(\ell; R_i, R_j, R_k) = [15R_i][26R_j][34R_k] - [16R_i][24R_j][35R_k] = 0. \quad (3)$$

By construction, $6 \in \ell_{156}$. Now, $\{1, 5, 1 + 5\}$ is a frame of reference for the line ℓ_{156}. This means that there exists a unique choice of $\alpha, \beta \in \mathbb{C}$ such that $\alpha 1 + \beta 5 = 6$. Here, it is important to remark that $1, \dots, 6$ are fixed representatives of the corresponding points in $\mathbb{P}^2_{\mathbb{C}}$, which makes the choice of α and β unique. If we plug this into $QS(\ell; R_i, R_j, R_k)$, we obtain

$$\begin{aligned} QS(\ell; R_i, R_j, R_k) &= [15R_i][26R_j][34R_k] - [16R_i][24R_j][35R_k] \\ &= [15R_i][26R_j][34R_k] - [1\,\boldsymbol{\alpha 1 + \beta 5}\,R_i][24R_j][35R_k] \\ &\overset{\text{multilin.}}{=} [15R_i][26R_j][34R_k] - \alpha[11R_i][24R_j][35R_k] - \beta[15R_i][24R_j][35R_k] \\ &= [15R_i][26R_j][34R_k] - \beta[15R_i][24R_j][35R_k]. \end{aligned}$$

In this way, we managed to have the same term as the first factor of both products. Via the same argument, the structure of the quadrilateral set also yields that $4 = \alpha' 2 + \beta' 6$ and $5 = \alpha'' 3 + \beta'' 4$ for a unique choice of $\alpha', \alpha'', \beta', \beta'' \in \mathbb{C}$. Finally, we have that

$$QS(\ell; R_i, R_j, R_k) = [15R_i][26R_j][34R_k] - \beta\beta'\beta''[15R_i][26R_j][34R_k] = (1 - \beta\beta'\beta'')[15R_i][26R_j][34R_k].$$

As a consequence, proving (3) is equivalent to show that $\beta\beta'\beta'' = 1$. On the other hand,

$$\begin{aligned} 5 &= \alpha'' 3 + \beta''\, 4 \\ &= \alpha'' 3 + \alpha'\beta'' 2 + \beta'\beta''\, 6 \\ &= \alpha'' 3 + \alpha'\beta'' 2 + \alpha\beta'\beta'' 1 + \beta\beta'\beta'' 5. \end{aligned}$$

Thus, $(1 - \beta\beta'\beta'')5 = \alpha'' 3 + \alpha'\beta'' 2 + \alpha\beta'\beta'' 1$. Here, on the l.h.s., there is another representation of point 5; whereas, on the r.h.s., there is a point in the line ℓ_{123}. As far as $5 \notin \ell_{123}$ by construction, the equality above holds if and only if both sides give $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$; that is if and only if $\beta\beta'\beta'' = 1$, as desired. □

Furthermore, projective transformations keep track of the vanishing of these polynomials.

Lemma 6. *The vanishing of a polynomial $QS(\ell; R_i, R_j, R_k) = 0$ is a projective invariant property.*

Proof. We prove the lemma for the line ℓ_{123}. We need to show that

$$QS(\ell_{123}; R_i, R_j, R_k) = [15R_i][26R_j][34R_j] - [16R_i][24R_j][35R_k] = 0$$

is a projective invariant property. Therefore, we consider $T \in GL(\mathbb{C}, 3)$ and $D \in \mathrm{diag}(\mathbb{C}, 3)$, and we write down $QS(T\ell_{123}D; TPD)$, as follows:

$$\begin{aligned} QS(T\ell_{123}D; TR_iD, TR_jD, TR_kD) &= [T1D\ T5D\ TR_iD][T2D\ T6D\ TR_jD][T3D\ T4D\ TR_kD] + \\ &\quad - [T1D\ T6D\ TR_iD][T2D\ T4D\ TR_jD][T3D\ T5D\ TR_kD] \\ &= \det T^3 \det D^3([15R_i][26R_j][34R_k] - [16R_i][24R_j][35R_k]). \end{aligned}$$

At this stage, we can see that

$$QS(T\ell_{123}D; TR_iD, TR_jD, TR_kD) = 0 \iff QS(\ell_{123}; R_i, R_j, R_k) = 0$$

which completes the proof. The same argument applies analogously to other choices of lines. □

In [25,26], the vanishing of the bracket polynomials $QS(\ell; R_3)$ is proved to characterize the liftability of six points in $\mathbb{P}^1_{\mathbb{C}}$ to a quadrilateral set. More generally, the whole family of QS-polynomials just introduced offers a characterization of the liftability of a 6-tuple of collinear points in $\mathbb{P}^2_{\mathbb{C}}$ to a quadrilateral set. In particular, we prove the following result.

Theorem 7. *Let r be a line in $\mathbb{P}^2_{\mathbb{C}}$ and $1, \ldots, 6$ distinct points of r. Consider the collection $\mathcal{L} = \{\ell_{123}, \ell_{156}, \ell_{246}, \ell_{345}\}$, where ℓ_{ijk} is the combinatorial line consisting of points i, j, and k. Then, the following statements are equivalent:*

i. *The points* $1, \ldots, 6$ *are the projective image of a quadrilateral set.*
ii. *The polynomials* $QS(\ell; P^1, P^2, P^3)$ *vanish for any* $\ell \in \mathcal{L}$ *and any* $P^1, P^2, P^3 \in \mathbb{P}^2_{\mathbb{C}}$.
iii. *The polynomials* $QS(\ell; R_i, R_j, R_k)$ *vanish for any* $\ell \in \mathcal{L}$ *and any* $(i, j, k) \in \{1, 2, 3\}^3$.
iv. *The polynomials* $QS(\ell_{123}; R_i, R_j, R_k)$ *vanish for any* $(i, j, k) \in \{1, 2, 3\}^3$.
v. *The polynomials* $QS(\ell_{123}; R_i, R_j, R_k)$ *vanish for any* $(i, j, k) \in \{1, 2, 3\}^3$ *with* $i \leq j \leq k$.

Proof. It is immediate to see that $(ii) \implies (iii) \implies (iv) \implies (v)$.

$(i) \implies (ii)$ By Lemma 6, the vanishing of the QS polynomials is a projective invariant property, thus when six collinear points are the projective image of a quadrilateral set, $QS(\ell; P^1, P^2, P^3) = 0$ where ℓ, P^1, P^2, and P^3 satisfy the assumptions of Theorem 6.

$(ii) \implies (i)$ Conversely, let $1, \ldots, 6$ be collinear points that make the polynomials $QS(\ell; P^1, P^2; P^3)$ vanish for $\ell \in \{\ell_{123}, \ell_{156}, \ell_{246}, \ell_{345}\}$. We then computationally verify that the vanishing of the QS polynomials is equivalent to requiring the collinearity matrix Λ_{QS} not to have maximal rank.

$$\Lambda_{QS} = \begin{pmatrix} [23] & -[13] & [12] & 0 & 0 & 0 \\ [56] & 0 & 0 & 0 & -[16] & [15] \\ 0 & [46] & 0 & -[26] & 0 & [24] \\ 0 & 0 & [45] & -[35] & [34] & 0 \end{pmatrix}$$

As a consequence, the linear system $\Lambda_{QS}(z_1 \ldots z_6)^t = (0 \ldots 0)^t$ has a solution space of dimension at least 3. In other words, by Construction 1, there exists a nondegenerate quadrilateral set whose image via the projection through P on the line r consists of exactly points $1, \ldots, 6$.

