On the choice of the PS–triangles

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Report TW 353, February 2003



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Abstract

In this paper we look at different bases for the space $S_2^1(\Delta_{PS})$ of C^1 continuous quadratic splines on Powell–Sabin triangle splits. First, by requiring a partition of unity, we describe a general framework that leads to the notion of PS–triangles and control triangles. This is a useful property that gives insight in the shape of the surface. The problem of choosing appropriate basis functions for a certain triangulation of the domain is now equivalent to choosing the corresponding PS–triangles. Then we consider two choices more in detail: a normalised B–spline basis, and a basis based on minimal determining sets.

The approach with minimal determining sets leads to bases that are stable as a function of the smallest angle in the triangulation. The basis functions are not only linearly independent but also locally linearly independent. In the normalised B–spline representation the basis functions form a convex partition of unity. Geometrically this can be interpreted as a PS–triangle containing a specific set of Bézier domain points. There is more than one triangle that satisfies this requirement. We show why the triangle with minimal area is a suitable choice and also give an alternative that is computationally more efficient.

Keywords : quadratic splines, Powell–Sabin split, normalised B–splines, minimal determining set, local stable bases, partition of unity, convexity.
AMS(MOS) Classification : 65D07, 65D17, 68U07.

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1 Introduction

Let Δ be a triangulation of a polygonal domain Ω in $\mathbb{R}^2$ with boundary $\delta\Omega$. The polynomial spline space $S_d^r(\Delta)$ is of the form

$$S_d^r(\Delta) = \{s \in C^r(\Omega) : s|_T \in \mathcal{P}_d \text{ for all } T \in \Delta\}, \tag{1.1}$$

where $d > r \geq 0$ are given integers and $\mathcal{P}_d$ is the space of bivariate polynomials of total degree $\leq d$.

A basis $\{B_i\}_{i=1}^n$ for a spline space is called local if there is constant l such that for each $1 \leq i \leq n$, there is a vertex V_i of Δ for which

$$\text{supp}(B_i) \in M_i^l. \tag{1.2}$$

Here M_i^0 is defined to be the set of all triangles surrounding a vertex V_i and M_i^l is the union of all M_j^0 where V_j is a vertex of M_i^{l-1}. $M_i^0 = M_i$ is also called the molecule of the vertex V_i. The number of triangles in the molecule is the molecule number m_i and a local basis function for which $supp(B_i) = M_i$ is called star-supported.

The basis is called stable if there exists constants K_1 and K_2 depending only on the smallest angle in Δ such that for all choices of the coefficient vector c

$$K_1\|c\|_\infty \leq \|\sum_{i=1}^n c_i B_i\|_\infty \leq K_2\|c\|_\infty \tag{1.3}$$

with $\|c\|_\infty = \max_i |c_i|$ and $\|f\|_\infty = \max_\Omega |f(x,y)|$.

Finding stable local bases for spline spaces $S_d^r(\Delta)$ is a non trivial task for $r > 0$, and for general triangulations can only be done when $d \geq 3r+2$ [8]. In this paper we study C^1 continuous quadratic splines, thus with $r = 1$ and $d = 2$. Because there exists no general solution, we restrict ourselves to Powell–Sabin triangulations Δ_{PS}. These are triangulations with a special structure obtained from an arbitrary triangulation Δ by splitting each triangle into six subtriangles.

We discuss two bases for this space. The first one is constructed using the concept of minimal determining sets. This basis is proven to be local and stable by Lai and Schumaker [9]. An additional property is that the basis functions are locally linearly independent. The second basis is a convex partition of unity first introduced by Dierckx [5]. It is easily shown that this is also a stable basis. There is more than one possibility that satisfies the conditions and we give two examples.

2 C^1 Quadratic Splines on Powell-Sabin Triangle Splits

2.1 Polynomials in the Bernstein–Bézier representation

Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, $|\lambda| = \lambda_1 + \lambda_2 + \lambda_3 = d$, $\lambda_i \in \{0, 1, \ldots, d\}$ using standard multi index notation. Consider a non degenerate triangle $\mathcal{T}(T_1, T_2, T_3)$ in a plane with its vertices having Cartesian coordinates $T_i(x_i, y_i), i = 1, 2, 3$. Any point $P(x,y)$ in that plane can be expressed in terms of barycentric coordinates $\tau = (\tau_1, \tau_2, \tau_3)$ with respect to $\mathcal{T}$: $P = \sum_{i=1}^3 \tau_i T_i$ where $|\tau| = 1$. These barycentric coordinates are the unique solution to the system

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}. \tag{2.1}$$

Any polynomial $p(x,y) \in \mathcal{P}_d$ over the triangle $\mathcal{T}$ has a unique Bézier representation [7]

$$p(x,y) = b_{\mathcal{T}}^d(\tau) = \sum_{|\lambda|=d} b_\lambda B_\lambda^d(\tau), \tag{2.2}$$

in which b_λ are called Bézier ordinates, and

$$\begin{aligned} B_\lambda^d(\tau) &= \frac{d!}{\lambda_1!\lambda_2!\lambda_3!} \tau_1^{\lambda_1} \tau_2^{\lambda_2} \tau_3^{\lambda_3}, \\ B_\lambda^d(\tau) &\geq 0, \qquad (x,y) \in \mathcal{T}, \\ \sum_{|\lambda|=d} B_\lambda^d(\tau) &= 1, \qquad (x,y) \in \mathcal{T} \end{aligned} \tag{2.3}$$

are the Bernstein–Bézier polynomials on the triangle.

