A generalized minimal partial realization problem

M. Van Barel *A. Bultheel*

Report TW 229, June 1995



Katholieke Universiteit Leuven
Department of Computer Science
Celestijnenlaan 200A – B-3001 Heverlee (Belgium)

A generalized minimal partial realization problem*

M. Van Barel *A. Bultheel*

Report TW 229, June 1995

Department of Computer Science, K.U. Leuven

Abstract

In this paper, all solutions of a generalized minimal partial realization problem are parametrized in an easy way using the basis matrix concept. This basis matrix can be computed in an efficient way and was also used to solve several other interpolation problems.

Keywords : minimal partial realization, basis matrix, polynomial vector interpolation
AMS(MOS) Classification : Primary : 41-++, Secondary : 93B15, 93B30.

*Submitted to Linear Algebra and its Applications

A generalized minimal partial realization problem*

Marc Van Barel and Adhemar Bultheel
Department of Computer Science
Katholieke Universiteit Leuven
Celestijnenlaan 200 A
B-3001 Heverlee, Belgium
e-mail: Marc.VanBarel@cs.kuleuven.ac.be

June 21, 1995

Abstract

In this paper, all solutions of a generalized minimal partial realization problem are parametrized in an easy way using the basis matrix concept. This basis matrix can be computed in an efficient way and was also used to solve several other interpolation problems.

short running title: generalized minimal partial realization
AMS primary: 41-++
AMS secondary: 93B15, 93B30

1 Introduction

The classical minimal partial realization problem plays an important role in linear system theory. Given a sequence of Markov parameters $M_k \in \mathbb{F}^{p\times q}$, $k = 0, 1, \ldots$, where $\mathbb{F}$ is an arbitrary field, and some $\sigma \in \mathbb{N}$, find all matrix rational functions Z such that with $M(z) := \sum_{k=0}^{\infty} M_k z^{-k}$

$$M(z) - Z(z) = O_-(z^\sigma)$$

such that Z has minimal McMillan degree. The McMillan degree is a measure for the complexity of the matrix rational function Z. For discrete-time systems, the Markov parameters are just the values of the impulse response. For more details, we refer the interested reader to the book of Kailath [11].

*This research was supported by the National Fund for Scientific Research (NFWO), project Lanczos, grant #2.0042.93.

We generalize this problem in several directions. *First,* we allow there to be a finite number of M_k with negative index, so that $M(z)$ is a formal Laurent series. *Second,* we allow some shifts, which give "weights" to the degrees of the rows of the polynomial matrices $N(z)$ and $D(z)$, where $ND^{-1} := Z$ is a right polynomial matrix fraction description of Z. These shifts will influence the order conditions on the difference $M - Z$ columnwise. *Third,* we introduce some row-wise parameters in the order conditions of $M - Z$, so that, together with the previous generalization, we actually get componentwise order conditions.

The paper is organised as follows. Section 2 introduces the concepts and notation necessary to define the generalized minimal partial realization problem in Section 3. To solve this problem, it is linearized in Section 4 and interpreted as a polynomial vector interpolation problem around ∞ in Section 5. Section 6 gives the algorithm to compute a basis matrix for this problem allowing to write down all polynomial vector solutions with a limited complexity in an easy way. In [13], the concept of a basis matrix is used to solve the rational interpolation problem. In [17], the algorithm of [13] is generalized also allowing confluent interpolation points, poles and ∞ as an interpolation point. The algorithm of [17] is based on the fact that all solutions of an equivalent linearized interpolation problem can be written in terms of some basis vectors of an $\mathbb{F}[z]$-submodule of the module of all polynomial vectors $\mathbb{F}[z]^n$. To add a new interpolation condition, we only have to update this set of basis vectors. This can be done in a fast and reliable way leading to $O(k^2)$ FLOPS for k interpolation conditions. For the pole-free problem, this was generalized to the vector case [4, 8, 14, 15] and also to the matrix case [6, 5, 16]. For the matrix rational interpolation problem with poles as interpolation points, we refer to [12]. Once we have the basis vectors, we can parametrize all solutions of the original rational interpolation problem having minimal complexity [16]. Section 7 applies this idea to the generalized minimal partial realization problem. Two examples are given in Section 8. The V and W polynomial matrices of [1] play the roles of basis matrices to solve the *classical* problem. Such basis matrices or *generating systems* play an important role in related problems of linear system theory. See, e.g., [2, 3].

Instead of updating the basis matrix when one new interpolation condition is added, one could impose several interpolation conditions at once and compute the new basis matrix in an efficient way [18, 20]. These so-called "look-ahead" procedures can be used to enhance the numerical stability of fast methods to compute Padé approximants and to solve related Hankel systems of linear equations [9]. For Toeplitz systems, we refer the reader to [10]. Generalized Sylvester matrices and related multi-dimensional Padé systems are handled in [7] while block Toeplitz systems and related interpolation problems are solved in [19].

2 Notation

We will use the following notation:

- $\mathbb{F}$ denotes a field (finite or infinite).
- $\mathbb{F}[z]$ is the set of all polynomials with coefficients belonging to the field $\mathbb{F}$.

- $\mathbb{F}(z^{-1})$ denotes the set of formal Laurent series with only a finite number of terms with positive powers. Similarly for $\mathbb{F}(z)$.
- $\mathbb{F}^{p\times q}$ is the set of all $(p \times q)$ matrices with elements from $\mathbb{F}$. If $q = 1$, we shall use the notation $\mathbb{F}^p$ instead of $\mathbb{F}^{p\times 1}$. Similarly for $\mathbb{F}[z]^{p\times q}$ and $\mathbb{F}(z^{-1})^{p\times q}$.

If A is a $(p \times q)$ matrix, we denote the element of row i and column j of A using the corresponding small letter, i.e., $a_{i,j}$, and write $A =: [a_{i,j}]_p^q$ or if p and q are clear from the context $A =: [a_{i,j}]$. If $\vec{\tau} \in \mathbb{Z}^p$ with $\mathbb{Z}$ the set of all integer numbers, we can construct the following diagonal matrix:

$$z^{\vec{\tau}} := \mathrm{diag}(z^{\tau_1}, z^{\tau_2}, \ldots, z^{\tau_p}).$$

Definition 2.1 (projection operators) *If $M \in \mathbb{F}(z^{-1})^{p\times q}$ and $\Sigma := [\sigma_{i,j}] \in \mathbb{Z}^{p\times q}$, we define the projection operators π_Σ^+ and π_Σ^- as follows:*

$$\pi_\Sigma^+ M := [\pi^+_{\sigma_{i,j}} m_{i,j}]_p^q$$

and

$$\pi_\Sigma^- M := [\pi^-_{\sigma_{i,j}} m_{i,j}]_p^q$$

with

$$\begin{aligned} \pi_\sigma^+ m &:= z^\sigma(\pi^+(z^{-\sigma} m)), \\ \pi_\sigma^- m &:= m - \pi_\sigma^+ m, \end{aligned}$$

and

$$\pi^+ m := \text{ the polynomial part of } m \in \mathbb{F}(z^{-1}).$$

With $\Sigma_1, \Sigma_2 \in \mathbb{Z}^{p\times q}$, we use

$$\pi_{\Sigma_1,\Sigma_2} M := \pi^+_{\Sigma_2}(\pi^-_{\Sigma_1+1} M)$$

to denote part of M.

