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A PHYSICS INFORMED NEURAL NETWORK (PINN) APPROACH FOR SOIL-PILE INTERACTION

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Abstract. *This work presents a reference solution for a typical soil-pile interaction problem, by means of a Physics Informed Neural Network (PINN). Advanced elastodynamic solutions for pile response can perform as scoping tools in early design stage and complement finite element simulations serving as "benchmark" solutions to allow the verification of more complex dynamic numerical models. Their intrinsic theoretical interest lies in tackling the Sturm-Liouville (SL) boundary value problem, which in presence of soil inhomogeneity is not straightforward, and can yield solutions only for specific types of soil inhomogeneity. Inspired by the recent advancements in scientific machine learning in a wide range of scientific disciplines, an application of a PINN to Soil-Structure Interaction (SSI) is presented herein. In this respect, eigenvalues and eigenfunctions of a SL operator, which arises in the classic elastic solution of a single axially loaded pile embedded in inhomogeneous soil deposit, are obtained. PINNs are physically motivated neural networks, in the sense that natural constraints such as physical laws, boundary conditions or other physical properties are embedded in either the cost function or the architecture of the network, to form a data-efficient universal function approximator. Moreover, their versatility (high adaptability to a wide range of problems), their straightforward extension to higher dimensions and, their meshfree nature (free from geometrical restrictions imposed by conventional numerical methods), can render them a valuable asset in the SSI toolkit of geotechnical engineers.*

Keywords: soil-pile interaction, PINNs, neural network, eigenfunctions, eigenvalues

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1 INTRODUCTION

The analysis of pile foundations under axial load is an important topic in geotechnical engineering, and consequently there are several methods available for this purpose, each with its advantages and limitations. The selection of a suitable method depends on the complexity of the problem in question, the accuracy of the analysis, and the soil type. Current, commonly used, methods for determining pile settlements can be classified into four main groups: (A) experimental methods, which can provide direct measurement of the pile capacity and settlement, and can be used to validate the results of theoretical methods; such method can be expensive and time-consuming. (B) semi-empirical methods, which are relatively simple methods, to predict pile settlements in a wide range of soil conditions, yet its accuracy depends on calibration with field data. (C) numerical methods, such as the widely used finite element method, which can account for soil non-linearity, and can handle a variety of structure-soil configurations. Such methods require the use of specialized software and significant computational resources, especially for dynamic analyses. (D) elastic analytical models, which assume that the soil and the pile behave elastically, and could be used for preliminary design calculations, accounting implicitly for the effects of soil non-linearity by pertinent modifications of soil stiffness. An additional benefit of such methods is that they can act as complementary tools to finite element simulations, serving as benchmark solutions to allow the verification of more complex dynamic numerical models, and safeguard the design process from systemic risk caused by uncertainty in simulations.

With reference to models in group (D), a promising family of analytical models are those often referred to as Tajimi formulations, and are associated with the approximate continuum models of Matsuo and Ohara [1] and Tajimi [2]. Such models treat the pile as a rod or beam, following the classical strength-of-materials solution and assume perfect bonding at the soil-pile interface. The soil is modelled as an approximate continuum of the Tajimi type; yet retains the three-dimensional features of the problem. The number of dependent variables is reduced by eliminating certain stress and displacement components in the governing equations in the soil medium. The solution is then expressed in terms of ”soil modes” or eigenfunctions along the vertical spatial coordinate. A number of closed form solutions for homogeneous soils have been obtained by means of this approach for piles (e.g., [3, 4, 5, 6, 7, 8, 9, 10, 11]) and retaining walls (e.g., [12, 13, 14]). A detailed description of the advantages of this family of models against numerical solutions and simplified approaches is presented in Anoyatis *et al.* [11], for axially-loaded piles embedded in inhomogeneous soils.

