1. Fourier Transform (Continuous time)

1.1. Signals with finite energy

A finite energy signal is a signal $f(t)$ for which

$$\int_{-\infty}^{\infty} |f(t)|^2 dt < \infty$$

Scalar product:

$$\langle f(t), g(t) \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(t)} g(t) dt$$

The *Hilbert* space of all these functions is $L^2(\mathbb{R})$ or L^2

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1.2. The Fourier Transform

Suppose $f(t) \in L^2$, then its Fourier-transform is

$$F(\omega) = \mathcal{F}\{f(t)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \mathrm{e}^{-i\omega t} dt$$

This is:

$$F(\omega) = \langle \mathrm{e}^{i\omega t}, f(t) \rangle$$

Inverse transform

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) \mathrm{e}^{i\omega t} d\omega = \left\langle e^{-i\omega t}, F(\omega) \right\rangle$$

Convolution $\rightarrow$ product

$$(f * g)(t) \;\; = \;\; \int_{-\infty}^{\infty} f(u) g(t-u) du = \int_{-\infty}^{\infty} f(t-u) g(u) du$$

then:

$$\mathcal{F}\{f * g\} = \mathcal{F}\{f(t)\} \mathcal{F}\{g(t)\}$$

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1.3. Parseval's identity

The Fourier transform preserves energy:

$$\|f(t)\| = \|F(\omega)\|$$

or:

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

This is Parseval's identity. It can be stated more generally:

$$\langle f(t), g(t) \rangle = \langle F(\omega), G(\omega) \rangle$$

The norm-equivalence follows for $g(t) = f(t)$.

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1.4. Heisenberg's uncertainty

In (t, ω) plane, signal concentrated at (t_0, ω_0)

$$t_0 = \frac{\int_{\mathbb{R}} |f(t)|^2 t dt}{\int_{\mathbb{R}} |f(t)|^2 dt}; \quad \omega_0 = \frac{\int_{\mathbb{R}} |F(\omega)|^2 \omega d\omega}{\int_{\mathbb{R}} |F(\omega)|^2 d\omega}$$

with variances

$$\sigma_t^2 = \frac{\int_{\mathbb{R}} |f(t)|^2 (t - t_0)^2 dt}{\int_{\mathbb{R}} |f(t)|^2 dt}; \quad \sigma_\omega^2 = \frac{\int_{\mathbb{R}} |F(\omega)|^2 (\omega - \omega_0)^2 d\omega}{\int_{\mathbb{R}} |F(\omega)|^2 d\omega}$$

then:

$$\sigma_t \sigma_\omega \geq \frac{1}{2} \text{ or } \sigma_t^2 \sigma_\omega^2 \geq \frac{1}{4}$$

Higher resolution in the time domain
$\Leftrightarrow$
Lower resolution in the frequency domain

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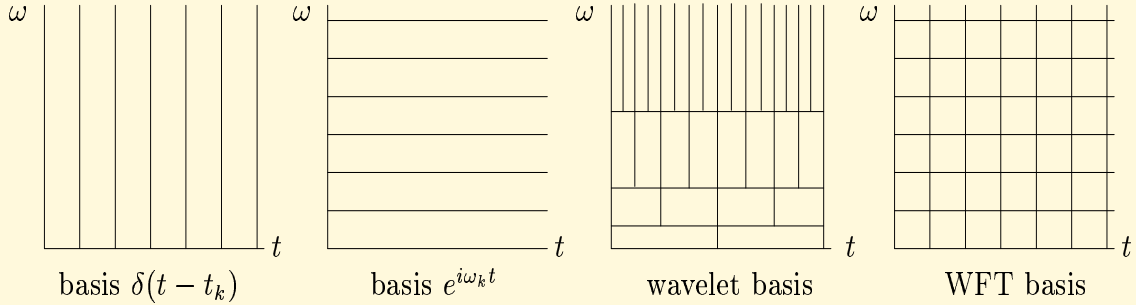
1.5. Time-frequency plane



1. basis functions $\delta(t-t_k)$ give 1st picture
2. basis functions $e^{i\omega_k t}$ give 2nd picture

how to get pictures like 3 or 4?
needs local basis in (t,ω)-plane

3. is more natural for human observation (sound/vision) = wavelets
4. corresponds to windowed or short time Fourier transform (WFT/STFT)

