

# Degrees of Relatedness

A Unified Framework for Parametricity, Irrelevance, Ad Hoc Polymorphism, Intersections, Unions and Algebra in Dependent Type Theory

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- Parametricity
  - Intuition
  - In System F
  - In System  $F\omega$
  - In DTT
- Degrees of relatedness
  - Intro & known modalities
  - Structural modality
  - $\cap$  and  $\cup$

# Parametricity, intuitively

In System F,  $F\omega$ , Haskell, . . . , **type parameters** are parametric.

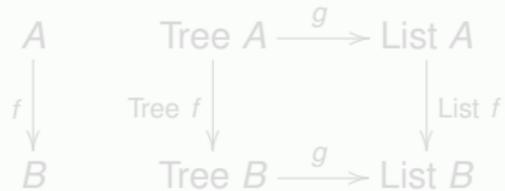
- Only used for type-checking,
- Not inspected (e.g. no pattern matching),
- Same algorithm on all types.

Enforced by the type system.

## Example

$g : \forall X. \text{Tree } X \rightarrow \text{List } X$

By parametricity:



irrespective of implementation.

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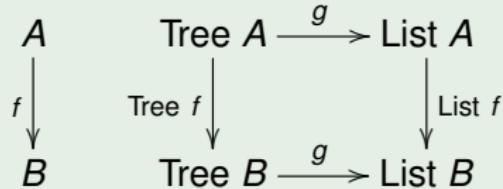
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# Intermezzo: Proof relevance

$R \in \text{Rel}(A, B)$  if

- $R \subseteq A \times B$ ,
- $R : A \times B \rightarrow \text{Prop}$ ,
- $R : A \times B \rightarrow \text{Set}$ ,

$R(a, b)$  if

- $(a, b) \in R$ ,
- $* \in R(a, b)$ ,
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## Example (Isomorphic groups)

$(\cong) \in \text{Rel}(\text{Grp}, \text{Grp})$

$\bar{x} \mapsto (\bar{x}, \bar{x}) : \mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ , so these groups are **isomorphic**.

## Example (Related sets)

$\text{Rel} \in \text{Rel}(\text{Set}, \text{Set})$

$\text{ElemOf} \in \text{Rel}(X, \text{List } X)$ , so these sets are **related**.

Always true, often in many ways!

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# Parametricity in **System F**

*(Reynolds, 1983)*

# System F: Type Judgements

 $X : *$ , $Y : *$  $\vdash$  $X \times Y : *$ 

Object semantics:

 $X \in \text{Set},$  $Y \in \text{Set}$  $\Rightarrow$  $X \times Y \in \text{Set}$ 

Relational semantics:

 $X_1 \in \text{Set},$  $Y_1 \in \text{Set}$  $\Rightarrow$  $X_1 \times Y_1 \in \text{Set}$  $X_2 \in \text{Set},$  $Y_2 \in \text{Set}$  $\Rightarrow$  $X_2 \times Y_2 \in \text{Set}$ 

Identity Extension Lemma (IEL)

... such that  $\text{Eq}_X \times \text{Eq}_Y \cong \text{Eq}_{X \times Y}$

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## Identity Extension Lemma (IEL)

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Type formers propagate relations:

$\bar{X} \times \bar{Y}$  Componentwise,

$\bar{X} \rightarrow \bar{Y}$  For all  $x_1, x_2$ :  $\bar{X}(x_1, x_2) \rightarrow \bar{Y}(f_1 x_1, f_2 x_2)$ ,

List  $\bar{X}$  Equal length,  $\bar{X}$ -related components,

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# Term Judgements

$$X : * \quad | \quad p : X, \quad q : X \quad \vdash \quad t[X, p, q] : X$$

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$$X \in \text{Set}, \quad p \in X, \quad q \in X \quad \Rightarrow \quad t[X, p, q] \in X$$

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## Free Theorem (Church Booleans)

Either  $t[X, p, q] = p$  or  $t[X, p, q] = q$ ,

i.e.  $\text{Bool} \cong \forall X. X \rightarrow X \rightarrow X$

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- Congruence (respected by everything),
- Extends equality: becomes equality in homogeneous case.

Term-relatedness à la Reynolds:

- ✓ Is a congruence (prev. slide)
  - ✓ Identity extension
- ⇒ **Is a notion of het. equality.**

Type-relatedness à la Reynolds is **NOT**:

To prove relatedness is to give  $\bar{X} \in \text{Rel}(X_1, X_2)$

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# Parametricity Summarized

**Open types** map **Rel-related types** to **Rel-related types**:

$$\begin{array}{c} X_1 \in \text{Set}, \\ \bar{X} \in \text{Rel} \\ \downarrow \\ X_2 \in \text{Set}, \end{array} \quad \begin{array}{c} Y_1 \in \text{Set}, \\ \bar{Y} \in \text{Rel} \\ \downarrow \\ Y_2 \in \text{Set} \end{array} \quad \Rightarrow \quad \begin{array}{c} X_1 \times Y_1 \in \text{Set} \\ \bar{X} \times \bar{Y} \in \text{Rel} \\ \downarrow \\ X_2 \times Y_2 \in \text{Set} \end{array}$$

