Logic-Sensitivity and Bitstring Semantics in the Square of Opposition

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Abstract

This paper explores the interplay between logic-sensitivity and bitstring semantics in the square of opposition. Bitstring semantics is a combinatorial technique for representing the formulas that appear in a logical diagram, while logic-sensitivity entails that such a diagram may depend, not only on the formulas involved, but also on the logic with respect to which they are interpreted. These two topics have already been studied extensively in logical geometry, and are thus well-understood by themselves. However, the precise details of their interplay turn out to be far more complicated. In particular, the paper describes an elegant and natural interaction between bitstrings and logic-sensitivity, which makes perfect sense when bitstrings are viewed as purely combinatorial entities. However, when we view bitstrings as semantically meaningful entities (which is actually the standard perspective, cf. the term 'bitstring *semantics*'!), this interaction does not seem to have a full and equally natural counterpart. The paper describes some attempts to address this situation, but all of them are ultimately found wanting. For now, it thus remains an open problem to capture this interaction between bitstrings and logic-sensitivity from a semantic (rather than merely a combinatorial) perspective.

Keywords: square of opposition, logical geometry, bitstring semantics, logic-sensitivity, combinatorial bitstrings, semantic bitstrings.

1 Introduction

The square of opposition has a rich and well-documented history in logic, and nowadays it is also used in various other disciplines that are concerned with reasoning, such as linguistics, psychology and computer science [13, 30, 37]. Among the oldest and most well-known examples are the squares of opposition for the categorical statements (from assertoric syllogistics) and for the modal expressions, which can both be traced back to the logical works of Aristotle.[1] Over the past decade, it has become increasingly clear that squares of opposition (and other, more complex diagrams) can be fruitfully studied as objects of independent interest. This has given rise to the flourishing research program of *logical geometry*,

[1] Charting the earliest roots of the assertoric and the modal squares has turned out to be quite complicated. See [4, 5, 6, 18, 20, 22, 24, 25, 33, 40] for more historical information.

which today has its own research topics (e.g., informational optimality [37]), its own mathematical tools and techniques (e.g., Aristotelian isomorphisms [9, 41]), and also its own internal dynamics, including — as will become clear in this paper — some challenging open problems.

In this paper we will investigate the interplay between logic-sensitivity and bitstring semantics in the square of opposition. On the one hand, logic-sensitivity is the phenomenon that various features of a given diagram (most importantly, its configuration of logical relations) strongly depend, not only on the concrete formulas involved, but also on the logic with respect to which these formulas are interpreted. On the other hand, bitstring semantics is a combinatorial technique that allows us to systematically compute compact representations of the formulas that appear in a given diagram, thus providing a concrete grip on its logical behavior. The wide-ranging significance of the present paper should be clear, since logic-sensitivity and bitstring semantics are not only among the most important research topics in logical geometry today [8, 13, 15, 38], but are also regularly discussed across the broad field of philosophical logic [2, 16, 23, 27, 28, 31, 32]. These two topics are well-understood by themselves, but the precise details of their interaction turn out to be far more complicated. In particular, even though bitstrings are purely combinatorial entities, they are typically also viewed as semantically meaningful — hence the term 'bitstring *semantics*'! However, when logic-sensitivity is taken into consideration, this close connection between the combinatorial and the semantic perspective on bitstrings seems to break down.

The paper will therefore make both a negative and a positive contribution. On the one hand, we will describe an elegant and natural interaction between bitstrings and logic-sensitivity: by deleting different bit positions (which corresponds to shifting between different logical systems), we can directly influence the resulting diagrams (e.g., their subalternation relations going in different directions). This makes perfect sense from the combinatorial perspective, but hitherto does not yet have an equally natural counterpart from the semantic perspective. On the other hand, we will present some attempts to come up with precisely such a semantic counterpart. Unfortunately, none of these attempts is entirely successful: those that arise quite naturally, do not *completely* match the combinatorial picture, while those that do fully match the combinatorial picture, feel rather *artificial*.[2] To sum up: we will show that the square of opposition exhibits an elegant interaction between bitstring semantics and logic-sensitivity, which can easily be described from the purely combinatorial perspective on bitstrings; however, we will also argue that for now, it remains an open problem in logical geometry to come up with a complete and equally natural description from the semantic perspective.

The paper is organized as follows. Sections 2 and 3 provide some background on resp. logic-sensitivity and bitstring semantics, in order to keep the paper rela-

[2] At the moment, we are deliberately being vague in distinguishing between 'natural' and 'artificial' attempts to come up with a semantic counterpart to the combinatorial picture. This distinction will be explained in more detail later in the paper.

tively self-contained. Sections 4 and 5 present the paper's key results. Section 4 adopts a purely combinatorial perspective on bitstrings, and goes on to describe an elegant interaction between bitstrings and logic-sensitivity in the square of opposition. Section 5 presents four attempts to describe this same interaction from a semantic perspective, but ultimately argues that this has to stay an open problem for now. Finally, Section 6 wraps things up, and discusses the broader significance of the paper's positive and negative results.

2 Logic-Sensitivity of Aristotelian Diagrams

Logic-sensitivity is a well-known phenomenon in logical geometry [8, 13] and beyond [2, 28, 31, 32]. In order to properly describe this phenomenon, we first need to introduce the notions of Aristotelian relations, diagrams and isomorphisms. This is done in Definitions 1, 2 and 3.

Definition 1. Let S be a logical system with Boolean connectives and a model-theoretic semantics $\models_{\mathsf{S}}$. The *Aristotelian relations for* S are defined as follows: two formulas $\varphi, \psi \in \mathcal{L}_{\mathsf{S}}$ are said to be

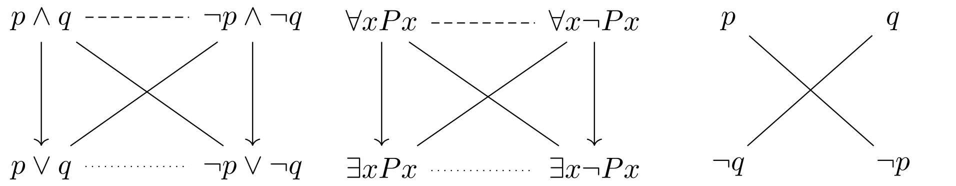
$$\begin{array}{llll} \mathsf{S}\text{-contradictory} & \text{iff} \quad \models_{\mathsf{S}} \neg(\varphi \wedge \psi) & \text{and} & \models_{\mathsf{S}} \varphi \vee \psi, \\ \mathsf{S}\text{-contrary} & \text{iff} \quad \models_{\mathsf{S}} \neg(\varphi \wedge \psi) & \text{and} & \not\models_{\mathsf{S}} \varphi \vee \psi, \\ \mathsf{S}\text{-subcontrary} & \text{iff} \quad \not\models_{\mathsf{S}} \neg(\varphi \wedge \psi) & \text{and} & \models_{\mathsf{S}} \varphi \vee \psi, \\ \text{in } \mathsf{S}\text{-subalternation} & \text{iff} \quad \models_{\mathsf{S}} \varphi \rightarrow \psi & \text{and} & \not\models_{\mathsf{S}} \psi \rightarrow \varphi. \end{array}$$

Furthermore, φ and ψ are said to be

$$\begin{array}{lllll} \mathsf{S}\text{-unconnected} & \text{iff} \quad \not\models_{\mathsf{S}} \neg(\varphi \wedge \psi) & \text{and} & \not\models_{\mathsf{S}} \varphi \vee \psi & \text{and} \\ & \quad\quad\; \not\models_{\mathsf{S}} \varphi \rightarrow \psi & \text{and} & \not\models_{\mathsf{S}} \psi \rightarrow \varphi. & \end{array}$$

Note that unconnectedness can be viewed as the *absence* of any Aristotelian relation: two (non-equivalent) formulas φ and ψ are unconnected iff they do not stand in a relation of contradiction, contrariety, subcontrariety or subalternation to each other. Furthermore, note that Definition 1 corresponds exactly with the traditional, more informal approach to the Aristotelian relations. For example, the clause $\models_{\mathsf{S}} \neg(\varphi \wedge \psi)$ says that there are no S-models $\mathbb{M}$ such that $\mathbb{M} \models \varphi \wedge \psi$, which corresponds to the idea that φ and ψ 'cannot be true together'. Similarly, the clause $\not\models \varphi \vee \psi$ corresponds to the idea that φ and ψ 'can be false together'.

It bears emphasizing that the Aristotelian relations hold up to logical equivalence; for example, if $\varphi \equiv_{\mathsf{S}} \varphi'$ and $\psi \equiv_{\mathsf{S}} \psi'$, then φ and ψ are S-contrary iff φ' and ψ' are S-contrary. Consequently, these relations could also be defined over the Lindenbaum-Tarski algebra of S, which is in line with the recent idea that logical geometry can be entirely developed in the context of Boolean algebras, rather than logical systems [12]. However, since this is not the main focus of this paper, we will simply work with formulas instead of their equivalence classes, while pointing out that we are working up to logical equivalence whenever salient.



(a) Classical square of opposition in CPL.

(b) Classical square of opposition in FOL.

(c) Degenerate square of opposition in CPL.

Figure 1: Three examples of Aristotelian diagrams. Full, dashed and dotted lines visualize contradiction, contrariety and subcontrariety, respectively; arrows visualize subalternations.

Definition 2. Let S be a logical system as in Definition 1, and let $\mathcal{F} \subseteq \mathcal{L}_{\mathsf{S}}$ be a finite fragment of formulas. An *Aristotelian diagram for* $(\mathcal{F}, \mathsf{S})$ is a directed vertex- and edge-labeled graph: its vertices are labeled by the formulas of $\mathcal{F}$, while its edges are labeled by the Aristotelian relations holding between those elements (relative to S). Specifically, if the vertices v, w are labeled by resp. $\varphi, \psi \in \mathcal{L}_{\mathsf{S}}$, which stand in the Aristotelian relation R (relative to S), then the edge from v to w is labeled by R.

The labeling of edges by means of Aristotelian relations is usually in accordance with the convention described in the caption of Figure 1, i.e., contrariety edges are visualized as dashed lines, etc. This figure shows three examples of Aristotelian diagrams: classical squares of opposition in classical propositional logic (CPL) and in first-order logic (FOL), and a so-called degenerate square of opposition in CPL.[3]

Definition 3. Consider Aristotelian diagrams for $(\mathcal{F}_1, \mathsf{S}_1)$ and $(\mathcal{F}_2, \mathsf{S}_2)$. An *Aristotelian isomorphism* $f\colon (\mathcal{F}_1, \mathsf{S}_1) \to (\mathcal{F}_2, \mathsf{S}_2)$ is a bijection $f\colon \mathcal{F}_1 \to \mathcal{F}_2$ such that for all $\varphi, \psi \in \mathcal{F}_1$ and for all Aristotelian relations R, it holds that $R_{\mathsf{S}_1}(\varphi, \psi)$ iff $R_{\mathsf{S}_2}(f(\varphi), f(\psi))$.