An added, less explored, yet powerful advantage of models within group (D) is that they can serve as a valuable tool in the development and application of Physics-Informed Neural Networks (PINNs); whenever available, analytical models, can provide a useful tool to assess PINN’s performance. PINNs is a framework that combines physics-based models with neural networks to solve complex problems. They were introduced in the seminal work of Raissi, Perdikaris and Karniadakis [15], and, since then, they have attracted the increasing attention of the Scientific Machine Learning community. PINNs can be seen as a powerful alternative to provide numerical solutions by means of training a neural network, which respects the physical laws of the problem at hand, as described by the pertinent partial differential equations (PDEs). PINNs’ ability to provide solutions to PDEs and eigenvalue problems has been tested to a series of works (e.g., [15, 16, 17, 18, 19]).

Evidently, since PDEs are fundamental to the description of a wide range of physical processes, PINNs have the potential to become a valuable tool for scientists and engineers, in

particular where less data are available. This is achieved due to PINNs being physically motivated and requiring no external data, since training can be performed in terms of generated data samples. Another key advantage compared to traditional numerical differential equation solvers is that PINNs are inherently mesh-free, and can generate solutions that are readily differentiable, contrary to solutions at specified grid points. Being mesh-free adds to the flexibility of PINNs, rendering them applicable to a wide range of problems, including those with complex geometries or multi-physics problems. Further, in the same spirit, PINNs can be more robust to changes in geometry or boundary conditions of a problem, as they are not tied to a specific mesh. This also means that the computational time needed to solve a problem using PINNs can be significantly reduced, rendering them as a computationally efficient alternative to available and commonly used finite element method. As a final remark, PINNs can help reduce modeling errors, in particular where the underlying physics of a system are not well understood. This is because PINNs can be intrinsically trained on experimental or simulation data, which allows for the experimental or simulation error to be accounted for in the model.

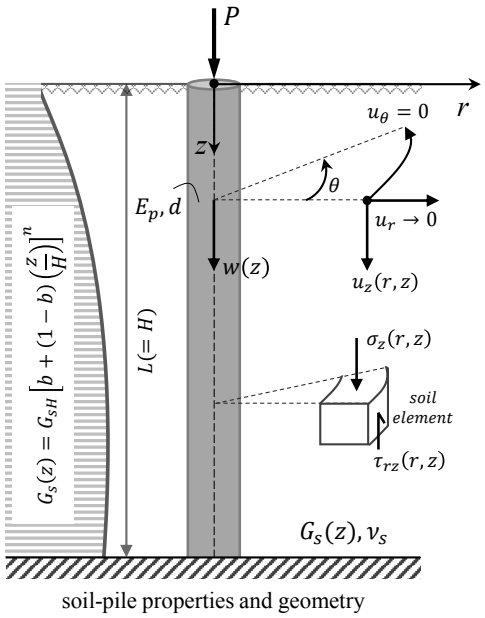

Figure 1: Stresses and displacements in a Tajimi type continuum.

Soil inhomogeneity is expressed via a depth-dependent soil shear modulus $G_s(z)$ which obeys a power law variation with depth; n and $b = (G_{s0}/G_{sH})^{1/n}$ are dimensionless inhomogeneity factors, G_{s0} and G_{sH} are the soil shear moduli at the surface and the base of the soil deposit, respectively.

$$G_s(z) = G_{sH}\left[b + (1-b)\frac{z}{H}\right]^n \tag{1}$$

As shown in [11], handling an inhomogeneous soil medium of the Tajimi type is not straightforward; application of the orthogonality identity of the soil modes is used to satisfy the perfect bonding condition at the pile-soil interface. The solution is expressed in the form of series and

can handle different types of soil inhomogeneity using natural soil "modes" (eigenfunctions), and associated "eigenvalues", to describe the attenuation of soil displacement with depth and radial distance from the pile. The series' coefficients are coupled and are obtained as solutions to a set of simultaneous algebraic equations of rank equal to the number of modes considered in the analysis. The eigenvalues are obtained as the roots of of an algebraic non-linear equation, which especially in presence of dynamic loads, becomes an intricate task [20]. Herein a PINN is constructed to obtain the eigenfunctions and eigenvalues of the problem (Fig. 1).