**Open terms** map **Rel-related types**  
and **het. equal values** to **het. equal values**:

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# Parametricity in **System F $\omega$** *(Atkey, 2012)*

**IEL:** Open types map preserve Eq.

If  $X \in \text{Set}$  (sem. of  $X : *$ ),  
then  $\text{Eq}_X \in \text{Rel}(X, X)$

If  $F \in \text{Set} \rightarrow \text{Set}$  (sem. of  $F : * \rightarrow *$ ),  
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| *                            | Set   | Rel  | $\text{Eq} : (T : *) \rightarrow \text{Rel}(T, T)$                         |
| $* \times *$                 | Set $\times$ Set  | Rel $\times$ Rel   | $\text{Eq} \times \text{Eq}$   |
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## Parametricity in Dependent Type Theory

DTT treats **types** and **terms** on equal footing, BUT

- Related **terms** are **het. equal**,
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Free Theorem (Yoneda lemma / Representation independence)

$$\forall(X : *).(X \rightarrow A) \rightarrow (X \rightarrow B) \quad \cong \quad A \rightarrow B$$

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# Let's have two relations

## System F

Values can be related:

$$(s : S) \sim^R (t : T)$$

**IEL:** if  $(s : A) \sim^A (t : A)$  then  $s = t$   
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Types can be related:

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which gives meaning to

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Things can be 0-related:

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Will also write  $s =^R t$  for  $s \sim_0^R t$ .

Things can be 1-related:

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# Continuity and Parametricity

## Continuity

$\text{List} : (\mathbf{con} \dashv \mathcal{U}_0) \rightarrow \mathcal{U}_0$

$$\begin{array}{ccc} X = Y & \xrightarrow{\quad} & \text{List } X = \text{List } Y \\ \downarrow \text{Eq} & & \downarrow \text{Eq} \\ X \sim_1 Y & \xrightarrow{\quad} & \text{List } X \sim_1 \text{List } Y \end{array}$$

In System F:  $\rightarrow$

## Parametricity

$\llbracket \quad : (\mathbf{par} \dashv X : \mathcal{U}_0) \rightarrow \text{List } X$

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## Degrees of Relatedness

- Level -1 types:  $\top$  (propositions)
- **Level 0 types:**  $= \rightarrow \top$
- **Level 1 types:**  $= \rightarrow \cap_1 \rightarrow \top$
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## Degrees of Relatedness

- **Level -1 types:**  $\top$  (propositions)
- **Level 0 types:**  $= \rightarrow \top$
- **Level 1 types:**  $= \rightarrow \cap_1 \rightarrow \top$
- **Level 2 types:**  $= \rightarrow \cap_1 \rightarrow \cap_2 \rightarrow \top$
- ...

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- **Depth -1 types:**  $\top$  (propositions)
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- ...

We can decouple **level** (predicativity) and **depth** (number of relations).

## Continuity: $1 \rightarrow 1$

$\text{List} : (\mathbf{con} \dashv \mathcal{U}_0) \rightarrow \mathcal{U}_0$

$$\begin{array}{ccc} X = Y & \xrightarrow{\hspace{2cm}} & \text{List } X = \text{List } Y \\ \downarrow \text{Eq} & \text{IEL} & \downarrow \text{Eq} \\ X \sim_1 Y & \xrightarrow{\hspace{2cm}} & \text{List } X \sim_1 \text{List } Y \end{array}$$

## Parametricity: $1 \rightarrow 0$

$[] : (\mathbf{par} \dashv X : \mathcal{U}_0) \rightarrow \text{List } X$

$$\begin{array}{c} X = Y \\ \downarrow \text{Eq} \\ R : X \sim_1 Y \xrightarrow{\hspace{2cm}} []_X =^{\text{List } R} []_Y \end{array}$$

## Definition

A **modality**  $\mu : c \rightarrow d$  is any diagram from  $c+1$  to  $d+1$  relations that preserves het. equality.

Bridges et al. (2016)

Bruijn et al. (2016), Bruijn, Chapman & Riley (2017)

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|                         | <b>Value-level objects</b><br>$a : A : \mathcal{U}_0$ can be   | <b>Type-level objects</b><br>$A : \kappa : \mathcal{U}_1$ can be  |
|-------------------------|--|---|
| 0-related<br>(het. eq.) | $(2 + 5 : \mathbb{N}) \sim_0^{\mathbb{N}} (7 : \mathbb{N})$<br>$([] : \text{List}_4 A) \sim_0^{\text{List}_\bullet A} ([] : \text{List}_6 A)$<br>$\exists R. (5 : \mathbb{N}) \sim_0^R (\text{true} : \text{Bool})$<br>$\forall R. \text{if}_X \sim_0^{\text{Bool} \rightarrow R \rightarrow R \rightarrow R} \text{if}_Y$ | $((\lambda X.X) \text{Bool} : \mathcal{U}_0) \sim_0^{\mathcal{U}_0} (\text{Bool} : \mathcal{U}_0)$<br>$([] : \text{List}_4 \kappa) \sim_0^{\text{List}_\bullet \kappa} ([] : \text{List}_6 \kappa)$<br>...  |
| 1-related               | n/a  | $((A : \mathcal{U}_0) \sim_1^{\mathcal{U}_0} (B : \mathcal{U}_0)) := \text{Rel}(A, B)$<br>$\mathbb{N} := \text{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \sim_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$<br>$\text{List}_\bullet A : \text{List}_4 A \sim_1^{\mathcal{U}_0} \text{List}_6 A$ |