To illustrate Definition 3, note that the two classical squares of opposition in Figure 1(a) and (b) are isomorphic to each other; a concrete Aristotelian isomorphism f from the former to the latter square is given by $f(p \wedge q) := \forall x Px$, $f(\neg p \wedge \neg q) := \forall x \neg Px$, $f(p \vee q) := \exists x Px$ and $f(\neg p \vee \neg q) := \exists x \neg Px$. By contrast, the degenerate square in Figure 1(c) is not isomorphic to either of the classical squares in Figure 1(a–b).

With these three notions defined, we are now in a position to describe the phenomenon of logic-sensitivity in Aristotelian diagrams. Given a fragment $\mathcal{F}$ of formulas coming from some logical language $\mathcal{L}$, it can happen that the Aristotelian

[3]The latter square is said to be 'degenerate' because in comparison to the classical squares, it lacks a contrariety, a subcontrariety and two subalternations, and thus largely consists of pairwise unconnected formulas (except for the contradictory pairs on its diagonals).

relations holding between those formulas, and thus ultimately the Aristotelian diagram for $\mathcal{F}$, strongly depend on the logic with respect to which these formulas are interpreted. More precisely, given a fragment $\mathcal{F} \subseteq \mathcal{L}$ and different logical systems S_1 and S_2 (which share the same language $\mathcal{L}$), we have a case of *logic-sensitivity* iff the identity function $id_{\mathcal{F}}$ is not an Aristotelian isomorphism between $(\mathcal{F}, \mathsf{S}_1)$ and $(\mathcal{F}, \mathsf{S}_2)$. We can distinguish between two types of cases of logic-sensitivity [13]: in the 'blatant' cases, there simply does not exist an Aristotelian isomorphism between $(\mathcal{F}, \mathsf{S}_1)$ and $(\mathcal{F}, \mathsf{S}_2)$ whatsoever, while in the more 'subtle' cases, such an Aristotelian isomorphism does exist, but $id_{\mathcal{F}}$ itself is not an Aristotelian isomorphism between $(\mathcal{F}, \mathsf{S}_1)$ and $(\mathcal{F}, \mathsf{S}_2)$.

For a concrete example, take the fragment $\mathcal{F}_m := \{\Box p, \Box\neg p, \Diamond p, \Diamond\neg p\}$, coming from the basic modal language $\mathcal{L}_\Box$, and consider the normal modal logics KD and K. It is easy to show that $\Box p$ and $\Box\neg p$ are KD-contrary, but K-unconnected. In general, we find that the Aristotelian diagram for $(\mathcal{F}_m, \mathsf{KD})$ is a classical square of opposition, as shown in Figure 2(a), while that for for $(\mathcal{F}_m, \mathsf{K})$ is a degenerate square of opposition, as shown in Figure 2(b).[4] These two diagrams are not Aristotelian isomorphic to each other, thus yielding a blatant case of logic-sensitivity.

However, we can push this even further. Consider the same fragment $\mathcal{F}_m$, but now relative to the modal logic KF. This logic is obtained by adding the axiom $\Diamond p \to \Box p$ to K, and is sound and complete with respect to the class of Kripke models $\langle W, R, V\rangle$ whose accessibility relation R is a partial function, i.e., for all worlds w, v, u, if wRv and wRu, then $v = u$.[5] It is easy to show that the Aristotelian diagram for $(\mathcal{F}_m, \mathsf{KF})$ is, once again, a classical square of opposition, as shown in Figure 2(c). This diagram is not isomorphic to the degenerate square for $(\mathcal{F}_m, \mathsf{K})$ in Figure 2(b), thus yielding another blatant case of logic-sensitivity. However, this diagram *is* isomorphic to the classical square for $(\mathcal{F}_m, \mathsf{KD})$ in Figure 2(a).[6] Nevertheless, the identity function $id_{\mathcal{F}_m}$ is not an Aristotelian isomorphism between $(\mathcal{F}_m, \mathsf{KD})$ and $(\mathcal{F}_m, \mathsf{KF})$,[7] thus yielding a more subtle case of logic-sensitivity.

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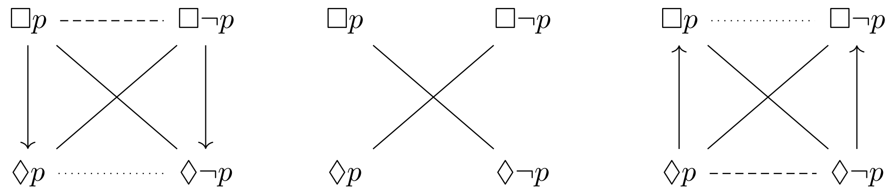
[4] The well-known problem of *existential import* in syllogistics can also be formulated in this way. Consider the fragment $\mathcal{F}_{cat}$ consisting of the four categorical statements $\forall x(Sx \to Px)$, $\forall x(Sx \to \neg Px)$, $\exists x(Sx \wedge Px)$ and $\exists x(Sx \wedge \neg Px)$, as well as the logics FOL (which lacks existential import, i.e., it is allowed that $I(S) = \emptyset$) and SYL (which has existential import, i.e., it is required that $I(S) \neq \emptyset$). Then the Aristotelian diagram for $(\mathcal{F}_{cat}, \mathsf{SYL})$ is a classical square, while the diagram for $(\mathcal{F}_{cat}, \mathsf{FOL})$ is a degenerate square [15]. This way of understanding existential import is quite common in (the historiography of) logic [1, 21, 29, 35].

[5] Philosophically speaking, KF is not very useful when the modalities are interpreted *statically* (e.g., as alethic, deontic or doxastic operators), but it comes about very naturally when describing the behavior of *dynamic* modalities [7, 17].

[6] A concrete Aristotelian isomorphism $f\colon (\mathcal{F}_m, \mathsf{KF}) \to (\mathcal{F}_m, \mathsf{KD})$ is defined by $f(\Box p) := \Diamond p$, $f(\Box\neg p) := \Diamond\neg p$, $f(\Diamond p) := \Box p$ and $f(\Diamond\neg p) := \Box\neg p$. Note that clearly $f \neq id_{\mathcal{F}_m}$.

[7] For example, upon comparing Figure 2(a) and (c), we observe that $\Box p$ and $\Box\neg p$ are KD-contrary, but KF-subcontrary; similarly, there is a KD-subalternation from $\Box p$ to $\Diamond p$, which flips into a KF-subalternation from $\Diamond p$ to $\Box p$.



(a) Classical square for $(\mathcal{F}_m, \mathsf{KD})$.

(b) Degenerate square for $(\mathcal{F}_m, \mathsf{K})$.

(c) Classical square for $(\mathcal{F}_m, \mathsf{KF})$.

Figure 2: Aristotelian diagrams for $\mathcal{F}_m$, relative to three modal logics.

3 Bitstring Semantics

The technique of bitstring semantics was initially developed within the specific context of logical geometry [38], but in recent years it has started to find applications across the disciplines of logic [9, 15, 16, 27], philosophy [10, 11, 23] and linguistics [34, 39]. In general, bitstring semantics allows us to systematically compute combinatorial representations of a given number of propositions, thus providing a concrete grip on their logical behavior. We will now summarize the main features of bitstring semantics; more details can be found in [15].

Definition 4. Consider a logical system S and fragment $\mathcal{F}$ as in Definition 2. The *partition induced by* $\mathcal{F}$ *in* S, denoted $\Pi_{\mathsf{S}}(\mathcal{F})$, is defined as follows:

$$\Pi_{\mathsf{S}}(\mathcal{F}) := \{ \bigwedge_{\varphi \in \mathcal{F}} \pm\varphi \mid \bigwedge_{\varphi \in \mathcal{F}} \pm\varphi \text{ is } \mathsf{S}\text{-consistent}\},$$

where $+\varphi = \varphi$ and $-\varphi = \neg\varphi$. The elements of $\Pi_{\mathsf{S}}(\mathcal{F})$ are called *anchor formulas*. The *Boolean closure of* $\mathcal{F}$ *in* S, denoted $\mathbb{B}_{\mathsf{S}}(\mathcal{F})$, is defined to be the smallest set $C \subseteq \mathcal{L}_{\mathsf{S}}$ such that (i) $\mathcal{F} \subseteq C$ and (ii) C is closed under the Boolean operations (up to logical equivalence), i.e., for all $\varphi, \psi \in C$, there exist $\alpha, \beta \in C$ such that $\alpha \equiv_{\mathsf{S}} \varphi \wedge \psi$ and $\beta \equiv_{\mathsf{S}} \neg\varphi$.

The set $\Pi_{\mathsf{S}}(\mathcal{F})$ is called a 'partition', because the anchor formulas are (i) jointly exhaustive, that is, $\models_{\mathsf{S}} \bigvee \Pi_{\mathsf{S}}(\mathcal{F})$, and (ii) mutually exclusive, that is, $\models_{\mathsf{S}} \neg(\alpha \wedge \beta)$ for distinct $\alpha, \beta \in \Pi_{\mathsf{S}}(\mathcal{F})$. The number of anchor formulas, $|\Pi_{\mathsf{S}}(\mathcal{F})|$, is called the 'Boolean complexity' of $\mathcal{F}$ (relative to S). Although anchor formulas are defined to be conjunctions of (negated) formulas from $\mathcal{F}$, in concrete applications they can often be simplified to much shorter, S-equivalent formulas. For a simple example, note that if we start from the fragment $\mathcal{F}^* := \{p \wedge q, p\}$, we can compute that $\Pi_{\mathsf{CPL}}(\mathcal{F}^*) = \{\alpha_1 := p \wedge q, \alpha_2 := p \wedge \neg q, \alpha_3 := \neg p\}$.[8]

[8] The full calculation looks as follows: $\alpha_1 := (p \wedge q) \wedge p \equiv_{\mathsf{CPL}} p \wedge q$, $\alpha_2 := \neg(p \wedge q) \wedge p \equiv_{\mathsf{CPL}} p \wedge \neg q$ and $\alpha_3 := \neg(p \wedge q) \wedge \neg p \equiv_{\mathsf{CPL}} \neg p$. The fourth conjunction that we should consider, i.e., $(p \wedge q) \wedge \neg p$, is CPL-inconsistent, and thus does not get included in $\Pi_{\mathsf{CPL}}(\mathcal{F}^*)$.