$(iii) \implies (ii)$ The implication is followed by the following equalities of ideals in $\mathbb{C}[x_1, \ldots, z_6]$:

$$\begin{aligned} &\langle [123], [156], [246], [345], QS(\ell; P^1, P^2, P^3) \text{ for } \ell \in \mathcal{L} \text{ and } P^1, P^2, P^3 \in \mathbb{P}^2_{\mathbb{C}} \rangle = \\ &\quad = \langle [123], [156], [246], [345], QS(\ell; R_i, R_j, R_k) \text{ for } \ell \in \mathcal{L} \text{ and } (i, j, k) \in \{1, 2, 3\}^3 \rangle. \end{aligned}$$

However, the equality of the two ideals holds by the multilinearity of determinants.

$(iv) \implies (iii)$ Similarly to the previous implication, we check that:

$$\begin{aligned} &\langle [123], [156], [246], [345], QS(\ell; R_i, R_j, R_k) \text{ for } \ell \in \mathcal{L} \text{ and } (i, j, k) \in \{1, 2, 3\}^3 \rangle = \\ &\quad = \langle [123], [156], [246], [345], QS(\ell_{123}; R_i, R_j, R_k) \text{ for } (i, j, k) \in \{1, 2, 3\}^3 \rangle. \end{aligned}$$

More generally, if we fix a line ℓ, the polynomials $QS(\ell; R_i, R_j, R_k)$ together with $[123]_X$, $[156]_X$, $[246]_X$, and $[345]_X$ generate the corresponding polynomials for the other lines.

Assuming that $QS(\ell_{123}; R_i, R_j, R_k) = 0$ for any $(i, j, k) \in \{1, 2, 3\}^3$ for any line $\ell \in \mathcal{L} \setminus \{\ell_{123}\}$, we construct a projection ϕ that sends ℓ_{123} to ℓ and keeps the quadrilateral set globally fixed (so that the intersection points in $\mathcal{P}$ are just permuted by ϕ). We can define these projections explicitly. The points $\{1, 2, 5, 4\}$ are a frame of reference for the projective plane and each permutation of these four points defines uniquely a projection ϕ that permutes the 3 diagonal points $(3, 6, 14 \wedge 25)$. If we add the requirement that $14 \wedge 25$ is fixed (which is necessary for the ϕ-stability of the quadrilateral set), we find the three desired projections. By Lemma 6 the vanishing of $QS(\ell_{123}; R_i, R_j, R_k) = 0$ is a projective invariant property. Hence, to obtain all the polynomials $QS(\ell; R_i, R_j, R_k)$, we can choose the generators among the polynomials $QS(\ell_{123}, R_i, R_j, R_k)$.

$(v) \implies (iv)$ We check the following equality of ideals in $\mathbb{C}[x_1, \ldots, z_6]$:

$$\begin{aligned} &\langle [123], [156], [246], [345], QS(\ell_{123}; R_i, R_j, R_k) \text{ for } (i, j, k) \in \{1, 2, 3\}^3 \rangle = \\ &= \langle [123], [156], [246], [345], QS(\ell_{123}; R_i, R_j, R_k) \text{ for } (i, j, k) \in \{1, 2, 3\}^3 \text{ with } i \leq j \leq k \rangle. \end{aligned}$$

We point out that a choice of (i,j,k) determines the letters appearing in the terms of $QS(\ell_{123};R_i,R_j,R_k)$. For example, $QS(\ell_{123};R_1,R_1,R_2)$ is the sum of monomials of degree 6, where each of them are the product of one x coordinate, two y coordinates, and three z coordinates. The same holds for $QS(\ell_{123};R_1,R_2,R_1)$ and $QS(\ell_{123};R_2,R_1,R_1)$. In spite of that,

$$QS(\ell_{123};R_1,R_1,R_2) \neq QS(\ell_{123};R_1,R_2,R_1) \neq QS(\ell_{123};R_2,R_1,R_1).$$

In other words, the selection of (i,j,k) determines the letter multidegree of the QS polynomial. However, multiple choices of (i,j,k) can yield the same letter multidegree. Specifically, having a generator per letter multidegree is sufficient to generate all the polynomials. The remaining QS-polynomials with the same multidegree are obtained by adding the corresponding generator to a polynomial combination of [123], [156], [246], and [345], with coefficients stored in Table 1. □

Table 1. This table allows one to reconstruct how the polynomials that have been removed from item (iii) to item (iv) can be written as a polynomial combination of the generators in (iv). Each block of rows corresponds to a different letter multidegree. In the first column, we select the excluded choices of R_i, R_j, R_k, and, in the second one, there is the generator of the ideal in (iv) having the same letter multidegree. Each of the excluded polynomials is the sum of the corresponding generator and a combination of [123], [156], [246], and [345] whose coefficients are specified in the corresponding entry of the table.

i,j,k	Gener.	Coeff. of [123]	Coeff. of [156]	Coeff. of [246]	Coeff. of [345]
1,2,1	1,1,2	$-y_6z_4z_5+y_5z_4z_6$	$y_3z_2z_4-y_2z_3z_4$	$-y_5z_1z_3+y_1z_3z_5$	$-y_6z_1z_2+y_1z_2z_6$
2,1,1	1,1,2	$-y_6z_4z_5+y_4z_5z_6$	$y_4z_2z_3-y_2z_3z_4$	$-y_3z_1z_5+y_1z_3z_5$	$-y_6z_1z_2+y_2z_1z_6$
1,3,1	1,1,3	$y_4y_6z_5-y_4y_5z_6$	$-y_3y_4z_2+y_2y_4z_3$	$y_3y_5z_1-y_1y_3z_5$	$y_2y_6z_1-y_1y_2z_6$
3,1,1	1,1,3	$y_5y_6z_4-y_4y_5z_6$	$-y_3y_4z_2+y_2y_3z_4$	$y_3y_5z_1-y_1y_5z_3$	$y_1y_6z_2-y_1y_2z_6$
2,1,2	1,2,2	$x_5z_4z_6-x_4z_5z_6$	$-x_4z_2z_3+x_3z_2z_4$	$-x_5z_1z_3+x_3z_1z_5$	$-x_2z_1z_6+x_1z_2z_6$
2,2,1	1,2,2	$x_6z_4z_5-x_4z_5z_6$	$-x_4z_2z_3+x_2z_3z_4$	$x_3z_1z_5-x_1z_3z_5$	$x_6z_1z_2-x_2z_1z_6$
1,3,2	1,2,3	$-x_4y_6z_5+x_4y_5z_6$	$x_4y_3z_2-x_4y_2z_3$	$-x_3y_5z_1+x_3y_1z_5$	$-x_2y_6z_1+x_2y_1z_6$
2,1,3	1,2,3	$-x_5y_4z_6+x_4y_5z_6$	$x_4y_3z_2-x_3y_4z_2$	$x_5y_3z_1-x_3y_5z_1$	$x_2y_1z_6-x_1y_2z_6$
2,3,1	1,2,3	$-x_6y_4z_5+x_4y_5z_6$	$x_4y_3z_2-x_2y_4z_3$	$-x_3y_5z_1+x_1y_3z_5$	$-x_6y_2z_1+x_2y_1z_6$
3,1,2	1,2,3	$-x_5y_6z_4+x_4y_5z_6$	$x_4y_3z_2-x_3y_2z_4$	$-x_3y_5z_1+x_5y_1z_3$	$-x_1y_6z_2+x_2y_1z_6$
3,2,1	1,2,3	$-x_6y_5z_4+x_4y_5z_6$	$x_4y_3z_2-x_2y_3z_4$	$-x_3y_5z_1+x_1y_5z_3$	$-x_6y_1z_2+x_2y_1z_6$
3,1,3	1,3,3	$x_5y_4y_6-x_4y_5y_6$	$-x_4y_2y_3+x_3y_2y_4$	$-x_5y_1y_3+x_3y_1y_5$	$-x_2y_1y_6+x_1y_2y_6$
3,3,1	1,3,3	$x_6y_4y_5-x_4y_5y_6$	$-x_4y_2y_3+x_2y_3y_4$	$x_3y_1y_5-x_1y_3y_5$	$x_6y_1y_2-x_2y_1y_6$
2,3,3	2,2,3	$x_4x_6z_5-x_4x_5z_6$	$-x_3x_4z_2+x_2x_4z_3$	$x_3x_5z_1-x_1x_3z_5$	$x_2x_6z_1-x_1x_2z_6$
3,2,2	2,2,3	$x_5x_6z_4-x_4x_5z_6$	$-x_3x_4z_2+x_2x_3z_4$	$x_3x_5z_1-x_1x_5z_3$	$x_1x_6z_2-x_1x_2z_6$
3,2,3	2,3,3	$-x_5x_6y_4+x_4x_6y_5$	$x_2x_4y_3-x_2x_3y_4$	$x_1x_5y_3-x_1x_3y_5$	$x_2x_6y_1-x_1x_6y_2$
3,3,2	2,3,3	$-x_5x_6y_4+x_4x_5y_6$	$x_3x_4y_2-x_2x_3y_4$	$-x_3x_5y_1+x_1x_5y_3$	$-x_1x_6y_2+x_1x_2y_6$

