EPSILON AND QD ALGORITHMS FOR THE MATRIX-PADÉ AND 2-D PADÉ PROBLEM

A. BULTHEEL

Report TW 57 May 1982

DEPARTEMENT COMPUTERWETENSCHAPPEN
K.U.Leuven
Celestijnenlaan 200 A, B-3030 Heverlee (Belgium)

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A. BULTHEEL
Department of Computer Science
K.U.Leuven (Belgium)

Abstract

QD relations and an ε-algorithm are given for the computation of the matrix-Padé table or the Padé table of 2-D rectangular approximants as defined by Bose-Basu. The results are derived in a uniform framework.

1. Introduction

A matrix version of the q-d and ε-algorithm was given by Wynn [1]. We give an alternative proof of these relations, allowing a generalization for the computation of 2-D rectangular Padé approximants in the sense of Bose-Basu [2]. In section 2, a generalized concept of continued fraction (CF) will be introduced, which will allow us to treat the matrix Padé and 2-D Padé problem in a unified framework. In section 3 we briefly review the definition of rectangular 2D-Padé approximants. The subsequent sections give the QD relations for staircases and sawteeth. As an introduction to the derivation of the ε-algorithm, given in section 7, we prove in section 6 the cross-relation for Padé approximants of both types considered.

2. Continued fractions

We shall denote a (right) continued fraction (CF) by

$$a_0 b_0^{-1} + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \ldots$$

and define it to be the ordered set of convergents $\{c_n\}_0^\infty$ with $c_n = P_n Q_n^{-1}$ where

$$P_{-1} = b_0,\ P_0 = a_0,\ P_{i+1} = P_{i-1}\, a_{i+1} + P_i\, b_{i+1}\ ,\ i \geqslant 0$$

$$Q_{-1} = 0\ ,\ Q_0 = b_0,\ Q_{i+1} = Q_{i-1}\, a_{i+1} + Q_i\, b_{i+1}\ ,\ i \geqslant 0.$$

This is clearly the classical definition if a_i and b_i, and consequently also P_i and Q_i are scalars. In the following applications we need a generalization where a_i, b_i can be matrices, rectangular in general, but of appropriate dimensions. In what follows, a_0 and b_0 will be either both square matrices or both row vectors. Consequently, P_n and Q_n will be matrices with the same number of rows as a_0 and b_0 respectively. a_i and b_i for $i > 0$ will be square matrices. The inversion and multiplication operations have to be understood in the following sense. If it is an operation with square matrices, it means the ordinary matrix inversion or product. These operations are understood to operate componentwise for row vectors : e.g. if $v = [v_1, v_2, \ldots, v_n]$, and $w = [w_1\ w_1 \ldots w_n]$, then $v^{-1} = [v_1^{-1}, v_2^{-1}, \ldots, v_n^{-1}]$ and $vw = [v_1 w_1, v_2 w_2, \ldots, v_n w_n]$ and if α is a scalar then $\alpha v = [\alpha v_1, \alpha v_2, \ldots, \alpha v_n]$.

3. Rectangular 2D-Padé approximants

As it was shown in [2],[4],[5],[6], the coefficients of the denominator of a rectangular 2D-Padé approximant satisfies a block Hankel system of linear equations. To make use of Rissanen's algorithm [7], auxiliary approximants are introduced, called horizontal or vertical approximants.

Let $F(x,y) = \sum_{i,j \in \mathbb{Z}_+^2} f_{ij} x^i y^j$ be given.

$$P_k^{[m_P,n_P/m_Q,n_Q]}(x,y) = \sum_{i,j \in \mathcal{P}} p_{ij} x^i y^j \ , \ \mathcal{P} = \{(i,j) \mid 0 \leq i \leq m_P \ , \ 0 \leq j \leq n_P\}$$

and

$$Q_k^{[m_P,n_P/m_Q,n_Q]}(x,y) = \sum_{i,j \in \mathcal{Q}} q_{ij} x^i y^j \ , \ \mathcal{Q} = \{(i,j) \mid 0 \leq i \leq m_Q \ , \ 0 \leq j \leq n_Q\}$$

are the numerator and denominators of Padé approximants satisfying

$$[F(x,y) Q_k^{[m_P,n_P/m_Q,n_Q]}(x,y)]_{\mathcal{P}} = P_k^{[m_P,n_P/m_Q,n_Q]}(x,y)$$

$$[F(x,y) Q_k^{[m_P,n_P/m_Q,n_Q]}(x,y)]_{J_k} = 0$$

where $[\sum_{i,j} g_{ij} x^i y^j]_S$ means $\sum_{(i,j) \in S} g_{ij} x^i y^j$.