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$(2+5 : \mathbb{N}) \sim_0^{\mathbb{N}} (7 : \mathbb{N})$   
because  
 $2+5 \equiv 7$

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$((\lambda X.X) \text{Bool} : \mathcal{U}_0) \sim_0^{\mathcal{U}_0} (\text{Bool} : \mathcal{U}_0)$   
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$([] : \text{List}_4 A) \sim_0^{\text{List}_\bullet A} ([] : \text{List}_6 A)$   
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 $\text{List}_\bullet A \in \text{Rel}(\text{List}_4 A, \text{List}_6 A)$

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$$([] : \text{List}_4 \kappa) \sim_0^{\text{List}_\bullet \kappa} ([] : \text{List}_6 \kappa)$$

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 for some  
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$(\text{if}_X : \text{Bool} \rightarrow X \rightarrow X \rightarrow X) \sim_0^{\text{Bool} \rightarrow R \rightarrow R \rightarrow R} (\text{if}_Y : \text{Bool} \rightarrow Y \rightarrow Y \rightarrow Y)$   
for all  
 $R \in \text{Rel}(X, Y)$

|                         | Value-level objects<br>$a : A : \mathcal{U}_0$ can be  | Type-level objects<br>$A : \kappa : \mathcal{U}_1$ can be   |
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$(a : A) \sim_i^R (b : B)$  is always w.r.t.  $R : (A : \mathcal{U}_n) \sim_{i+1}^{\mathcal{U}_n} (B : \mathcal{U}_n)$

$$((A : \mathcal{U}_0) \sim_1^{\mathcal{U}_0} (B : \mathcal{U}_0)) := \text{Rel}(A, B)$$

|                         | <b>Value-level objects</b><br>$a : A : \mathcal{U}_0$ can be   | <b>Type-level objects</b><br>$A : \kappa : \mathcal{U}_1$ can be  |
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See paper for  $\sim_2$  (as of **kind-level**) and higher.

Four laws:

- **Reflexivity:**  $(a : A) \sim_i^A (a : A)$
- **Weakening:**  $((a : A) \sim_i^R (b : B)) \rightarrow ((a : A) \sim_{i+1}^R (b : B))$
- **Dependency:**  $(a : A) \sim_i^R (b : B)$  presumes  $R : A \sim_{i+1}^{\mathcal{U}_n} B$
- **Identity extension:**  $(a : A) \sim_0^A (b : A)$  means  $a \equiv b : A$ .

# Ad hoc polymorphism

Law of excluded middle (**wrong**):

$$\text{lem} : (\text{par} \downarrow X : \mathcal{U}) \rightarrow X \uplus (X \rightarrow \text{Empty})$$

**Free Theorem** (contradiction!)

$$((\text{par} \downarrow X : \mathcal{U}) \rightarrow X) \uplus ((\text{par} \downarrow X : \mathcal{U}) \rightarrow X \rightarrow \text{Empty})$$

Ad hoc:  $1 \rightarrow 0$

$$\text{lem} : (\text{hoc} \downarrow X : \mathcal{U}) \rightarrow X \uplus (X \rightarrow \text{Empty})$$

$$\begin{array}{ccc} X = Y & \longrightarrow & \text{lem } X = \text{lem } Y \\ \downarrow & & \\ X \sim_1 Y & & \end{array}$$

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**Irrelevance** := ignored by definitional equality

Sized lists:

- $\text{nil}_X : (\text{irr} \downarrow n : \mathbb{N}) \rightarrow (\text{irr} \downarrow 0 < n) \rightarrow \text{List}_n X,$
- $\text{cons}_X : (\text{irr} \downarrow m n : \mathbb{N}) \rightarrow (\text{irr} \downarrow m < n) \rightarrow X \rightarrow \text{List}_m X \rightarrow \text{List}_n X$

Two ways to annotate  $[a]$ :

- $\text{as}_2 := \text{cons}_A 2 \_ \_ \_ \_ a(\text{nil}_A 2 \_ \_ \_ \_ ) : \text{List}_5 A, \quad \text{nil}_A 2 \_ \_ \_ \_ : \text{List}_2 A$
- $\text{as}_3 := \text{cons}_A 3 \_ \_ \_ \_ a(\text{nil}_A 3 \_ \_ \_ \_ ) : \text{List}_5 A, \quad \text{nil}_A 3 \_ \_ \_ \_ : \text{List}_3 A$
- $\text{cons}_A * * \_ \_ a(\text{nil}_A * * \_ \_ ) : \text{List}_5 A \Rightarrow \text{as}_2 \equiv \text{as}_3,$

Irrelevance is a **dependent generalization** of **constancy**.