The domain point ξ_λ associated with the Bézier ordinate b_λ is the point in the (x,y) plane with barycentric coordinates $(\frac{\lambda_1}{d}, \frac{\lambda_2}{d}, \frac{\lambda_3}{d})$ or

$$\xi_\lambda = \frac{\lambda_1 T_1 + \lambda_2 T_2 + \lambda_3 T_3}{d}, \quad |\lambda| = d. \tag{2.4}$$

The disk of domain points of the vertex T_1 is defined by

$$D_n^{\mathcal{T}}(T_1) = \{\xi_\lambda \mid \lambda_1 \geq d - n\}, \tag{2.5}$$

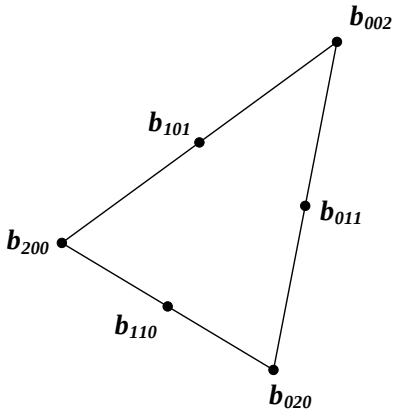


Figure 1: Positions of Bézier ordinates for $d = 2$.

with similar definitions at the other vertices.
The Cartesian coordinates (X_λ, Y_λ) of a domain point ξ_λ are the coefficients b_λ in the representation (2.2) of the specific polynomials $p(x,y) = x$ and $p(x,y) = y$

$$\begin{aligned} x &= \sum_{|\lambda|=d} X_\lambda B_\lambda^d(\tau), \\ y &= \sum_{|\lambda|=d} Y_\lambda B_\lambda^d(\tau). \end{aligned} \tag{2.6}$$

The points (ξ_λ, b_λ) are the control points for the surface $z = b_{\mathcal{T}}^d(\tau)$ and the piecewise linear interpolant to these points is the Bézier net or control net. This is displayed schematically in figure 1 for the case $d = 2$. The domain points ξ_λ are marked with dots. The control net mimics the shape of the surface and is tangent to the polynomial surface at the three vertices of the triangle.

Continuity conditions between triangles can be expressed as relations between the Bézier ordinates. Let $b_{\mathcal{T}}^d(\tau)$ a polynomial with Bézier ordinates b_{ijk} on the triangle $\mathcal{T}(T_1, T_2, T_3)$, and $c_{\mathcal{T}^*}^d(\tau)$ a polynomial with Bézier ordinates c_{ijk} on the triangle $\mathcal{T}^*(T_1^*, T_2, T_3)$ where T_1^* has barycentric coordinates λ with respect to $\mathcal{T}(T_1, T_2, T_3)$. A necessary and sufficient condition for $b_{\mathcal{T}}^d(\tau)$ and $c_{\mathcal{T}^*}^d(\tau)$ to be C^r continuous across the common boundary is

$$c_{ijk} = b_{0jk}^i(\lambda), \quad i = 0, 1, \ldots, r, \quad i + j + k = d. \tag{2.7}$$

with

$$\begin{aligned} &b_{ijk}^0(\tau) = b_{ijk}, \\ &b_{ijk}^r(\tau) = \tau_1 b_{(i+1)jk}^{r-1}(\tau) + \tau_2 b_{i(j+1)k}^{r-1}(\tau) + \tau_3 b_{ij(k+1)}^{r-1}(\tau), \quad r = 1, \ldots, d, \quad i + j + k + r = d. \end{aligned} \tag{2.8}$$

In the case of C^1 continuity this becomes

$$C^0 : \; c_{0jk} = b_{0jk}, \tag{2.9}$$

$$C^1 : \; c_{1jk} = \lambda_1 b_{1jk} + \lambda_2 b_{0(j+1)k} + \lambda_3 b_{0j(k+1)}. \tag{2.10}$$

Representing complex shapes, however, requires to use patch complexes with a great number of Bézier triangles. Keeping up continuity conditions between all the neighbouring patches then results, in general, in nontrivial relations between their Bézier ordinates.

2.2 PS–refinement

Let the triangulation Δ of $\Omega \subset \mathbb{R}^2$ be constituted of triangles $\rho_j,\ j = 1, \ldots, t$, and have vertices V_k with Cartesian coordinates $(x_k, y_k),\ k = 1, \ldots, n$. Then the Powell–Sabin refinement Δ_{PS} of Δ divides each triangle ρ_j into six smaller triangles with a common vertex Z_j as follows (figure 2) :

(1) Choose an interior point Z_j in each triangle ρ_j, so that if two triangles ρ_i and ρ_j have a common edge, then the line joining these interior points Z_i and Z_j intersects the common edge at a point R_{ij} between its vertices. Choosing Z_j as the incentre of each triangle ρ_j ensures the existence of the points R_{ij}. Other choices may be more appropriate from the practical point of view.

(2) Join each point Z_j to the vertices of ρ_j.

(3) For each edge of the triangle ρ_j
- which belongs to the boundary $\delta\Omega$, join Z_j to an arbitrary point of the edge.
- which is common to a triangle ρ_i, join Z_j to R_{ij}.

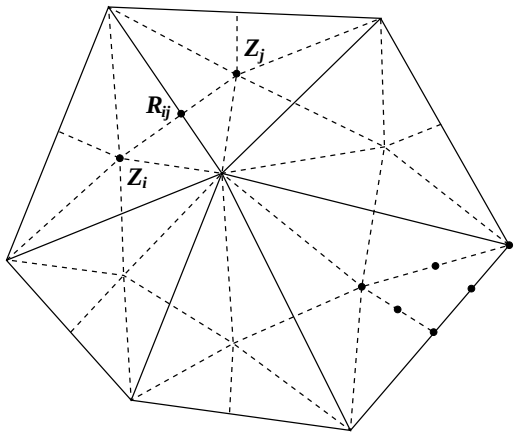


Figure 2. PS–refinement. Each triangle ρ_j is split in to six smaller triangles with a common vertex Z_j.