If $J_k = \{(i,j) \mid m_P+1 \leq i \leq m_P+m_Q+1, \ n_P+1 \leq j \leq n_P+n_Q+1\} \setminus$

$$\{(m_P+m_Q+1-k, \ n_P+n_Q+1)\}$$

and $q_{m_Q-k,0} = 1$, $k = 0,1,\ldots,m_Q$, then

$P_k^{[m_P,n_P/m_Q,n_Q]}(x,y) \, (Q_k^{[m_P,n_P/m_Q,n_Q]}(x,y))^{-1}$ is called the k-th horizontal approximant, and if

$$J_k = \{(i,j) \mid m_P+1 \leq i \leq m_P+m_Q+1, \ n_P+1 \leq j \leq n_P+n_Q+1\} \setminus$$

$$\{(m_P+m_Q+1, n_P+n_Q+1-k)\}$$

and $q_{0,n_Q-k} = 1$, $k = 0,1,\ldots,n_Q$, then

$P_k^{[m_P,n_P/m_Q,n_Q]}(x,y)\,(Q_k^{[m_P,n_P/m_Q,n_Q]}(x,y))^{-1}$ is called the k-th vertical approximant.

Define a $1\times(m_Q+1)$ row vector $[m_P,n_P/m_Q,n_Q]_H$ with as k-th element the k-th horizontal approximant and similarly a vector $[m_P,n_P/m_Q,n_Q]_V$ of vertical approximants, being of dimension $1\times(n_Q+1)$. The last element in both vectors is the 2D-Padé approximant of type $(m_P,n_P/m_Q,n_Q)$. If we are willing to compute the denominators of these vectors simultaneously, then it is possible to use all the recursions that exist for block Hankel or Toeplitz matrices [8]. E.g., keeping the degree of x constant, we can compute a diagonal with respect to y as it was done in [2,6], but also staircases, sawteeth or other paths can be computed in this way. Similarly we can keep the degree of y constant and compute such paths with respect to x. For more details see [5].

4. QD algorithms for staircases in Padé tables

The matrix Padé problem for a normal table was treated e.g. in [3]. An algorithm has been described to compute a descending staircase in the Padé table :

$$T_k = \{[k+j/j],\ [k+j+1/j]\}_{j=0}^{\infty} \quad \text{for} \quad k \geqslant 0.$$

The approximants were given by the successive convergents of a CF of the form

$$[k/0] + \frac{z^{k+1} f_{k+1}}{\lfloor I} - \frac{z\, a_1^{(k)}}{\lfloor I} - \frac{z\, b_1^{(k)}}{\lfloor I} - \frac{z\, a_2^{(k)}}{\lfloor I} - \ldots \quad . \tag{4.1}$$

f_{k+1} is the (k+1)-st coefficient of the given power series $F(z) = \sum_0^{\infty} f_i z^i$ and $[k/0] = P^{[k/0]}(z) = \sum_0^{k} f_i z^i$.

For the 2D-Padé approximants we consider the elements $T_k^H = \{[m_P,k+j/m_Q,j]_H\ ,\ [m_P,k+j+1/m_Q,j]_H\}_{j=0}^{\infty}$ where m_P and m_Q, the degrees in x of numerator and denominator, are fixed. Since m_P and m_Q are left unchanged, we simplify the notation as $T_k^H(m_P,m_Q) = \{[k+j/j],\ [k+j+1/j]\}_{j=0}^{\infty}$. The elements in $T_k^H(m_P,m_Q)$ are the successive convergents of a CF of the form

$$[k/0] + \frac{y^{k+1} C_0^{(k)}}{\lfloor I} - \frac{y\, a_1^{(k)}}{\lfloor I} - \frac{y\, b_1^{(k)}}{\lfloor I} - \ldots \tag{4.2}$$