2 THE ANALYTICAL MODEL

Key to the Tajimi approach in axial mode is the physically motivated assumption that the vertical normal and vertical shear stresses in the soil are controlled exclusively by the vertical soil displacement component. This reduces the two equations of classical axisymmetric elasticity to one [11] (Fig. 1). Thus, the equilibrium of vertical forces acting on a soil element in axisymmetric mode yields the following partial differential

$$\frac{\partial(\tau\, r)}{\partial r} \;+\; r\,\frac{\partial \sigma}{\partial z} \;=\; 0 \tag{2}$$

where $\tau = \tau_{rz}(r,z)$ is the vertical shear stress and $\sigma = \sigma_z(r,z)$ is the vertical normal stress. The stress-displacement relations in an axisymmetric Tajimi type continuum are obtained as

$$\sigma(r,z) \approx -\eta_s^2\, G_s(z)\,\frac{\partial u}{\partial z} \tag{3}$$

and

$$\tau(r,z) \approx -G_s(z)\,\frac{\partial u}{\partial r} \tag{4}$$

where $u = u(r,z)$ is the vertical soil displacement and $\eta_s = \sqrt{(2-\nu_s)/(1-2\nu_s)}$ is a dimensionless compressibility parameter, which accounts, indirectly, for the effect of vanishing the horizontal soil displacement on soil stresses, and depends solely on Poisson ratio ν_s [4]. Substituting Equations (3) and (4) into (2), the equilibrium equation is written in terms of displacements. The solution is then obtained using the method of separation of variables, which enables the decomposition of the governing partial differential equation into the pair of ordinary differential equations. The equation which involves the independent variable z is a Sturm-Liouville (SL) equation with variable coefficients, and its solution depends on the functional form $G_s(z)$.

$$\frac{d}{dz}\left(\frac{G_s(z)}{G_{sH}}\,\frac{d\Phi}{dz}\right) \;+\; a^2\left(\frac{G_s(z)}{G_{sH}}\right)\Phi \;=\; 0, \quad z \in [0,H] \tag{5}$$

Finding the distinct values a_m ($m = 1, 2, 3, \dots$), for which nontrivial solutions exist, is part of the SL theory. a_m's are referred to as the eigenvalues of the boundary-value problem. Solutions to Equation (5) are the eigenfunctions $\Phi = \Phi_m(z)$ (i.e., soil modes), which satisfy the boundary conditions of the problem i.e., stress-free soil surface and zero displacements at the base. In the next section a PINN is constructed to obtain the eigenfunctions and eigenvalues of the SL problem, by considering the properties of SL operators. According to the *Universal Approximation Theorem*, any continuous function can be approximated using a neural network [21]. In this spirit, Hornik *et al.* demonstrated that neural networks can approximate any measurable function [22]. The special case of linear variation of shear modulus with depth and zero shear modulus at the surface is investigated in this work i.e., $b = 0$ and $n = 1$ in Equation (1).

3 THE NEURAL NETWORK

A fully-connected feed-forward Neural Network (NN) i.e., $\mathcal{N}^L : R \rightarrow R$ is employed to solve the SL problem. The NN transforms the input data to an output which approximates each eigenfunction Φ_m. An input $x \in R$ is successively propagated through L layers of neurons. Each neuron applies an affine transformation, followed by a nonlinear activation function. The training points are uniformly generated within the interval $[0, H]$, to effectively create an one-dimensional grid. The grid points created within each training iteration are slightly perturbed to capture extra points within the interval. This allows the network to train with additional points on top of the initial uniformly distributed points. A Neural Network with three hidden layers of 30 neurons each, and a trigonometric activation function $\sin x$ are selected. The input of each hidden layer is further fed directly to the next hidden layer. This way a Residual Network (ResNet) structure is created, where each layer is itself a block. The ResNet structure is quite ordinary in the context of deep learning. It usually aims to fix the problem of vanishing or exploding gradient, which is common in image recognition tasks [23]. Despite the radically different nature of the problem at hand, the ResNet structure turned out to be fruitful. The output of the NN can be written as:

$$N(z) = \mathcal{N}^L(z; \theta),\ \theta \in \Theta \tag{6}$$

The set of all weights span the parameter space Θ, whilst the tunable parameters θ update iteratively during training. The training strategy lies in tuning the parameters θ and minimise a Residual Loss L_θ on the given data:

$$\text{Find } \theta^* \in \Theta :\ \theta^* = \underset{\theta}{\operatorname{argmin}}\, L_\theta \tag{7}$$