Now we consider the space of piecewise C^1 continuous quadratic polynomials on Δ_{PS}:

$$S_2^1(\Delta_{PS}) := \{s \in C^1(\Omega) : s|_{\mathcal{T}} \in \mathcal{P}_2,\ \mathcal{T} \in \Delta_{PS}\}. \tag{2.11}$$

Each of the $6t$ triangles resulting from the PS–refinement becomes the domain triangle of a quadratic Bernstein–Bézier polynomial, i.e. we choose $d = 2$ in equation (2.2) and (2.3), as indicated for one subtriangle in figure 2. Powell and Sabin [11] proved that the dimension of the space $S_2^1(\Delta_{PS})$ equals $3n$: there exists a unique solution $s(x, y) \in S_2^1(\Delta_{PS})$ for the interpolation problem

$$s(V_k) = f_k, \quad \frac{\partial s}{\partial x}(V_k) = f_{x,k}, \quad \frac{\partial s}{\partial y}(V_k) = f_{y,k}, \quad k = 1, \ldots, n. \tag{2.12}$$

So given the function and derivative values at each vertex V_k, the Bézier ordinates on the domain subtriangles are uniquely defined and the continuity conditions between subtriangles are automatically fulfilled.

2.3 Basis functions

Because the dimension of $S_2^1(\Delta_{PS})$ equals $3n$, we look for a representation of a Powell–Sabin spline with $3n$ basis functions $B_i(x,y)$

$$s(x,y) = \sum_{i=1}^{3n} c_i B_i(x,y). \tag{2.13}$$

Dierckx [5] suggested a basis in which three basis functions $B_i^j, j = 1,2,3$ are associated with each vertex V_i. The representation (2.13) becomes

$$s(x,y) = \sum_{i=1}^{n}\sum_{j=1}^{3} c_{ij} B_i^j(x,y). \tag{2.14}$$

Such a basis function $B_i^j(x,y)$ is the solution of the interpolation problem (2.12) with all $(f_k, f_{x,k}, f_{y,k})$ $= (0,0,0)$ except for $(f_i, f_{x,i}, f_{y,i}) = (\alpha_{ij}, \beta_{ij}, \gamma_{ij})$. $B_i^j(x,y)$ obviously vanishes outside the molecule M_i of the vertex V_i: the basis functions are local and star-supported.

A historical choice of Shi et al. [12] is

$$\begin{aligned}(\alpha_{i1}, \beta_{i1}, \gamma_{i1}) &= (\frac{1}{4}, 0, \epsilon),\\ (\alpha_{i2}, \beta_{i2}, \gamma_{i2}) &= (\frac{1}{4}, \epsilon, 0),\\ (\alpha_{i3}, \beta_{i3}, \gamma_{i3}) &= (\frac{1}{2}, \epsilon, \epsilon).\end{aligned} \tag{2.15}$$

where $\epsilon \in [1/4h, 1/2h]$ and h is the length of the longest edge of Δ. The resulting basis functions form a convex partition of unity, but the basis can be very poor from a numerical point of view. For instance, if h is large, then ϵ will be small. Hence, if there are molecules in the triangulation that are large and small, the three basis function corresponding to one vertex will tend to be numerically linearly dependent. Therefor, in the subsequent sections, we look for choices for $(\alpha_i, \beta_i, \gamma_i)$ that only depend on the molecule M_i and not on the whole triangulation Δ.

We define the control points of a surface $z = s(x,y)$ as

$$C_{ij} = (Q_{ij}, c_{ij}) = (X_{ij}, Y_{ij}, c_{ij}) \tag{2.16}$$

in which the Cartesian coordinates (X_{ij}, Y_{ij}) of the domain point Q_{ij} are the coefficients c_{ij} in the representation (2.14) of the specific polynomials $s(x,y) = x$ and $s(x,y) = y$

$$\begin{aligned}x &= \sum_{i=1}^{n}\sum_{j=1}^{3} X_{ij} B_i^j(x,y)\\ y &= \sum_{i=1}^{n}\sum_{j=1}^{3} Y_{ij} B_i^j(x,y).\end{aligned} \tag{2.17}$$

The three control points for each vertex together form a triangle which we call the control triangle T_i

$$T_i(C_{i1}, C_{i2}, C_{i3}). \tag{2.18}$$

The projection of the control triangle in the (x,y) plane is called the PS–triangle t_i

$$t_i(Q_{i1}, Q_{i2}, Q_{i3}). \tag{2.19}$$

In figure 3(a) we show one triangle (V_1, V_2, V_3) of the molecule M_1 of the vertex V_1. Let the different points in (V_1, V_2, V_3) have barycentric coordinates

$$\begin{array}{lll} V_1 = (1,0,0) & R_{12} = (\lambda_{12}, 1-\lambda_{12}, 0) & Z = (a,b,c). \\ V_2 = (0,1,0) & R_{23} = (0, \mu_{23}, 1-\mu_{23}) & \\ V_3 = (0,0,1) & R_{31} = (\nu_{13}, 0, 1-\nu_{13}) & \end{array} \tag{2.20}$$