Applying the characterization in Theorem 7, we now compute a minimal generating set for I_{QS}.

Theorem 8. *The associated ideal I_{QS} of the quadrilateral set is minimally generated as:*

$$I_{QS} = \langle [123]_X, [156]_X, [246]_X, [345]_X, QS(\ell_{123};R_i,R_j,R_k)\,\forall\, i \le j \le k, \text{ with } i,j,k \in \{1,2,3\}\rangle. \tag{4}$$

Proof. For the ease of notation, we denote I for the ideal on the right-hand side of (4). By Theorem 6, we have that $I \subset I_{QS}$. By numerical computations available on GitHub: https://github.com/ollieclarke8787/PointAndLineConfigurations (accessed in February 2024).

We verify that the natural GRevLex term order gives a square-free initial ideal for I. Thus, the ideal I is radical, and verifying that $I \supset I_{QS}$ is equivalent to showing that $V(I) \subset V(I_{QS})$.

Let $A \in \mathbb{C}^{18}$ be a point in $V(I)$. The coordinates $A_1^1, A_1^2, A_1^3, \ldots, A_6^1, A_6^2, A_6^3$ of A can be seen as the (x, y, z)-coordinates of 6 points in the projective plane, which can be represented by the 3×6 matrix:

$$A = \begin{pmatrix} A_1^1 & A_2^1 & \ldots & A_6^1 \\ A_1^2 & A_2^2 & \ldots & A_6^2 \\ A_1^3 & A_2^3 & \ldots & A_6^3 \end{pmatrix}$$

The columns of A generate a realizable matroid M_A which corresponds to a point-line configuration $\mathcal{C}_A$ (as the columns of A are coordinates of points). Since $A \in V(I)$, A satisfies the determinantal collinearity conditions. Hence, A is a point in the combinatorial closure of the matroid QS. Corollary 1 ensures that after a suitable perturbation, A becomes either a realization of the matroid corresponding to a 6-pointed line or a realization of QS.

The fact that $A \in V(I)$ implies that the coordinates of A satisfy also the QS polynomials, implying that, if the points of A lie on a line, then A is the projective image of a quadrilateral set and therefore becomes the realization of a quad-set with an arbitrary small lifting. This holds also if only some of the points in C_A are loops. To check this is enough to consider the case where we have 5 points on a line and a loop. We want to verify the existence of a sixth point in the line such that $1, \ldots, 6$ is the projective image of a quadrilateral set. To this purpose, following Figure 11, we consider the two lines of the quadrilateral set that are not involved by the loop point and represent them as collinear points in the fibers of the nonloops. At this stage, the sixth point of the quadrilateral set is uniquely determined by the intersection of the corresponding ghost lines, and its projection on ℓ is the point we wanted. Therefore, A is in the Euclidean closure of Γ_{QS}. Hence, $A \in V(I_{QS})$.

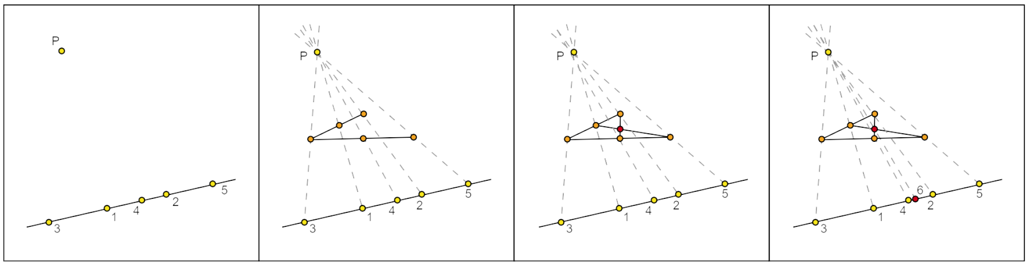

Figure 11. From left to right, this figure justifies the existence of at least a choice of 6 completing $1, \ldots, 5$ to a projective image of a quadrilateral set.

We now prove the minimality of the generating set. In particular, we show that all the polynomials we used to generate I are independent. By reasons of degree and point multidegree, the generators of degree 3 are polynomially independent over R. Hence, it is enough to check that generators of degree 6 having different letter multidegree are polynomially independent over $\mathbb{C}[x_1, \ldots, z_6]$. The fact that all the generators have point multidegree $(1, 1, 1, 1, 1, 1)$ and letter multidegree fixed by the partition of variables plays a central role in this proof.

First, we note that any choice of $(i, j, k) \in 1, 2, 3^3$ results in certain variables not appearing in the polynomial $QS(\ell_{123}; R_i, R_j, R_k)$. In Table 2 below, each row corresponds to a choice of (i, j, k) with $i \leq j \leq k$ and indicates the variables that do not appear in the corresponding generator.

Table 2. For each generator of degree 6, identified by a choice of i, j, and k in the first column, the table shows which variables do not appear in the polynomial.

(i,j,k)	1	2	3	4	5	6
$(1,1,1)$	x_1	x_2	x_3	x_4	x_5	x_6
$(2,2,2)$	y_1	y_2	y_3	y_4	y_5	y_6
$(3,3,3)$	z_1	z_2	z_3	z_4	z_5	z_6
$(1,1,2)$	x_1	x_2	y_3	-	-	x_6
$(1,1,3)$	x_1	x_2	z_3	-	-	x_6
$(1,2,2)$	x_1	y_2	y_3	y_4	-	-
$(1,3,3)$	x_1	z_2	z_3	z_4	-	-
$(2,2,3)$	y_1	y_2	z_3	-	-	y_6
$(2,3,3)$	y_1	z_2	z_3	z_4	-	-
$(1,2,3)$	x_1	y_2	z_3	-	-	-

Keeping this in mind, we can deduce that the generators of letter multidegree $(3,3,0)$, $(3,0,3)$, $(0,3,3)$, namely, the ones corresponding to the first three rows in Table 2, are polynomially independent with respect to the others. For ease of computation, let us focus on the specific case of $QS(\ell_{123};R_1,R_1,R_1)$, which has a letter multidegree of $(0,3,3)$ and does not contain any x variables. Since all the other generators are the sum of monomials containing x variables, any polynomial combination of them would result in the sum of monomials with a multidegree of (a,b,c) where $a>0$. This demonstrates that $QS(\ell_{123};R_1,R_1,R_1)$ is not contained in the ideal generated by the other generators. An analogous argument applies to the generators with letter multidegrees of $(3,0,3)$ and $(3,3,0)$.

Now, let us consider the other generators of degree 6. We show preliminarily the following claim.

Claim 1. *In the notation above, none of the generators having a permutation of* $(1,2,3)$ *as letter multidegree, i.e., rows 4 to 9 in Table 2, is contained in* $\langle[123]_X,[156]_X,[246]_X,[345]_X\rangle$.