Definition 2.2 (order) *If $\pi_\Sigma^+ M = 0$, then we write*

$$M = O_-(z^{\sigma_{i,j}-1}) \quad \text{or} \quad M = O_-(z^{\Sigma-1}).$$

Similarly, if $\pi_\Sigma^- M = 0$, we use the following notation

$$M = O_+(z^{\sigma_{i,j}}) \quad \text{or} \quad M = O_+(z^{\Sigma}).$$

Definition 2.3 (coefficients) *For $m \in \mathbb{F}(z^{-1})$, we define the ith coefficient of m as*

$$coeff(m, i) := \pi_{i,i} m.$$

For $M \in \mathbb{F}(z^{-1})^{p\times q}$ and $\Sigma \in \mathbb{Z}^{p\times q}$, we define the Σth coefficient of M as:

$$coeff(M, \Sigma) := \pi_{\Sigma,\Sigma} M.$$

Definition 2.4 (strict order) *We write*

$$M = O_+^s(z^\Sigma)$$

iff

$$M = O_+(z^\Sigma)$$

and

$$coeff(M, \Sigma)$$

has full rank. We write

$$M = O_-^s(z^\Sigma)$$

iff

$$M = O_-(z^\Sigma)$$

and

$$coeff(M, \Sigma)$$

has full rank.

Definition 2.5 ($\vec{\tau}$-degree, $\vec{\tau}$-column reduced) *Given $\vec{\tau} \in \mathbb{Z}^n$, any nonzero $\vec{p} \in \mathbb{F}(z^{-1})^n$ can be written as*

$$\vec{p}(z) = z^{\vec{\tau}} z^\alpha O_-^s(z^0).$$

We call α the $\vec{\tau}$-degree of $\vec{p}$. Note that α can be negative. ($\vec{\tau}$-deg $\vec{0} := -\infty$)
The $\vec{\tau}$-degree of $P \in \mathbb{F}(z^{-1})^{n\times q}$ is defined as the maximum of the degree of all possible $(m \times m)$ determinants taken from the matrix $z^{-\vec{\tau}}P$ with $m := \min\{n, q\}$.
A matrix $P \in \mathbb{F}(z^{-1})^{n\times q}$ is called $\vec{\tau}$-column reduced iff it is of the form

$$P(z) = z^{\vec{\tau}} O_-^s(z^0) z^{\vec{\alpha}}$$

with $\vec{\alpha} \in \mathbb{Z}^q$. It is clear that α_j, $j = 1, 2, \ldots, q$ denotes the $\vec{\tau}$-degree of the jth column of P.

Lemma 2.1 *Given $\vec{\tau} \in \mathbb{Z}^n$, $n \geq q$, the matrix $P \in \mathbb{F}(z^{-1})^{n\times q}$ is $\vec{\tau}$-column reduced iff $\vec{\tau}$-deg $P = \sum_{j=1}^q \alpha_j =: |\vec{\alpha}|$ with α_j the $\vec{\tau}$-degree of the jth column of P.*

Definition 2.6 ($\vec{\tau}$-McMillan degree of a right coprime matrix fraction description) *Given $\vec{\tau} \in \mathbb{Z}^{p+q}$, take the $(p+q) \times q$ polynomial matrix*

$$z^{-\vec{\tau}} \begin{bmatrix} N \\ D \end{bmatrix}.$$

The maximum of the degree of all possible $(q \times q)$ determinants taken from this matrix is the $\vec{\tau}$-McMillan degree of the right coprime polynomial matrix fraction description (RCPMFD) ND^{-1}.

Any matrix rational function Z can be written as a right coprime matrix fraction description (RCPMFD) $Z = ND^{-1}$, i.e., with $N \in \mathbb{F}[z]^{p\times q}$, $D \in \mathbb{F}[z]^{q\times q}$ with $\det D \not\equiv 0$, and N and D right coprime. Moreover, all RCPMFDs of Z can be represented as $Z = (NU)(DU)^{-1}$ with $U \in \mathbb{F}[z]^{q\times q}$ unimodular, i.e., $\det U \in \mathbb{F} \setminus \{0\}$. With U unimodular, the $\vec{\tau}$-degrees of $[N^T \;\; D^T]^T$ and $[(NU)^T \;\; (DU)^T]^T$ are equal.

Definition 2.7 ($\vec{\tau}$-McMillan degree of a matrix rational function) *Hence, we define the $\vec{\tau}$-McMillan degree of a matrix rational function Z as the $\vec{\tau}$-McMillan degree of one of its possible RCPMFDs ND^{-1}, i.e., the $\vec{\tau}$-degree of $[N^T \quad D^T]^T$.*

Theorem 2.2 *Any matrix $P \in \mathbb{F}(z^{-1})^{n\times q}$ can be made $\vec{\tau}$-column reduced by right multiplication with a unimodular matrix U. Hence, any $(p\times q)$ matrix rational function Z can be written as $Z = ND^{-1}$ with*

$$P := \begin{bmatrix} N \\ D \end{bmatrix} \in \mathbb{F}[z]^{(p+q)\times q} \text{ } \vec{\tau}\text{-column reduced.}$$

In this case, the $\vec{\tau}$-McMillan degree of Z is equal to $|\vec{\alpha}|$ where $\vec{\alpha}$ contains the $\vec{\tau}$-degrees of the columns of P.

3 Generalized minimal partial realization problem

We generalize the classical minimal partial realization (MPR) problem by introducing the $\vec{\tau}$-McMillan degree and allowing a polynomial part for the given series of Markov parameters.

Definition 3.1 (generalized minimal partial realization (GMPR)) *Given the formal Laurent series $M \in \mathbb{F}(z^{-1})^{p\times q}$, the shift parameters $\vec{\tau}_N \in \mathbb{Z}^p$ and $\vec{\tau}_D \in \mathbb{Z}^q$, the matrix $\Sigma \in \mathbb{Z}^{p\times q}$ such that Σ can be written as:*

$$\Sigma = [-\gamma_i - \tau_{D,j}]_p^q = [\sigma_{i,j}]_p^q \quad \text{with } \vec{\gamma} \in \mathbb{Z}^p$$

(the reason for this condition will become clear later), we look for all $(p \times q)$ matrix rational functions Z such that

$$M(z) - Z(z) = O_-(z^{\Sigma}) \tag{1}$$

and the $\vec{\tau}$-McMillan degree of Z is minimal with

$$\vec{\tau} := \begin{bmatrix} \vec{\tau}_N \\ \vec{\tau}_D \end{bmatrix}.$$

Note that we get the classical problem by taking $\vec{\tau} = \vec{0}$ and $M = O_-(z^0)$.