It can be shown that every formula in $\mathbb{B}_{\mathsf{S}}(\mathcal{F})$ is logically equivalent to a disjunction of anchor formulas: for every $\varphi \in \mathbb{B}_{\mathsf{S}}(\mathcal{F})$ we have

$$\varphi \equiv_{\mathsf{S}} \bigvee \{\alpha \in \Pi_{\mathsf{S}}(\mathcal{F}) \mid \models_{\mathsf{S}} \alpha \to \varphi\}.$$

The bitstring semantics $\beta_{\mathsf{S}}^{\mathcal{F}} \colon \mathbb{B}_{\mathsf{S}}(\mathcal{F}) \to \{0,1\}^{|\Pi_{\mathsf{S}}(\mathcal{F})|}$ maps every formula $\varphi \in \mathbb{B}_{\mathsf{S}}(\mathcal{F})$ onto its bitstring representation $\beta_{\mathsf{S}}^{\mathcal{F}}(\varphi)$, which is a sequence of $|\Pi_{\mathsf{S}}(\mathcal{F})|$ bits that keeps track of which anchor formulas enter into this disjunction. In particular, the bitstring $\beta_{\mathsf{S}}^{\mathcal{F}}(\varphi)$ will have the value 1 in its i^{th} bit position iff $\models_{\mathsf{S}} \alpha_i \to \varphi$. Continuing our example about $\mathcal{F}^*$ from above, note that $p \equiv_{\mathsf{CPL}} (p \wedge q) \vee (p \wedge \neg q) = \alpha_1 \vee \alpha_2$, and thus we represent p by the bitstring 110; formally: $\beta_{\mathsf{CPL}}^{\mathcal{F}^*}(p) = 110$.

The Aristotelian relations and unconnectedness can straightforwardly be generalized to bitstrings. In particular, meets and joins of bitstrings are calculated in a bitwise fashion, and two bitstrings $b_1, b_2 \in \{0,1\}^n$ are said to be

n-contradictory	iff	$b_1 \wedge b_2 = 0 \cdots 0$	and	$b_1 \vee b_2 = 1 \cdots 1$,
n-contrary	iff	$b_1 \wedge b_2 = 0 \cdots 0$	and	$b_1 \vee b_2 \neq 1 \cdots 1$,
n-subcontrary	iff	$b_1 \wedge b_2 \neq 0 \cdots 0$	and	$b_1 \vee b_2 = 1 \cdots 1$,
in n-subalternation	iff	$b_1 \wedge b_2 = b_1$	and	$b_1 \wedge b_2 \neq b_2$.

Furthermore, b_1 and b_2 are said to be

n-unconnected	iff	$b_1 \wedge b_2 \neq 0 \cdots 0$	and	$b_1 \vee b_2 \neq 1 \cdots 1$	and
		$b_1 \wedge b_2 \neq b_1$	and	$b_1 \wedge b_2 \neq b_2$.	

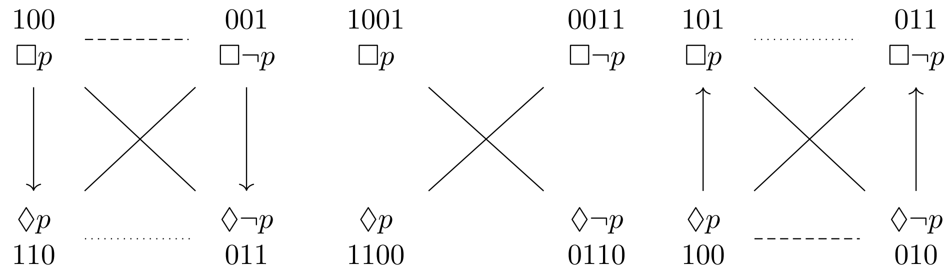
It can be shown that $\beta_{\mathsf{S}}^{\mathcal{F}}$ is a Boolean isomorphism between the Boolean algebras $\mathbb{B}_{\mathsf{S}}(\mathcal{F})$ and $\{0,1\}^{|\Pi_{\mathsf{S}}(\mathcal{F})|}$, and therefore also preserves all the Aristotelian relations [19]. Returning one more time to our example about $\mathcal{F}^*$, we observe that the formulas $p \wedge q$ and $\neg p$ are CPL-contrary, while their bitstring representations $\beta_{\mathsf{CPL}}^{\mathcal{F}^*}(p \wedge q) = 100$ and $\beta_{\mathsf{CPL}}^{\mathcal{F}^*}(\neg p) = 001$ are 3-contrary.

For a more substantial example, we again turn to the modal fragment $\mathcal{F}_m = \{\Box p, \Box\neg p, \Diamond p, \Diamond\neg p\}$ that was introduced in Section 2. One can compute that

- $\Pi_{\mathsf{K}}(\mathcal{F}_m) = \{\Box p \wedge \Diamond p,\ \Diamond p \wedge \Diamond\neg p,\ \Box\neg p \wedge \Diamond\neg p,\ \Box p \wedge \Box\neg p\}$,
- $\Pi_{\mathsf{KD}}(\mathcal{F}_m) = \{\Box p,\ \Diamond p \wedge \Diamond\neg p,\ \Box\neg p\}$,
- $\Pi_{\mathsf{KF}}(\mathcal{F}_m) = \{\Diamond p,\ \Diamond\neg p,\ \Box p \wedge \Box\neg p\}$.

If we compare $\Pi_{\mathsf{K}}(\mathcal{F}_m)$ with $\Pi_{\mathsf{KD}}(\mathcal{F}_m)$, we notice that the first and third anchor formulas of $\Pi_{\mathsf{K}}(\mathcal{F}_m)$ have been simplified (viz., from $\Box p \wedge \Diamond p$ to $\Box p$ and from $\Box\neg p \wedge \Diamond\neg p$ to $\Box\neg p$, respectively), while its fourth anchor formula (viz., $\Box p \wedge \Box\neg p$) is KD-inconsistent, and is thus absent from $\Pi_{\mathsf{KD}}(\mathcal{F}_m)$ altogether.[9] In terms of bitstring representations, this means that the KD-bitstrings can be viewed as the result

[9] Partitions are unordered sets, so strictly speaking, it does not make sense to talk about the 'first', 'second', etc. anchor formulas of a partition. This can easily be solved by turning the unordered set into an ordered tuple. However, for ease of formulation, we will simply refer to the anchor formulas in the order in which they are listed when we write down the partition. For example, in light of how $\Pi_{\mathsf{K}}(\mathcal{F}_m)$ was written down in the main text, we will unequivocally call $\Box p \wedge \Diamond p$ the *first* anchor formula of this partition, $\Diamond p \wedge \Diamond\neg p$ its *second* anchor formula, and so on.



(a) Classical square for $(\mathcal{F}_m, \mathsf{KD})$.

(b) Degenerate square for $(\mathcal{F}_m, \mathsf{K})$.

(c) Classical square for $(\mathcal{F}_m, \mathsf{KF})$.

Figure 3: Aristotelian diagrams for $\mathcal{F}_m$, relative to three modal logics, including bitstring representations.

of systematically deleting the fourth bit from the K-bitstrings. For example, since $\Box p \equiv_{\mathsf{K}} (\Box p \wedge \Diamond p) \vee (\Box p \wedge \Box\neg p)$ and $\Box p \equiv_{\mathsf{KD}} \Box p$, we have $\beta_{\mathsf{K}}^{\mathcal{F}_m}(\Box p) = 1001$ and $\beta_{\mathsf{KD}}^{\mathcal{F}_m}(\Box p) = 100$; in general, compare the bitstring representations of $\mathcal{F}_m$ relative to K and KD in Figure 3(a–b). In a completely analogous fashion, the second anchor formula of $\Pi_{\mathsf{K}}(\mathcal{F}_m)$ (viz., $\Diamond p \wedge \Diamond\neg p$) is KF-inconsistent, and hence the KF-bitstrings can be viewed as the result of systematically deleting the second bit from the K-bitstrings; for example, we have $\beta_{\mathsf{K}}^{\mathcal{F}_m}(\Box p) = 1001$ and $\beta_{\mathsf{KF}}^{\mathcal{F}_m}(\Box p) = 101$. Again, in general, compare the bitstring representations of $\mathcal{F}_m$ relative to K and KF in Figure 3(b–c).

These observations match with some well-known results from logical geometry [9, 15], viz., that degenerate and classical squares of opposition have Boolean complexities 4 and 3, respectively (i.e., representing them requires bitstrings of length at least 4 and at least 3, respectively). Recalling the topic of logic-sensitivity from Section 2, we thus observe that the underlying logical system not only has an impact on the type of Aristotelian diagram (viz., degenerate square vs. classical square), but also on Boolean complexity (viz., 4 vs. 3).

4 Purely Combinatorial Bitstrings

Until now (both in this paper and, more generally, in the entire research program of logical geometry), bitstrings have been treated as thoroughly *semantic* entities. Each bitstring is seen as the representation of some formula from some fragment relative to some logical system; for example, $100 = \beta_{\mathsf{CPL}}^{\mathcal{F}^*}(p \wedge q)$ and $001 = \beta_{\mathsf{CPL}}^{\mathcal{F}^*}(\neg p)$. Bitstrings capture key semantic properties of the formulas that they represent, such as their Aristotelian relations; for example, the 3-contrariety between 100 and 001 corresponds to the CPL-contrariety between $p \wedge q$ and $\neg p$. Finally, even bitstring operations like deleting the fourth bit position are semantically motivated; for example, in terms of the fourth anchor formula $\Box p \wedge \Box\neg p \in \Pi_{\mathsf{K}}(\mathcal{F}_m)$ being K-consistent but KD-inconsistent — formally: $\beta_{\mathsf{K}}^{\mathcal{F}_m}(\Box p \wedge \Box\neg p) = 0001$ but

$\beta_{\mathsf{KD}}^{\mathcal{F}_m}(\Box p \wedge \Box\neg p) = 000$.