Proof of the Claim. If $QS(\ell_{123};R_1,R_1,R_2)$ were in the ideal $\langle[123]_X,[156]_X,[246]_X,[345]_X\rangle$, it would be a polynomial combination of the four generators with at least one nonzero coefficient of letter multidegree $(0,1,2)$. Three x variables never appear, namely x_1,x_2, and x_6; these appear in $[123]_X,[156]_X,[246]_X$. If any of these variables appear in a monomial of the combination, this monomial must be canceled out using the other degree 3 generators containing the same variable.

Now, assume by contradiction that $[123]_X$ is multiplied by a nonzero coefficient $y_lz_mz_n$ of point multidegree $(0,0,0,1,1,1)$. Here $l\neq 4,5$, for reasons of point multidegree. So the only possibility is $y_6z_4z_5$. Then, two x_1 monomials must be cancelled: $x_1y_2y_6z_3z_4z_5 - x_1y_3y_6z_2z_4z_5$. For this purpose, we have to multiply $[156]_X$ by $y_2z_3z_4+y_3z_2z_4$. The second term gives rise to an x_6 monomial that cannot be canceled for reasons of point multidegree. A symmetric argument works starting from $[156]_X,[246]_X$. Thus, the coefficients of $[123]_X,[156]_X,[246]_X$ have to be zero. The only possibility left is that $QS(\ell_{123};R_1,R_1,R_2)$ is the product of $[345]_X$ by a homogeneous polynomial of degree 3, letter multidegree $(0,1,2)$ and point multidegree $(1,1,0,0,0,1)$. But this can be excluded because such a multiplication cannot give rise to any of the monomials $x_ay_by_cz_4z_5z_d$ appearing in $QS(\ell_{123};R_1,R_1,R_2)$.

By symmetry, an equivalent strategy can be performed with the generators of letter multidegree $(1,1,3),(1,2,2),(1,3,3),(2,2,3)$ and $(2,3,3)$. □

Now, on the contrary, assume that a 6-degree generator is a polynomial combination of the others, i.e.,

$$g=\sum_{i=1}^{4}p_ig_{3,i}+\sum_{i=1}^{9}k_ig_{6,i}$$

where $p_i \in R$, $k_i \in \mathbf{k}$, the $g_{3,i}$'s are the generators of degree 3 and the $g_{6,i}$'s are the generators of degree 6. By reason of the letter multidegree, none of the monomials of g can arise from the generators of 6-th degree. In addition, the product of a monomial for a degree 3 generator either gives rise to monomials of the same letter multidegree of g or to monomials of the same letter multidegree of one of the other generators. This implies, in turn, that both g and $\sum_{i=1}^{9} k_i g_{6,i}$ are in $\langle [123]_X, [156]_X, [246]_X, [345]_X \rangle$.

If g has a letter multidegree of $(1,2,3)$ (possibly permuted), the statement follows because it leads to a contradiction with the claim. If g is the generator with a letter multidegree of $(2,2,2)$, then we can conclude due to the fact that $\sum_{i=1}^{9} k_i g_{6,i} \in \langle [123]_X, [156]_X, [246]_X, [345]_X \rangle$. Indeed, since the $g_{6,i}$'s all have different letter multidegrees, this implies that at least one of the $g_{6,i}$'s is contained in $\langle [123]_X, [156]_X, [246]_X, [345]_X \rangle$, contradicting Claim 1. □

As an immediate consequence of Corollary 2 and Theorem 8, we have the following:

Corollary 2. *The circuit variety $V_{\mathscr{C}(QS)}$ decomposes irreducibly as*

$$V_{\mathscr{C}(QS)} = V_0 \cup V_{QS} = V(\langle [ijk] \mid \{i,j,k\} \subseteq \{1,\ldots,6\} \rangle) \cup V(I_{QS}).$$

5. The 3 × 4 Grid Matroid

In this section, we focus on the 3×4 grid matroid and its defining equations. This example is chosen based on the work of [24], where a generating set for the matroid ideal is computed using a specialized algorithm tailored for this configuration. Here, we give a geometric description of the generators.

In Example 5, we introduced the simple matroid G_4^3, whose point-line configuration is a 3×4 grid. Such matroid is realizable over $\mathbb{C}$ and any realization is represented as a 3×12 matrix whose columns are coordinates of points in $\mathbb{P}^2_{\mathbb{C}}$, with respect to a reference $\{R_1, R_2, R_3, U\}$. In view of Proposition 1, we study the ideal $I_{G_4^3} = \sqrt{I_{\mathscr{C}(G_4^3)} : J_{G_4^3}^{\infty}}$ of the algebraic variety $V_{G_4^3}$.

Notation 9.

- *Let c_i be a column in $\mathcal{L}_{G_4^3}$, whose points are P_i^1, P_i^2 and P_i^3, where the upper index labels the row. Then, the points in $G_4^3 \setminus c_i$ are in natural bijection with a 3×3 matrix.*
- *Let Σ be the set of permutation matrices σ over three elements. For $j = 1,2,3$, we denote as σ^j the nonzero entry of the $j-$th row and as P_σ^j the corresponding point in $G_4^3 \setminus c_i$.*
- *For $k \in \{1,2,3,4\} \setminus \{i\} = \{k_1, k_2, k_3\}$, each column c_k has a single point paired with a nonzero entry of σ. We label $Q_k^{\sigma,1}$ and $Q_k^{\sigma,2}$ the points in $\mathcal{P}_{G_4^3}$ which belong to $c_k \setminus \{P_\sigma^j\}$. In particular, $Q_k^{\sigma,1}$ is the point with the lowest index.*
- *Let $P^1, \ldots, P^6$ be six points in $\mathbb{P}^2_{\mathbb{C}}$, not necessarily distinct. We introduce the polynomials:*

$$G_4^3(c_i; P^1, \ldots, P^6) = \sum_{\sigma \in \Sigma} \operatorname{sgn}(\sigma)[P_i^1 P_\sigma^1 P^1][P_i^2 P_\sigma^2 P^2][P_i^3 P_\sigma^3 P^3][Q_{k_1}^{\sigma,1} Q_{k_1}^{\sigma,2} P^4][Q_{k_2}^{\sigma,1} Q_{k_2}^{\sigma,2} P^5][Q_{k_3}^{\sigma,1} Q_{k_3}^{\sigma,2} P^6],$$

$$G_4^3(c_i; P) = \sum_{\sigma \in \Sigma} \operatorname{sgn}(\sigma)[P_i^1 P_\sigma^1 P][P_i^2 P_\sigma^2 P][P_i^3 P_\sigma^3 P][Q_{k_1}^{\sigma,1} Q_{k_1}^{\sigma,2} P][Q_{k_2}^{\sigma,1} Q_{k_2}^{\sigma,2} P][Q_{k_3}^{\sigma,1} Q_{k_3}^{\sigma,2} P].$$

The G_4^3 polynomials are defined by a sum over all permutation matrices, making them independent of the bijection between the points of $G_4^3 \setminus c_i$ and the entries of a 3×3 matrix. This choice only affects the sign of the polynomials. We focus on configurations that cause the G_4^3 polynomials to vanish.