4 Linearization of the problem

By Theorem 2.2, we can always represent Z by a $\vec{\tau}$-column reduced RCPMFD

$$Z = ND^{-1}$$

with $N \in \mathbb{F}[z]^{p\times q}$, $D \in \mathbb{F}[z]^{q\times q}$ and

$$P := \begin{bmatrix} N \\ D \end{bmatrix} \quad \vec{\tau}\text{-column reduced.}$$

By taking this representation, making the $\vec{\tau}$-McMillan degree of Z minimal is equivalent to taking the $\vec{\tau}$-degree of P minimal, i.e., the sum of the $\vec{\tau}$-degrees of the columns of P minimal.

Special case

We shall first handle the special case when M has the form

$$M = O_-(z^{\tau_{N,i}-\tau_{D,j}-1}). \tag{2}$$

In Section 7, we will handle the general case dropping this condition on M. In the classical case with $\vec{\tau} = \vec{0}$, this reduces to M being strictly proper.

Theorem 4.1 *If $M = O_-(z^{\tau_{N,i}-\tau_{D,j}-1})$ and $\Sigma \leq [\tau_{N,i} - \tau_{D,j} - 1]$ (elementwise), any $\vec{\tau}$-column reduced RCPMFD ND^{-1} for the solution Z of the GMPR problem has the form*

$$\begin{bmatrix} N \\ D \end{bmatrix} = \begin{bmatrix} z^{\vec{\tau}_N} & 0 \\ 0 & z^{\vec{\tau}_D} \end{bmatrix} \begin{bmatrix} N_h \\ D_h \end{bmatrix} z^{\vec{\alpha}}$$

with $D_h = O_-^s(z^0)$ and $N_h = O_-(z^{-1})$.

proof. Because $\Sigma \leq [\tau_{N,i} - \tau_{D,j} - 1]$, the interpolation conditions (1) imply that

$$M - ND^{-1} = O_-(z^{\tau_{N,i}-\tau_{D,j}-1})$$

or multiplying to the right by D

$$N - MD = O_-(z^{\tau_{N,i}-\tau_{D,j}-1})D$$

which can also be written as:

$$[I \quad -M] \begin{bmatrix} N \\ D \end{bmatrix} = O_-(z^{\tau_{N,i}-\tau_{D,j}-1})D. \tag{3}$$

Because $ND^{-1} = Z$ is a $\vec{\tau}$-column reduced RCPMFD for Z, we can write

$$\begin{bmatrix} N \\ D \end{bmatrix} = \begin{bmatrix} z^{\vec{\tau}_N} & 0 \\ 0 & z^{\vec{\tau}_D} \end{bmatrix} \begin{bmatrix} N_h \\ D_h \end{bmatrix} z^{\vec{\alpha}}$$

with

$$\begin{bmatrix} N_h \\ D_h \end{bmatrix} = O_-^s(z^0).$$

We can rewrite (3) as

$$[I \quad -M] \begin{bmatrix} z^{\vec{\tau}_N} & 0 \\ 0 & z^{\vec{\tau}_D} \end{bmatrix} \begin{bmatrix} N_h \\ D_h \end{bmatrix} z^{\vec{\alpha}} = O_-(z^{\tau_{N,i}-\tau_{D,j}-1}) z^{\vec{\tau}_D} D_h z^{\vec{\alpha}}$$

which is equivalent to:

$$[z^{\vec{\tau}_N} \quad -Mz^{\vec{\tau}_D}] \begin{bmatrix} N_h \\ D_h \end{bmatrix} z^{\vec{\alpha}} = O_-(z^{\tau_{N,i}-1}) D_h z^{\vec{\alpha}}.$$

Multiplying to the left by $z^{-\vec{\tau}_N}$ and to the right by $z^{-\vec{\alpha}}$, gives us the equivalent relation

$$[I \quad -z^{-\vec{\tau}_N} M z^{\vec{\tau}_D}] \begin{bmatrix} N_h \\ D_h \end{bmatrix} = O_-(z^{-1}) D_h. \tag{4}$$

From (2), we get that $-z^{-\vec{\tau}_N} M z^{\vec{\tau}_D} = O_-(z^{-1})$. So, (4) gives

$$N_h = O_-(z^{-1}) D_h = O_-(z^{-1}).$$

Hence, $D_h = O_-^s(z^0)$. □

The previous theorem shows that if M has the form

$$M = O_-(z^{\tau_{N,i} - \tau_{D,j} - 1}),$$

we can look for a RCPMFD ND^{-1} satisfying the interpolation conditions (1) with D being $\vec{\tau}_D$-column reduced.
This implies that

$$P := \begin{bmatrix} N \\ D \end{bmatrix}$$

will be $\vec{\tau}$-column reduced. Minimizing the $\vec{\tau}$-degree of P is equivalent to minimizing the $\vec{\tau}_D$-degree of D.

Theorem 4.2 *If D is $\vec{\tau}_D$-column reduced with column degrees $\vec{\alpha}$, the original interpolation conditions (1) are equivalent to*

$$N - MD = O_-(z^{-\gamma_i + \alpha_j}).$$

proof. If D is $\vec{\tau}_D$-column reduced, it can be written as

$$D = z^{\vec{\tau}_D} D_h z^{\vec{\alpha}}, \qquad \text{with } D_h = O_-^s(z^0).$$

Multiplying the original interpolation conditions (1)

$$M - Z = O_-(z^{-\gamma_i - \tau_{D,j}})$$

to the right by D gives us:

$$\begin{aligned} N - MD &= O_-(z^{-\gamma_i - \tau_{D,j}}) z^{\vec{\tau}_D} D_h z^{\vec{\alpha}} \\ &= O_-(z^{-\gamma_i + \alpha_j}). \end{aligned}$$

Because $D^{-1} = z^{-\vec{\alpha}} D_h^{-1} z^{-\vec{\tau}_D}$ with $D_h^{-1} = O_-^s(z^0)$ we can go all the way back. □

To solve the GMPRP, we have to look for polynomial vectors (the columns of D)

$$\vec{p} = \begin{bmatrix} \vec{n} \\ \vec{d} \end{bmatrix}$$

such that

$$F(z)\vec{p}(z) = O_-(z^{-\gamma_i + \alpha})$$

with $\alpha := \vec{\tau}$-degree of $\vec{p}$ and $F := [I \quad -M]$.
We shall solve this interpolation problem around ∞ using a module theoretic framework.

5 Polynomial vector interpolation problem around ∞

Definition 5.1 (polynomial vector interpolation problem around ∞) *Take*

$$F \in \mathbb{F}(z^{-1})^{p \times n}, \quad \vec{\tau} \in \mathbb{Z}^n, \quad \vec{\gamma} \in \mathbb{Z}^p.$$

For each $\alpha \in \mathbb{Z}$, the polynomial vector interpolation problem looks for all polynomial vectors $\vec{p} \in \mathbb{F}[z]^n$ satisfying the interpolation conditions

$$F\vec{p} = O_-(z^{-\gamma_i+\alpha}) \tag{5}$$

with the additional condition that $\alpha \geq \vec{\tau}\text{-deg}\,\vec{p}$.