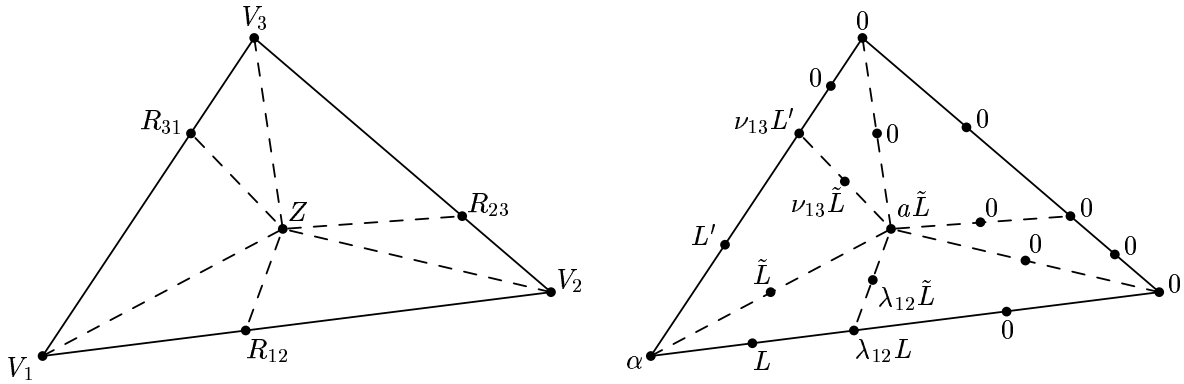


Figure 3. Bézier ordinates of a basis function $B_1(x,y)$ on one triangle of its molecule M_1.

From [6] we recall figure 3(b) in which the Bézier ordinates of a basis function $B_1(x,y)$ corresponding to the triplet (α, β, γ) are indicated on the triangle of figure 3(a).

$$\begin{aligned} L &= \alpha + \frac{1-\lambda_{12}}{2}\left(\beta(x_2 - x_1) + \gamma(y_2 - y_1)\right) \\ L' &= \alpha + \frac{1-\nu_{12}}{2}\left(\beta(x_3 - x_1) + \gamma(y_3 - y_1)\right) \\ \tilde{L} &= \alpha + \frac{b}{2}\left(\beta(x_2 - x_1) + \gamma(y_2 - y_1)\right) + \frac{c}{2}\left(\beta(x_3 - x_1) + \gamma(y_3 - y_1)\right). \end{aligned} \tag{2.21}$$

3 PS–triangles and control triangles

We normalise the basis functions such that they form a partition of unity (reproduction of a constant)

$$\sum_{i=1}^{n}\sum_{j=1}^{3} B_i^j(x,y) = 1 \qquad (x,y) \in \Omega. \tag{3.1}$$

The representation (2.14) is then an affine combination of the coeficients c_{ij}.

Because of the locality of the basis functions only the three $B_i^j(x,y), j = 1,2,3$ have a nonvanishing function value and derivative values at V_i. The necessary conditions for (3.1) are then

$$\begin{aligned} \alpha_{i1} + \alpha_{i2} + \alpha_{i3} &= 1, \\ \beta_{i1} + \beta_{i2} + \beta_{i3} &= 0, \\ \gamma_{i1} + \gamma_{i2} + \gamma_{i3} &= 0. \end{aligned} \tag{3.2}$$

In [5] is is shown that using the conditions (3.2) and the representation (2.14) we find the values for the (X, Y) coordinates of the control points defined in (2.17) as the solution of the systems

$$\begin{cases} X_{i1}\alpha_{i1} + X_{i2}\alpha_{i2} + X_{i3}\alpha_{i3} = x_i \\ X_{i1}\beta_{i1} + X_{i2}\beta_{i2} + X_{i3}\beta_{i3} = 1 \\ X_{i1}\gamma_{i1} + X_{i2}\gamma_{i2} + X_{i3}\gamma_{i3} = 0 \end{cases} \tag{3.3}$$

and

$$\begin{cases} Y_{i1}\alpha_{i1} + Y_{i2}\alpha_{i2} + Y_{i3}\alpha_{i3} = y_i \\ Y_{i1}\beta_{i1} + Y_{i2}\beta_{i2} + Y_{i3}\beta_{i3} = 0 \\ Y_{i1}\gamma_{i1} + Y_{i2}\gamma_{i2} + Y_{i3}\gamma_{i3} = 1. \end{cases} \tag{3.4}$$

From the above formulas we see that V_i lies inside its PS–triangle t_i

$$V_i = \alpha_{i1}Q_{i1} + \alpha_{i2}Q_{i2} + \alpha_{i3}Q_{i3} \tag{3.5}$$

The area of the PS–triangle t_i is

$$A(t_i(Q_{i1}, Q_{i2}, Q_{i3})) = \frac{1}{2|\beta_{i1}\gamma_{i2} - \gamma_{i1}\beta_{i2}|} \tag{3.6}$$

It is easily proven that now the control triangle T_i is tangent to the surface $z = s(x, y)$ at V_i. The tangent point is $(x_i, y_i, s(V_i))$. This is an interesting property to gain insight in the shape of the surface. Using the control triangles we can interactively change the shape of a PS–spline surface, a change in T_l will only affect the patches related to the molecule M_l.

With the systems of equations (3.3) and (3.4) we can compute the PS–triangles given the triplets (α, β, γ) and vice versa. The problem of choosing the triplets $(\alpha_{ij}, \beta_{ij}, \gamma_{ij})$ in order to find basis functions $B_i^j(x, y)$ can now alternatively be formulated as choosing a suitable PS–triangle.

4 Minimal determining sets

Lai and Schumaker [9] and Alfeld and Schumaker [3] constructed stable local bases for certain spline spaces $S_d^r(\Delta_{PS})$ on Powell–Sabin splits. The construction is based on the principle of minimal determing sets and we first review some aspects of this idea. Then we zoom in on their solution for the space $S_2^1(\Delta_{PS})$ and show that this corresponds to a special choice of PS–triangle in our framework.

A good introduction of the concept of a minimal determining set is given by Alfeld in [1]. We first look at $S_d^0(\Delta_{PS})$, the space of continuous splines of degree d on the PS–refinement Δ_{PS}. Let $\mathcal{D}$ the union of the domain points associated with each triangle of Δ_{PS}. As observed in [2] for general triangulations, there is a 1-1 correspondence between the set of splines in $S_d^0(\Delta_{PS})$ and the set of coefficients $\{b_\lambda(s)\}_{\xi_\lambda \in \mathcal{D}}$ or in other words, each spline $s \in S_d^0(\Delta_{PS})$ is uniquely determined by its Bézier ordinates $\{b_\lambda(s)\}_{\xi_\lambda \in \mathcal{D}}$.