Theorem 9. *Let $\mathcal{C}$ be a 3×4 grid configuration in $\mathbb{P}^2_{\mathbb{C}}$. Then, for any column c_i and any six points $P^1, \ldots, P^6$ in general position with respect to any couple of points of the configuration, $G_4^3(c_i; P^1, \ldots, P^6)$ = 0.*

Proof. Due to the multilinearity of determinants, the statement follows if $G_4^3(c_i; R_{i_1}, \dots, R_{i_6}) = 0$, for any $i = 1, \dots 4$ and any $(i_1, \dots, i_6) \in \{1,2,3\}^6$, where we have fixed $\{R_1, R_2, R_3, R_1 + R_2 + R_3\}$ as the frame of reference on $\mathbb{P}^2_{\mathbb{C}}$. We now prove the statement for a particular choice of i. It can be easily repeated for other possible choices. Let us assume $i = 1$. We want to show that

$$\begin{aligned} G_4^3(c_1; R_{i_1}, \dots, R_{i_6}) = {} & [1\,4\,R_{i_1}][2\,8\,R_{i_2}][3\,12\,R_{i_3}][5\,6\,R_{i_4}][7\,9\,R_{i_5}][10\,11\,R_{i_6}] + \\ & + [1\,7\,R_{i_1}][2\,11\,R_{i_2}][3\,6\,R_{i_3}][4\,5\,R_{i_4}][8\,9\,R_{i_5}][10\,12\,R_{i_6}] + \\ & + [1\,10\,R_{i_1}][2\,5\,R_{i_2}][3\,9\,R_{i_3}][4\,6\,R_{i_4}][7\,8\,R_{i_5}][11\,12\,R_{i_6}] + \\ & - [1\,4\,R_{i_1}][2\,11\,R_{i_2}][3\,9\,R_{i_3}][5\,6\,R_{i_4}][7\,8\,R_{i_5}][10\,12\,R_{i_6}] + \\ & - [1\,7\,R_{i_1}][2\,5\,R_{i_2}][3\,12\,R_{i_3}][4\,6\,R_{i_4}][8\,9\,R_{i_5}][10\,11\,R_{i_6}] + \\ & - [1\,10\,R_{i_1}][2\,8\,R_{i_2}][3\,6\,R_{i_3}][4\,5\,R_{i_4}][7\,9\,R_{i_5}][11\,12\,R_{i_6}] \\ = {} & 0. \end{aligned} \tag{5}$$

We rewrite all terms of the polynomial as multiples of the first one. So, we consider the following:

- $\{1, 4, 1 + 4\}$ is a basis for the projective line r_1. Thus, there exists a unique choice of $a, a', b, b' \in \mathbb{C}$ such that: (i) $7 = a0 + b4$ and (ii) $10 = a'0 + b'4$.
- $\{2, 8, 2 + 8\}$ is a basis for the projective line r_2. Thus, there exists a unique choice of $c, c', d, d' \in \mathbb{C}$ such that: (iii) $5 = c2 + d8$ and (iv) $11 = c'2 + d'8$.
- $\{3, 12, 3 + 12\}$ is a basis for the projective line r_3. Thus, there exists a unique choice of $e, e', f, f' \in \mathbb{C}$ such that: (v) $6 = e3 + f12$ and (vi) $9 = e'3 + f'12$.
- $\{5, 6, 5 + 6\}$ is a basis for the projective line c_2. Thus, there exists a unique choice of $\alpha, \beta \in \mathbb{C}$ such that: (vii) $4 = \alpha 5 + \beta 6$.
- $\{7, 9, 7 + 9\}$ is a basis for the projective line c_3. Thus, there exists a unique choice of $\gamma, \delta \in \mathbb{C}$ such that: $(viii)$ $8 = \gamma 7 + \delta 9$.
- $\{10, 11, 10 + 11\}$ is a basis for the projective line c_4. Thus, there exists a unique choice of $\varepsilon, \zeta \in \mathbb{C}$ such that: (ix) $12 = \varepsilon 10 + \zeta 11$.

Here, it is important to remark that $0, \dots, 12$ are fixed representatives of the corresponding points in $\mathbb{P}^2_{\mathbb{C}}$, which makes the choice of the coefficients in (i)–(ix) unique. Now, we exploit (i)–(ix) to modify the columns of the matrices showing up in the second, third, fourth, fifth and sixth summands of $G_4^3(c_1; R_{i_1}, \dots, R_{i_6})$. By multi-linearity, we obtain:

$$\begin{aligned} G_4^3(c_1; R_{i_1}, \dots, R_{i_6}) = {} & 1 \cdot [1\,4\,R_{i_1}][2\,8\,R_{i_2}][3\,12\,R_{i_3}][5\,6\,R_{i_4}][7\,9\,R_{i_5}][10\,11\,R_{i_6}] + \\ & - bd'f\beta\gamma\zeta[1\,4\,R_{i_1}][2\,8\,R_{i_2}][3\,12\,R_{i_3}][5\,6\,R_{i_4}][7\,9\,R_{i_5}][10\,11\,R_{i_6}] + \\ & - b'df'\alpha\delta\varepsilon[1\,4\,R_{i_1}][2\,8\,R_{i_2}][3\,12\,R_{i_3}][5\,6\,R_{i_4}][7\,9\,R_{i_5}][10\,11\,R_{i_6}] + \\ & - d'f'\delta\zeta[1\,4\,R_{i_1}][2\,8\,R_{i_2}][3\,12\,R_{i_3}][5\,6\,R_{i_4}][7\,9\,R_{i_5}][10\,11\,R_{i_6}] + \\ & - bd\alpha\gamma[1\,4\,R_{i_1}][2\,8\,R_{i_2}][3\,12\,R_{i_3}][5\,6\,R_{i_4}][7\,9\,R_{i_5}][10\,11\,R_{i_6}] + \\ & - b'f\beta\varepsilon[1\,4\,R_{i_1}][2\,8\,R_{i_2}][3\,12\,R_{i_3}][5\,6\,R_{i_4}][7\,9\,R_{i_5}][10\,11\,R_{i_6}] \\ = {} & (1 - bd'f\beta\gamma\zeta - b'df'\alpha\delta\varepsilon - d'f'\delta\zeta - bd\alpha\gamma - b'f\beta\varepsilon) \cdot \\ & \cdot [1\,4\,R_{i_1}][2\,8\,R_{i_2}][3\,12\,R_{i_3}][5\,6\,R_{i_4}][7\,9\,R_{i_5}][10\,11\,R_{i_6}] \end{aligned}$$

As a consequence, the claim is equivalent to $bd'f\beta\gamma\zeta + b'df'\alpha\delta\varepsilon + d'f'\delta\zeta + bd\alpha\gamma + b'f\beta\varepsilon = 1$. However,

$$\begin{aligned}
4 &\overset{(vii)}{=} \underline{\alpha 5} + \underline{\beta 6} \\
&\overset{(iii,v)}{=} c\alpha 2 + e\beta 3 + \underline{d\alpha 8} + \underline{f\beta 12} \\
&\overset{(viii,ix)}{=} c\alpha 2 + e\beta 3 + \underline{d\alpha\gamma 7} + \underline{d\alpha\delta 9} + \underline{ef\beta 10} + \underline{f\beta\zeta 11} \\
&\overset{(i,ii,iv,vi)}{=} *_1 1 + *_2 2 + *_3 3 + (bd\alpha\gamma + b' f\beta\varepsilon)4 + \underline{d' f\beta\zeta 8} + \underline{df'\alpha\delta 12} \\
&\overset{(viii,ix)}{=} *_1 1 + *_2 2 + *_3 3 + (bd\alpha\gamma + b' f\beta\varepsilon)4 + \underline{d' f\beta\gamma\zeta 7} + \underline{d' f\beta\delta\zeta 9} + \underline{df'\alpha\delta\varepsilon 10} + \underline{df'\alpha\delta\zeta 11} \\
&\overset{(i,ii,iv,vi)}{=} *_1 1 + *_2 2 + *_3 3 + (bd' f\beta\gamma\zeta + b'df'\alpha\delta\varepsilon + bd\alpha\gamma + b' f\beta\varepsilon)4 + \underline{dd'f'\alpha\delta\zeta 8} + \underline{d'ff'\beta\gamma\zeta 12} \\
&\overset{(iii,v)}{=} *_1 1 + *_2 2 + *_3 3 + (bd' f\beta\gamma\zeta + b'df'\alpha\delta\varepsilon + bd\alpha\gamma + b' f\beta\varepsilon)4 + \underline{d'f'\alpha\delta\zeta 5 + d'f'\beta\delta\zeta 6} \\
&\overset{(viii)}{=} *_1 1 + *_2 2 + *_3 3 + (bd' f\beta\gamma\zeta + b'df'\alpha\delta\varepsilon + d'f'\delta\zeta + bd\alpha\gamma + b' f\beta\varepsilon)4,
\end{aligned}$$

which implies that:

$$(1 - bd' f\beta\gamma\zeta - b'df'\alpha\delta\varepsilon - d'f'\delta\zeta - bd\alpha\gamma - b' f\beta\varepsilon)4 = *_1 1 + *_2 2 + *_3 3.$$

Here, on the l.h.s., there is another representation of point 6; conversely, on the r.h.s., there is a point in the line c_1. Since by construction $4 \notin c_1$, the equality above may hold if and only if both sides give $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, that is if and only if $bd' f\beta\gamma\zeta + b'df'\alpha\delta\varepsilon + d'f'\delta\zeta + bd\alpha\gamma + b' f\beta\varepsilon = 1$. The thesis follows. □

Furthermore, projections keep track of the vanishing of these polynomials.