WFT is older than wavelets

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1.6. Windowed Fourier Transform (WFT)

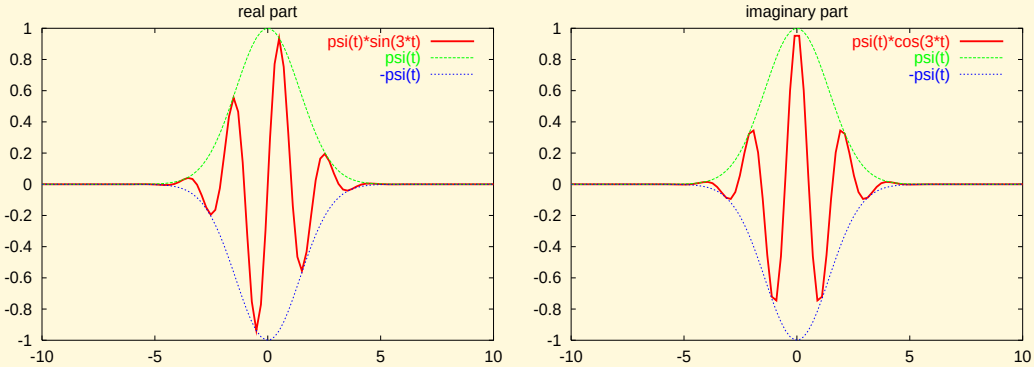
Use a "windowed" instead of a "pure" sine/cosine:

$$\mathcal{F}_\psi(f(t)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\overline{\mathrm{e}^{i\omega t}\psi(x-b)}dt$$

ψ is a (localizing) window function: for example a Gaussian:

$$\psi(t) = \mathrm{e}^{-t^2/2\sigma_0}$$

This leads to the *Gabor transform*



However, this approach has a drawback: the time window is constant for all frequencies: for high frequencies, we have more oscillations in the window.

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1.7. The continuous wavelet transform (CWT)

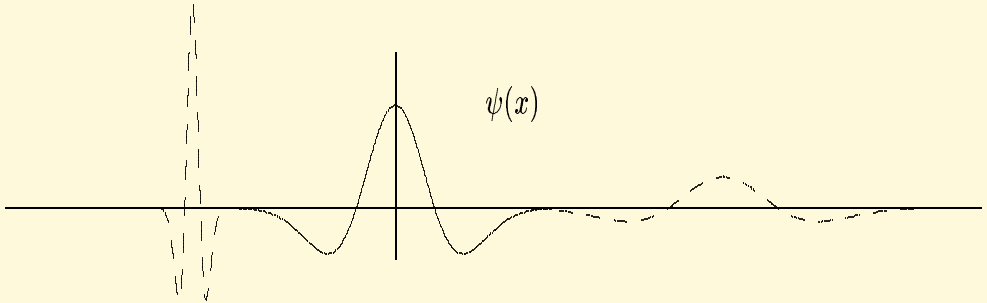
(sneak preview)

$$\mathcal{W}_\psi(a,b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(t) \overline{\psi\left(\frac{t-b}{a}\right)} dt$$

This is an *overcomplete* function representation.
(from one dimension, we transform into a 2D space!!).

- the translation factor $b \rightarrow$ shifts along t-axis
- the scaling factor $a \rightarrow$ shifts along ω-axis
  shorter time support $\leftrightarrow$ higher frequencies

for example mexican hat



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2. Multi-resolution

2.1. Objective and Definition

→ approximation of finite energy, continuous time signals.
multi-resolution analysis:

$$\left\{ \begin{array}{l} \text{successive (”multi”)} \\ \text{approximations (”resolutions”)} \end{array} \right\}$$

of a function.
Successive basis functions all have same shape:
→ translations and dilations of one function.