Codomain  $\text{List}_\square A$  must be **shape-irrelevant**.

Abel & Scherer (2012), example 2.8

Abel, Vezzosi & Winterhalter (2017)

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# Shape-Irrelevance and Irrelevance

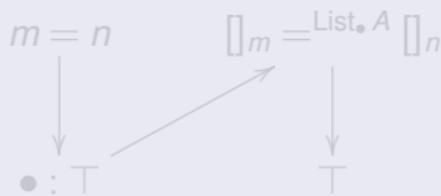
Shape-irrelevance:  $0 \rightarrow 1$

$$\text{List}_{\square} A : (\text{shi} + \mathbb{N}) \rightarrow \mathcal{U}_0$$



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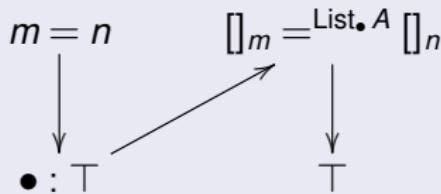
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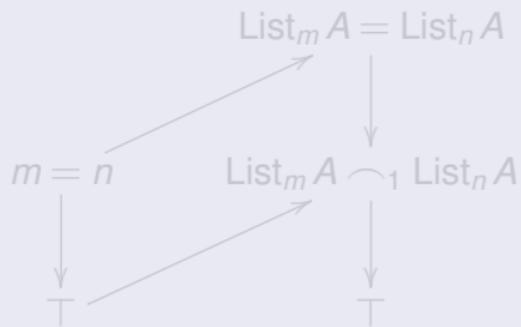
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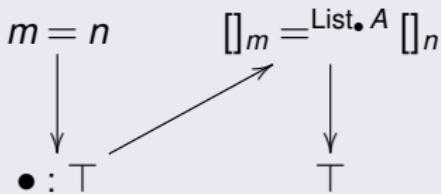
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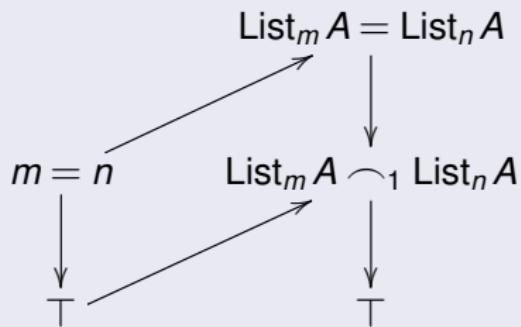
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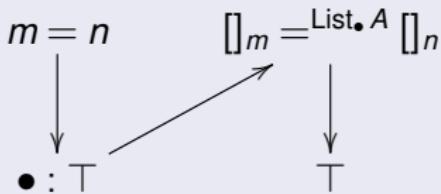
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# Composition of Modalities

Given

- $f : (\mu \dashv A) \rightarrow B,$
- $g : (\nu \dashv B) \rightarrow C,$

what is the modality of  $g \circ f : (\nu \circ \mu \dashv A) \rightarrow C?$

Example

$\text{if}_{(\text{List}_4 A)} : \text{Bool} \rightarrow \text{List}_4 A \rightarrow \text{List}_4 A \rightarrow \text{List}_4 A$

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- We can ignore **irrelevant** parts.
- *if* uses first arg. parametrically.
- List uses size index **shape-irrelevantly**.