Lemma 7. *The vanishing of the polynomial $G_4^3(c_r; R_{i_1}, \dots, R_{i_6})$ is a projective invariant property.*

Proof. We prove the lemma for the column c_1. We need to show that

$$G_4^3(c_1; P^1, \dots, P^6) = \sum_{\sigma\in\Sigma} \operatorname{sgn}(\sigma)[1P_\sigma^1 P^1][2P_\sigma^2 P^2][3P_\sigma^3 P^3][Q_{k_1}^{\sigma,1} Q_{k_1}^{\sigma,2} P^4][Q_{k_2}^{\sigma,1} Q_{k_2}^{\sigma,2} P^5][Q_{k_3}^{\sigma,1} Q_{k_3}^{\sigma,2} P^6] = 0$$

is a projective invariant property. We consider $T \in GL(\mathbb{C}, 3)$ and $D \in \operatorname{diag}(\mathbb{C}, 3)$ and we write down

$$\begin{aligned}
G_4^3(Tc_1D; TP^1D, \dots, TP^6D) = \sum_{\sigma\in\Sigma} &\operatorname{sgn}(\sigma)[T1D\ TP_\sigma^1 D\ TP^1D][T2D\ TP_\sigma^2 D\ TP^2D]\cdot \\
&\cdot [T3D\ TP_\sigma^3 D\ TP^3D][TQ_{k_1}^{\sigma,1}D\ TQ_{k_1}^{\sigma,2}D\ TP^4D]\cdot \\
&\cdot [TQ_{k_2}^{\sigma,1}D\ TQ_{k_2}^{\sigma,2}D\ TP^5D][TQ_{k_3}^{\sigma,1}D\ TQ_{k_3}^{\sigma,2}D\ TP^6D] \\
= \det T^3 \det D^3 \Bigg(&\sum_{\sigma\in\Sigma} \operatorname{sgn}(\sigma)[1P_\sigma^1 P^1][2P_\sigma^2 P^2][3P_\sigma^3 P^3]\cdot \\
&\cdot [Q_{k_1}^{\sigma,1} Q_{k_1}^{\sigma,2} P^4][Q_{k_2}^{\sigma,1} Q_{k_2}^{\sigma,2} P^5][Q_{k_3}^{\sigma,1} Q_{k_3}^{\sigma,2} P^6]\Bigg).
\end{aligned}$$

We observe that

$$G_4^3(Tc_1D; TP^1D, \dots, TP^6D) = 0 \quad \Longleftrightarrow \quad G_4^3(c_1; P^1, \dots, P^6) = 0,$$

which completes the proof for c_1. The analogous proof works for other choices of c_i. □

The family of G_4^3 polynomials introduced in Theorem 9 characterizes the liftability of 12-tuples of collinear points to a 3×4 grid. Specifically, the following holds.

Theorem 10. *Let r be a line in $\mathbb{P}^2_{\mathbb{C}}$ and $1, \ldots, 12$ distinct points of r. Consider the collection $\mathcal{L} = \{c_1 = \{1,2,3\}, c_2 = \{4,5,6\}, c_3 = \{7,8,9\}, c_4 = \{10,11,12\}, r_1 = \{1,4,7,10\}, r_2 = \{2,5,8,11\}, r_3 = \{3,6,9,12\}\}$, where each of the tuples is the combinatorial line consisting of the points thereby contained. Then, the following statements are equivalent:*

- *i.* *The points $1, \ldots, 12$ are projective image of a 3×4 grid.*
- *ii.* *The polynomials $G^3_4(c_i; P^1, \ldots, P^6) = 0$ vanish for any $c_i \in \mathcal{L}$ and any $P^1, \ldots, P^6 \in \mathbb{P}^2_{\mathbb{C}}$.*
- *iii.* *The polynomials $G^3_4(c_i; R_i, \ldots, R_n)$ vanish for any $c_i \in \mathcal{L}$ and any $(i, j, k, l, m, n) \in \{1,2,3\}^6$.*
- *iv.* *The polynomials $G^3_4(c_1; R_i, \ldots, R_n)$ vanish for any $(i, j, k, l, m, n) \in \{1,2,3\}^6$.*
- *v.* *The polynomials $G^3_4(c_1; R_i, \ldots, R_n)$ vanish for any $(i, j, k, l, m, n) \in \{1,2,3\}^6$ with $i \leq j \leq k \leq l \leq m \leq n$.*

Proof. It is immediate to see that $(ii) \implies (iii) \implies (iv) \implies (v)$.

$(i) \implies (ii)$ By Lemma 7, the vanishing of the G^3_4 polynomials is a projective invariant property, and so, if twelve collinear points are the projective image of a 3×4 grid, then $G^3_4(c_i; P^1, \ldots, P^6) = 0$ where $c_i, P^1, \ldots, P^6$ satisfy the assumptions of Theorem 9.

$(ii) \implies (i)$ We now consider twelve collinear points $1, \ldots, 12$ which satisfy the vanishing of the polynomials $G^3_4(c_i; P^1, \ldots, P^6)$, for any choice of $c_i \in \mathcal{L}$, and any choice of $P^1, \ldots, P^6 \in \mathbb{P}^2_{\mathbb{C}}$. Let P be a point which does not lie on the line of points $1, \ldots, 12$.

Claim 2. *The condition that the polynomials $G^3_4(c_i; P^1, \ldots, P^6)$ vanish for any choice of $c_i \in \mathcal{L}$ and any choice of $P^1, \ldots, P^6 \in \mathbb{P}^2_{\mathbb{C}}$ implies that the collinearity matrix $\Lambda_{G^3_4}$ has rank ≤ 9.*

$$\Lambda_{G^3_4} = \begin{pmatrix}
[2\,3] & -[1\,3] & [1\,2] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & [5\,6] & -[4\,6] & [4\,5] & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & [8\,9] & -[7\,9] & [7\,8] & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & [11\,12] & -[10\,12] & [10\,11] \\
[4\,7] & 0 & 0 & -[1\,7] & 0 & 0 & [1\,4] & 0 & 0 & 0 & 0 & 0 \\
[4\,10] & 0 & 0 & -[1\,10] & 0 & 0 & 0 & 0 & 0 & [1\,4] & 0 & 0 \\
[7\,10] & 0 & 0 & 0 & 0 & 0 & -[1\,10] & 0 & 0 & [1\,7] & 0 & 0 \\
0 & 0 & 0 & [7\,10] & 0 & 0 & -[4\,10] & 0 & 0 & [4\,7] & 0 & 0 \\
0 & [5\,8] & 0 & 0 & -[2\,8] & 0 & 0 & [2\,5] & 0 & 0 & 0 & 0 \\
0 & [5\,11] & 0 & 0 & -[2\,11] & 0 & 0 & 0 & 0 & 0 & [2\,5] & 0 \\
0 & [8\,11] & 0 & 0 & 0 & 0 & 0 & -[2\,11] & 0 & 0 & [2\,8] & 0 \\
0 & 0 & 0 & 0 & [8\,11] & 0 & 0 & -[5\,11] & 0 & 0 & [5\,8] & 0 \\
0 & 0 & [6\,9] & 0 & 0 & -[3\,9] & 0 & 0 & [3\,6] & 0 & 0 & 0 \\
0 & 0 & [6\,12] & 0 & 0 & -[3\,12] & 0 & 0 & 0 & 0 & 0 & [3\,6] \\
0 & 0 & [9\,12] & 0 & 0 & 0 & 0 & 0 & -[3\,12] & 0 & 0 & [3\,9] \\
0 & 0 & 0 & 0 & 0 & [9\,12] & 0 & 0 & -[6\,12] & 0 & 0 & [6\,9]
\end{pmatrix}$$