with $[k/0] = [m_P,k/m_Q,0]_H$ and $y^{k+1}C_0^{(k)} = \mathrm{diag}([m_P,k+1/m_Q,0]_H - [m_P,k/m_Q,0]_H)$. The $a_i^{(k)}$ and $b_i^{(k)}$ are square matrices of size $(m_Q+1)\times(m_Q+1)$. (4.2) is formally the same as (4.1). Of course a CF of the same form exists for the

case where the degree of y is kept constant. It produces the elements of the staircase

$$T_k^v = \{[k+j,n_P/j,n_Q]_V,[k+j+1,n_P/j,n_Q]_V\}_{j=0} \cdot$$

The proof of the QD relations is as in the scalar case now.

The matrix QD algorithm goes as follows :

$$\begin{aligned} a_{i+1}^{(k)} &= (b_i^{(k)})^{-1}\, a_i^{(k+1)}\, b_i^{(k+1)} \\ b_i^{(k)} &= b_{i-1}^{(k+1)} + a_i^{(k+1)} - a_i^{(k)} \end{aligned} \qquad i,k \geqslant 1 \qquad (4.3)$$

with initial conditions

$$b_0^{(k)} = 0 \ , \ a_i^{(k)} = (c_0^{(k)})^{-1} (c_0^{(k+1)}) \quad , \quad k \geqslant 1$$

Proof

An even contraction of

$$[k+1/0] + \cfrac{z^{k+2} c_0^{(k+1)}}{\left|\, I} - \cfrac{z\, a_1^{(k+1)}}{\left|\, I} - \cfrac{z\, b_1^{(k+1)}}{\left|\, I} - \cfrac{z\, a_2^{(k+1)}}{\left|\, I} - \ \ldots$$

gives :

$$[k+1/0] + \cfrac{z^{k+2} c_0^{(k+1)}}{\left|\, I - z\, a_1^{(k+1)}} + \sum_{i \geqslant 1} \cfrac{-z^2 a_i^{(k+1)} b_i^{(k+1)}}{\left|\, I - z(a_{i+1}^{(k+1)} + b_i^{(k+1)})}\ , \qquad (4.5)$$

while an odd contraction of

$$[k/0] + \cfrac{z^{k+1} c_0^{(k)}}{\left|\, I} - \cfrac{z\, a_1^{(k)}}{\left|\, I} - \cfrac{z\, b_1^{(k)}}{\left|\, I} - \cfrac{z\, a_2^{(k)}}{\left|\, I} - \ \ldots$$

gives

$$[k+1/0] + \cfrac{z^{k+2} c_0^{(k)} a_1^{(k)}}{\left|\, I - z(a_1^{(k)} + b_1^{(k)})} + \sum_{i \geqslant 2} \cfrac{-z^2 b_{i-1}^{(k)} a_i^{(k)}}{\left|\, I - z(a_i^{(k)} + b_i^{(k)})}\ . \qquad (4.6)$$

The convergents of (4.5) and (4.6) are $[k+j+1/j]$, $j = 0,1,\ldots$ and they have the same normalization. Since this is true for all z, we can equate the corresponding coefficients, from which (4.3) directly follows.

□

For ascending staircases $U_k = \{[k-j/j],[k-j-1/j]\}_{j=0}^{k}$, the CF expansion is of the form

$$[k/0] + \cfrac{z^k c_0^{(k)}}{I} - \cfrac{a_1^{(k)}}{zI} - \cfrac{b_1^{(k)}}{I} - \cfrac{a_2^{(k)}}{zI} - \cdots$$

with $z^k c_0^{(k)} = [k-1/0]-[k/0] = f_k z^k$ in the matrix Padé case and $z^k c_0^{(k)}$ $= \mathrm{diag}([m_P,k-j/m_Q,0]_H - [m_P,k/m_Q,0]_H)$ or

$= \mathrm{diag}([k-1,n_P/0,n_Q]_V - [k,n_P/0,n_Q]_V)$

in the 2D-Padé case. Replace z by ξ^{-1}, then the CF is equivalent with

$$[k/0] + \cfrac{\xi^{-k} c_0^{(k)}}{I} - \cfrac{\xi\, a_1^{(k)}}{I} - \cfrac{\xi\, b_1^{(k)}}{I} - \cfrac{\xi\, a_2^{(k)}}{I} - \cdots$$

which is of the same form as treated before. The QD relations are therefore the same.