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The precise mathematical definition is as follows:

A *multi-resolution analysis* (MRA) is a sequence of closed subspaces V_j ($j \in \mathbb{Z}$) of $L^2(\mathbb{R})$ with the following properties

1. $V_j \subset V_{j+1}$
2. (a) $\bigcup_{j=-\infty}^{+\infty} V_j$ is dense in $L^2(\mathbb{R})$ i.e.: $\overline{\bigcup_{j=-\infty}^{+\infty} V_j} = L^2(\mathbb{R})$

   (b) $\bigcap_{j=-\infty}^{+\infty} V_j = \{0\}$ (0 is constant function)
3. (shift invariance)
   $v(t) \in V_0 \Leftrightarrow v(t-k) \in V_0, \forall k \in \mathbb{Z}$
4. (scale invariance)
   $v(t) \in V_j \Leftrightarrow v(2t) \in V_{j+1}, \forall j \in \mathbb{Z}$
5. (shift invariant basis)
   $\exists \varphi(t) \in V_0 : \{\varphi(t-k)\}_{k\in\mathbb{Z}}$ is a stable basis (Riesz-basis) for V_0

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2.2. The dilation equation

(refinement equation, two-scale equation)

The function $\varphi(t)$ is called *scaling function* or father function.

Dilation equation:

$$\boxed{\exists\{c_k\}_{k\in\mathbb{Z}} : \varphi(t) = \sum_{k\in\mathbb{Z}} c_k \varphi(2t-k)}$$

Immediate consequence:

$$\int_{-\infty}^{\infty} \varphi(t)dt = \sum_{k\in\mathbb{Z}} c_k \int_{-\infty}^{\infty} \varphi(2t-k)dt = \sum_{k\in\mathbb{Z}} c_k \frac{1}{2} \int_{-\infty}^{\infty} \varphi(t)dt$$

Hence, choosing

$$\int_{-\infty}^{\infty} \varphi(t)dt = \theta \neq 0 \Leftrightarrow \Phi(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(t)dt = \frac{\theta}{\sqrt{2\pi}}$$

we get

$$\sum_{k\in\mathbb{Z}} c_k = 2$$

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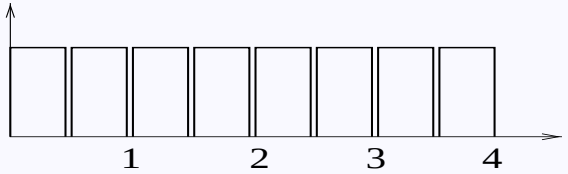
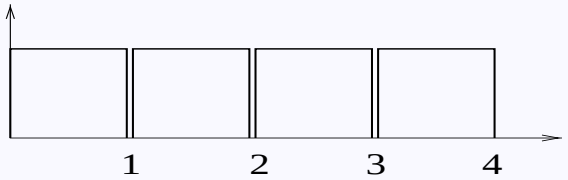
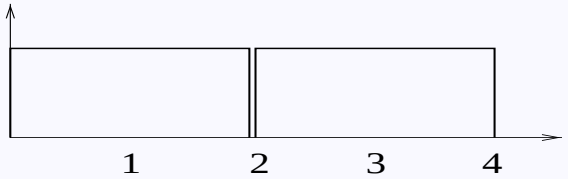
2.3. Example



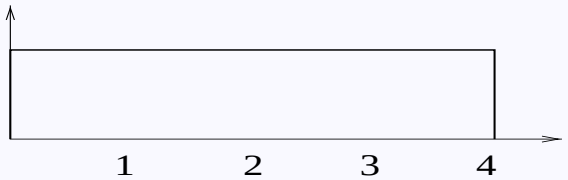
V_1



V_0



V_{-1}



V_{-2}

Terminology: j is called the *resolution level*
2^j is called the *scale*

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2.4. Solution of the dilation equation

2.4.1. By iteration

Numerical solution in time domain:

$$\varphi(t) = \sum_{k\in\mathbb{Z}} c_k \varphi(2t-k)$$

Hence, start with (for instance)

$$\varphi^{(0)}(t) = \chi_{[0,1]}(t)$$

(Box function on $[0,1]$) and iterate

$$\varphi^{(i)}(t) = \sum_{k\in\mathbb{Z}} c_k \varphi^{(i-1)}(2t-k)$$

until convergence.

This algorithm is proposed in continuous time. While implementing it (e.g. in MATLAB), be careful with discretisation!!

Theorem
if $c_n = 0$ *for* $n < n^-$ *and for* $n > n^+$, *then* $\mathrm{supp}(\varphi) \subset [n^-, n^+]$

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2.4.2. By recursion

The iteration procedure starts with an initial value in *all* points t (in practice only in points of discretisation)
Each iteration step computes a new function value in each point.