Proof of the Claim. We aim to prove that all 10 minors of the matrix $\Lambda_{G^3_4}$ vanish when evaluated at the coordinates of points 1 through 12. In considering the 10 minors of the matrix $\Lambda_{G^3_4}$, we note that each line contains the variables of three collinear points. Specifically, we can introduce the following partition of the rows of the matrix $\Lambda_{G^3_4}$:

- Rows $\mathscr{C} = \{I, II, III, IV\}$ correspond to the columns of the grid.
- Rows $\mathscr{R}_1 = \{V, VI, VII, VIII\}$ correspond to the first row of the grid.
- Rows $\mathscr{R}_2 = \{IX, X, XI, XII\}$ correspond to the second row of the grid.
- Rows $\mathscr{R}_3 = \{XIII, XIV, XV, XVI\}$ correspond to the third row of the grid.

The submatrices of $\Lambda_{G^3_4}$ formed by the rows in one of the $\mathscr{R}i$'s and the corresponding nonzero columns have rank 2. Thus, whenever a 10×10 submatrix of ΛG^3_4 contains three rows of one of the $\mathscr{R}_i$'s, the corresponding minor is automatically 0. Consequently, the only minors that can possibly be nonzero are those generated by a selection of rows including the four rows from $\mathscr{C}$ and two rows from each $\mathscr{R}_i$. Furthermore, the choice of two rows in each of the $\mathscr{R}_i$s does not affect the final computation of the 10 minors. This is because the determinant of a matrix remains unchanged if one replaces two rows with linear combinations of them.

To sum up, after having chosen a 10×10 submatrix of $\Lambda_{G_4^3}$ there are only three possible patterns for its nonzero entries, up to switching its columns and rows.

$$
\begin{pmatrix}
\circ &&&& \circ &&&& \circ & \\
& \circ &&&& \circ &&&& \circ \\
&& \circ &&&& \circ &&& \\
&&& \circ &&&& \circ && \\
\circ & \circ & \circ & \circ &&&&&& \\
\circ & \circ & \circ & \circ &&&&&& \\
&&&& \circ & \circ & \circ & \circ && \\
&&&& \circ & \circ & \circ & \circ && \\
&&&&&&&& \circ & \circ \\
&&&&&&&& \circ & \circ
\end{pmatrix}
\quad
\begin{pmatrix}
\circ &&&& \circ &&& \circ && \\
& \circ &&&& \circ &&& \circ & \\
&& \circ &&&& \circ &&& \circ \\
&&& \circ &&&&&& \\
\circ & \circ & \circ & \circ &&&&&& \\
\circ & \circ & \circ & \circ &&&&&& \\
&&&& \circ & \circ & \circ &&& \\
&&&& \circ & \circ & \circ &&& \\
&&&&&&& \circ & \circ & \circ \\
&&&&&&& \circ & \circ & \circ
\end{pmatrix}
\quad
\begin{pmatrix}
\circ &&&& \circ &&& \circ && \\
& \circ &&&& \circ &&& \circ & \\
&& \circ &&&& \circ &&& \\
&&& \circ &&&&&& \circ \\
\circ & \circ & \circ & \circ &&&&&& \\
\circ & \circ & \circ & \circ &&&&&& \\
&&&& \circ & \circ & \circ &&& \\
&&&& \circ & \circ & \circ &&& \\
&&&&&&& \circ & \circ & \circ \\
&&&&&&& \circ & \circ & \circ
\end{pmatrix}
$$

In each of the matrices among the patterns above, there are three clear blocks of columns. Each of these blocks corresponds to one of the rows of the grid. Since the rows of the grid play a symmetric role, up to relabeling the points, we can associate the leftmost block with r_1, the central block with r_2, and the rightmost block with r_3. This reduces the study cases to 6 minors having the first pattern, 4 minors having the second pattern, and 12 minors having the third pattern.

These 10 minors are all contained in the ideal generated by the polynomials $G_4^3(c_i; (0,0,1)^t)$, as shown in the computations available on GitHub (accessed in 10 February 2024). $\square$

It follows from the claim that the linear system $\Lambda_{G_4^3}(z_1 \dots z_{12})^t = (0 \dots 0)^t$ has a solution space of dimension at least 3. In other words, it is possible to choose $z_1, \dots, z_{12}$ such that

$$
\begin{cases}
[2\,3]\cdot z_1-[1\,3]\cdot z_2+[1\,2]\cdot z_3=[5\,6]\cdot z_4-[4\,6]\cdot z_5+[4\,5]\cdot z_6=[8\,9]\cdot z_7-[7\,9]\cdot z_8+[7\,8]\cdot z_9=[11\,12]\cdot z_{10}-[10\,12]\cdot z_{11}+[10\,11]\cdot z_{12}=0 \\
[4\,7]\cdot z_1-[1\,7]\cdot z_4+[1\,4]\cdot z_7=[4\,10]\cdot z_1-[1\,10]\cdot z_4+[1\,4]\cdot z_{10}=[7\,10]\cdot z_1-[1\,10]\cdot z_7+[1\,7]\cdot z_{10}=[7\,10]\cdot z_4-[4\,10]\cdot z_7+[4\,7]\cdot z_{10}=0 \\
[5\,8]\cdot z_2-[2\,8]\cdot z_5+[2\,5]\cdot z_8=[5\,11]\cdot z_2-[2\,11]\cdot z_5+[2\,5]\cdot z_{11}=[8\,11]\cdot z_2-[2\,11]\cdot z_8+[2\,8]\cdot z_{11}=[8\,11]\cdot z_5-[5\,11]\cdot z_8+[5\,8]\cdot z_{11}=0 \\
[6\,9]\cdot z_3-[3\,9]\cdot z_6+[3\,6]\cdot z_9=[6\,12]\cdot z_3-[3\,12]\cdot z_6+[3\,6]\cdot z_{12}=[9\,12]\cdot z_3-[3\,12]\cdot z_9+[3\,9]\cdot z_{12}=[9\,12]\cdot z_6-[6\,12]\cdot z_9+[6\,9]\cdot z_{12}=0
\end{cases}
$$

and such that the points $\begin{pmatrix} x_i \\ 1 \\ z_i \end{pmatrix}$ for $i = 1, \dots, 12$ span the whole projective plane. This ensures the existence of a nondegenerate 3×4 grid whose image via the projection through P on the line r consists exactly of points $1, \dots, 12$.

$(iii) \implies (ii)$ The implication follows because the polynomials in (ii) and (iii) generate the same ideal by the multilinearity of determinants.

$(v) \implies (iii)$ The implication follows from direct computation. For each polynomial $g = G_4^3(c_i; R_i, \dots, R_n)$, we verify that g belongs to the ideal generated by 3 minors arising from the collinearity constraints and the polynomials in (v). The code is available on GitHub (accessed in 10 February 2024). $\square$

This characterization is crucial to provide a minimal generating set for the ideal $I_{G_4^3}$.