5. QD relations for sawteeth

Instead of the staircases considered in previous sections, we can also look at sawteeth like

$$S_k^+ = \{[k/j],[k+1/j]\}_{j=0}^{\infty} .$$

A continued fraction expansion following this sawtooth has the form of a Perron fraction [9] :

$$[k/0] + \cfrac{z^{k+1} c_0^{(k)}}{I} - \cfrac{z\, I}{a_1^{(k)}} - \cfrac{b_1^{(k)}}{I} - \cfrac{z\, I}{a_2^{(k)}} - \cfrac{b_2^{(k)}}{I} - \cdots$$

with $z^{k+1} c_0^{(k)} = [k+1/0] - [k/0]$ in the matrix-Padé case and a corresponding diagonal expression for the 2D-case.

A contraction over the odd convergents gives

$$[k+1/0] + \cfrac{z^{k+2} c_0^{(k)}}{a_1^{(k)} - b_1^{(k)} - zI} + \sum_{j \geq 1} \cfrac{-b_j^{(k)}\, z}{a_{j+1}^{(k)} - b_{j+1}^{(k)} - zI}$$

A contraction over the even convergents of the CF for S_{k+1}^+ gives

$$[k+1/0] - \cfrac{z^{k+2} c_0^{(k+1)} a_1^{(k+1)}}{a_1^{(k+1)} - zI} + \sum_{j \geq 1} \cfrac{-(a_j^{(k+1)})^{-1} b_j^{(k+1)} a_{j+1}^{(k+1)} z}{(I-(a_j^{(k+1)})^{-1} b_j^{(k+1)}) a_{j+1}^{(k+1)} - zI}$$

Equating corresponding coefficients gives

$$\begin{cases} a_1^{(k+1)} = -\,(c_0^{(k+1)})^{-1}\, c_0^{(k)}, \\ b_0^{(k)} = 0 \end{cases} \qquad \begin{cases} a_j^{(k+1)} = (b_{j-1}^{(k+1)})^{-1}\, a_{j-1}^{(k+1)} b_{j-1}^{(k)} \\ b_j^{(k)} = a_j^{(k)} = a_j^{(k)} - (I-(a_{j-1}^{(k+1)})^{-1}\, b_{j-1}^{(k+1)})\, a_j^{(k+1)} \end{cases}$$

or, after substituting $(a_{j-1}^{(k+1)})^{-1}\, b_{j-1}^{(k+1)} = b_{j-1}^{(k)}\, (a_j^{(k+1)})^{-1}$

$$\begin{cases} a_j^{(k+1)} = (b_{j-1}^{(k+1)})^{-1}\, a_{j-1}^{(k+1)}\, b_{j-1}^{(k)} \\ b_j^{(k)} = a_j^{(k)} - a_j^{(k+1)} + b_{j-1}^{(k)} \end{cases} \qquad \begin{cases} a_1^{(k+1)} = -(c_0^{(k+1)})\, c_0^{(k)} \\ b_0^{(k)} = 0 \end{cases}$$

To fill up the whole table of $a_j^{(k)}$, $b_j^{(k)}$, we can border it with the boundary condition $a_\ell^{(-1)} = 0$, $\forall \ell$ so that all the elements in the table

$$\begin{array}{llll} & a_1^{(-1)} = 0 & a_2^{(-1)} = 0 & a_3^{(-1)} = 0 \ \ldots \\ b_0^{(-1)} = 0 & b_1^{(-1)} & b_2^{(-1)} & \\ & a_1^{(0)} = -(c_0^{(0)})^{-1}\, c_0^{(-1)} & a_2^{(0)} & a_3^{(0)} \ \ldots \\ b_0^{(0)} = 0 & b_1^{(0)} & b_2^{(0)} & \\ & a_1^{(1)} = -(c_0^{(1)})^{-1}\, c_0^{(0)} & a_2^{(1)} & a_3^{(1)} \ \ldots \\ b_0^{(1)} = 0 & b_1^{(1)} & b_2^{(1)} & \\ & a_1^{(2)} = -(c_0^{(2)})^{-1}\, c_0^{(1)} & a_2^{(2)} & a_3^{(2)} \ \ldots \\ b_0^{(2)} = 0 & b_1^{(2)} & b_2^{(2)} & \\ \vdots \quad \vdots & \vdots \quad \vdots & \vdots \quad \vdots & \vdots \quad \vdots \end{array}$$

are defined.