Recursion starts from the *exact* function values in *some* points and uses the dilation equation to compute other points, exactly (no function value iteration, the iteration is now on the number of points)

We have:

$$\varphi(\frac{t}{2}) = \sum_{k \in \mathbb{Z}} c_k \varphi(t-k)$$

If we know $\varphi(t)$ for all integer t, then we can compute it for all $t = k/2$. Repeating this yields $\varphi(t)$ for all dyadic points $t = k/2^j$.

We know $\varphi(k) = 0$ if $k \notin \{n^-, \ldots n^+\}$.

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Example: $n^- = 0, n^+ = 5$. Then we have

$$\begin{bmatrix} \varphi(0) \\ \varphi(1) \\ \varphi(2) \\ \varphi(3) \\ \varphi(4) \\ \varphi(5) \end{bmatrix} = \begin{bmatrix} c_0 & & & & & \\ c_2 & c_1 & c_0 & & & \\ c_4 & c_3 & c_2 & c_1 & c_0 & \\ & c_5 & c_4 & c_3 & c_2 & c_1 \\ & & & c_5 & c_4 & c_3 \\ & & & & & c_5 \end{bmatrix} \cdot \begin{bmatrix} \varphi(0) \\ \varphi(1) \\ \varphi(2) \\ \varphi(3) \\ \varphi(4) \\ \varphi(5) \end{bmatrix}$$

The matrix must have eigenvalue 1 and the function values are the components of the corresponding eigenvector.

We call this matrix C.

From this, it follows: $\varphi(n^-) = 0 = \varphi(n^+)$ (unless $c_{n^-} = 1$ or $c_{n^+} = 1$)

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2.4.3. Cascade algorithm

Recall $V_j \subset V_{j+1}$.
Suppose $f(t) = \sum_k s_{j,k}\varphi(2^j t - k) = \sum_l s_{j+1,l}\varphi(2^{j+1}t - l)$
Use $\varphi(x) = \sum_i c_i \varphi(2x - i)$ to compute $s_{j+1,k}$ from $s_{j,k}$ Result:

$$s_{j+1,l} = \sum_k c_{l-2k} s_{j,k}$$

Now, start the iteration with $s_{0,k} = \delta_k$.
This corresponds to $f(t) = \varphi(t)$.
If $j \to \infty$, the support of $\varphi(2^j t - k)$ gets smaller and smaller.
At a certain moment it can only be represented by one pixel.
The $s_{j,k}$ at this level are an approximation for $\varphi(k/2^j)$.

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2.4.4. Fourier domain solution

Define the Fourier transform $\Phi(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(t) \mathrm{e}^{-i\omega t}\, dt$
Then the dilation equation becomes in the Fourier domain

$$\Phi(\omega) = C(\frac{\omega}{2})\Phi(\frac{\omega}{2}) = \cdots = \Phi(\omega/2^k) \prod_{j=1}^{k} C(\omega/2^j)$$

with $C(\omega) = \sum_k c_k \mathrm{e}^{-i\omega k}$.

Since $\lim_{k\to\infty} \omega/2^k = 0$, and $\Phi(0) = \frac{\theta}{\sqrt{2\pi}}$, we have:

$$\Phi(\omega) = \frac{\theta}{\sqrt{2\pi}} \prod_{j=1}^{\infty} C(\omega/2^j)$$

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3. Properties of the father function

Theorem: If $\varphi(t)$ is Riemann-integrable, then

$$\boxed{\sum_{k\in\mathbb{Z}} \varphi(t-k) = \theta}$$

This is called *partition of unity.*

Theorem:

$$\boxed{\sum_{k\in\mathbb{Z}} \varphi(t-k) = \theta \Leftrightarrow \sum_{n\in\mathbb{Z}} c_{2n} = \sum_{n\in\mathbb{Z}} c_{2n+1} \Leftrightarrow C(\pi) = 0}$$

We have

$$C(\omega) = \frac{1}{2}\sum_{k\in\mathbb{Z}} c_k \mathrm{e}^{-ik\omega}$$

and since $\sum_{k\in\mathbb{Z}}(-1)^k c_k = 0$ this becomes: $\boxed{C(\pi) = 0}$
and since $\sum_{k\in\mathbb{Z}} c_k = 2$ also $\boxed{C(0) = 1}$

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4. Orthogonality

L^2 is a unitary space with dot/inner/scalar product:

$$\langle f(t), g(t) \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(t)} g(t) dt$$

This allows to define the concept of orthogonality:

$$f(t) \perp g(t) \Leftrightarrow \langle f(t), g(t) \rangle = 0$$

If $\varphi(t-k) \perp \varphi(t-l), \forall k \neq l$, we have an *orthogonal basis*.