The ADAM optimizer, a learning rate equal to 10^{-2}, and a loss function with four different terms are adopted to train the Neural Network. All four terms are essential for the successful training of the network. Each one accomplishes a specific task, which is explained in detail in the next subsections.

3.1 The parametric function

To enforce the boundary conditions, a parametric trick, commonly used in PINNs is adopted (e.g., [19, 24]): for Dirichlet and homogeneous boundary conditions, the output N is multiplied with a suitable function $g(z)$. In the same fashion, to implement the mixed boundary conditions for the problem examined, Φ's are described as follows:

$$\Phi(z) \ = \ \tanh(z - H) \left[(\tanh(z - H) - 2\, \tanh(-H))\, N \ + \ \tanh(-H)\, \tanh(z)\, \frac{\partial N}{\partial z} \right] \tag{8}$$

This way the boundary conditions are enforced directly to the output. This is commonly referred to as hard constraint, in contrast to soft constraints imposed via loss functions.

3.2 The loss function

The following loss function is adopted:

$$L \ = \ \lambda_{\mathrm{DE}}\, L_{\mathrm{DE}} \ + \ \lambda_{\mathrm{norm}}\, L_{\mathrm{norm}} \ + \ \lambda_{\mathrm{Bol}}\, L_{\mathrm{Bol}} \ + \ \lambda_{\mathrm{Slide}}\, L_{\mathrm{Slide}} \tag{9}$$

where DE is the differential equation error, $norm$ is the normalization error, Bol refers to the nodes error, and $Slide$ is the slide loss.

Differential equation error The following loss function is considered, which complies with the requirement for Φ to satisfy Equation (5)

$$L_{\mathrm{DE}} \;=\; \int_0^H \left[\frac{d}{dz}\left(\frac{G_s(z)}{G_{sH}}\,\frac{d\Phi}{dz}\right) \;+\; a^2\,\frac{G_s(z)}{G_{sH}}\,\Phi\right]^2 dz \tag{10}$$

where a^2 is given by the Rayleigh quotient:

$$a^2 \;=\; \frac{\displaystyle\int_0^H G_s(z)\,\left(\frac{d\Phi}{dz}\right)^2 dz}{\displaystyle\int_0^H G_s(z)\,\Phi^2\,dz}. \tag{11}$$

Normalization error The normalization loss enforces a non-zero solution and the eigenfunction's squared integral to be finite. This way it prevents the network from discovering trivial eigenfunctions and eigenvalues:

$$L_{\mathrm{norm}} \;=\; \left(1 - \int_0^H \frac{G_s(z)}{G_{sH}}\,\Phi^2\,dz\right)^2 \tag{12}$$

Nodes error In principle, the aforementioned steps should help discover the first eigenfunction. The challenge lies in guiding the NN to discover the higher eigenfunctions. In the work of Jin *et al.* this is achieved by considering the inherent orthogonality identity of the eigenfunctions of SL problems [19]. In this work a more efficient alternative is adopted. According to the SL theorem, the m-th eigenfunction Φ_m has exactly m nodes in $(0, H)$ [25]. For a regular SL problem it can be proven that there is a sign alternation of Φ_{m+1} between two successive nodes of $\Phi_m, \forall m \in N$. This means that Φ_{m+1} has a node within that interval [25]. It can be shown that for Neumann boundary conditions, there is at least one node of Φ_{m+1} between the corresponding endpoint and the closest Φ_m root. In this work, an approach inspired by the root-finding Bisection method is employed to discover the nodes accounting for the alternating positive and negative behavior of the eigenfunction [26]. Thus, one can set:

$$L_{\mathrm{Bol}}^{m,m+1} \;=\; \sum_{i=0}^{m+1} \Phi_{m+1}(z_i)\,\Phi_{m+1}(z_{i+1}) \;+\; \left|\Phi_{m+1}(z_i)\,\Phi_{m+1}(z_{i+1})\right| \tag{13}$$

where $z_1, \ldots, z_m$ are the nodes of Φ_m, and $\{z_0, z_{m+1}\} = \{0+\varepsilon, H-\varepsilon\}, \varepsilon \ll 1$. By minimising the loss term expressed by Equation (13) to zero, the network reproduces a function with a node within each desired interval.