For the sawteeth $S_k^- = \{[k+1/j],[k/j]\}_{j=0}^{\infty}$, we have the CF

$$[k+1/0] + \cfrac{z^{k+1} c_0^{(k)}}{I} - \cfrac{I}{z\, a_1^{(k)}} - \cfrac{z\, b_1^{(k)}}{I} - \cfrac{I}{z\, a_2^{(k)}} - \cfrac{z\, b_2^{(k)}}{I} - \ \ldots$$

With the transformation $z \to \xi^{-1}$, we can bring this in an equivalent form of the CF for S_k^+. The QD relations are again the same.
Relations of this type have been considered recently by McCabe [9].

6. The cross relation for CF's

In [1] the ε-algorithm was derived in a general context having a relation with the matrix-Padé problem.
We shall give a more direct proof, based on the cross-relation which is classical for the scalar case, but the matrix version has not been given earlier as far as we know.

$$([m+1/\ell] - [m/\ell])^{-1} + ([m-1/\ell] - [m/\ell])^{-1} =$$
$$([m/\ell+1] - [m/\ell])^{-1} + ([m/\ell-1] - [m/\ell])^{-1} \qquad (6.1)$$

$$\begin{array}{c} [m-1/\ell] \\ | \\ [m/\ell-1] - [m/\ell] - [m/\ell+1] \\ | \\ [m+1/\ell] \end{array}$$

We give a proof that is valid for the general definition of CF's we adopted.
The inverse and product of a square matrix is in the classical sense, the inverse and product of a vector has to be taken componentwise.

Lemma

Let $c_k = P_k Q_k^{-1}$, $k = 0,1,2,\ldots$ be the convergents of the CF $a_0 b_0^{-1} + \dfrac{a_1|}{|b_1} + \dfrac{a_2|}{|b_2} + \ldots$
then

$$c_{n+1} - c_n = (-1)^n (c_1 - c_0) Q_0 a_2 Q_2^{-1} \ \ldots \ Q_{n-2} a_n Q_n^{-1} Q_{n-1} a_{n+1} Q_{n+1}^{-1} \qquad (6.2)$$

$$c_{n+2} - c_n = (-1)^n (c_1 - c_0) Q_0 a_2 Q_2^{-1} \ \ldots \ Q_{n-2} a_n Q_n Q_{n-1} a_{n+1} b_{n+2} Q_{n+2}^{-1} \qquad (6.3)$$

$$(c_{n+1} - c_n)^{-1} + (c_n - c_{n-1})^{-1} = Q_n b_{n+1} b_{n+2} Q_{n+2}^{-1} (c_{n+2} - c_n)^{-1} \qquad (6.4)$$

Proof

1) $c_{n+1} - c_n = P_{n+1}(Q_{n+1})^{-1} - P_n(Q_n)^{-1} = [P_{n+1} - P_nQ_n^{-1}Q_{n+1}]Q_{n+1}^{-1}$.

Substitute $P_{n+1} = P_nb_{n+1} + P_{n-1}a_{n+1}$ and

$Q_{n+1} = Q_nb_{n+1} + Q_{n-1}a_{n+1}$

then

$$c_{n+1} - c_n = [P_{n-1} - P_nQ_n^{-1}Q_{n-1}]a_{n+1}Q_{n+1}^{-1}$$

$$= -[P_n - P_{n-1}Q_{n-1}^{-1}Q_n]Q_n^{-1}Q_{n-1}a_{n+1}Q_{n+1}^{-1}.$$

Continuing like this we obtain

$$c_{n+1} - c_n = (-1)^n(P_1Q_1^{-1} - P_0Q_0^{-1})Q_0a_2Q_2^{-1} \ldots Q_{n-2}a_nQ_n^{-1}Q_{n-1}a_{n+1}Q_{n+1}^{-1}$$

which is the identity (6.2).