If

$$f(t) = \sum_{k \in \mathbb{Z}} \alpha_k \varphi(t-k)$$

we can easily find α_m: multiply both sides by $\overline{\varphi(t-m)}$ and integrate to get

$$\alpha_m = \frac{\langle \varphi(t-m), f(t) \rangle}{\|\varphi(t-m)\|^2} = \frac{\langle \varphi(t-m), f(t) \rangle}{\|\varphi(t)\|^2}$$

since it is easy to see that $\|\varphi(t-m)\| = \|\varphi(t)\|$.

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5. Normalisation

theorem: $\varphi(t-m)$ is orthogonal basis then

$$\left|\int_{-\infty}^{\infty}\varphi(s)ds\right|^2=\int_{-\infty}^{\infty}|\varphi(s)|^2ds$$

$\Rightarrow$ If we choose $\int_{-\infty}^{\infty}\varphi(s)ds=\theta$, then $\|\varphi(t)\|=\theta$

If $\varphi(t-m)$ is an ortho*normal* basis for V_0

$\Rightarrow$ $\varphi(2^jt-m)$ is an ortho*gonal* basis for V_j

$\Rightarrow$ $\varphi_{j,k}(t)=2^{j/2}\varphi(2^jt-k)$ is an ortho*normal* basis for V_j, $j\in\mathbb{Z}$

So $h_k=\frac{c_k}{\sqrt{2}}$ is an *normalised* sequence:

$$\sum_{k\in\mathbb{Z}}|h_k|^2=1\quad\text{and}\quad\sum_{k\in\mathbb{Z}}h_k=\sqrt{2}$$

The dilation equation $\varphi(t)=\sum_{k\in\mathbb{Z}}c_k\varphi(2t-k)$ becomes:

$$\varphi_{j,m}(t)=\sum_{k\in\mathbb{Z}}h_k\varphi_{j+1,2m+k}(t).$$

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Orthogonality of the basis, can be expressed as

$$\boxed{\sum_{n\in\mathbb{Z}} c_{n-2k}\overline{c_n} = 2\delta_k} \qquad (*)$$

The coefficients have *double shift orthogonality*.
The inverse is also true: coefficients with double shift orthogonality lead to orthogonal scaling functions.

In the frequency domain, this is:

$$\boxed{|C(\omega)|^2 + |C(\omega+\pi)|^2 = 1}$$

Corollary: Orthogonal scaling functions with compact support must have an even number of non-zero coefficients in the dilation equation.

Furthermore for $\Phi(\omega)$ we have:

Theorem: If $\{\varphi_{0,k}\}$ is orthonormal $\Leftrightarrow$

$$\boxed{\sum_{k\in\mathbb{Z}} |\Phi(\omega+2k\pi)|^2 = \frac{1}{\sqrt{2\pi}}}$$

$\Leftrightarrow$ *partition of unity property.*

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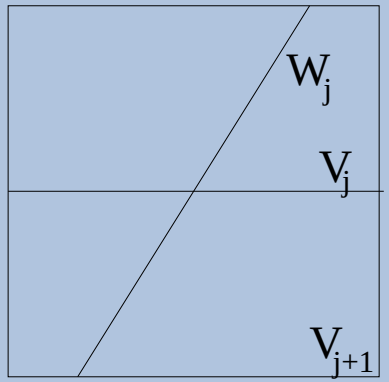
6. Wavelets



Recall:

$$V_j \subset V_{j+1}$$

We now try to catch the information present in V_{j+1}, but lost in V_j: Suppose we have W_j, such that:

$$\begin{array}{ccc} V_{j+1} & = & V_j \oplus W_j \\ & \Updownarrow & \\ V_{j+1} = V_j + W_j & \text{and} & V_j \cap W_j = \{\mathbf{0}\} \end{array}$$

V_{j+1} is the direct sum of V_j and W_j.