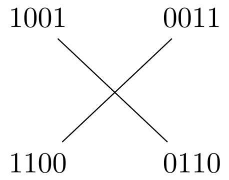
In this section, we will temporarily take a step back, and treat bitstrings as purely *combinatorial* entities. We will thus no longer view them as representing some formula from some logical fragment relative to some logical system. From this purely combinatorial perspective, we can still study Aristotelian relations between bitstrings; for example, it still holds that 100 and 001 are 3-contrary to each other; the only difference is that this 3-contrariety is no longer viewed as corresponding to some contrariety between formulas that are represented by these bitstrings. Similarly, we can still study bitstring operations like $d_4\colon \{0,1\}^4 \to \{0,1\}^3$, i.e., deleting the fourth bit position, although this is no longer viewed as corresponding to an anchor formula being consistent in one logic but inconsistent in another one.

Let us now have another look at some of our earlier observations, from this purely combinatorial perspective. Consider the three Aristotelian diagrams in Figure 3, but focus exclusively on the bitstrings, while ignoring the formulas that they (used to) represent. We observe that the classical squares in Figure 3(a) and (c) have bitstrings of length 3, which are obtained by systematically deleting the *fourth*, resp. the *second* bit position from the bitstrings of length 4 in the degenerate square in Figure 3(b). From our current, purely combinatorial perspective on bitstrings, however, all bit positions are equally important, since we are only interested in bitstrings up to a permutation of their bit positions. Since there is nothing special about the second or the fourth bit position, the following question thus naturally arises: what happens if we delete the first or third bit position from the length-4 bitstrings in the degenerate square in Figure 3(b)?

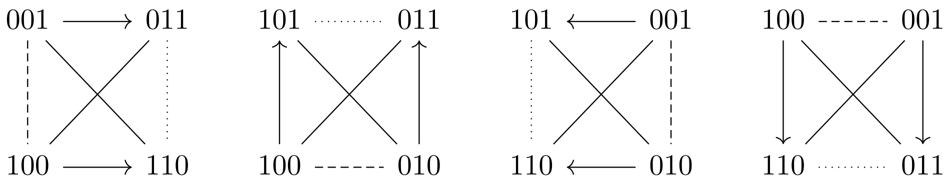
The situation is completely described in Figure 4. Part (a) of this figure shows the same degenerate square with bitstrings of length 4 as in Figure 3(b). Each deletion of a bit position yields bitstrings of length 3, which make up a classical square, cf. Figure 4(b–e). In particular, every (deletion of a) bit position corresponds to a different direction of the subalternation arrows in the resulting classical square:

- Deleting the *first* bit position yields a classical square with the subalternations going *from left to right*; cf. Figure 4(b).
- Deleting the *second* bit position yields a classical square with the subalternations going *upwards*; cf. Figure 4(c). — Also recall Figure 3(c).
- Deleting the *third* bit position yields a classical square with the subalternations going *from right to left*; cf. Figure 4(d).
- Deleting the *fourth* bit position yields a classical square with the subalternations going *downwards*; cf. Figure 4(e). — Also recall Figure 3(a).

We thus observe a systematic correspondence between some logical/algebraic features of Aristotelian diagrams (viz., the bit position that gets deleted) and some



(a) Degenerate square.



(b) Class. square after deleting bit position 1.

(c) Class. square after deleting bit position 2.

(d) Class. square after deleting bit position 3.

(e) Class. square after deleting bit position 4.

Figure 4: Five Aristotelian diagrams with purely combinatorial bitstrings.

of their visual/geometric properties (viz., the direction of the subalternation arrows). This observation fits nicely in a broader pattern of *logical-geometrical correspondences* in Aristotelian diagrams, which are studied extensively in logical geometry [14, 36], and which ultimately even explain the very name of this research program. Another point of interest concerns the link with the *informativity of logical relations* [37]. For example, consider the pair of 4-unconnected bitstrings $(1001, 0011)$ at the upper edge of the degenerate square in Figure 4(a). Deleting the first, second, third or fourth bit position yields resp. the 3-left-implication $(001, 011)$,[10] the 3-subcontrariety $(101, 011)$, the 3-right-implication $(101, 001)$ and the 3-contrariety $(100, 001)$ at the upper edges of the four classical squares in Figure 4(b–e). This corresponds with the facts that (i) left-implication, subcontrariety, right-implication and contrariety are precisely the four relations that are one level above unconnectedness in the informativity ordering of logical relations [37, Figure 7], and that (ii) deleting one bit position can trigger an increase of at most one level in the informativity of the resulting relations.[11]

The way we have described the correspondence between deleting bit positions and the direction of the resulting subalternations is, in a sense, conventional. After all, if we permute the four bit positions in the bitstrings that appear in the degener-

[10] 'Left-implication' [37] corresponds to the relation that is called 'subalternation' in this paper, while 'right-implication' corresponds to the converse of this relation. Furthermore, our relation of unconnectedness corresponds to the intersection of what are called 'non-contradiction' and 'non-implication' in [37]; also cf. [38, Figure 2].

[11] While bitstring semantics and the informativity ordering of logical relations are both well-studied in logical geometry [15, 37], their interplay has hitherto not yet been systematically explored. The observations made here, regarding bit deletions and increasing informativity, constitute a modest first step in that direction.

ate square in Figure 4(a), then deleting, say, the first bit position might no longer correspond to the subalternations going from left to right, as in Figure 4(b), but rather to them going upwards, downwards or from right to left. Nevertheless, as soon as a specific ordering on the bit positions is fixed, it is always the case that we find a one-to-one correspondence between deleting the four bit positions and the four directions of the resulting subalternations. There is another, equally harmless, element of conventionality in the way we have described the correspondence thus far, viz., in terms of the direction of the resulting subalternations. We could equally well have told this story in terms of the other resulting Aristotelian relations. For example, we could also say that deleting the first, second, third or fourth bit position yields a classical square with a contrariety at resp. its left, lower, right and upper edge, as can be seen in Figure 4(b–e).

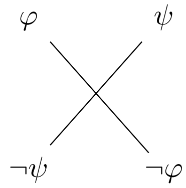
5 An Open Problem and Some Attempted Solutions

In the previous section we described a natural correspondence between deleting bit positions and the direction of the resulting subalternations, but we did so from a purely *combinatorial* perspective on bitstrings. However, as we have emphasized throughout this paper, bitstrings are standardly viewed as thoroughly *semantic* entities, which immediately suggests the question: to what extent can this combinatorial story be re-told from the semantic perspective on bitstrings? We formulate this as the following open problem:

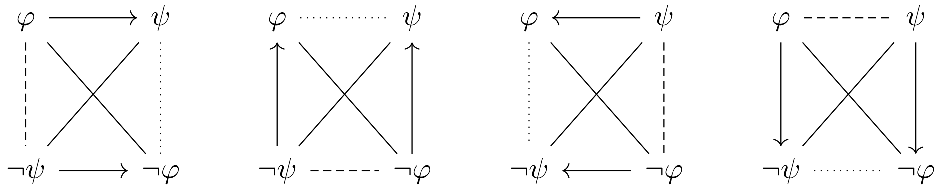
Open Problem 1. Find a language $\mathcal{L}$, a fragment $\mathcal{F} = \{\varphi, \psi, \neg\varphi, \neg\psi\} \subseteq \mathcal{L}$, and five logical systems S_0, S_1, S_2, S_3, S_4 for that same language $\mathcal{L}$, such that:

- The Aristotelian diagram for $(\mathcal{F}, \mathsf{S}_0)$ is a degenerate square; cf. Figure 5(a).
- The Aristotelian diagram for $(\mathcal{F}, \mathsf{S}_1)$ is a classical square, with a subalternation arrow from φ to ψ, i.e., going *from left to right*; cf. Figure 5(b).
- The Aristotelian diagram for $(\mathcal{F}, \mathsf{S}_2)$ is a classical square, with a subalternation arrow from $\neg\varphi$ to ψ, i.e., going *upwards*; cf. Figure 5(c).
- The Aristotelian diagram for $(\mathcal{F}, \mathsf{S}_3)$ is a classical square, with a subalternation arrow from $\neg\varphi$ to $\neg\psi$, i.e., going *from right to left*; cf. Figure 5(d).
- The Aristotelian diagram for $(\mathcal{F}, \mathsf{S}_4)$ is a classical square, with a subalternation arrow from φ to $\neg\psi$, i.e., going *downwards*; cf. Figure 5(e).

Ideally, we would also like the logical systems S_0, S_1, S_2, S_3 and S_4 to be *independently motivated*, i.e., they should already have been studied because of their mathematical interest (e.g., K as the minimal normal modal logic) and/or philosophical applications (e.g., KD as a system of deontic or doxastic logic). We are thus less interested in *ad hoc*, artificially constructed logical systems, whose sole purpose would be to provide a solution to Open Problem 1.



(a) Degenerate square for $(\mathcal{F}, \mathsf{S}_0)$.



(b) Class. square for $(\mathcal{F}, \mathsf{S}_1)$.

(c) Class. square for $(\mathcal{F}, \mathsf{S}_2)$.

(d) Class. square for $(\mathcal{F}, \mathsf{S}_3)$.

(e) Class. square for $(\mathcal{F}, \mathsf{S}_4)$.

Figure 5: Five Aristotelian diagrams for our open problem.

Note that the degenerate square for $(\mathcal{F}, \mathsf{S}_0)$ is not Aristotelian isomorphic to any of the classical squares for $(\mathcal{F}, \mathsf{S}_n)$, for $1 \leq n \leq 4$. These constitute four blatant cases of logic-sensitivity. Furthermore, note that for any $1 \leq m < n \leq 4$, the diagrams for $(\mathcal{F}, \mathsf{S}_m)$ and $(\mathcal{F}, \mathsf{S}_n)$ are both classical squares of opposition (and are thus Aristotelian isomorphic to each other), but the identity function $id_{\mathcal{F}}$ is not an Aristotelian isomorphism between $(\mathcal{F}, \mathsf{S}_m)$ and $(\mathcal{F}, \mathsf{S}_n)$. This yields six additional, more subtle cases of logic-sensitivity.

Furthermore, note that if we manage to solve Open Problem 1, we will indeed have succeeded in 're-telling the story' from the previous section, but now in terms of formulas and logical systems (i.e., in terms of semantic bitstrings), rather than in terms of purely combinatorial bitstrings. After all, for such a fragment $\mathcal{F} \subseteq \mathcal{L}$ and logical systems $\mathsf{S}_0, \mathsf{S}_1, \mathsf{S}_2, \mathsf{S}_3, \mathsf{S}_4$, we have:

- $\Pi_{\mathsf{S}_0}(\mathcal{F}) = \{\varphi \wedge \neg\psi,\ \ \neg\varphi \wedge \neg\psi,\ \ \neg\varphi \wedge \psi,\ \ \varphi \wedge \psi\}$

  Since $\varphi \equiv_{\mathsf{S}_0} (\varphi \wedge \neg\psi) \vee (\varphi \wedge \psi)$, we have $\beta^{\mathcal{F}}_{\mathsf{S}_0}(\varphi) = 1001$. Similarly, we find that $\beta^{\mathcal{F}}_{\mathsf{S}_0}(\psi) = 0011$, $\beta^{\mathcal{F}}_{\mathsf{S}_0}(\neg\psi) = 1100$ and $\beta^{\mathcal{F}}_{\mathsf{S}_0}(\neg\varphi) = 0110$. Compare the formulas in Figure 5(a) with the bitstrings in Figure 4(a).