For the subspace $S_2^r(\Delta_{PS}) \subset S_2^0(\Delta_{PS})$ with $r < d$, additional smoothness conditions between the triangles need to be satisfied. In the case of C^1 continuity these conditions are given by (2.10). It is clear that we cannot independently choose all the coefficients $\{b_\lambda\}_{\xi_\lambda \in \mathcal{D}}$ anymore. A determining set $\mathcal{M}$ is now defined as a subset of the set of domain points $\mathcal{D}$ such that if $b_\lambda = 0$ for all $\xi_\lambda \in \mathcal{M}$, then $b_\lambda = 0$ for all $\xi_\lambda \in \mathcal{D}$. The set $\mathcal{M}$ is called a minimal determining set if there is no smaller determining set. So every spline $s \in S_d^1(\Delta_{PS})$ is uniquely determined by the restricted set of Bézier coefficients $\{b_\lambda(s)\}_{\xi_\lambda \in \mathcal{M}}$. The dimension of a spline space equals the number of points in a minimal

determining set $\mathcal{M}$.

For each $\xi_\lambda \in \mathcal{M}$, let B_λ the unique spline for which

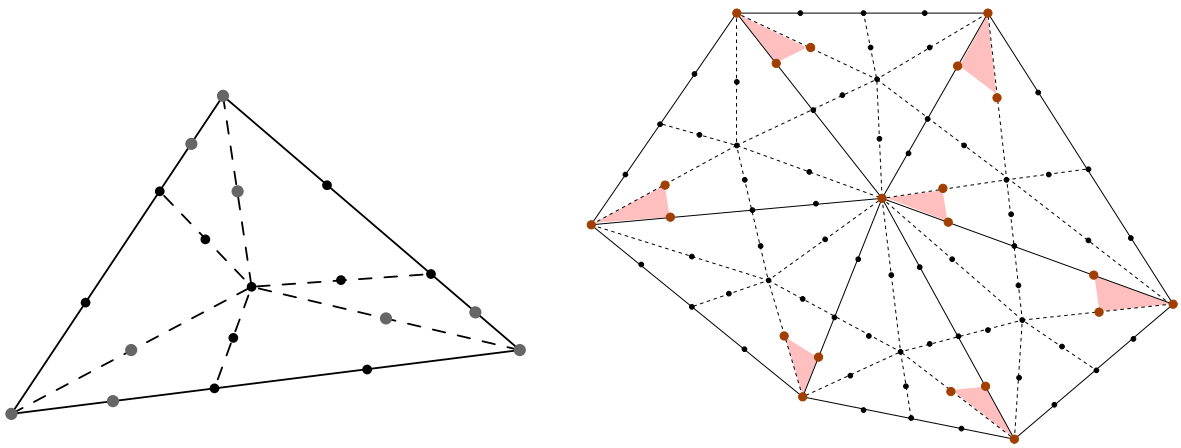
$$b_\eta(B_\lambda) = \delta_{\eta,\lambda}, \quad \text{for all } \xi_\eta \in \mathcal{M}. \tag{4.1}$$

So this spline B_λ is defined by setting one Bézier ordinate b_λ in the minimal determining set equal to one and the others to zero. The Bézier ordinates that are not in the minimal determining set are defined by the continuity conditions. The set $\mathcal{B} = \{B_\lambda\}_{\xi_\lambda \in \mathcal{M}}$ is a linearly independent set and forms a basis for the spline space. However, considerable care is needed in choosing $\mathcal{M}$ to insure that the basis $\mathcal{B}$ is stable and local.

Lai and Schumaker [9] give minimal determining sets for certain choices for d and r that lead to stable and star-supported bases. In the case of C^1 continuous quadratic splines they suggest the following set. For each vertex V_i choose a triangle $\mathcal{T}_i$ of Δ_{PS} attached to V_i and include the 1-disk of domain points $D_1^{\mathcal{T}_i}(V_i)$ in the minimal determining set $\mathcal{M}$

$$\mathcal{M} = \bigcup D_1^{\mathcal{T}_i}(V_i) \quad \text{for } i = 1 \ldots n. \tag{4.2}$$

In figure 4 two examples are shown. The domain points that are part of the minimal determining set are indicated with larger dots.



(a) (b)

Figure 4. The domain points that together form a minimal determining set for these examples are indicated with larger dots.

In figure 5 we give the Bézier ordinates of one basis function of the minimal determining set of figure 4(a). One Bézier ordinate corresponding to a domain point in the minimal determining set is set to one, the others equal zero. The Bézier ordinates of which the corresponding domain points are not contained in the minimal determining set are determined by the C^1 continuity condition (2.10). Comparing these Bézier ordinates with figure 3 gives us the values for (α, β, γ) or using (3.3) and (3.4) the corners $Q(X, Y)$ of the PS–triangle. These corners turn out to be the domain points in the minimal determining set. The PS–triangles are shaded in figure 4(b).

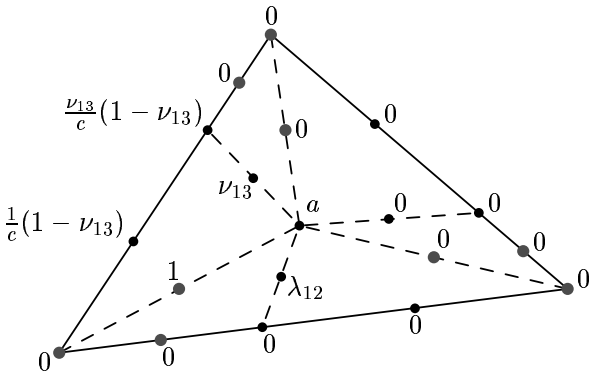


Figure 5. Bézier ordinates of one basis function. One Bézier ordinate with domain point in the minimal determining set is set to one.