Slide loss The last term is expressed by the following equation:

$$L_{\mathrm{slide}}^{m+1} \;=\; \frac{a_m^2}{a_{m+1}^2}. \tag{14}$$

In fact this is not a conventional loss term; it is not possible to minimize such term, since that would require the eigenvalue to become infinite. Instead, it is used herein to guide the neural network to search for eigenvalues greater than the previously obtained ones. This becomes

increasingly important for eigenvalues of higher modes. It is common knowledge that finding functions with high frequency components is not an easy task for neural networks [27]. For the problem at hand, such term creates some kind of inverse-squared slide in terms of the eigenvalue, which along with the other loss terms, enables the network to capture the more complicated, excited eigenfunctions.

Each loss term is multiplied by a suitable factor to ensure that they balance well with each other. Then the network parameters are updated to minimize all of them. The integrals in the loss functions are calculated using the Monte Carlo integration [28]. This way the Mean Squared Error (MSE) is reduced, following previous works on PINNs [15, 19, 29].

4 RESULTS

The neural network architecture consists of 3 blocks with 30 neurons each. Results are presented for the special case of $b = 0$ and $n = 1$, which correspond to piles in Gibson soil with zero stiffness at the soil surface. The terms in Equation (9) are selected as: $\lambda_{\mathrm{DE}} = 1$, $\lambda_{\mathrm{norm}} = 1$, $\lambda_{\mathrm{Bol}} = 10^5$ and $\lambda_s = 10^4$. Comparisons of the neural network predictions with results obtained from the semi-analytical model by Anoyatis *et al.* [11], demonstrate the predictive power of the proposed method.

Table 1 presents the results of the first fifteen eigenvalues a_m. The comparisons between the actual eigenvalues a_{true} obtained from the semi-analytical model by Anoyatis *et al.* [11], and the eigenvalues a_{NN} obtained from the proposed neural network are in excellent agreement.

m-th eigenvalue	a_{true}	a_{NN}	$\frac{a_{\mathrm{NN}} - a_{true}}{a_{true}} \times 10^{-5}$
1	2.405	2.405	0.0
2	5.520	5.521	18
3	8.654	8.655	11
4	11.792	11.793	0.8
5	14.931	14.932	0.6
6	18.071	18.073	11
7	21.212	21.213	0.4
8	24.352	24.355	12
9	27.493	27.495	0.7
10	30.635	30.637	0.6
11	33.776	33.778	0.5
12	36.917	36.920	0.8
13	40.058	40.062	0.9
14	43.200	43.204	0.9
15	46.341	46.345	0.8

Table 1: Results of the first fifteen eigenvalues a_n. Comparison between the actual eigenvalues a_{true} obtained from the semi-analytical model by Anoyatis *et al.* [11], and the eigenvalues a_{NN} obtained from the proposed neural network. $b = 0$, $n = 1$ in Eq. (1)

Figures 2 and 3 plot the eigenfunctions for the 1st-2nd-3rd and 11th-12th-13th soil modes, respectively. The predictions from the proposed PINN are compared against results from the

analytical model [11]. An excellent agreement is observed for all cases investigated, which demonstrate the accuracy of the proposed method.