It turns out that for every $\vec{p}$ there exists an α_p such that $\vec{p}$ satisfies (5) for $\alpha \geq \alpha_p$. Hence, the set $\mathcal{P}$ of all polynomial vectors $\vec{p}$ satisfying (5) with a corresponding α_p is the complete set of polynomial vectors, i.e., $\mathcal{P} = \mathbb{F}[z]^n$. Note that $\mathcal{P}$ is an $\mathbb{F}[z]$-module which can be generated by n basis vectors. Because $\mathcal{P}$ is the complete $\mathbb{F}[z]$-module $\mathbb{F}[z]^n$, we know that the columns of any unimodular $(n \times n)$ matrix form a set of basis vectors. In the sequel, we want to construct a basis matrix (unimodular) with additional information such that it is easy to derive all solution vectors $\vec{p}$ satisfying (5) for a given α with the additional condition that $\alpha \geq \vec{\tau}\text{-deg}\,\vec{p}$. The additional information we have to keep with every basis vector $\vec{b}_j$ is an integer number α_j. We use the notation:

$$B := \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \dots & \vec{b}_n \\ (\alpha_1) & (\alpha_2) & \dots & (\alpha_n) \end{bmatrix}.$$

In the next section, we shall design an algorithm that computes such a basis matrix B. We shall show that all polynomial vectors $\vec{p}$ satisfying (5) with $\alpha \geq \vec{\tau}\text{-deg}\,\vec{p}$ can be represented as

$$\vec{p}(z) = \sum_{j=1}^{n} a_j(z)\vec{b}_j(z) \quad \text{with} \quad \begin{cases} \deg a_j \leq \alpha - \alpha_j \\ a_j \in \mathbb{F}[z] \end{cases}.$$

6 Algorithm

To compute such a basis, we construct a recursive algorithm. In each step, we increment one of the γ_i, i.e., we add one new interpolation condition.
We can always find $s_i \in \mathbb{Z}$ such that $F(z) = O_-(z^{s_i-\tau_j})$. Then we can start by putting all $-\gamma_i = s_i$. The initial basis matrix B is

$$B := \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \dots & \vec{e}_n \\ (\alpha_1) & (\alpha_2) & \dots & (\alpha_n) \end{bmatrix}$$

with $\vec{e}_j$ the jth identity vector and α_j the $\vec{\tau}$-degree of $\vec{e}_j$.
Given $\alpha \in \mathbb{Z}$, any polynomial vector $\vec{p}$ with $\alpha \geq \vec{\tau}\text{-deg}\,\vec{p}$ satisfies (5). Hence, all solutions can be written as

$$\vec{p} = \sum_{i=1}^{n} a_i\vec{b}_i, \qquad (\vec{b}_i = \vec{e}_i)$$

with $\deg a_i \leq \alpha - \alpha_i$. Note also that $\alpha_i \geq \vec{\tau}\text{-deg}\,\vec{b}_i$, $i = 1, 2, \ldots n$.
Suppose now that we have a basis matrix B such that $\alpha_i \geq \vec{\tau}\text{-deg}\,\vec{b}_i$ and, for each α, that every $\vec{p}$ satisfying (5) with $\alpha \geq \vec{\tau}\text{-deg}\,\vec{p}$ can be written as

$$\vec{p} = \sum_{i=1}^{n} a_i \vec{b}_i \quad \text{with } \deg a_i \leq \alpha - \alpha_i.$$

The interpolation conditions for the basis vectors can be written as:

$$FB = z^{-\vec{\gamma}}(R + O_-(z^{-1}))z^{\vec{\alpha}} \text{ with } R \in \mathbb{F}^{p \times n}. \tag{6}$$

We call R the residual.
Let us determine which part of F has to be known to compute the residual of every polynomial vector $\vec{p}$ having $\vec{\tau}$-degree $\leq \alpha$.
We can write $\vec{p}$ as $\vec{p} = z^{\vec{\tau}}\vec{p}_h z^{\alpha}$ with $\vec{p}_h = O_-(z^0)$. The residual $\vec{r}$ of $\vec{p}$ is defined by

$$\begin{aligned} F\vec{p} &= z^{-\vec{\gamma}}(\vec{r} + O_-(z^{-1}))z^{\alpha} \text{ or} \\ Fz^{\vec{\tau}}\vec{p}_h z^{\alpha} &= z^{-\vec{\gamma}}(\vec{r} + O_-(z^{-1}))z^{\alpha} \text{ or} \\ (z^{\vec{\gamma}}Fz^{\vec{\tau}})\vec{p}_h &= \vec{r} + O_-(z^{-1}). \end{aligned}$$

Hence, to compute the residual $\vec{r}$, we only need the polynomial part of $z^{\vec{\gamma}}Fz^{\vec{\tau}}$ or $\pi^{+}_{-\gamma_i - \tau_j}F$ has to be given.
Suppose we want to compute a basis matrix B' by incrementing γ_i, i.e., $\gamma'_j = \gamma_j$, $j \neq i$ and $\gamma'_i = \gamma_i + 1$.
There are two possibilities:

- $r_{i,j} = 0$, $j = 1, 2, \ldots, n$, i.e., all relevant residuals are zero. We can take $B' = B$ and $\vec{\alpha}' = \vec{\alpha}$. It is clear that $\alpha'_i \geq \vec{\tau}\text{-deg}\,\vec{b}'_i$. We can write (6) as

  $$FB' = z^{-\vec{\gamma}'}(R' + O_-(z^{-1}))z^{\vec{\alpha}'} \text{ with } R' \in \mathbb{F}^{p \times n}.$$

  Consider the set

  $$\mathcal{P}'_\alpha := \{\vec{p} \mid \vec{p} = \sum_{i=1}^{n} a_i(z)\vec{b}'_i(z), \deg a_i(z) \leq \alpha - \alpha'_i\}$$

  and the set

  $$\mathcal{I}'_\alpha := \{\vec{p} \mid \alpha \geq \vec{\tau}\text{-deg}\,\vec{p}, \vec{p} \text{ satisfies (5) for } \vec{\gamma}' \text{ instead of } \vec{\gamma}\}.$$

  We have to prove that $\mathcal{P}'_\alpha \equiv \mathcal{I}'_\alpha$.
  Take an element $\vec{p}$ of $\mathcal{P}'_\alpha$. Because $\alpha'_i \geq \vec{\tau}\text{-deg}\,\vec{b}'_i$, it is clear that $\alpha \geq \vec{\tau}\text{-deg}\,\vec{p}$ and $\vec{p}$ satisfies (5) for $\vec{\gamma}'$ instead of $\vec{\gamma}$. Hence, $\vec{p} \in \mathcal{I}'_\alpha$.
  On the other hand, take $\vec{p} \in \mathcal{I}'_\alpha$. We know that

  $$F\vec{p} = O_-(z^{-\gamma'_i + \alpha}) = O_-(z^{-\gamma_i + \alpha})$$

  with $\alpha \geq \vec{\tau}\text{-deg}\,\vec{p}$.
  Because

  $$\mathcal{I}'_\alpha \subset \mathcal{I}_\alpha = \mathcal{P}_\alpha = \mathcal{P}'_\alpha \subset \mathcal{I}'_\alpha,$$

  we get that

  $$\mathcal{I}'_\alpha = \mathcal{P}'_\alpha = \mathcal{I}_\alpha = \mathcal{P}_\alpha.$$