- $\Pi_{\mathsf{S}_1}(\mathcal{F}) = \{\neg\psi,\ \ \neg\varphi \wedge \psi,\ \ \varphi\}$

  If we compare $\Pi_{\mathsf{S}_1}(\mathcal{F})$ with $\Pi_{\mathsf{S}_0}(\mathcal{F})$, we find that the second and fourth anchor formulas have been simplified (viz., from $\neg\varphi \wedge \neg\psi$ to $\neg\psi$ and from $\varphi \wedge \psi$ to φ, respectively), while its first anchor formula (viz., $\varphi \wedge \neg\psi$) is S_1-inconsistent, and is thus absent from $\Pi_{\mathsf{S}_1}(\mathcal{F})$ altogether.

An easy computation yields $\beta^{\mathcal{F}}_{\mathsf{S}_1}(\varphi) = 001$, $\beta^{\mathcal{F}}_{\mathsf{S}_1}(\psi) = 011$, $\beta^{\mathcal{F}}_{\mathsf{S}_1}(\neg\psi) = 100$ and $\beta^{\mathcal{F}}_{\mathsf{S}_1}(\neg\varphi) = 110$. Compare Figure 5(b) with Figure 4(b).

- $\Pi_{\mathsf{S}_2}(\mathcal{F}) = \{\neg\psi,\ \neg\varphi,\ \varphi \wedge \psi\}$

  If we compare $\Pi_{\mathsf{S}_2}(\mathcal{F})$ with $\Pi_{\mathsf{S}_0}(\mathcal{F})$, we find that the first and third anchor formulas have been simplified (viz., from $\varphi \wedge \neg\psi$ to $\neg\psi$ and from $\neg\varphi \wedge \psi$ to $\neg\varphi$, respectively), while its second anchor formula (viz., $\neg\varphi \wedge \neg\psi$) is S_2-inconsistent, and is thus absent from $\Pi_{\mathsf{S}_2}(\mathcal{F})$ altogether.

  An easy computation yields $\beta^{\mathcal{F}}_{\mathsf{S}_2}(\varphi) = 101$, $\beta^{\mathcal{F}}_{\mathsf{S}_2}(\psi) = 011$, $\beta^{\mathcal{F}}_{\mathsf{S}_2}(\neg\psi) = 100$ and $\beta^{\mathcal{F}}_{\mathsf{S}_2}(\neg\varphi) = 010$. Compare Figure 5(c) with Figure 4(c).

- $\Pi_{\mathsf{S}_3}(\mathcal{F}) = \{\varphi \wedge \neg\psi,\ \neg\varphi,\ \psi\}$

  If we compare $\Pi_{\mathsf{S}_3}(\mathcal{F})$ with $\Pi_{\mathsf{S}_0}(\mathcal{F})$, we find that the second and fourth anchor formulas have been simplified (viz., from $\neg\varphi \wedge \neg\psi$ to $\neg\varphi$ and from $\varphi \wedge \psi$ to ψ, respectively), while its third anchor formula (viz., $\neg\varphi \wedge \psi$) is S_3-inconsistent, and is thus absent from $\Pi_{\mathsf{S}_3}(\mathcal{F})$ altogether.

  An easy computation yields $\beta^{\mathcal{F}}_{\mathsf{S}_3}(\varphi) = 101$, $\beta^{\mathcal{F}}_{\mathsf{S}_3}(\psi) = 001$, $\beta^{\mathcal{F}}_{\mathsf{S}_3}(\neg\psi) = 110$ and $\beta^{\mathcal{F}}_{\mathsf{S}_3}(\neg\varphi) = 010$. Compare Figure 5(d) with Figure 4(d).

- $\Pi_{\mathsf{S}_4}(\mathcal{F}) = \{\varphi,\ \neg\varphi \wedge \neg\psi,\ \psi\}$

  If we compare $\Pi_{\mathsf{S}_4}(\mathcal{F})$ with $\Pi_{\mathsf{S}_0}(\mathcal{F})$, we find that the first and third anchor formulas have been simplified (viz., from $\varphi \wedge \neg\psi$ to φ and from $\neg\varphi \wedge \psi$ to ψ, respectively), while its fourth anchor formula (viz., $\varphi \wedge \psi$) is S_4-inconsistent, and is thus absent from $\Pi_{\mathsf{S}_4}(\mathcal{F})$ altogether.

  An easy computation yields $\beta^{\mathcal{F}}_{\mathsf{S}_4}(\varphi) = 100$, $\beta^{\mathcal{F}}_{\mathsf{S}_4}(\psi) = 001$, $\beta^{\mathcal{F}}_{\mathsf{S}_4}(\neg\psi) = 110$ and $\beta^{\mathcal{F}}_{\mathsf{S}_4}(\neg\varphi) = 011$. Compare Figure 5(e) with Figure 4(e).

We now turn to the fundamental question: *can Open Problem 1 be solved*? The initial prospects look quite good. As a first attempt, note that the concrete example that we discussed in Sections 2 and 3 constitutes a very natural, albeit partial solution of Open Problem 1: we can take $\mathcal{L}$ to be the basic modal language $\mathcal{L}_\Box$, $\mathcal{F}$ to be the fragment $\mathcal{F}_m = \{\Box p, \Box\neg p, \Diamond p, \Diamond\neg p\}$ and S_0, S_2 and S_4 to be the normal modal logics K, KF and KD, respectively. The observations made in Sections 2 and 3 (as summarized in Figure 3) show that this language, fragment and logical systems indeed have the properties that they are required to have. However, this only constitutes a *partial* solution to Open Problem 1, because we did not specify concrete logical systems for S_1 and S_3.

We can somewhat improve on this situation. For our second attempt to solve Open Problem 1, we continue to work in the basic modal language $\mathcal{L}_\Box$, but now we take $\mathcal{F}$ to be the fragment $\mathcal{F}^*_m := \{\lambda, \mu, \neg\mu, \neg\lambda\}$, where $\lambda := (\Box p \to \Box\Box p) \wedge (\neg\Box p \to \Box\neg\Box p)$ and $\mu := \Diamond\top \wedge (\Box p \to \Box\Box p)$, and we take S_0, S_1, S_2 and S_3 to be the normal modal logics K, KD, $\mathsf{K4}$ and $\mathsf{K5}$, respectively. We can now prove the following:

- The Aristotelian diagram for $(\mathcal{F}, \mathsf{S}_0) = (\mathcal{F}^*_m, \mathsf{K})$ is a degenerate square, and $\Pi_{\mathsf{K}}(\mathcal{F}^*_m) = \{\lambda \wedge \neg\mu,\ \ \neg\lambda \wedge \neg\mu,\ \ \neg\lambda \wedge \mu,\ \ \lambda \wedge \mu\}$, where
  $\lambda \wedge \neg\mu \equiv_{\mathsf{K}} \Box\bot$,
  $\neg\lambda \wedge \neg\mu \equiv_{\mathsf{K}} \Box p \wedge \neg\Box\Box p$,
  $\neg\lambda \wedge \mu \equiv_{\mathsf{K}} (\Box p \to \Box\Box p) \wedge \neg\Box p \wedge \neg\Box\neg\Box p$ and
  $\lambda \wedge \mu \equiv_{\mathsf{K}} \Diamond\top \wedge (\Box p \to \Box\Box p) \wedge (\neg\Box p \to \Box\neg\Box p)$.
- The Aristotelian diagram for $(\mathcal{F}, \mathsf{S}_1) = (\mathcal{F}^*_m, \mathsf{KD})$ is a classical square, with a subalternation from λ to μ, and $\Pi_{\mathsf{KD}}(\mathcal{F}^*_m) = \{\neg\lambda \wedge \neg\mu,\ \ \neg\lambda \wedge \mu,\ \ \lambda \wedge \mu\}$.
  In comparison with $\Pi_{\mathsf{K}}(\mathcal{F}^*_m)$, note that the first anchor formula (viz., $\lambda \wedge \neg\mu \equiv_{\mathsf{K}} \Box\bot$) is KD-inconsistent, and is thus absent from $\Pi_{\mathsf{KD}}(\mathcal{F}^*_m)$.
- The Aristotelian diagram for $(\mathcal{F}, \mathsf{S}_2) = (\mathcal{F}^*_m, \mathsf{K4})$ is a classical square, with a subalternation from $\neg\lambda$ to μ and $\Pi_{\mathsf{K4}}(\mathcal{F}^*_m) = \{\lambda \wedge \neg\mu,\ \ \neg\lambda \wedge \mu,\ \ \lambda \wedge \mu\}$.
  In comparison with $\Pi_{\mathsf{K}}(\mathcal{F}^*_m)$, note that the second anchor formula (viz., $\neg\lambda \wedge \neg\mu \equiv_{\mathsf{K}} \Box p \wedge \neg\Box\Box p$) is K4-inconsistent, and is thus absent from $\Pi_{\mathsf{K4}}(\mathcal{F}^*_m)$.
- The Aristotelian diagram for $(\mathcal{F}, \mathsf{S}_3) = (\mathcal{F}^*_m, \mathsf{K5})$ is a classical square, with a subalternation from $\neg\lambda$ to $\neg\mu$, and $\Pi_{\mathsf{K5}}(\mathcal{F}^*_m) = \{\lambda \wedge \neg\mu,\ \ \neg\lambda \wedge \neg\mu,\ \ \lambda \wedge \mu\}$.
  In comparison with $\Pi_{\mathsf{K}}(\mathcal{F}^*_m)$, note that the third anchor formula (viz., $\neg\lambda \wedge \mu \equiv_{\mathsf{K}} (\Box p \to \Box\Box p) \wedge \neg\Box p \wedge \neg\Box\neg\Box p$) is K5-inconsistent, and is thus absent from $\Pi_{\mathsf{K5}}(\mathcal{F}^*_m)$.