2) $c_{n+2} - c_{n+1} = (-1)^{n+1}(c_1-c_0)Q_0a_2Q_2^{-1} \ldots Q_{n-1}a_{n+1}Q_{n+1}^{-1}Q_na_{n+2}Q_{n+2}^{-1}$

$c_{n+1} - c_n = (-1)^n(c_1-c_0)Q_0a_2Q_2^{-1} \ldots Q_{n-1}a_{n+1}Q_{n+1}^{-1}$

so that by adding together we obtain :

$$c_{n+2} - c_n = (-1)^n(c_1-c_0)Q_0a_2Q_2^{-1} \ldots Q_{n+1}a_{n+1}Q_{n+1}^{-1}\ (\mathbb{I} - Q_na_{n+2}Q_{n+2}^{-1}).$$

(We use the notation $\mathbb{I}$ to indicate the identity matrix or [11...1] as is appropriate).

Because

$$Q_{n+2} = Q_{n+1}b_{n+2} + Q_na_{n+2}$$

we can replace the last factor by $Q_{n+1}b_{n+2}Q_{n+2}^{-1}$, which gives (6.3).

3) $(c_{n+1} - c_n)^{-1} = \{(-1)^n(c_1-c_0)Q_0a_2Q_2^{-1} \ldots Q_{n-1}a_{n+1}Q_{n+1}^{-1}\}^{-1}$

$(c_n - c_{n-1})^{-1} = Q_{n-1}a_{n+1}Q_{n+1}^{-1}(c_{n+1}-c_n)^{-1}$

so that by adding together

$$(c_{n+1} - c_n)^{-1} + (c_n - c_{n-1})^{-1} = (\mathbb{I} - Q_{n-1}a_{n+1}Q_{n+1}^{-1})\{(-1)^n(c_1-c_0)Q_0a_2\ldots Q_{n+1}^{-1}\}^{-1}$$

$$= Q_nb_{n+1}Q_{n+1}^{-1}\{(-1)^n(c_1-c_0)Q_0a_2\ldots Q_{n+1}^{-1}\}^{-1}.$$

Inversion of (6.3) gives

$$(c_{n+2} - c_n)^{-1} = Q_{n+2}(b_{n+2})^{-1}\{(-1)^n(c_1-c_0)Q_0a_2 \ldots Q_{n-1}a_{n+1}\}^{-1} .$$

After substitution of this we obtain :

$$(c_{n+1} - c_n)^{-1} + (c_n - c_{n-1})^{-1} = Q_n b_{n+1} Q_{n+1}^{-1} Q_{n+1} b_{n+2} Q_{n+2}^{-1} (c_{n+2}-c_n)^{-1}$$

which gives (6.4).

□

Proof of (6.1)

(1) is now easily verified. The CF for staircase T_{k-1} is

$$[k-1/0] + \frac{z^{k+1}\ c_0^{(k-1)}|}{|\ I} - \frac{z\ a_1^{(k-1)}|}{|\ I} - \frac{z\ b_1^{(k-1)}|}{|\ I} - \ldots$$

so that the coefficients $b_{n+1} = b_{n+2} = I$ in identity (6.4).

Four successive convergents are given by

$$c_{2\ell} = [k-1+\ell/\ell],\ c_{2\ell+1} = [k+\ell/\ell],\ c_{2\ell+2} = [k+\ell/\ell+1],\ c_{2\ell+3} = [k+\ell+1/\ell+1]$$

so that (6.4) gives

$$([k+\ell/\ell+1] - [k+\ell/\ell])^{-1} + ([k+\ell/\ell] - [k+\ell-1/\ell])^{-1} =$$

$$Q^{[k+\ell/\ell]}(Q^{[k+\ell+1/\ell+1]})^{-1}([k+\ell+1/\ell+1] - [k+\ell/\ell])^{-1} \qquad (6.5)$$

For the staircase T_k, four successive convergents are

$$c_{2\ell-1} = [k+\ell/\ell-1],\ c_{2\ell} = [k+\ell/\ell]\ c_{2\ell+1} = [k+1+\ell/\ell],\ c_{2\ell+2} = [k+\ell+1/\ell+1]$$

so that from (6.4) again

$$([k+\ell+1/\ell] - [k+\ell/\ell])^{-1} + ([k+\ell/\ell] - [k+\ell/\ell-1])^{-1} =$$

$$Q^{[k+\ell/\ell]}(Q^{[k+\ell+1/\ell+1]})^{-1}([k+\ell+1/\ell+1] - [k+\ell/\ell])^{-1} \qquad (6.6)$$

The right hand sides of (6.5) and (6.6) coincide, so that also the left hand sides are equal. If we replace $k+\ell$ by m then the required identity (6.1) follows.