- $\exists j$ such that $r_{i,j} \neq 0$, i.e., not all relevant residuals are zero.
  Take k such that

$$\alpha_k := \min\{\alpha_j \mid r_{i,j} \neq 0, j = 1, 2, \ldots, n\}.$$

We compute the new basis matrix B' as

$$B' = B\ V$$

with (only the nonzero elements are displayed)

$$V := \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ v_1 & v_2 & \ldots & v_{k-1} & v_k & v_{k+1} & \ldots & v_n \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{bmatrix}$$

with

$$v_j := -\frac{r_{i,j}}{r_{i,k}} z^{\alpha_j - \alpha_k}.$$

We compute $\vec{\alpha}'$ as $\vec{\alpha}' = \vec{\alpha} + \vec{e}_k$.
It is clear that $\alpha'_j \geq \vec{\tau}\text{-deg}\ \vec{b}_j$, $j = 1, 2, \ldots, n$.
Note that B' is also unimodular because B and V are unimodular. Because every column of B' satisfies the interpolation conditions, i.e.,

$$FB' = O_-(z^{-\gamma'_i + \alpha'_j}),$$

we get that

$$FB'\vec{a} = O_-(z^{-\gamma'_i + \alpha})$$

with $\vec{a} = [a_1, a_2, \ldots, a_n]^T$ and $\deg a_i \leq \alpha - \alpha'_i$.
Because

$$B' = z^{\vec{\tau}} B_h z^{\vec{\alpha}'}$$

with

$$B_h = O_-(z^0),$$

we get that

$$\begin{aligned} B'\vec{a} &= z^{\vec{\tau}} B_h z^{\vec{\alpha}'} z^{-\vec{\alpha}'} z^{\alpha} \vec{a}_h \text{ with } \vec{a}_h = O_-(z^0) \\ &= z^{\vec{\tau}} B_h \vec{a}_h z^{\alpha} \end{aligned}$$

or $\alpha \geq \vec{\tau}$-deg of $\vec{p} = B'\vec{a}$.
Hence, every element $\vec{p}$ of $\mathcal{P}'_\alpha$ is also an element of $\mathcal{I}'_\alpha$, i.e., $\mathcal{P}'_\alpha \subset \mathcal{I}'_\alpha$.
To prove that $\mathcal{P}'_\alpha = \mathcal{I}'_\alpha$, we consider the equivalent linear algebra problem and

look at the dimension of the solution spaces. We assume that $\mathcal{P}_\alpha = \mathcal{I}_\alpha$ and prove that $\mathcal{P}'_\alpha = \mathcal{I}'_\alpha$.
Consider the set $\mathcal{I}_\alpha$. Using, e.g., the classical basis $1, z, z^2, \ldots$, we can represent any $\vec{p}$ having $\vec{\tau}$-degree $\leq \alpha$ by using $\sum_{i=1}^{n} \max\{0, \tau_i + \alpha + 1\}$ coordinates.
The interpolation conditions (5) can be transformed into a set of linear equations with the coordinates as unknowns. Because $\mathcal{I}_\alpha = \mathcal{P}_\alpha$, we know that the dimension of the solution set is given by:

$$\sum_{i=1}^{n} \max\{\alpha - \alpha_i + 1, 0\}$$

and a basis is given by the coordinate representation of all polynomial vectors of the form $z^\delta \vec{b}_i$ with $0 \leq \delta \leq \alpha - \alpha_i$.
Indeed, these vectors are $\mathbb{F}$-linearly independent. Suppose they are not. Then, there exists a nonzero constant vector $\vec{a}$ such that

$$[\vec{b}_1, z\vec{b}_1, \ldots, z^{\alpha-\alpha_1}\vec{b}_1 | \ldots | \vec{b}_n, z\vec{b}_n, \ldots, z^{\alpha-\alpha_n}\vec{b}_n]\vec{a} = \vec{0}$$

or

$$B\vec{a}(z) = \vec{0} \quad \text{or} \quad \vec{a}(z) = B^{-1}\vec{0} = \vec{0} \quad \text{or} \quad \vec{a}(z) = \vec{0}$$

which is contradicting our assumption that $\vec{a} \neq \vec{0}$.
Incrementing γ_i is equivalent to adding one new linear equation. Therefore, the dimension of the solution space for a fixed α is decreased by one iff the new linear equation is linearly independent of the previous equations.
We have that $\mathcal{P}'_\alpha = \mathcal{I}'_\alpha = \mathcal{P}_\alpha = \mathcal{I}_\alpha$ for $\alpha < \alpha_k$.
For $\alpha \geq \alpha_k$, we know that $\dim \mathcal{I}'_\alpha = \dim \mathcal{I}_\alpha - 1$. We have already shown that $\mathcal{P}'_\alpha \subset \mathcal{I}'_\alpha$ but written in terms of coordinates $\dim \mathcal{P}'_\alpha = \dim \mathcal{P}_\alpha - 1$. Therefore, $\mathcal{P}'_\alpha = \mathcal{I}'_\alpha$.

Let us summarize the algorithm.

Algorithm 6.1 *Given* $F \in \mathbb{F}(z^{-1})^{p \times n}$, $\vec{\tau} \in \mathbb{Z}^n$, $\vec{\gamma}* \in \mathbb{Z}^p$.

Initialization:

Choose $s_i \geq -\gamma_i^*$, $i = 1, 2, \ldots, n$ *such that* $F = O_-(z^{s_i - \tau_j})$
$B \leftarrow I_n$ $(n \times n)$ *identity matrix*
$\vec{\alpha} \leftarrow -\vec{\tau}$ *(the* $\vec{\tau}$*-degrees of the columns of* B*)*
$-\vec{\gamma} \leftarrow \vec{s}$

While $\vec{\gamma} \neq \vec{\gamma}^*$ *(not handled all interpolation conditions)*

- *Choose* i *such that* $\gamma_i + 1 \leq \gamma_i^*$; $\gamma_i \leftarrow \gamma_i + 1$.
- *Get the residuals* $r_{i,j}$, $j = 1, 2, \ldots, n$, *i.e., the* i*th row of* R *defined by (6)*
- **If** $\exists j : r_{i,j} \neq 0$ **then**
  - *Compute* k *such that*

$$\alpha_k = \min\{\alpha_j \mid r_{i,j} \neq 0\}$$

- *Construct the unimodular V matrix, i.e., the identity matrix I_n except for the kth row, which has elements*

$$v_j = -\frac{r_{i,j}}{r_{i,k}} z^{\alpha_j - \alpha_k}$$

- $B \leftarrow BV$
- $\vec{\alpha} \leftarrow \vec{\alpha} + \vec{e}_k$

The previous algorithm generates a basis matrix B with additional information $\vec{\alpha}$ such that it is easy to represent all elements of set $\mathcal{I}_\alpha$ as elements of $\mathcal{P}_\alpha$.
Note that the algorithm requires $O(\|\vec{s} + \vec{\gamma}^*\|^2)$ FLOPS. Algorithm 6.1 could be considered a Levinson-type algorithm in the sense that only the first part of the residual series is computed, the part which is needed to continue the computations. On the other hand, one could also consider a Schur-type algorithm in which the complete residual series is adapted in each step. The residuals, needed to continue the computations, can be read off directly as part of the residual series.