The logics K, KD, K4 and K5 are well-motivated systems of normal modal logic. However, the fragment $\mathcal{F}^*_m$ looks a bit messy, especially in comparison to the elegant fragment $\mathcal{F}_m$ that we dealt with in the first attempted solution. Finally, and most importantly, this second attempt still does not constitute a *complete* solution to Open Problem 1, since we have not specified a concrete logical system for S_4.

We now describe a third attempt to solve Open Problem 1, which will be a bit more involved. First of all, we introduce the language $\mathcal{L}^{\circ\bullet}_{\Box\blacksquare}$, which has, next to the usual Boolean connectives, four unary connectives $\circ$, $\bullet$, $\Box$ and $\blacksquare$. It is thus defined by the following BNF:

$$\varphi \ ::= \ p \mid \varphi \wedge \varphi \mid \neg\varphi \mid \circ\,\varphi \mid \bullet\,\varphi \mid \Box\varphi \mid \blacksquare\varphi$$

The other Boolean connectives and the duals of $\Box$ and $\blacksquare$ are defined as usual; in particular, we have $\Diamond\varphi := \neg\Box\neg\varphi$ and $\blacklozenge\varphi := \neg\blacksquare\neg\varphi$. We will work with the fragment $\mathcal{F}^{\circ\bullet}_{\Box\blacksquare}$, which is defined as follows:

$$\mathcal{F}^{\circ\bullet}_{\Box\blacksquare} := \{\Box\blacksquare p, \Box \circ \blacklozenge p, \Diamond\blacksquare \bullet p, \Diamond \circ \blacklozenge \bullet p\}$$

Finally, S^*_0, S^*_1, S^*_2, S^*_3 and S^*_4 are systems of bimodal logic, which are interpreted on Kripke models $\langle W, R^{\Box}, R^{\blacksquare}, V\rangle$. The modal operators $\Box$ and $\blacksquare$ are interpreted as usual, in terms of the relations $R^{\Box}$ and $R^{\blacksquare}$, respectively. The full semantics of these logics is summarized by the following table (for now, ignore the row for S_5^* at the bottom of the table):

	$\circ$	$\Box$	$\bullet$	$\blacksquare$
S_0^*	$\neg$	K	*id*	*id*
S_1^*	*id*	*id*	$\neg$	KD
S_2^*	$\neg$	KF	*id*	*id*
S_3^*	*id*	*id*	$\neg$	KF
S_4^*	$\neg$	KD	*id*	*id*
S_5^*	*id*	*id*	$\neg$	K

For all $\star \in \{\circ, \Box, \bullet, \blacksquare\}$ and all $\mathsf{S} \in \{\mathsf{S}_0^*, \mathsf{S}_1^*, \mathsf{S}_2^*, \mathsf{S}_3^*, \mathsf{S}_4^*, \mathsf{S}_5^*\}$, the cell $\begin{array}{c|c} & \star \\ \hline \mathsf{S} & id \end{array}$ in the table means that $\star$ is the identity connective in S, i.e., $\star\varphi \equiv_{\mathsf{S}} \varphi$. In case $\star$ is a modal operator, $\Box$ or $\blacksquare$, such an *id*-cell moreover entails that $\Box\varphi \equiv_{\mathsf{S}} \varphi \equiv_{\mathsf{S}} \Diamond\varphi$ or $\blacksquare\varphi \equiv_{\mathsf{S}} \varphi \equiv_{\mathsf{S}} \blacklozenge\varphi$, respectively. Furthermore, for $\star \in \{\circ, \bullet\}$, the cell $\begin{array}{c|c} & \star \\ \hline \mathsf{S} & \neg \end{array}$ means that $\star$ is a classical negation in S, i.e., $\star\varphi \equiv_{\mathsf{S}} \neg\varphi$. Finally, for $\star \in \{\Box, \blacksquare\}$, the cell $\begin{array}{c|c} & \star \\ \hline \mathsf{S} & \mathsf{K} \end{array}$ means that $\star$ is a K-type modal operator in S (and similarly for KD and KF).

For example, the first row of this table states that S_0^* is interpreted on Kripke models $\mathbb{M} = \langle W, R^{\Box}, R^{\blacksquare}, V\rangle$ such that $R^{\Box}$ is a binary relation on W, $R^{\blacksquare}$ is the identity relation on W (i.e., $wR^{\blacksquare}v$ iff $w = v$, for all $w, v \in W$), and the semantics for $\circ$ and $\bullet$ is: $\mathbb{M}, w \models \circ\varphi$ iff $\mathbb{M}, w \not\models \varphi$, while $\mathbb{M}, w \models \bullet\varphi$ iff $\mathbb{M}, w \models \varphi$. Similarly, the second row of the table states that S_1^* is interpreted on Kripke models $\mathbb{M} = \langle W, R^{\Box}, R^{\blacksquare}, V\rangle$ such that $R^{\Box}$ is the identity relation on W, $R^{\blacksquare}$ is a serial relation on W, and the semantics for $\circ$ and $\bullet$ is: $\mathbb{M}, w \models \circ\varphi$ iff $\mathbb{M}, w \models \varphi$, while $\mathbb{M}, w \models \bullet\varphi$ iff $\mathbb{M}, w \not\models \varphi$. The semantics of the remaining logical systems is completely analogous.

Note that in none of the logics considered here, $\circ$ and $\bullet$ are really necessary, in the sense that every formula containing $\circ$ or $\bullet$ can be rewritten as an equivalent formula that does not contain $\circ$ or $\bullet$. However, this equivalent formula will look different in the different logical systems. For example, $\circ p \equiv_{\mathsf{S}} \neg p$ and $\bullet p \equiv_{\mathsf{S}} p$ for $\mathsf{S} \in \{\mathsf{S}_0^*, \mathsf{S}_2^*, \mathsf{S}_4^*\}$, while $\circ p \equiv_{\mathsf{S}} p$ and $\bullet p \equiv_{\mathsf{S}} \neg p$ for $\mathsf{S} \in \{\mathsf{S}_1^*, \mathsf{S}_3^*, \mathsf{S}_5^*\}$.

We can now prove the following:

- In S_0^*, the fragment $\mathcal{F}^{\circ\bullet}_{\Box\blacksquare}$ simplifies to $\{\Box p, \Box\neg p, \Diamond p, \Diamond\neg p\}$, and since $\Box$ is a K-modality in S_0^*, the Aristotelian diagram for $(\mathcal{F}^{\circ\bullet}_{\Box\blacksquare}, \mathsf{S}_0^*)$ is a degenerate square. The partition $\Pi_{\mathsf{S}_0{}^*}(\mathcal{F}^{\circ\bullet}_{\Box\blacksquare})$ contains the following anchor formulas:
  - $\Box\blacksquare p \wedge \Diamond\blacksquare\bullet p \equiv_{\mathsf{S}_0^*} \Box p \wedge \Diamond p$
  - $\Diamond\blacksquare\bullet p \wedge \Diamond\circ\blacklozenge\bullet p \equiv_{\mathsf{S}_0^*} \Diamond p \wedge \Diamond\neg p$

– $\Box \circ \blacklozenge p \wedge \Diamond \circ \blacklozenge \bullet p \equiv_{\mathsf{S}_0^*} \Box\neg p \wedge \Diamond\neg p$
– $\Box\blacksquare p \wedge \Box \circ \blacklozenge p \equiv_{\mathsf{S}_0^*} \Box p \wedge \Box\neg p$

- In S_1^*, the fragment $\mathcal{F}^{\circ\bullet}_{\Box\blacksquare}$ simplifies to $\{\blacksquare p, \blacklozenge p, \blacksquare\neg p, \blacklozenge\neg p\}$, and since $\blacksquare$ is a KD-modality in S_1^*, the Aristotelian diagram for $(\mathcal{F}^{\circ\bullet}_{\Box\blacksquare}, \mathsf{S}_1^*)$ is a classical square, with a subalternation from $\blacksquare p$ to $\blacklozenge p$, i.e., (before simplification) from $\Box\blacksquare p$ to $\Box \circ \blacklozenge p$. The partition $\Pi_{\mathsf{S}_1^*}(\mathcal{F}^{\circ\bullet}_{\Box\blacksquare})$ contains the following anchor formulas:

  – $\Diamond\blacksquare \bullet p \wedge \Diamond \circ \blacklozenge \bullet p \equiv_{\mathsf{S}_1^*} \blacksquare\neg p \wedge \blacklozenge\neg p \equiv_{\mathsf{S}_1^*} \blacksquare\neg p$
  – $\Box \circ \blacklozenge p \wedge \Diamond \circ \blacklozenge \bullet p \equiv_{\mathsf{S}_1^*} \blacklozenge p \wedge \blacklozenge\neg p$
  – $\Box\blacksquare p \wedge \Box \circ \blacklozenge p \equiv_{\mathsf{S}_1^*} \blacksquare p \wedge \blacklozenge p \equiv_{\mathsf{S}_1^*} \blacksquare p$

  In comparison with $\Pi_{\mathsf{S}_0^*}(\mathcal{F}^{\circ\bullet}_{\Box\blacksquare})$, note that the first anchor formula (viz., $\Box\blacksquare p \wedge \Diamond\blacksquare\bullet p \equiv_{\mathsf{S}_1^*} \blacksquare p \wedge \blacksquare\neg p$) is S_1^*-inconsistent, and is thus absent from $\Pi_{\mathsf{S}_1^*}(\mathcal{F}^{\circ\bullet}_{\Box\blacksquare})$.

- In S_2^*, the fragment $\mathcal{F}^{\circ\bullet}_{\Box\blacksquare}$ simplifies to $\{\Box p, \Box\neg p, \Diamond p, \Diamond\neg p\}$, and since $\Box$ is a KF-modality in S_2^*, the Aristotelian diagram for $(\mathcal{F}^{\circ\bullet}_{\Box\blacksquare}, \mathsf{S}_2^*)$ is a classical square, with a subalternation from $\Diamond p$ to $\Box p$, i.e., (before simplification) from $\Diamond\blacksquare \bullet p$ to $\Box\blacksquare p$. The partition $\Pi_{\mathsf{S}_2^*}(\mathcal{F}^{\circ\bullet}_{\Box\blacksquare})$ contains the following anchor formulas:

  – $\Box\blacksquare p \wedge \Diamond\blacksquare \bullet p \equiv_{\mathsf{S}_2^*} \Box p \wedge \Diamond p \equiv_{\mathsf{S}_2^*} \Diamond p$
  – $\Box \circ \blacklozenge p \wedge \Diamond \circ \blacklozenge \bullet p \equiv_{\mathsf{S}_2^*} \Box\neg p \wedge \Diamond\neg p \equiv_{\mathsf{S}_2^*} \Diamond\neg p$
  – $\Box\blacksquare p \wedge \Box \circ \blacklozenge p \equiv_{\mathsf{S}_2^*} \Box p \wedge \Box\neg p$

  In comparison with $\Pi_{\mathsf{S}_0^*}(\mathcal{F}^{\circ\bullet}_{\Box\blacksquare})$, note that the second anchor formula (viz., $\Diamond\blacksquare \bullet p \wedge \Diamond \circ \blacklozenge \bullet p \equiv_{\mathsf{S}_2^*} \Diamond p \wedge \Diamond\neg p$) is S_2^*-inconsistent, and is thus absent from $\Pi_{\mathsf{S}_2^*}(\mathcal{F}^{\circ\bullet}_{\Box\blacksquare})$.