$V_j + W_j$ is a *linear sum*: all elements of $V_j + W_j$ can be written as a sum of an element from V_j and an element from W_j. If $V_j \cap W_j = \{\mathbf{0}\}$, this decomposition is unique.

special case $V_j \perp W_j$: orthogonal sum; W_j is the orthogonal complement of V_j in V_{j+1}.

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Theorem: *If $\{\phi(t-k)\}_{k\in\mathbb{Z}}$ constitute an orthogonal basis for V_0, then there exists one function $\psi(t)$ such that $\{\psi(t-k)\}_{k\in\mathbb{Z}}$ forms an orthogonal basis for the orthogonal complement W_0 of V_0 in V_1.*

We call this function *mother function* or *wavelet function*.

We know:

$$V_J = V_{J-1} \oplus W_{J-1} = V_L \oplus \bigoplus_{j=L}^{J-1} W_j$$

A function

$$f(t) = \sum_{k=0}^{2^J-1} s_k \varphi(2^J t - k)$$

can be written as: (suppose $L \leq 0$)

$$f(t) = \sum_{j=L}^{J-1} \sum_{k=0}^{2^j-1} w_{j,k} \psi(2^j t - k) + \sum_{k=0}^{2^L-1} s_k \varphi(2^L t - k)$$

This decomposition is a *discrete wavelet expansion*

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7. The wavelet equation

Since $W_0 \subset V_1$, we have $\boxed{\psi(x) = \sum_{k\in\mathbb{Z}} d_k \varphi(2x-k)}$

Orthogonality

$$\int_{-\infty}^{\infty} \overline{\psi(x)}\varphi(x-k)dx = 0 \quad \& \quad \int_{-\infty}^{\infty} \overline{\psi(x)}\psi(x-k)dx = \delta_k$$

implies:

$$\boxed{\sum_{n\in\mathbb{Z}} c_{n-2k}\overline{d_n} = 0} \quad \boxed{\sum_{n\in\mathbb{Z}} d_{n-2k}\overline{d_n} = 2\delta_k}$$

A possible solution is:

$$d_k = (-1)^k \overline{c}_{1-k}$$

If we put both sequences c_k and d_k under each other

$\boldsymbol{c}$	$\ldots$	c_{-3}	c_{-2}	c_{-1}	c_0	c_1	c_2	c_3	c_4	$\ldots$
$\boldsymbol{d}$	$\ldots$	$-c_4$	c_3	$-c_2$	c_1	$-c_0$	c_{-1}	$-c_{-2}$	c_{-3}	$\ldots$

we immediately see that their inner product is zero.

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8. The integral of the mother function

Since

$$\psi(x) = \sum_{k\in\mathbb{Z}} d_k \varphi(2x-k)$$

we have (in the orthogonal case):

$$\int_{-\infty}^{\infty} \psi(x)dx = \sum_{k\in\mathbb{Z}} (-1)^k \bar{c}_{1-k} \int_{-\infty}^{\infty} \varphi(2x-k)dx$$

Since

$$\sum_{k\in\mathbb{Z}} (-1)^k c_{1-k} = 0$$

this is:

$$\int_{-\infty}^{\infty} \psi(x)dx = 0$$

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9. Normalisation of the wavelets

Suppose the fully orthogonal case and call:

$$\psi_{j,m}(t) = 2^{j/2}\psi(2^j t - m)$$

Then this is an orthonormal basis for L^2. Also:

$$\begin{aligned}
\psi_{j,m}(t) &= 2^{j/2}\psi(2^j t - m) \\
&= 2^{j/2}\sum_{k\in\mathbb{Z}} d_k \varphi(2^{j+1}t - 2m - k) \\
&= \sum_{k\in\mathbb{Z}} \frac{d_k}{\sqrt{2}}\, 2^{(j+1)/2}\varphi(2^{j+1}t - 2m - k) \\
&= \sum_{k\in\mathbb{Z}} g_k\, \varphi_{j+1,2m+k}(t)
\end{aligned}$$

where $g_k = \frac{d_k}{\sqrt{2}}$ is a normalised sequence:

$$\sum_{k\in\mathbb{Z}} |g_k|^2 = \sum_{k\in\mathbb{Z}} |h_k|^2 = 1 \quad \text{and} \quad \sum_{k\in\mathbb{Z}} g_k = 0, \quad \sum_{k\in\mathbb{Z}} h_k = \sqrt{2}$$

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