So  $\text{par} \circ \text{shi} = \text{irr}?$

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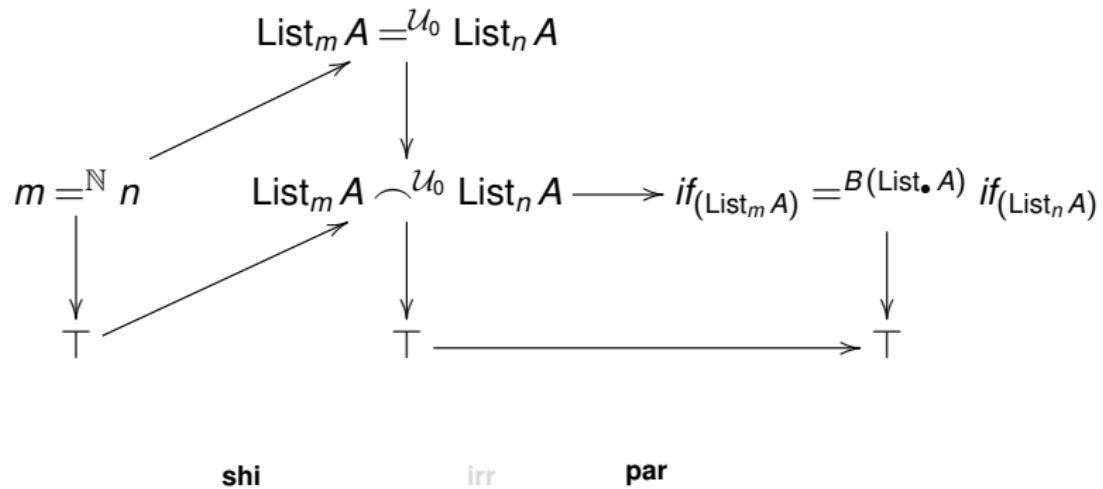
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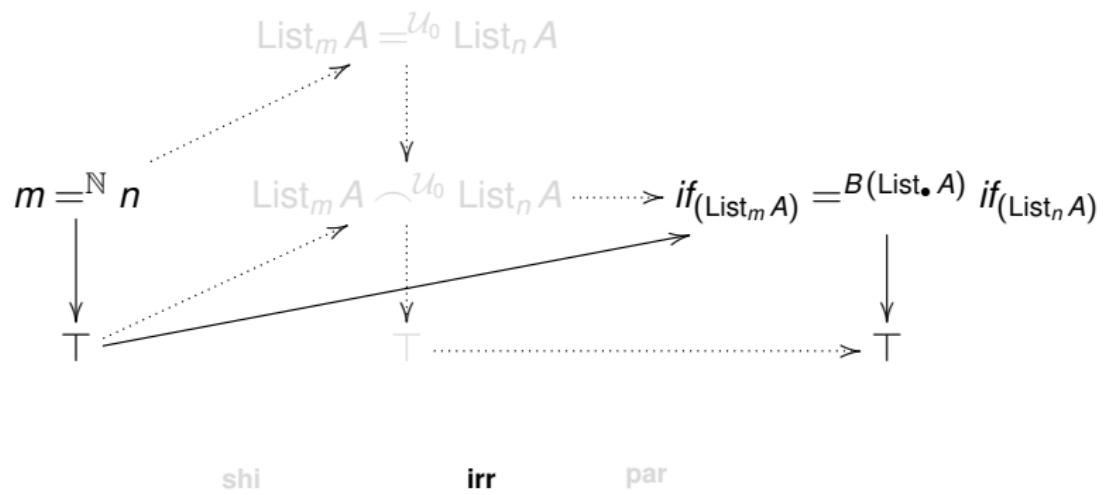
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# All modalities at lowest levels

| $(\mu \dashv A) \rightarrow B$ | $B : \mathcal{U}_0$<br>values | $B : \mathcal{U}_1$<br>types                    | $B : \mathcal{U}_n$             |
|--------------------------------|-------------------------------|---|---------------------------------|
| $A : \mathcal{U}_0$<br>values  | <b>hoc, irr</b>               | <b>hoc, shi, irr</b>                            |                                 |
| $A : \mathcal{U}_1$<br>types   | <b>hoc, par, irr</b>          | <b>hoc, con, shi,<br/>par, shi&amp;par, irr</b> |                                 |
| $A : \mathcal{U}_m$            |                               |   | $\frac{(m+n+2)!}{(m+1)!(n+1)!}$ |

# Algebra: The Structural Modality

Least fixpoint  $\text{Mu}F$  of functor  $F : \text{Type} \rightarrow \text{Type}$ :

- $\cong$  inductive type  $A$  with constructor  $\alpha : FA \rightarrow A$
- $\cong$  initial  $F$ -algebra (i.e. type  $X$  with  $\xi : FX \rightarrow X$ )
- $\cong$  Church-encoding:

$$\forall \underbrace{X}_{\text{carrier}}. \underbrace{(FX \rightarrow X)}_{F\text{-algebra-structure}} \rightarrow X$$

View  $\forall$  as **limit** operator.

Limit of everything is initial object.

Church encoding:  $\text{Mu}F$  is limit of all  $F$ -algebras.

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# Curry the Church-encoding?

Currying = collecting arguments in a (dependent) tuple.

System F

$$\begin{aligned} & \forall X.(FX \rightarrow X) \rightarrow X \\ & \cong^? (\exists X.FX \rightarrow X) \rightarrow X \end{aligned}$$

Scope error:  $X$  is out of scope!

DTT

$$\begin{aligned} & (\mathbf{par} \downarrow X : \mathcal{U}) \rightarrow (FX \rightarrow X) \rightarrow X \\ & \cong^? \left( \hat{X} : (\mathbf{par} \downarrow X : \mathcal{U}) \times \right. \\ & \quad \left. (FX \rightarrow X) \right) \rightarrow (\text{fst } \hat{X}) \end{aligned}$$

Unsound:  $\text{fst}$  disrespects equality.

$$\top : \text{Bool} \hookrightarrow_1^{\mathcal{U}} \mathbb{N}$$

$$\bullet : \text{true} =^\top 4$$

$$(\top, \bullet) : (\text{Bool}, \text{true}) = (\mathbb{N}, 4)$$

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# Structurality to the rescue