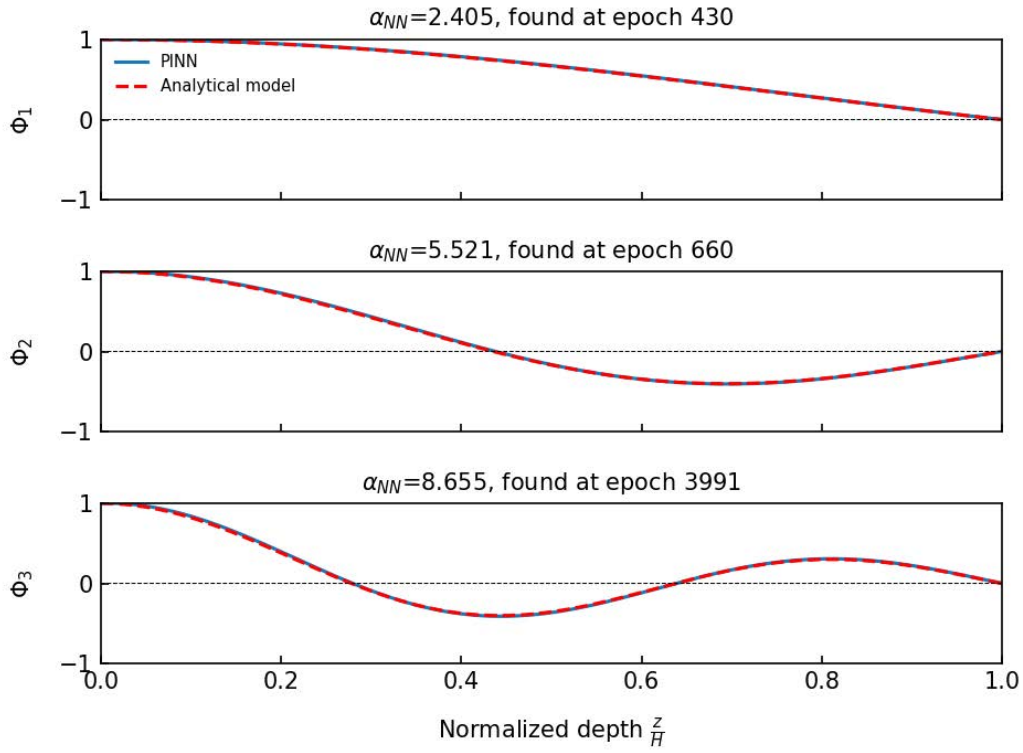

Figure 2: Profiles of the first three eigenfunctions Φ_1, Φ_2, Φ_3. Comparisons between the proposed neural network approach and results from the analytical model by Anoyatis *et al.* [11].

Figures 4 presents the evolution of loss functions and eigenvalues with epochs, for the first three modes ($m = 1, 2, 3$, sub-figures (a) and (c)) and higher modes ($m = 11, 12, 13$, sub-figures (b) and (d)). In machine learning an epoch refers to the one entire passing of training data through the algorithm. It can be seen as a hyperparameter that determines the process of training the machine learning model. Each time a data set passes through an algorithm, completes an epoch. In both cases examined (low and high modes), the neural network detects the correct eigenvalue, and starts to build the corresponding suitable eigenfunction. It is evident that higher modes require training using a larger number of epochs.

5 SUMMARY AND CONCLUDING REMARKS

This study investigates the use of Physics-Informed Neural Networks (PINNs) as a method for solving classical problems in geotechnical engineering. For this reason, a soil-pile interaction problem for which an analytical model of the Tajimi type for the soil has been recently developed by some of the Authors, is selected [11]. The analytical approach extends the model proposed by Mylonakis [3], following the seminal work of Nogami & Novak [4] for end-bearing piles, to vertically inhomogeneous soils. Fundamental to the semi-analytical approach is the solution of the boundary value problem; the calculation of the eigenvalues and the corresponding eigenfunctions (soil modes). The eigenvalues are obtained as the roots of of an algebraic nonlinear equation, which especially in dynamic regime can be a challenging task [20].