A basis $\mathcal{B}$ is said to be locally linearly independent [4] provided that for every $\mathcal{T} \in \Delta_{PS}$, the subset $\{B_i\}_{i\in\Sigma_\mathcal{T}}$ are linearly independent on $\mathcal{T}$ where

$$\Sigma_\mathcal{T} = \{i : \mathcal{T} \subseteq \mathrm{supp}(B_i)\}. \tag{4.3}$$

Since $S_2^1(\Delta_{PS})$ contains the space $\mathcal{P}_2$ of quadratic polynomials, $\mathcal{B}$ being locally linearly independent is equivalent to the condition

$$\#\Sigma_\mathcal{T} = \dim \mathcal{P}_2 = 6 \text{ for every } \mathcal{T} \in \Delta_{PS}. \tag{4.4}$$

A simple count learns that $\mathcal{B} = \{B_\lambda\}_{\xi_\lambda\in\mathcal{M}}$ is indeed a locally linearly independent set.

5 Convex partition of unity

In section 3 the basis functions form a partition of unity. However, in practical applications it is often better to have a convex partition of unity. Apart from (3.1) we also want the basis functions to be positive

$$B_i^j(x,y) \geq 0 \quad (x,y) \in \Omega. \tag{5.1}$$

Due to this extra condition we have the interesting property that a PS–spline patch lies in the convex hull of the nine control points of the triangle. Nonnegative basis functions are also usefull for local editing.

The Bernstein–Bézier polynomials (2.3) form a stable basis for the space of bivariate polynomials [10]. The Bézier control points of a Powell–Sabin surface expanded in a basis that is a convex partition of unity, can be computed as convex stable combinations of the PS–spline control points [5], which proves the stability condition (1.3).

Because the Bernstein polynomials are positive (2.3), it is necessary and sufficient that the Bézier ordinates of the basis functions are nonnegative. From figure 3(b) we find the conditions

$$\alpha_{ij} \geq 0, \quad L \geq 0, \quad L' \geq 0, \quad \tilde{L} \geq 0, \tag{5.2}$$

with L, L' and $\tilde{L}$ as defined in (2.21).

In [5] it is proven that these conditions are equivalent to requesting that the PS–points of a vertex lie inside its PS–triangle. The PS–points are defined as the Bézier domain points ξ_λ that

immediately surround the vertex and the vertex itself. An example with one triangle is shown in figure 6. The PS–points are the vertex V_1 and the points S, $\tilde{S}$ and S'.

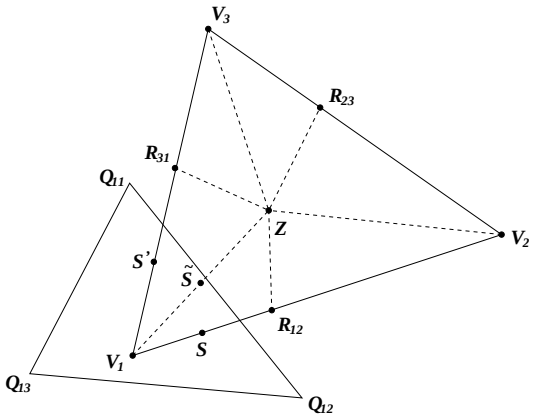


Figure 6: PS–points and PS–triangle.

There are several possible PS–triangles that contain all the PS–points of a vertex and that lead to basis function in a convex partition of unity. In the next subsections we look at two special solutions.

5.1 Optimal solution

A first possibility is the PS–triangle with the smallest area. In that case the control points will be close to the surface which is useful for example for local editing. To minimise the area (3.6) we have to maximise

$$\beta_{i1}\gamma_{i2} - \gamma_{i1}\beta_{i2} \;\rightarrow\; \max \tag{5.3}$$

subject to the constraints (5.2).

This corresponds to making the triplets $(\alpha_{ij}, \beta_{ij}, \gamma_{ij}), j = 1,2,3$ as linearly independent as possible since, taking account of (3.2)

$$\begin{vmatrix} \alpha_{i1} & \alpha_{i2} & \alpha_{i3} \\ \beta_{i1} & \beta_{i2} & \beta_{i3} \\ \gamma_{i1} & \gamma_{i2} & \gamma_{i3} \end{vmatrix} = \beta_{i1}\gamma_{i2} - \gamma_{i1}\beta_{i2}. \tag{5.4}$$

This corresponds to linearly independent basis functions B_i^1, B_i^2, B_i^3, because each triplet defines a basis function. Alternatively, we can show that the equation

$$a_1 B_i^1(x,y) + a_2 B_i^2(x,y) + a_3 B_i^3(x,y) = 0 \tag{5.5}$$

has only the trivial solution for unknowns a_1, a_2 and a_3. Plugging in three well chosen points gives a linear system of equations in a_1, a_2 and a_3. The determinant of this system has to be different from zero. For example, choosing the PS–points in one of the triangles of the molecule to evaluate equation (5.5) leads to the following expression for the determinant

$$\frac{1}{4}\left[\beta_{i1}\gamma_{i2} - \gamma_{i1}\beta_{i2}\right]\left[(x_{12}y_{13} - x_{13}y_{12})\left(1 - \lambda_{12} - \lambda_{13} - b - c + \lambda_{12}\lambda_{13} + b\lambda_{13} + c\lambda_{12}\right)\right]. \tag{5.6}$$

In this expression we recognise three factors.