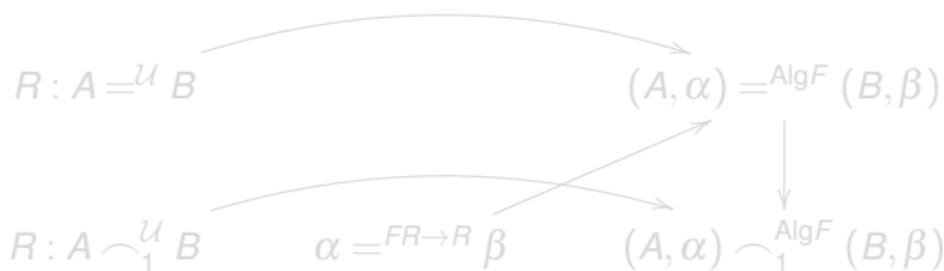
$(\mathbf{par} : X : \mathcal{U}) \times (\mathcal{F}X \rightarrow X)$  is a **datatype**:

- **par** takes type  $X$  to value level,
- related types are identified,

$\mathcal{F}$ -algebras are **type-level** objects:

- $X$  should stay at the type-level,
- structure  $\xi : \mathcal{F}X \rightarrow X$  should be taken to the type-level

$$\text{Alg}\mathcal{F} = (X : \mathcal{U}) \times (\mathbf{str} : \mathcal{F}X \rightarrow X)$$



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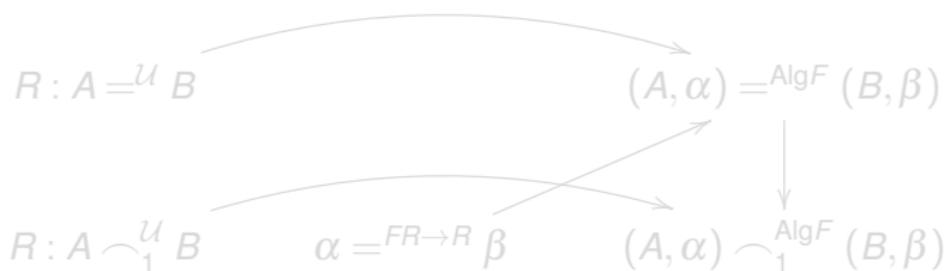
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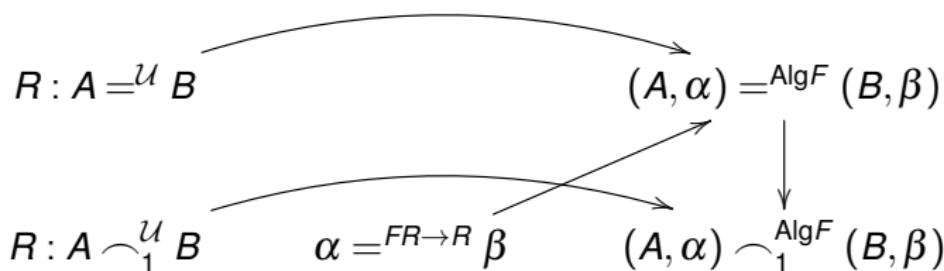
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**str** :  $d \rightarrow d + 1$

**Structurality:** how algebras depend on their structure.

Example ( $F$ -algebras)

$$\text{Alg}F = (X : \mathcal{U}) \times (\text{str} : FX \rightarrow X)$$

$$(\text{par} : X : \mathcal{U}) \rightarrow (FX \rightarrow X) \rightarrow X \quad \cong \quad (\text{par} : \hat{X} : \text{Alg}F) \rightarrow \text{fst } \hat{X}$$

$$\text{par} \circ \text{con} = \text{par}$$

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# Unions and Intersections

$$S \times T \cong (b : \text{Bool}) \rightarrow \text{if}(b, S, T)$$

requires  $R \sqsubset S \rightsquigarrow_1^U T$

$$(s, t) \leftrightarrow \lambda b. \text{if}(b, s, t)$$

requires  $\_ \sqsubset s =^R t$

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Definitional relatedness:

- Syntax: Type-checker can generate evidence.
- Semantics using erasure: Either hacky or WIP.

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$$S \cap T \cong (\text{irr} \mid b : \text{Bool}) \rightarrow \text{if}_{\text{shi}}(b, S, T) \quad \text{requires } R \sqsubseteq S \cap_1^U T$$

$$(s, t) \leftrightarrow \lambda b. \text{if}_{\text{irr}}(b, s, t) \quad \text{requires } \_ \sqsubseteq s =^R t$$

$$\text{fst} \leftrightarrow \lambda f. f \text{true}$$

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$$S \uplus T \cong (b : \text{Bool}) \times \text{if}(b, S, T) \quad \text{requires } R \sqsubseteq S \uplus_1^U T$$

$$\text{inl } s \leftrightarrow (\text{true}, s)$$

$$\text{inr } t \leftrightarrow (\text{false}, t)$$

$$\text{case}(q, f, g) \leftrightarrow \text{if}(\text{fst } q, f, g)(\text{snd } q) \quad \text{requires } \_ \sqsubseteq f =^{R \rightarrow C} g$$

Definitional relatedness:

- Syntax: Type-checker can generate evidence.
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## The Cubical Model

Modelling multimode DTT:

- For every **mode** (depth)  $d$ , pick a **model**  $\mathcal{D}$  of MLTT.
- For every **modality**  $\mu : c \rightarrow d$ , pick a **model morphism**  $\mu : \mathcal{C} \rightarrow \mathcal{D}$ .