Theorem 11. *Let G_4^3 be the simple matroid underlying the grid configuration with 3 rows and 4 columns, and let $I_{G_4^3}$ denote the ideal of the matroid variety. Then,*

$$
I_{G_4^3} = \langle [1\,2\,3], \dots, [10\,11\,12], G_4^3(c_1; R_i, R_j, R_k, R_l, R_m, R_n)\ \forall\, i \le j \le k \le l \le m \le n, \\ \text{with } i, j, k, l, m, n \in \{1,2,3\} \rangle,
$$

where $[1\,2\,3], \dots, [10\,11\,12]$ are the collinearities given by the grid.

Proof. For ease of notation, we denote I for the ideal on the right-hand side of the above equation. By Theorem 9, we have that $I \subset I_{G_4^3}$. We prove the converse inclusion in the following 3 steps.

$\boldsymbol{V(I) = V(I_{G_4^3})}$. We prove this equality by double inclusion. Since $I \subset I_{G_4^3}$, we have that $V_{G_4^3} \subseteq V(I)$. We prove $V(I) \subset V(G_4^3)$. Let $A \in \mathbb{C}^{36}$ be a point in $V(I)$. The coordinates

$A_1^1, A_1^2, A_1^3, \ldots, A_{12}^1, A_{12}^2, A_{12}^3$ of A can be seen as the (x, y, z)-coordinates of 12 points in the projective plane. These can be represented with the following 3×12 matrix:

$$A = \begin{pmatrix} A_1^1 & A_2^1 & \ldots & A_{12}^1 \\ A_1^2 & A_2^2 & \ldots & A_{12}^2 \\ A_1^3 & A_2^3 & \ldots & A_{12}^3 \end{pmatrix}.$$

The columns of A generate a realizable matroid M_A which corresponds to a point-line configuration $\mathcal{C}_A$ (as the columns of A are coordinates of points). Since $A \in V(I)$, A satisfies the determinantal collinearity conditions. Hence, A is a point in the combinatorial closure of the matroid associated with the grid configuration G_4^3. Corollary 1 ensures that A is close either to a realization of the matroid corresponding to a line with 12 marked points or to a realization of G_4^3.

The fact that $A \in V(I)$ implies that the coordinates of A satisfy also the G_4^3 polynomials, implying that, if the points of A lie on a line, then they can be lifted to a 3×4 grid. This is true also in the case of having 11 points on a line and a loop, indeed we can always find a 12th point on the line such that $1, \ldots, 12$ are the projective image of a 3×4 grid. Among the 11 points there is a 3×3 subgrid which can always be lifted by Remark 10. The two other points are determined by the intersection of this grid with the fibres and the last one comes consequently (see Figure 12). Now, via the perturbation procedure, we conclude that A is in the Euclidean closure of $\Gamma_{G_4^3}$. Hence, $A \in V(I_{G_4^3})$.

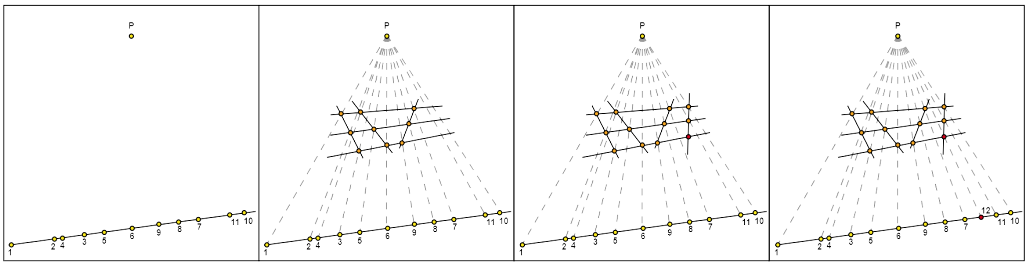

Figure 12. From left to right, this figure justifies the existence of at least a choice of 12 completing $1, \ldots, 11$ to a projective image of a 3×4 grid.

The ideal $I_{G_4^3}$ is radical. The circuit ideal of the 3×4 grid is radical by the explicit computation in [24], where the authors show that the circuit ideal is the intersection of two prime ideals, hence it is radical. We also have that $I_{G_4^3} = \sqrt{I}$. In addition, in the proof of Theorem 10, we have computed in *Macaulay2* that I is equal to one of the prime ideals in their decomposition, by reducing their generators modulo the G_4^3 polynomials, which all reduce to zero. Thus, as I is a radical ideal, $I_{G_4^3} = I$.

Minimal generating set. To prove the minimality of the generating set, we show that none of the generators belong to the ideal generated by the others. This is verified numerically, and the code is available on GitHub (accessed in 10 February 2024). □

As a direct consequence of Corollary 1 and Theorem 11, we have

Corollary 3. *The circuit variety $V_{\mathscr{C}(G_4^3)}$ decomposes irreducibly as*

$$V_{\mathscr{C}(G_4^3)} = V_0 \cup V_{G_4^3} = V(\langle [ijk] \mid \{i, j, k\} \subseteq \{1, \ldots, 12\} \rangle) \cup V(I_{G_4^3}).$$

6. Conclusions

We conclude this paper by outlining several potential extensions of our work. Our main focus has been the challenging task of identifying the generators of the matroid ideal I_M. Classical tools like the Grassmann–Cayley algebra can be used to construct such polynomials. However, a complete characterization of these polynomials remains elusive, primarily due to the saturation step involved in constructing I_M, which encodes matroid independence relations and may introduce additional polynomials beyond those generated by circuit relations [7,18].

In this paper, we introduced a new geometric notion, "liftability", which we used to study the varieties associated with matroids and their defining equations. We showed that when a matroid is liftable, the circuit variety and matroid variety coincide. In cases where M is quasi-liftable, the circuit variety can be expressed as the union of the matroid variety and the variety of a line. Finding an irreducible decomposition of matroid varieties in general, however, remains a difficult problem, and proving irreducibility is known only for specific cases. For instance, the matroid varieties of positroids are irreducible [31], possessing unique combinatorial structures [32–34]. Other families of matroids for which irreducibility has been studied include forest-type point-line configurations, as well as nilpotent, solvable [35], and paving matroids [36]. An interesting problem is to identify additional families of matroids for which this property can be characterized.

In this paper, we presented two explicit examples of matroids—namely, the 3×4 grid and the quadrilateral set—where we computed all their defining equations and determined their irreducible decompositions. However, Theorem 1 applies to a broader class of matroids beyond these examples. A key question for future investigation is to identify other matroids that are liftable (Definition 13) or quasi-liftable (Definition 14). One important open problem is to establish a combinatorial criterion that guarantees the applicability of our theorem. Another promising direction is to develop sufficient conditions for the liftability of higher-rank matroids, as suggested by our results in Section 3.

Notably, the grid matroids naturally arise in the study of conditional independence models in statistics While our results are specifically demonstrated for the 3×4 grid, they may offer insights into more general cases of $s \times t$ grids. It would be valuable to explore the existence of nontrivial polynomials for higher-dimensional grids, as well as the irreducible decompositions of their varieties.

Author Contributions: The authors (O.C., G.M. and F.M.) equally contributed to the conceptualization, methodology, writing, and resource acquisition. Software development was solely carried out by O.C. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the grants G023721N and G0F5921N (Odysseus Programme) from the Research Foundation—Flanders (FWO), the iBOF/23/064 from KU Leuven and the UiT Aurora project MASCOT.

Data Availability Statement: All computational data are available on Oliver Clarke's GitHub repository: (https://github.com/ollieclarke8787/PointAndLineConfigurations).

Acknowledgments: G.M. would like to thank Emiliano Liwski for helpful discussions about the proofs of Lemma 5 and Theorem 5.

Conflicts of Interest: The authors declare no conflict of interest.

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