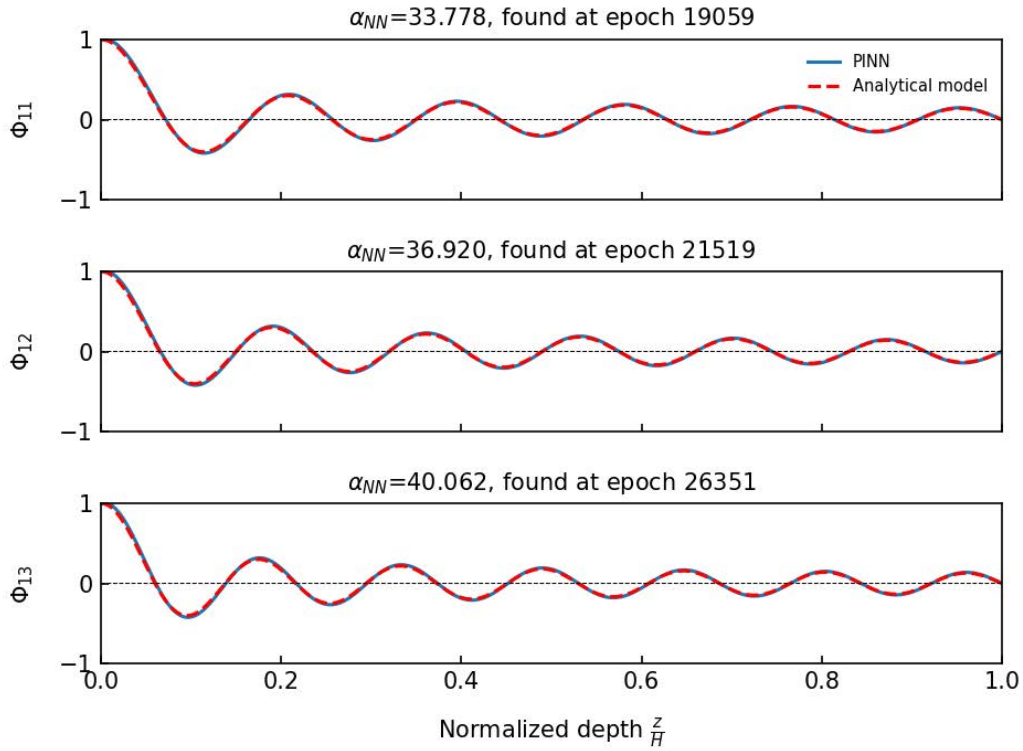

Figure 3: Profiles of higher eigenfunctions Φ_{11}, Φ_{12}, Φ_{13}. Comparisons between the proposed neural network approach and results from the analytical model by Anoyatis *et al.* [11].

Herein a PINN is constructed to obtain the eigenfunctions and eigenvalues of the problem for an axially-loaded, end-bearing pile embedded in Gibson soil, by considering the properties of Sturm-Liouville operators. The ADAM optimizer, a learning rate equal to 10^{-2}, and a loss function with four different terms are adopted to train the neural network. For the cases examined, it has been demonstrated that the PINN can be seen as a powerful alternative to provide numerical solutions by means of training a neural network, which respects the physical laws of the soil-pile interaction problem, as described by the governing partial differential equation. The predictive power of the proposed neural network based approach is demonstrated via the excellent agreement between proposed PINN results and results obtained from the existing analytical method in terms of both eigenvalues and eigenfunctions. It is also evident that higher modes require training using a larger number of epochs.

Overall, it is shown that analytical approach can be a valuable tool in the development and application of PINNs, allowing physical laws and relationships to be incorporated into machine learning models, which in turn lead to accurate and efficient predictions. Combining the power of neural networks with the precision of analytical models, render PINNs an attractive and promising alternative to predict the behavior of more complex foundation systems i.e., complex geometry, soil non-linearity, dynamic analyses without the need for expensive simulations or (limited) experimental data. It is though fair to mention that PINNs are a relatively new method and further systematic exploration is required. Nevertheless they are likely to play an increasingly important role in the development of advanced methodologies and scientific understanding in the coming years.