7 Solution of the GMPR problem

Let us look how we can use the algorithm of the previous section to solve the GMPR problem.
We look for a numerator column $\vec{n}$ and corresponding denominator column $\vec{d}$ such that:

$$[I \quad -M] \begin{bmatrix} \vec{n} \\ \vec{d} \end{bmatrix} = O_-(z^{-\gamma_i + \alpha})$$

with

$$\alpha = \vec{\tau}\text{-deg} \begin{bmatrix} \vec{n} \\ \vec{d} \end{bmatrix}.$$

Rewriting this in terms of the previous algorithm, we get

$$F = [I \quad -M] \in \mathbb{F}(z^{-1})^{p \times n} \quad \text{with } n = p + q.$$

To initialize, we have to find $\vec{s}$ such that

$$F = O_-(z^{s_i - \tau_j}).$$

Special case

Remember that we assumed M to be of the form

$$M = O_-(z^{\tau_{N,i} - \tau_{D,j} - 1}).$$

Hence,

$$\begin{aligned} F = [I \quad -M] &= z^{\vec{\tau}_N}[I \quad -z^{-\vec{\tau}_N} M z^{\vec{\tau}_D}] \begin{bmatrix} z^{-\vec{\tau}_N} & 0 \\ 0 & z^{-\vec{\tau}_D} \end{bmatrix} \\ &= z^{\vec{\tau}_N}[O_-^s(z^0)]z^{-\vec{\tau}} \\ &= O_-^s[z^{\tau_{N,i}-\tau_j}] \\ &= O_-[z^{\tau_{N,i}-\tau_j}]. \end{aligned}$$

Therefore, we can start the algorithm by choosing

$$\vec{s} = \vec{\tau}_N.$$

To get a solution of smallest $\vec{\tau}$-McMillan degree, we take q basis vectors $\vec{b}_{i(j)}$, $j = 1, 2, \ldots, q$ having smallest $\alpha_{i(j)}$ with $\alpha_{i(j)} = \vec{\tau}\text{-deg}\, \vec{b}_{i(j)}$.
Because $\alpha_{i(j)} = \vec{\tau}\text{-deg}\, \vec{b}_{i(j)}$, we know that the $(q \times q)$ matrix D of

$$\begin{bmatrix} N \\ D \end{bmatrix} := [\vec{b}_{i(1)}, \vec{b}_{i(2)}, \ldots, \vec{b}_{i(q)}]$$

is $\vec{\tau}_D$-column reduced. Hence,

$$M - Z = O_-(z^{-\gamma_i - \tau_{D,j}}).$$

In analogy to the classical MPR problem, we get the following result.

Theorem 7.1 *All solutions of the GMPR problem can be represented as follows. The set of basis vectors generated by Algorithm 6.1 is divided in the two subsets:* $B_1 := [\vec{b}_{i(1)}, \vec{b}_{i(2)}, \ldots, \vec{b}_{i(q)}]$ *where* $\vec{b}_{i(j)}$, $j = 1, 2, \ldots, q$ *are* q *basis vectors having smallest* $\alpha_{i(j)}$ *with* $\alpha_{i(j)} = \vec{\tau}\text{-deg}\, \vec{b}_{i(j)}$, *and the remaining basis vectors:* $B_2 := [\vec{b}'_1, \vec{b}'_2, \ldots, \vec{b}'_p]$. *We construct two polynomial matrices* $N \in \mathbb{F}[z]^{p \times q}$ *and* $D \in \mathbb{F}[z]^{q \times q}$

$$\begin{bmatrix} N \\ D \end{bmatrix} := B_1 + B_2\ \Pi \quad \textit{with}\ \Pi \in \mathbb{F}[z]^{p \times q}.$$

The polynomial matrix Π *contains the parameters with the condition that* $\deg \pi_{i,j} \leq \alpha_{i(j)} - \alpha'_i$ *(*$\alpha'_i := \alpha$ *connected with* $\vec{b}'_i$*). For a different* Π, $Z = ND^{-1}$ *will be a different matrix rational function solving the GMPR problem.*

General case

Let us look now at the general case when $M \in \mathbb{F}(z^{-1})^{p \times q}$ but has not to satisfy the constraint $M = O_-(z^{\tau_{N,i} - \tau_{D,j} - 1})$.
We split up M in a "polynomial part" M^+ and a "strictly proper part" M^-:

$$M = M^+ + M^-$$

with $M^+ := \pi^+_{\tau_{N,i} - \tau_{D,j}} M$ and $M^- := \pi^-_{\tau_{N,i} - \tau_{D,j}} M$. Note that for the classical case, i.e., $\vec{\tau}_N = \vec{0}$ and $\vec{\tau}_D = \vec{0}$, M^+ and M^- are the polynomial and the strictly proper part of

M respectively.
Any $(p \times q)$ matrix rational function Z can be split up in a similar way, i.e., $Z = Z^+ + Z^-$ with

$$Z^+ := \pi^+_{\tau_{N,i}-\tau_{D,j}} Z \quad \text{and} \quad Z^- := \pi^-_{\tau_{N,i}-\tau_{D,j}} Z.$$

Because Z^- is also rational and satisfies

$$Z^- = O_-(z^{\tau_{N,i}-\tau_{D,j}-1}),$$

we can represent Z^- using a $\vec{\tau}$-column reduced RCPMFD

$$Z^- = ND^{-1}$$

with D $\vec{\tau}_D$-column reduced.
The $\vec{\tau}$-McMillan degree of Z is equal to the $\vec{\tau}$-degree of

$$\begin{bmatrix} Z^+D + N \\ D \end{bmatrix} \tag{7}$$

if $(Z^+D + N)D^{-1}$ is a RCPMFD for Z.