- In S_3^*, the fragment $\mathcal{F}^{\circ\bullet}_{\Box\blacksquare}$ simplifies to $\{\blacksquare p, \blacklozenge p, \blacksquare\neg p, \blacklozenge\neg p\}$, and since $\blacksquare$ is a KF-modality in S_3^*, the Aristotelian diagram for $(\mathcal{F}^{\circ\bullet}_{\Box\blacksquare}, \mathsf{S}_3^*)$ is a classical square, with a subalternation from $\blacklozenge p$ to $\blacksquare p$, i.e., (before simplification) from $\Box \circ \blacklozenge p$ to $\Box\blacksquare p$. The partition $\Pi_{\mathsf{S}_3^*}(\mathcal{F}^{\circ\bullet}_{\Box\blacksquare})$ contains the following anchor formulas:

  – $\Box\blacksquare p \wedge \Diamond\blacksquare \bullet p \equiv_{\mathsf{S}_3^*} \blacksquare p \wedge \blacksquare\neg p$
  – $\Diamond\blacksquare \bullet p \wedge \Diamond \circ \blacklozenge \bullet p \equiv_{\mathsf{S}_3^*} \blacksquare\neg p \wedge \blacklozenge\neg p \equiv_{\mathsf{S}_3^*} \blacklozenge\neg p$
  – $\Box\blacksquare p \wedge \Box \circ \blacklozenge p \equiv_{\mathsf{S}_3^*} \blacksquare p \wedge \blacklozenge p \equiv_{\mathsf{S}_3^*} \blacklozenge p$

  In comparison with $\Pi_{\mathsf{S}_0^*}(\mathcal{F}^{\circ\bullet}_{\Box\blacksquare})$, note that the third anchor formula (viz., $\Box \circ \blacklozenge p \wedge \Diamond \circ \blacklozenge \bullet p \equiv_{\mathsf{S}_3^*} \blacklozenge p \wedge \blacklozenge\neg p$) is S_3^*-inconsistent, and is thus absent from $\Pi_{\mathsf{S}_3^*}(\mathcal{F}^{\circ\bullet}_{\Box\blacksquare})$.

- In S_4^*, the fragment $\mathcal{F}^{\circ\bullet}_{\Box\blacksquare}$ simplifies to $\{\Box p, \Box\neg p, \Diamond p, \Diamond\neg p\}$, and since $\Box$ is a KD-modality in S_4^*, the Aristotelian diagram for $(\mathcal{F}^{\circ\bullet}_{\Box\blacksquare}, \mathsf{S}_4^*)$ is a classical square, with a subalternation from $\Box p$ to $\Diamond p$, i.e., (before simplification) from $\Box\blacksquare p$ to $\Diamond\blacksquare\bullet p$. The partition $\Pi_{\mathsf{S}_4^*}(\mathcal{F}^{\circ\bullet}_{\Box\blacksquare})$ contains the following anchor formulas:

  - $\Box\blacksquare p \wedge \Diamond\blacksquare\bullet p \equiv_{\mathsf{S}_4^*} \Box p \wedge \Diamond p \equiv_{\mathsf{S}_4^*} \Box p$
  - $\Diamond\blacksquare\bullet p \wedge \Diamond\circ\blacklozenge\bullet p \equiv_{\mathsf{S}_4^*} \Diamond p \wedge \Diamond\neg p$
  - $\Box\circ\blacklozenge p \wedge \Diamond\circ\blacklozenge\bullet p \equiv_{\mathsf{S}_4^*} \Box\neg p \wedge \Diamond\neg p \equiv_{\mathsf{S}_4^*} \Box\neg p$

  In comparison with $\Pi_{\mathsf{S}_0^*}(\mathcal{F}^{\circ\bullet}_{\Box\blacksquare})$, note that the fourth anchor formula (viz., $\Box\blacksquare p \wedge \Box\circ\blacklozenge p \equiv_{\mathsf{S}_4^*} \Box p \wedge \Box\neg p$) is S_4^*-inconsistent, and is thus absent from $\Pi_{\mathsf{S}_4^*}(\mathcal{F}^{\circ\bullet}_{\Box\blacksquare})$.

Taken together, these considerations show that $\langle \mathcal{L}^{\circ\bullet}_{\Box\blacksquare}, \mathcal{F}^{\circ\bullet}_{\Box\blacksquare}, \mathsf{S}_0^*, \mathsf{S}_1^*, \mathsf{S}_2^*, \mathsf{S}_3^*, \mathsf{S}_4^*\rangle$ constitutes a *complete* solution to Open Problem 1. Furthermore, continuing along these lines, we easily obtain another solution. In particular, consider one final system, S_5^*, which was already included in the overview table before. As one can see in that table, S_5^* is the exact 'mirror image' of S_0^*. In this new logic S_5^*, the fragment $\mathcal{F}^{\circ\bullet}_{\Box\blacksquare}$ simplifies to $\{\blacksquare p, \blacksquare\neg p, \blacklozenge p, \blacklozenge\neg p\}$, and since $\blacksquare$ is a K-modality in S_5^*, the Aristotelian diagram for $(\mathcal{F}^{\circ\bullet}_{\Box\blacksquare}, \mathsf{S}_5^*)$ is a degenerate square. Replacing S_0^* with S_5^*, it is easy to check that $\langle \mathcal{L}^{\circ\bullet}_{\Box\blacksquare}, \mathcal{F}^{\circ\bullet}_{\Box\blacksquare}, \mathsf{S}_5^*, \mathsf{S}_1^*, \mathsf{S}_2^*, \mathsf{S}_3^*, \mathsf{S}_4^*\rangle$ constitutes another complete solution to Open Problem 1.

These two solutions are said to be *complete*, because they specify a concrete logical system for *every* 'variable' S_0, S_1, S_2 S_3 and S_4 that occurs in the statement of Open Problem 1. However, neither of these solutions is fully satisfactory, because the concrete logics that they are based on, i.e., S_0^* to S_5^*, are strongly *ad hoc*. These logics have been specifically constructed to provide a complete solution to Open Problem 1, but seem to lack any independent philosophical or mathematical motivation. To appreciate the highly artificial nature of these logics, note how the connectives $\circ$ and $\bullet$ switch back and forth between behaving like classical negation in some of the logics involved, and behaving like the identity connective in others.

We conclude our discussion by emphasizing that the introduction of S_5^* is also motivated by considerations regarding deductive strength. Consider the standard way of comparing logics' deductive strength: given two logics S, S' over the same language $\mathcal{L}$, we say that S' is *at least as strong as* S (notation: $\mathsf{S} \leq \mathsf{S}'$) iff for all $\varphi \in \mathcal{L}$: if $\models_{\mathsf{S}} \varphi$ then $\models_{\mathsf{S}'} \varphi$. This also yields notions of 'strictly stronger than' and 'incomparability': $\mathsf{S} < \mathsf{S}'$ iff $\mathsf{S} \leq \mathsf{S}'$ but not $\mathsf{S}' \leq \mathsf{S}$, while $\mathsf{S} \,||\, \mathsf{S}'$ iff neither $\mathsf{S} \leq \mathsf{S}'$ nor $\mathsf{S}' \leq \mathsf{S}$. In the first two (partial) solutions to Open Problem 1, S_0 is strictly weaker than all other logics, which are themselves mutually incomparable. For example, in the first partial solution, we have $\mathsf{K} < \mathsf{KF}$, $\mathsf{K} < \mathsf{KD}$ and $\mathsf{KF} \,||\, \mathsf{KD}$, as is visualized Figure 6.[12] In the third (complete) solution, this is no longer the

[12]Similarly, in the second partial solution, we have $\mathsf{K} < \mathsf{KD}$, $\mathsf{K} < \mathsf{K4}$, $\mathsf{K} < \mathsf{K5}$ and $\mathsf{KD} \,||\, \mathsf{K4} \,||\, \mathsf{K5} \,||\, \mathsf{KD}$.

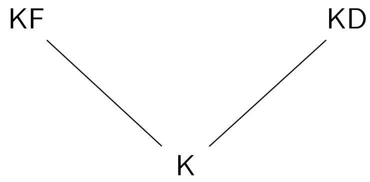


Figure 6: Ordering of K, KF and KD, according to their deductive strength.

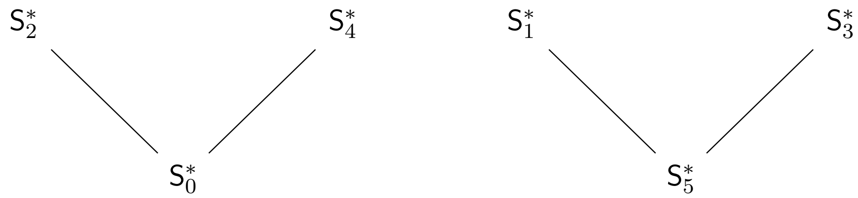


Figure 7: Ordering of the logics S_0^* to S_5^*, according to their deductive strength.

case. In particular, although S_2^* and S_4^* are strictly stronger than S_0^* and $\mathsf{S}_m^* \mid\mid \mathsf{S}_n^*$ for all distinct $m, n \in \{1, 2, 3, 4\}$, the logics S_1^* and S_3^* are *not* strictly stronger than S_0^*, but rather incomparable with it. By introducing the logic S_5^*, we can restore this to a more symmetric situation, as is visualized in Figure 7. In particular, we now have $\mathsf{S}_i^* < \mathsf{S}_j^*$, $\mathsf{S}_i^* < \mathsf{S}_k^*$ and $\mathsf{S}_j^* \mid\mid \mathsf{S}_k^*$ for $(i, j, k) \in \{(0, 2, 4), (5, 1, 3)\}$, and furthermore $\mathsf{S}_m^* \mid\mid \mathsf{S}_n^*$ for all $m \in \{0, 2, 4\}$ and $n \in \{1, 3, 5\}$. In this sense, the third and fourth solution together can be seen as the result of 'reduplicating' the highly natural (but partial) first solution (based on K, KF and KD), namely once for the $\Box$-modality and once for the $\blacksquare$-modality.