(1) The first factor, which we expected to see, is inversely proportional with the area of the PS–triangle. So the smaller the PS–triangle, the smaller the expression for the determinant.

(2) The second factor depends on the triangulation and equals the dot product of the vectors $\overrightarrow{V_1V_2}$ and $\overrightarrow{V_1V_3}$. This is the length of these vectors multiplied with the cosine of the angle between them. The appearance of this factor indicates that triangles with a small angle in a corner are less favorable.
Also note that if the triangles of Δ become smaller, this factor will become smaller too. This happens for example when doing subdivision [13], ie. finding a representation of a PS–spline on a refinement of the initial triangulation Δ. That indicates why it is useful to rescale, or make smaller, the PS–triangles in the original vertices after subdivision.

(3) The third factor is the influence of the PS–refinement. The more regular this refinement is, the smaller this factor is.

5.2 Practical solution

To determine the PS–triangles for all the vertices in a triangulation we need to solve the quadratic programming problem (5.3) with constraints (5.2) for each vertex. In this section we describe a practical algorithm to find a suitable PS–triangle that contains the PS–points without solving an expensive optimisation problem. When the molecule number m_i equals three, there is a trivial solution for the PS–triangle t_i. For vertices with a higher molecule number, we reduce the number of PS–points by taking them together in a certain way, until we have the trivial case again. We also discuss a degenerate case in which the algorithm breaks down and illustrate the importance of the sequence in which the points are handled.

5.2.1 *Trivial case*

When the molecule number m_i equals three, the six PS–points of V_i form a triangle. This is the smallest triangle that contains the PS–points and thus it is an obvious choice for the PS–triangle t_i. No extra computation is needed to find this triangle. An example is shown in figure 7.

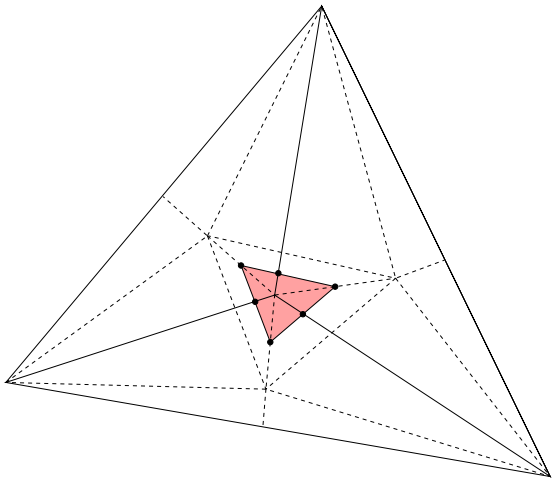


Figure 7: Trivial case: for molecule $m_i = 3$, the PS–points form a triangle.

5.2.2 *Reduce other configurations to trivial case*

When there are more then three triangles in the molecule M_i, the PS–points do not form a triangle but a polygon with m_i corners. We now change the m_i-gon into a (m_i-1)-gon that contains the

m_i-gon. Therefor we choose an edge of the original polygon and add a corner at the intersection of the neighbouring edges. We drop the original edge and its two corners.

We repeat this procedure until we become a triangle that contains the original polygon and thus contains the PS–points of V_i. We use this triangle as the PS–triangle t_i. An example of the subsequent steps is shown in figure 8. The middle vertex has six triangles in its molecule. In three steps we find a PS–triangle that contains all the PS–points.

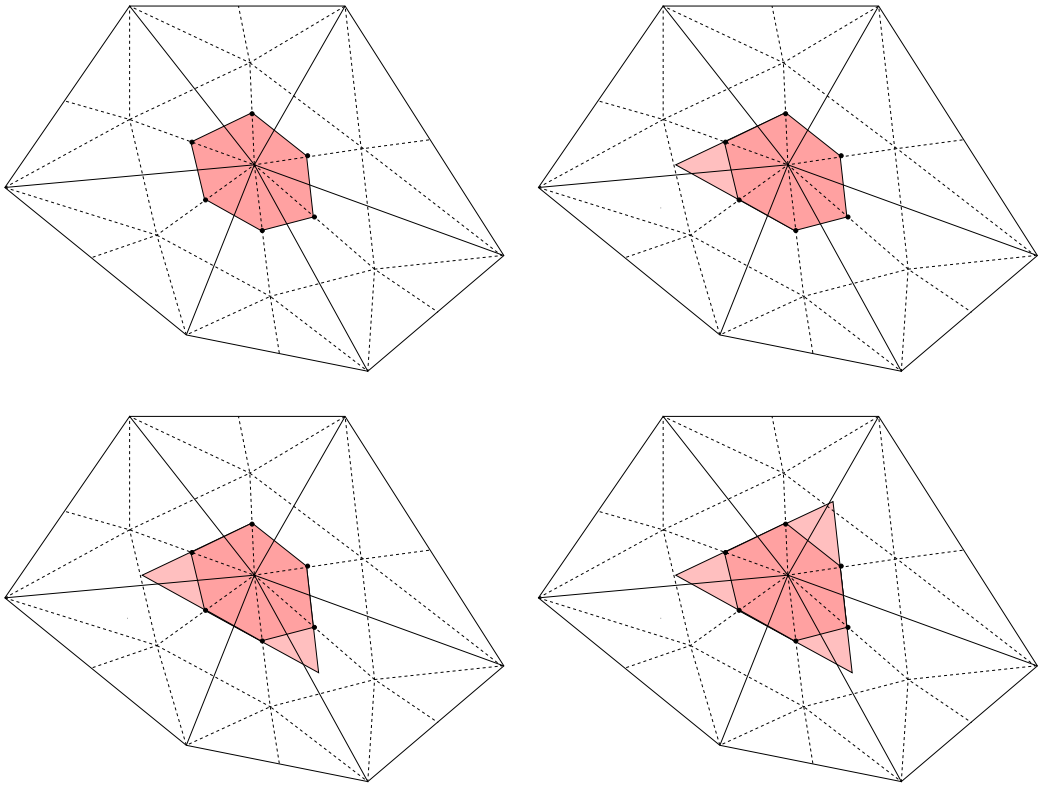


Figure 8. In each step we remove an edge from the polygon formed by the PS–points in such a way that the resulting triangle contains the original polygon and can be used as PS–triangle.