□

7. The ε-algorithm

From the cross-relation for Padé approximants it is now easy to derive the ε-algorithm of Wynn.

Define $\varepsilon_{2\ell}^{(k)} = [k+\ell/\ell]$, then the cross rule is written as

$$(\varepsilon_{2\ell}^{(k+1)} - \varepsilon_{2\ell}^{(k)})^{-1} + (\varepsilon_{2\ell}^{(k-1)} - \varepsilon_{2\ell}^{(k)})^{-1} = (\varepsilon_{2\ell+2}^{(k+1)} - \varepsilon_{2\ell}^{(k)})^{-1} + (\varepsilon_{2\ell-3}^{(k+1)} - \varepsilon_{2\ell}^{(k)})^{-1}$$

$$\varepsilon_0^{(k)} = P^{[k/0]}(Q^{[k/0]})^{-1} \quad , \quad \varepsilon_{-2}^{(k)} = P^{[k/-1]}(Q^{[k/-1]})^{-1}$$

$$\varepsilon_{2k}^{(-k)} = P^{[0/k]}(Q^{[0/k]})^{-1} \quad , \quad \varepsilon_{2k}^{(-k-1)} = 0$$

Auxiliary quantities with an odd subscript are defined by

$$(\varepsilon_{2\ell}^{(k-1)} - \varepsilon_{2\ell}^{(k)})^{-1} = \varepsilon_{2\ell-1}^{(k)} - \varepsilon_{2\ell+1}^{(k-1)}$$

$$(\varepsilon_{2\ell}^{(k+1)} - \varepsilon_{2\ell}^{(k)})^{-1} = \varepsilon_{2\ell+1}^{(k)} - \varepsilon_{2\ell-1}^{(k+1)}$$

$$(\varepsilon_{2\ell-2}^{(k+1)} - \varepsilon_{2\ell}^{(k)})^{-1} = \varepsilon_{2\ell-1}^{(k)} - \varepsilon_{2\ell-1}^{(k+1)}$$

$$(\varepsilon_{2\ell+2}^{(k-1)} - \varepsilon_{2\ell}^{(k)})^{-1} = \varepsilon_{2\ell+1}^{(k)} - \varepsilon_{2\ell+1}^{(k-1)}$$

Then, clearly for all k and ℓ we have

$$\varepsilon_{\ell+1}^{(k)} = \varepsilon_{\ell-1}^{(k+1)} + (\varepsilon_{\ell}^{(k+1)} - \varepsilon_{\ell}^{(k)})^{-1}$$

with the same boundary conditions as above, except for $\varepsilon_{-2}^{(k)}$, which can be replaced by $\varepsilon_{-1}^{(k)} = 0$.

This gives an algorithm which is formally the same as in the scalar case.

8. Conclusion

For a normal Padé table, there is no significant difference between the theory of scalar or matrix Padé approximants. Very similar results are obtained although some care has to be taken in the proofs. Rectangular 2D-Padé approximants in the sense of Bose-Basu and Lévy-Morf-Kung can be computed in a similar way. The key to both is the solution of block Hankel or block Toeplitz systems. The recursive solutions that exist there give directly continued fraction expansions

that correspond to staircases or sawteeth. Once this is observed, QD relations and the ε-altorithm are easily derived.
Other forms of the ε-algorithm for the 2D case have been given in the literature. Watson [10] applies the 1-D ε-algorithm to both variables successively. The corresponding approximation is not the Padé type as it is normally defined.
In [11] A. Cuyt gives an ε-algorithm for the abstract multivariate Padé approximation. The derivation is based on determinantal identities.

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