6 Conclusion

In this paper we have explored the interface between logic-sensitivity and bitstring semantics in the square of opposition. Although these two topics have already been studied extensively in logical geometry, their interaction continues to present us with challenging problems. In particular, a story that was very easy to tell in terms of 'purely combinatorial bitstrings', turns out to be much harder to tell in terms of formulas and logical systems, i.e., in terms of 'semantic bitstrings'. We have therefore presented this as an open problem, and discussed four attempted solutions $\langle \mathcal{L}, \mathcal{F}, \mathsf{S}_0, \mathsf{S}_1, \mathsf{S}_2, \mathsf{S}_3, \mathsf{S}_4 \rangle$, which are summarized here:

	$\mathcal{L}$	$\mathcal{F}$	S_0	S_1	S_2	S_3	S_4
attempt 1	$\mathcal{L}_\Box$	$\mathcal{F}_m$	K	—	KF	—	KD
attempt 2	$\mathcal{L}_\Box$	$\mathcal{F}_m^*$	K	KD	$\mathsf{K4}$	$\mathsf{K5}$	—
attempt 3	$\mathcal{L}_{\Box\blacksquare}^{\circ\bullet}$	$\mathcal{F}_{\Box\blacksquare}^{\circ\bullet}$	S_0^*	S_1^*	S_2^*	S_3^*	S_4^*
attempt 4	$\mathcal{L}_{\Box\blacksquare}^{\circ\bullet}$	$\mathcal{F}_{\Box\blacksquare}^{\circ\bullet}$	S_5^*	S_1^*	S_2^*	S_3^*	S_4^*

Unfortunately, none of these attempts is entirely successful: the first two are based on well-motivated logics such as K, KD and $\mathsf{K4}$, but they are *incomplete*; by contrast, the last two are complete, but they are based on the logics S_0^* to S_5^*, which are ad hoc and feel rather *artificial*. A fully satisfactory solution, which is both complete and natural, has not been found until now, and thus has to be left for future research. In ongoing work, we are also studying similar interactions between logic-sensitivity and bitstring semantics in other Aristotelian diagrams beyond the square, e.g., in various *hexagons* of opposition.

These results will steadily gain a broader relevance, as the literature on logic-sensitivity and bitstring semantics continues to grow larger and more diverse. For example, there is some preliminary work that applies bitstring semantics to various topological interpretations of the Region Connection Calculus (RCC) [3, 26, 42]. Here, too, it will be crucial to investigate how the resulting bitstrings depend on the precise axioms of the underlying topological interpretation. This investigation is still in its infancy, but since it concerns the interaction between bitstring semantics and logic-sensitivity, it will clearly have to take into account the insights presented in this paper.

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References

[1] M. Brown. Generalized quantifiers and the square of opposition. *Notre Dame Journal of Formal Logic*, 25:303–322, 1984.

[2] R. Christensen. The logic of Δ. *Thought*, 2:350–356, 2013.

[3] A. Cohn, B. Bennett, J. Gooday, and N. M. Gotts. Qualitative spatial representation and reasoning with the Region Connection Calculus. *GeoInformatica*, 1:275–316, 1997.

[4] M. Correia. Boethius on the square of opposition. In J.-Y. Béziau and D. Jacquette, editors, *Around and Beyond the Square of Opposition*, pages 41–52. Springer, Basel, 2012.

[5] M. Correia. Aristotle's squares of opposition. *South American Journal of Logic*, 3:313–326, 2017.

[6] M. Correia. Logic in Apuleius and Boethius. *Revista Portuguesa de Filosofia*, 73:1035–1052, 2017.

[7] L. Demey. Structures of oppositions for public announcement logic. In J.-Y. Béziau and D. Jacquette, editors, *Around and Beyond the Square of Opposition*, pages 313–339. Springer, Basel, 2012.

[8] L. Demey. Interactively illustrating the context-sensitivity of Aristotelian diagrams. In H. Christiansen, I. Stojanovic, and G. Papadopoulos, editors, *Modeling and Using Context*, LNCS 9405, pages 331–345. Springer, Berlin/Heidelberg, 2015.

[9] L. Demey. Computing the maximal Boolean complexity of families of Aristotelian diagrams. *Journal of Logic and Computation*, 28:1323–1339, 2018.

[10] L. Demey. Aristotelian diagrams in the debate on future contingents. *Sophia*, 58:321 – 329, 2019.

[11] L. Demey. Boolean considerations on John Buridan's octagons of oppositions. *History and Philosophy of Logic*, 40:116–134, 2019.

[12] L. Demey. Metalogic, metalanguage and logical geometry. *Logique et Analyse*, 248:453–478, 2019.

[13] L. Demey. Logic-sensitivity of Aristotelian diagrams in non-normal modal logics. *Axioms*, 10(128):1 – 25, 2021.

[14] L. Demey and H. Smessaert. Logical and geometrical distance in polyhedral Aristotelian diagrams in knowledge representation. *Symmetry*, 9(10)(204), 2017.

[15] L. Demey and H. Smessaert. Combinatorial bitstring semantics for arbitrary logical fragments. *Journal of Philosophical Logic*, 47:325–363, 2018.

[16] P. Falcão. Visualizing polymorphisms and counter-polymorphisms in S5 modal logic. In V. Giardino, S. Linker, R. Burns, F. Bellucci, J.-M. Boucheix, and P. Viana, editors, *Diagrammatic Representation and Inference*, LNCS 13462, pages 296–311, Berlin/Heidelberg, 2022. Springer.

[17] J. D. García-Cruz. Aristotelian relations in PDL: The hypercube of dynamic oppositions. *South American Journal of Logic*, 3:389 – 414, 2017.

[18] C. Geudens and L. Demey. On the Aristotelian roots of the modal square of opposition. *Logique et Analyse*, 255:313 – 348, 2021.

[19] S. Givant and P. Halmos. *Introduction to Boolean Algebras*. Springer, New York, NY, 2009.

[20] W. L. Gombocz. Apuleius is better still: a correction to the square of opposition. *Phronesis*, 43:124–131, 1990.

[21] K. Klement. New logic and the seeds of analytic philosophy. Boole, Frege. In J. A. Shand, editor, *A Companion to Nineteenth-Century Philosophy*, pages 454–479. Wiley-Blackwell, Hoboken, NJ, 2019.

[22] J. Lemaire. Is Aristotle the father of the square of opposition? In J.-Y. Béziau and S. Gerogiorgakis, editors, *New Dimensions of the Square of Opposition*, pages 33 – 69. Philosophia Verlag, Munich, 2017.

[23] J. Lemanski and F. Schang. A bitstring semantics for calculus CL. In J.-Y. Beziau and I. Vandoulakis, editors, *The Exoteric Square of Opposition*, pages 171–193. Springer, Cham, 2022.

[24] D. Londey and C. Johanson. Apuleius and the square of opposition. *Phronesis*, 29:165–173, 1984.

[25] D. Londey and C. Johanson. *The Logic of Apuleius*. Brill, Leiden, 1987.

[26] C. Lutz and F. Wolter. Modal logics of topological relations. *Logical Methods in Computer Science*, 2:1–41, 2006.

[27] A. Moktefi and F. Schang. Another side of categorical propositions: The Keynes-Johnson octagon of oppositions. *History and Philosophy of Logic*, pages 1–17, forthcoming.

[28] E. J. Nelson. The square of opposition. *The Monist*, 42:269–278, 1932.

[29] A. Orenstein. Geach, Aristotle and predicate logics. *Philosophical Investigations*, 38:96–114, 2015.

[30] T. Parsons. The traditional square of opposition. In E. N. Zalta, editor, *Stanford Encyclopedia of Philosophy*. CSLI, Stanford, CA, 2017.

[31] C. Pizzi. Generalization and composition of modal squares of opposition. *Logica Universalis*, 10:313–325, 2016.

[32] C. Pizzi. Contingency logics and modal squares of opposition. In J.-Y. Béziau and S. Gerogiorgakis, editors, *New Dimensions of the Square of Opposition*, pages 201–220. Philosophia Verlag, Munich, 2017.

[33] L. Pozzi. *Studi di Logica Antica e Medioevale*. Liviana Editrice, Padova, 1974.

[34] K. Roelandt. *The Meaning of* Most*. Proportional and Comparative Interpretations in Dutch*. LOT Publications, Utrecht, 2016.

[35] P. Simons. Judging correctly: Brentano and the reform of elementary logic. In D. Jacquette, editor, *The Cambridge Compansion to Brentano*, pages 45–65. Cambridge University Press, Cambridge, 2004.

[36] H. Smessaert and L. Demey. Logical and geometrical complementarities between Aristotelian diagrams. In T. Dwyer, H. Purchase, and A. Delaney, editors, *Diagrammatic Representation and Inference*, LNCS 8578, pages 246–260. Springer, Berlin/Heidelberg, 2014.

[37] H. Smessaert and L. Demey. Logical geometries and information in the square of opposition. *Journal of Logic, Language and Information*, 23:527–565, 2014.

[38] H. Smessaert and L. Demey. The unreasonable effectiveness of bitstrings in logical geometry. In J.-Y. Béziau and G. Basti, editors, *The Square of Opposition: A Cornerstone of Thought*, pages 197–214. Springer, Basel, 2017.

[39] H. Smessaert and L. Demey. On the logical geometry of geometric angles. *Logica Universalis*, 16:581–601, 2022.

[40] M. W. Sullivan. *Apuleian Logic. The Nature, Sources, and Influence of Apuleius's Peri hermeneias*. North-Holland, Amsterdam, 1967.

[41] L. Vignero. Combining and relating Aristotelian diagrams. In A. Basu, G. Stapleton, S. Linker, C. Legg, E. Manalo, and P. Viana, editors, *Diagrammatic Representation and Inference*, LNCS 12909, pages 221–228. Springer, Berlin/Heidelberg, 2021.

[42] F. Wolter and M. Zakharyaschev. Qualitative spatio-temporal representation and reasoning: a computational perspective. In G. Lakemeyer and B. Nebel, editors, *Exploring Artificial Intelligence in the New Millenium*, pages 175–216. Morgan Kauffman, San Francisco, CA, 2002.