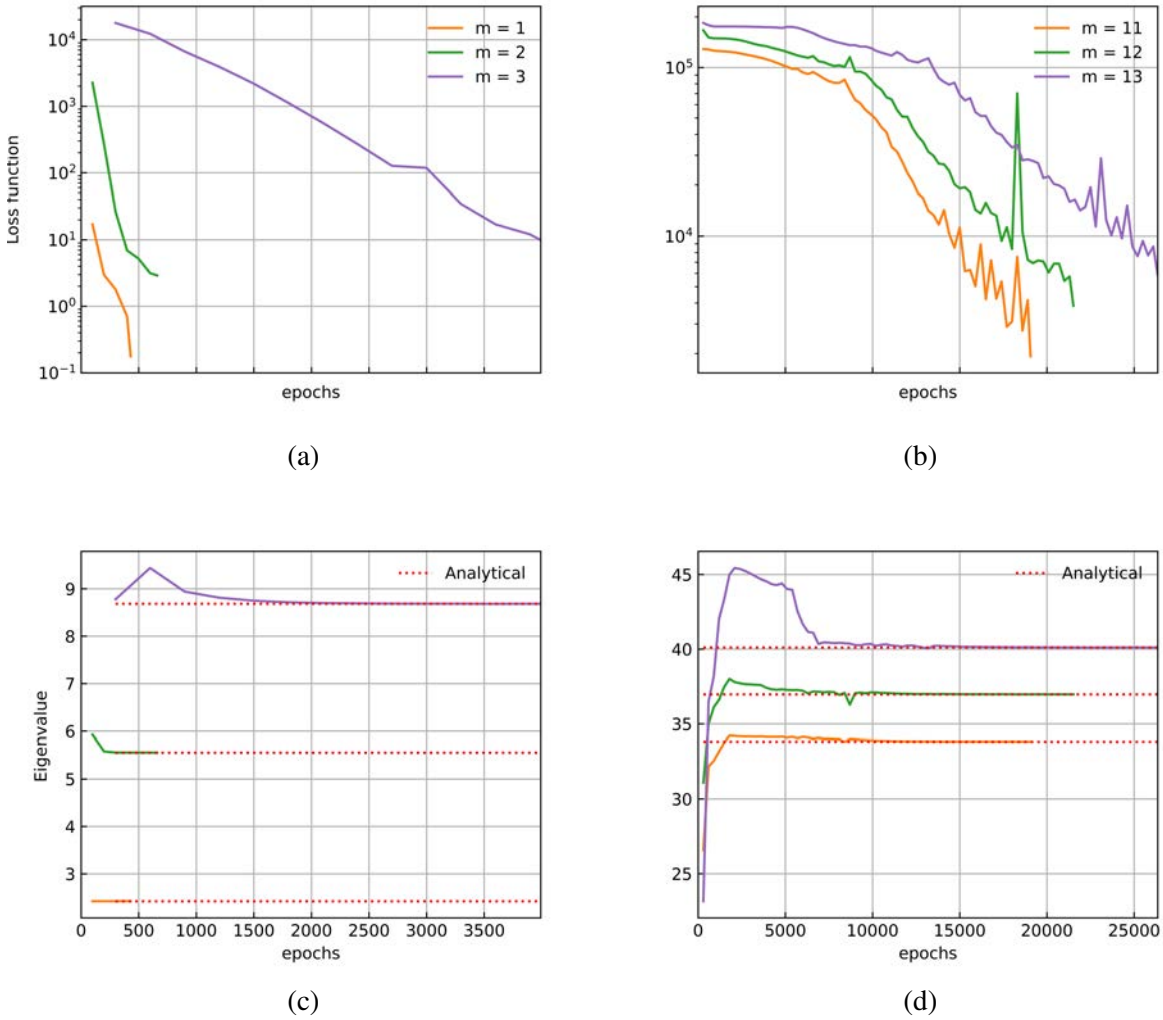

Figure 4: Evolution of loss functions and eigenvalues a_m's during training, measured in epochs. (a)-(c) $m = 1, 2, 3$; (b)-(d) $m = 11, 12, 13$. Subfigures (c) and (d) show comparisons against the actual eigenvalues a_{true}, as obtained from the analytical solution by Anoyatis *et al.* [11].

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