5.2.3 *Degenerate case*

Depending on the initial configuration it is possible that the polygon has two pairs of parallel edges. In that case we cannot apply the above described procedure because the edges do not intersect. Figure 9 shows such a degenerate example. We now choose for the PS–triangle two of the edges of the polygon and take the third edge such that it contains the unused corner.

5.2.4 *Sequence of operations*

The order in which the edges of the polygon are removed can be important. Figure 10 repeats the second and third picture of the example of figure 8, but another edge is chosen to be removed in the second step. This leads to a polygon with edges that are almost parallel, so almost the degenerate case. This problem is easily avoided by giving priority to edges that are (almost) parallel to other edges in the polygon.

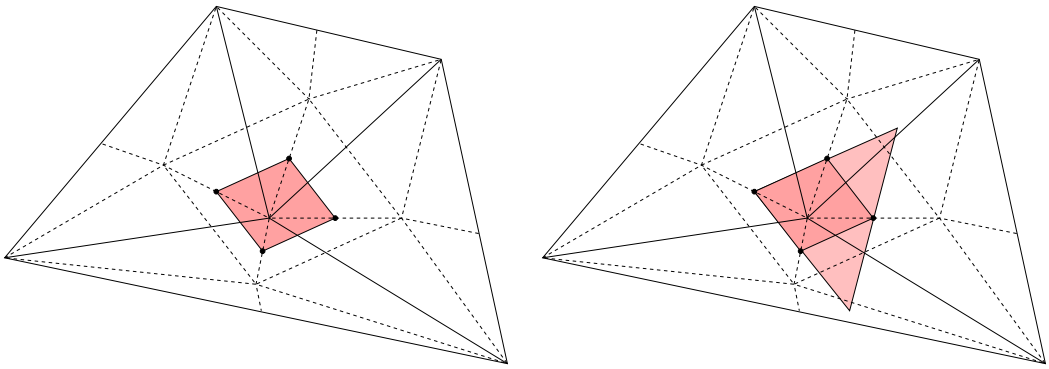


Figure 9. Degenerate case: the edges of the polygon are parallel, no intersection can be found.

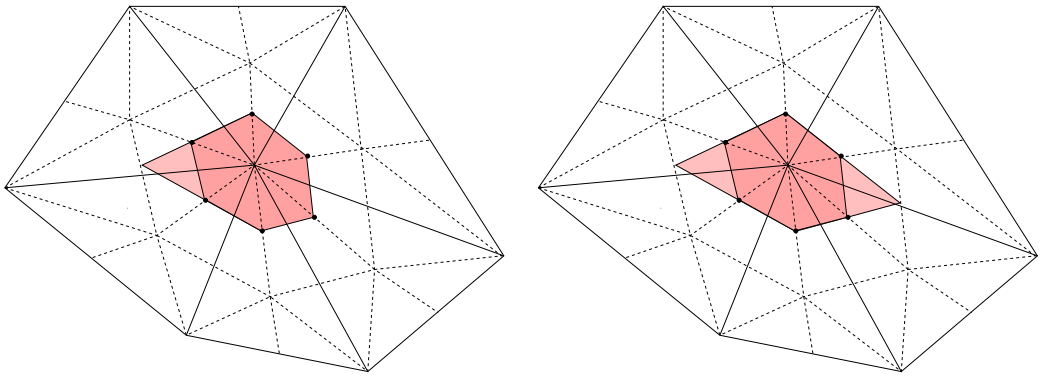


Figure 10: The order in which the edges of the polygon are removed is important.

5.2.5 *Boundary*

At the boundary it can be necessary to use the vertex itself as a corner of the polygon. An example of this situation is shown in figure 11 for the leftmost vertex. Indeed, if the vertex, which is also a PS–point, is not used it is not contained in the resulting PS–triangle. This adaptation of the algorithm is not needed when the vertex lies correctly inside the polygon as shown on the right of figure 11.

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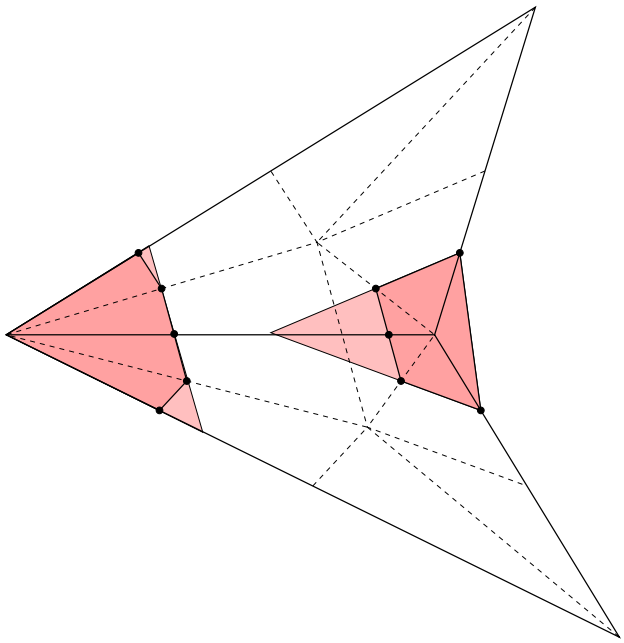


Figure 11. At the boundary of the domain it can be necessary to use the vertex also as a corner of the polygon.

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