Lemma 7.2 *If*

$$\tau_{N,i} \geq \tau_{D,j}, \quad i = 1, 2, \ldots, p \quad \textit{and} \quad j = 1, 2, \ldots, q, \tag{8}$$

then (7) is a RCPMFD for Z.

proof. Because ND^{-1} is a RCPMFD for Z^-, we know that N and D are polynomial matrices.
Using (8), Z^+ turns out to be polynomial and, hence, also $Z^+D + N$ is polynomial.
Now, we have to show that $Z^+D + N$ and D are right coprime, i.e.,

$$\begin{bmatrix} Z^+D + N \\ D \end{bmatrix}$$

has full rank for all possible values of z.
Suppose that there exists a right common divisor R of $Z^+D + N$ and D, i.e.,

$$Z^+D + N = PR$$

and

$$D = D'R$$

with P and D' polynomial matrices.
It follows that

$$(Z^+D + N)R^{-1} = Z^+DR^{-1} + NR^{-1} = Z^+D' + NR^{-1} = P$$

is polynomial.
Because also Z^+D' is polynomial, NR^{-1} is polynomial or R is also a right common

divisor of N.
Hence, N and D are not right coprime which contradicts our assumption. □

Let us look in more detail at the $\vec{\tau}$-degree of (7) which is equal to the $\vec{0}$-degree of

$$\begin{aligned}
&\begin{bmatrix} z^{-\vec{\tau}_N} & 0 \\ 0 & z^{-\vec{\tau}_D} \end{bmatrix} \begin{bmatrix} Z^+ + ND^{-1} \\ I \end{bmatrix} D \\
&\qquad = \begin{bmatrix} z^{-\vec{\tau}_N} Z^+ z^{\vec{\tau}_D} + z^{-\vec{\tau}_N} N D^{-1} z^{\vec{\tau}_D} \\ I \end{bmatrix} z^{-\vec{\tau}_D} D \\
&\qquad = \begin{bmatrix} P^+ + P^- \\ I \end{bmatrix} z^{-\vec{\tau}_D} D \qquad (9)
\end{aligned}$$

with

$$P^+ = z^{-\vec{\tau}_N} Z^+ z^{\vec{\tau}_D} = O_+(z^0)$$

and

$$P^- = z^{-\vec{\tau}_N} N D^{-1} z^{\vec{\tau}_D} = O_-(z^{-1}).$$

Hence, the $\vec{\tau}$-McMillan degree of Z is equal to the $\vec{0}$-degree of

$$\begin{bmatrix} P^+ + P^- \\ I \end{bmatrix}$$

added to the $\vec{\tau}_D$-degree of D. Note that

$$\vec{0}\text{-deg} \begin{bmatrix} P^+ + P^- \\ I \end{bmatrix} = \vec{0}\text{-deg} \begin{bmatrix} P^+ \\ I \end{bmatrix}$$

because $P^- = O_-(z^{-1})$.
Hence, to find all matrix rational functions Z solving the GMPR problem, we have to find all D having minimal $\vec{\tau}_D$-degree and all N such that $ND^{-1} - Z^- = O_-(z^{\Sigma})$ with $\Sigma \leq [\tau_{N,i} - \tau_{D,j} - 1]$ (elementwise). All solutions can be written as

$$Z^+ + ND^{-1}, \qquad \text{with } Z^+ = M^+.$$

If the condition $\Sigma \leq [\tau_{N,i} - \tau_{D,j} - 1]$ is not true for some of the elements of Σ, we also have to fill in the free parameters of Z^+ such that

$$\vec{0}\text{-deg} \begin{bmatrix} z^{-\vec{\tau}_N} Z^+ z^{\vec{\tau}_D} \\ I \end{bmatrix} \quad \text{is minimal.}$$

We can do this by making the matrix

$$\begin{bmatrix} z^{-\vec{\tau}_N} Z^+ z^{\vec{\tau}_D} \\ I \end{bmatrix} \quad \text{column reduced}$$

with the sum of the column degrees as small as possible. This could lead to certain conditions on the free parameters as in Example 8.2 of Section 8.

To make a polynomial matrix column reduced, we can use the following procedure. Consider the following elementary polynomial transformation matrices:

- $E_{i,j}$:= the unit matrix in which column i and column j are interchanged;
- $E_{i,j}(q(z))$:= the unit matrix in which column j is replaced by column j plus $q(z) \in \mathbb{F}[z]$ times column i;
- $E_i(k)$:= the unit matrix where column i is multiplied by $k \in \mathbb{F} \setminus \{0\}$.

Suppose we have a polynomial $(p \times q)$ matrix $A(z)$ having full column rank. The coefficient of degree k of element $a_{i,j}(z)$ is denoted by $a_{i,j,k}$. The degrees of the columns of $A(z)$ are denoted by $\alpha_1, \ldots, \alpha_n$.

To make this polynomial matrix column reduced, i.e., where the highest degree coefficients of the columns are linearly independent, we can use the following algorithm:

Algorithm 7.1
While *not column reduced*

- *Consider the set of linear equations*
$$\begin{bmatrix} a_{1,1,\alpha_1} & \cdots & a_{1,q,\alpha_q} \\ \vdots & & \vdots \\ a_{p,1,\alpha_1} & \cdots & a_{p,q,\alpha_q} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_q \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$
which always has a nontrivial solution c_i.
- *Define* l *such that* $l = \max\{\alpha_i | c_i \neq 0\}$.
- *Replace* A *by* $A\Pi_{i=1,i\neq l}^{q} E_{i,l}(-c_i z^{\alpha_l - \alpha_i}/c_l)$.

The idea of the previous algorithm is to make the highest degree coefficient of column l of A zero by subtracting shifted versions of the other columns. As long as the free parameters are not appearing in the highest degree coefficients, Algorithm 7.1 can be run straightforwardly. Otherwise, it is sometimes possible to make the highest degree coefficients linearly dependent by posing conditions on the free parameters.

8 Examples

Example 8.1 Take $\vec{\tau}_N = [1,0]^T$, $\vec{\tau}_D = [0,-1]^T$, $\vec{\gamma} = [3,4]^T$

$$M(z) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} z^2 + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z^1 + \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} z^0 + \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} z^{-1} + \begin{bmatrix} 1 & * \\ 0 & -1 \end{bmatrix} z^{-2} + \begin{bmatrix} * & * \\ 2 & * \end{bmatrix} z^{-3} + \ldots$$

The matrix $[\tau_{N,i} - \tau_{D,j}]$ is equal to

$$[\tau_{N,i} - \tau_{D,j}] = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

Therefore, the "polynomial" part Z^+ is

$$Z^+ = \pi^+_{\tau_{N,i}-\tau_{D,j}} M = \begin{bmatrix} z^2 & 0 \\ 1 & 0 \end{bmatrix}.$$

Using the algorithm of section (6.1), we get the following basis matrix

$$B = \begin{bmatrix} 1 & -3z^2-3 & -1 & -z^2-2z-1 \\ 0 & +2 & 0 & 0 \\ 0 & -z^2 & -1 & -z^2 \\ 0 & -2z & 0 & -2 \\ (3) & (2) & (1) & (2) \end{bmatrix} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \vec{b}_4 \\ (\alpha_1) & (\alpha_2) & (\alpha_3) & (\alpha_4) \end{bmatrix}$$

Therefore all matrix rational functions Z which solve the GMPR problem, can be written as a RCPMFD $Z = ND^{-1} + Z^+$ with

$$\begin{bmatrix} N \\ D \end{bmatrix} = [\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4] \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ az+b & cz+d \\ 0 & 1 \end{bmatrix}.$$

Z has minimal $\vec{\tau}$-McMillan degree equal to

$$\vec{0}\text{-deg} \begin{bmatrix} z^1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} + \vec{\tau}_D\text{-deg}\, D = 1 + 4 = 5.$$