Presheaf models:

- Every presheaf cat. models MLTT with  $\Pi, \Sigma, \mathcal{U}, =$  with UIP, ...

Degrees of relatedness:

- Model  $d$  using **refl. graphs** with edges labelled  $0, 1, \dots, d$ ,
- Internal parametricity  $\Rightarrow$  iterated graphs (= **cubical sets**),
- Restrict to **discrete** types: those that satisfy IEL,
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Fixing modal  $\Sigma$ -type:

- **Problem:** by default  $(\text{Bool}, \text{true}) \neq (\mathbb{N}, 4) : (\text{par} \downarrow X : \mathcal{U}) \times X$ .
- **Solution:** take a quotient. Prove that eliminator still works.  
(fst becomes unsound.)

Fixing the universe:

- **Problem:** default Hofmann-Streicher universe  $\mathcal{U}^{HS}$  is not discrete.
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- **Solution:** Define
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# Conclusion

- Unified framework (type system + presheaf model) for:
  - **parametricity**
  - **continuity**
  - **. irrelevance**
  - **.. shape-irrelevance**
  - **ad hoc polym.**
  - novel **structural** modality
- Understanding of
  - **composition**,
  - **corresp. term  $\leftrightarrow$  type** (e.g. **irr  $\leftrightarrow$  shi**),
  - **different nature of types** (depth)
- Type-checking time **erasure of irrelevant subterms**
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## Try it out!

parametric branch of Agda (by Andrea Vezzosi)

- Implements ParamDTT (depth 1 fragment)
- With Glue/Weld, you can prove free theorems internally
- [github.com/agda/agda/tree/parametric](https://github.com/agda/agda/tree/parametric)
- [github.com/Saizan/parametric-demo](https://github.com/Saizan/parametric-demo)
- All 6 modalities  $\mu : 1 \rightarrow 1$  are available, unlike in  
Nuyts, Vezzosi & Devriese (2017)

Degrees of Relatedness: In progress.

Asymmetric relations:

- Proof-relevant subtyping,
- Directed type theory (synthetic category theory),
- Directed univalence :  $(A \curvearrowright_1 B) \simeq (A \rightarrow B)$ .

Erasure-based presheaf models for def. relatedness,  $\cap$  and  $\cup$ .

## Take home message

Describe function behaviour as action on degree of relatedness.  
**str, con, par, hoc, shi, irr** are instances of this.

Thanks!

Questions?

# Comparison with HoTT

| Degrees of Relatedness                                   | HoTT  |
|--|---|
| functions <b>act</b> on $\sim_i$                         | functions <b>preserve</b> $\simeq$                            |
| equality as $\sim_0$                                     | equality as $\simeq$  |
| relational HITs <sup>1</sup>                             | groupoidal HITs   |
| depth: $\mathcal{U}_\ell^d : \mathcal{U}_{\ell+1}^{d+1}$ | $h$ -level: $\mathcal{U}_\ell^h : \mathcal{U}_{\ell+1}^{h+1}$ |

---

<sup>1</sup>future work

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| 0-related<br>(het. eq.) | $(2+5:\mathbb{N}) \sim_0^{\mathbb{N}} (7:\mathbb{N})$<br>$([] : \text{List}_4 A) \sim_0^{\text{List}_4 A} ([] : \text{List}_6 A)$<br>$\exists R. (5:\mathbb{N}) \sim_0^R (\text{true} : \text{Bool})$<br>$\forall R. \text{if}_X \sim_0^{\text{Bool} \rightarrow R \rightarrow R \rightarrow R} \text{if}_Y$ | $((\lambda X.X) \text{Bool} : \mathcal{U}_0) \sim_0^{\mathcal{U}_0} (\text{Bool} : \mathcal{U}_0)$<br>$([] : \text{List}_4 \kappa) \sim_0^{\text{List}_6 \kappa} ([] : \text{List}_6 \kappa)$<br>...   | $((\lambda \xi.\xi) \kappa : \mathcal{U}_1) \sim_0^{\mathcal{U}_1} (\kappa : \mathcal{U}_1)$<br>$([] : \text{List}_4 \mathcal{A}) \sim_0^{\text{List}_6 \mathcal{A}} ([] : \text{List}_6 \mathcal{A})$<br>...   |
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| 1-related               | n/a  | $((A : \mathcal{U}_0) \sim_1^{\mathcal{U}_0} (B : \mathcal{U}_0)) := \text{Rel}(A, B)$<br>$\mathbb{N} := \text{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \sim_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$<br>$\text{List}_* A : \text{List}_4 A \sim_1^{\mathcal{U}_0} \text{List}_6 A$<br>$R : (G : \text{Grp}) \sim_1^{\text{Grp}} (H : \text{Grp})$<br>$R : (G : \text{Grp}) \sim_1^V (M : \text{Mon})$ | $((\kappa : \mathcal{U}_1) \sim_1^{\mathcal{U}_1} (\lambda : \mathcal{U}_1)) := \text{Rel}(\kappa, \lambda)^{* \rightarrow *}$<br>$\mathcal{U}_0 : (\mathcal{U}_0 : \mathcal{U}_1) \sim_1^{\mathcal{U}_1} (\mathcal{U}_0 : \mathcal{U}_1)$<br>$\text{List}_* \kappa : \text{List}_4 \kappa \sim_1^{\mathcal{U}_1} \text{List}_6 \kappa$<br>$p : (\alpha : \text{Cat}) \sim_1^{\text{Cat}} (\beta : \text{Cat})$ |
| 2-related               | n/a  | n/a  | $V : (\text{Grp} : \mathcal{U}_1) \sim_2^{\mathcal{U}_1} (\text{Mon} : \mathcal{U}_1)$  |