$\diamond$

Example 8.2 Take $\vec{\tau}_N = [1,0]^T$, $\vec{\tau}_D = [0,-1]^T$, $\Sigma = \begin{bmatrix} 2 & 3 \\ -2 & -1 \end{bmatrix}$,

$$M(z) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z^4 + \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} z^3 + \begin{bmatrix} * & * \\ 1 & -1 \end{bmatrix} z^2 + \begin{bmatrix} * & * \\ 1 & 0 \end{bmatrix} z^1 + \begin{bmatrix} * & * \\ 0 & 0 \end{bmatrix} z^0 + \begin{bmatrix} * & * \\ 1 & * \end{bmatrix} z^{-1} + \dots$$

The matrix $[\tau_{N,i} - \tau_{D,j}]$ is equal to

$$[\tau_{N,i} - \tau_{D,j}] = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

Therefore, the "polynomial" part Z^+ is

$$Z^+ = \pi^+_{\tau_{N,i}-\tau_{D,j}} M = \begin{bmatrix} z^3+az^2+bz & z^4+cz^3+dz^2 \\ z^2+z & z^3-z^2 \end{bmatrix}.$$

Hence,

$$\vec{0}\text{-deg} \begin{bmatrix} P^+ \\ I \end{bmatrix}$$

is equal to 2 when $a - c = 2$. Otherwise, it is equal to 3. Using Algorithm 6.1 to solve the GMPR problem for M^- with $\Sigma' = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}$, we get the following basis matrix

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & z & 1 & 0 \\ 0 & 0 & 0 & 1 \\ (0) & (1) & (1) & (1) \end{bmatrix} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \vec{b}_4 \\ (\alpha_1) & (\alpha_2) & (\alpha_3) & (\alpha_4) \end{bmatrix}$$

Therefore all matrix rational functions Z which solve the GMPR problem, can be written as a RCPMFD $Z = ND^{-1} + Z^+$ with $a = c+2$, $b, c, d, e, f, g, h, i, j$ arbitrary, and

$$\begin{bmatrix} N \\ D \end{bmatrix} = [\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4] \begin{bmatrix} ez + f & gz + h \\ 1 & 0 \\ i & j \\ 0 & 1 \end{bmatrix}.$$

Z has minimal $\vec{\tau}$-McMillan degree equal to

$$2 + \vec{\tau}_D\text{-deg}\, D = 2 + 2 = 4.$$

$\diamond$

9 Conclusion

In this paper, we have shown that all solutions of the GMPR problem can be parametrized in a nice way

- if $\tau_{N,i} \geq \tau_{D,j}$, $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, q$, and
- if the order conditions have the form $-\gamma_i - \tau_{D,j}$.

If the free parameters start to influence the flow of algorithm 7.1, it is possible that additional conditions have to be imposed on the free parameters to make the $\vec{\tau}$-McMillan degree minimal.

References

[1] A.C. Antoulas. On recursiveness and related topics in linear systems. *IEEE Trans. Automat. Control*, AC-31:1121–1134, 1986.

[2] A.C. Antoulas. Recursive modeling of discrete-time time series. In P. Van Dooren and B.F. Wyman, editors, *Proc. IMA Workshop on Linear Algebra in Systems and Control*, volume 62 of *IMA Volumes in Mathematics and its Applications*, 1993.

[3] A.C. Antoulas and J.C. Willems. A behavioral approach to linear exact modeling. *IEEE Trans. Automat. Control*, AC-38(12):1776–1802, 1993.

[4] B. Beckermann. A reliable method for computing M-Padé approximants on arbitrary staircases. *J. Comput. Appl. Math.*, 40:19–42, 1990.

[5] B. Beckermann and G. Labahn. A uniform approach for Hermite-Padé and simultaneous Padé approximants and their matrix-type generalizations. *Numerical Algorithms*, 3:45–54, 1992.

[6] B. Beckermann and G. Labahn. A uniform approach for the fast computation of matrix-type Padé approximants. *SIAM J. Matrix Anal. Appl.*, 15(3), 1994.

[7] S. Cabay, A.R. Jones, and G. Labahn. A stable algorithm for multi-dimensional Padé systems and the inversion of generalized Sylvester matrices. Technical Report TR 94-07, Department of Computing Science, The University of Alberta, Edmonton, Alberta, Canada, March 1994. To appear in SIMAX.

[8] S. Cabay, G. Labahn, and B. Beckermann. On the theory and computation of non-perfect Padé-Hermite approximants. *J. Comput. Appl. Math.*, 39:295–313, 1992.

[9] S. Cabay and R. Meleshko. A weakly stable algorithm for Padé approximants and the inversion of Hankel matrices. *SIAM J. Matrix Anal. Appl.*, 14:735–765, 1993.

[10] M.H. Gutknecht. Stable row recurrences for the Padé table and generically superfast lookahead solvers for non-Hermitian Toeplitz systems. *Linear Algebra Appl.*, 188/189:351–422, 1993.

[11] T. Kailath. *Linear Systems.* Prentice-Hall, 1980.

[12] M. Van Barel, B. Beckermann, A. Bultheel, and G. Labahn. Matrix rational interpolation with poles as interpolation points. In A.M. Cuyt, editor, *Nonlinear numerical methods and rational approximation II*, pages 137–148. Kluwer, 1994. Proceedings of the International Conference, Antwerp, September, 1993.

[13] M. Van Barel and A. Bultheel. A new approach to the rational interpolation problem. *J. Comput. Appl. Math.*, 32(1-2):281–289, 1990. Special Issue Proceedings of the International Conference on Extrapolation and Rational Approximation, Luminy, France, September 24-30, 1989.

[14] M. Van Barel and A. Bultheel. A new approach to the rational interpolation problem: the vector case. *J. Comput. Appl. Math.*, 33(3):331–346, 1990.

[15] M. Van Barel and A. Bultheel. The computation of non-perfect Padé-Hermite approximants. *Numer. Algorithms*, 1:285–304, 1991.

[16] M. Van Barel and A. Bultheel. A general module theoretic framework for vector M-Padé and matrix rational interpolation. *Numer. Algorithms*, 3:451–461, 1992. Special Issue Proceedings of the International Conference on Extrapolation and Rational Approximation, Tenerife, Jan. 1992.

[17] M. Van Barel and A. Bultheel. A new formal approach to the rational interpolation problem. *Numer. Math.*, 62:87–122, 1992.

[18] M. Van Barel and A. Bultheel. The "look-ahead" philosophy applied to matrix rational interpolation problems. In U. Helmke, R. Mennicken, and J. Saurer, editors, *Systems and networks: Mathematical theory and applications, Volume II: Invited and contributed papers*, volume 79 of *Mathematical Research*, pages 891–894. Akademie Verlag, 1994.

[19] M. Van Barel and A. Bultheel. A look-ahead algorithm for the solution of block Toeplitz systems. Technical Report TW224, Department of Computer Science, K.U. Leuven, March (Revised May) 1995. (34 pages).

[20] M. Van Barel and A. Bultheel. A look-ahead method to compute vector Padé-Hermite approximants. *Constr. Approx.*, 1995. To appear (23 pages).