$(5 : \mathbb{N}) \sim_0^R (\text{true} : \text{Bool})$   
 for some  
 $R \in \text{Rel}(\mathbb{N}, \text{Bool})$

|   | Value-level objects<br>$a : A : \mathcal{U}_0$ can be  | Type-level objects<br>$A : \kappa : \mathcal{U}_1$ can be  | Kind-level objects<br>$\kappa : \mathcal{A} : \mathcal{U}_2$ can be   |
|---|--|--|---|
| 0-related<br>(het. eq.)   | $(2+5:\mathbb{N}) \sim_0^{\mathbb{N}} (7:\mathbb{N})$<br>$([] : \text{List}_4 A) \sim_0^{\text{List}_6 A} ([] : \text{List}_6 A)$<br>$\exists R. (5:\mathbb{N}) \sim_0^R (\text{true} : \text{Bool})$<br>$\forall R. \text{if}_X \sim_0^{\text{Bool} \rightarrow R \rightarrow R \rightarrow R} \text{if}_Y$ | $((\lambda X.X) \text{Bool} : \mathcal{U}_0) \sim_0^{\mathcal{U}_0} (\text{Bool} : \mathcal{U}_0)$<br>$([] : \text{List}_4 \kappa) \sim_0^{\text{List}_6 \kappa} ([] : \text{List}_6 \kappa)$<br>$\dots$   | $((\lambda \xi.\xi) \kappa : \mathcal{U}_1) \sim_0^{\mathcal{U}_1} (\kappa : \mathcal{U}_1)$<br>$([] : \text{List}_4 \mathcal{A}) \sim_0^{\text{List}_6 \mathcal{A}} ([] : \text{List}_6 \mathcal{A})$<br>$\dots$   |
| 1-related   | n/a  | $((A : \mathcal{U}_0) \sim_1^{\mathcal{U}_0} (B : \mathcal{U}_0)) := \text{Rel}(A, B)$<br>$\mathbb{N} := \text{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \sim_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$<br>$\text{List}_* A : \text{List}_4 A \sim_1^{\mathcal{U}_0} \text{List}_6 A$<br>$R : (G : \text{Grp}) \sim_1^{\text{Grp}} (H : \text{Grp})$<br>$R : (G : \text{Grp}) \sim_1^V (M : \text{Mon})$ | $((\kappa : \mathcal{U}_1) \sim_1^{\mathcal{U}_1} (\lambda : \mathcal{U}_1)) := \text{Rel}(\kappa, \lambda)^{* \rightarrow *}$<br>$\mathcal{U}_0 : (\mathcal{U}_0 : \mathcal{U}_1) \sim_1^{\mathcal{U}_1} (\mathcal{U}_0 : \mathcal{U}_1)$<br>$\text{List}_* \kappa : \text{List}_4 \kappa \sim_1^{\mathcal{U}_1} \text{List}_6 \kappa$<br>$p : (\alpha : \text{Cat}) \sim_1^{\text{Cat}} (\beta : \text{Cat})$ |
| 2-related   | n/a  | n/a  | $V : (\text{Grp} : \mathcal{U}_1) \sim_2^{\mathcal{U}_1} (\text{Mon} : \mathcal{U}_1)$  |
| $(\text{if}_X : \text{Bool} \rightarrow X \rightarrow X \rightarrow X) \sim_0^{\text{Bool} \rightarrow R \rightarrow R \rightarrow R} (\text{if}_Y : \text{Bool} \rightarrow Y \rightarrow Y \rightarrow Y)$<br>for all<br>$R \in \text{Rel}(X, Y)$ |  |  |   |

|                         | Value-level objects<br>$a : A : \mathcal{U}_0$ can be  | Type-level objects<br>$A : \kappa : \mathcal{U}_1$ can be  | Kind-level objects<br>$\kappa : \mathcal{A} : \mathcal{U}_2$ can be   |
|-------------------------|--|--|---|
| 0-related<br>(het. eq.) | $(2+5:\mathbb{N}) \sim_0^{\mathbb{N}} (7:\mathbb{N})$<br>$([] : \text{List}_4 A) \sim_0^{\text{List}_4 A} ([] : \text{List}_6 A)$<br>$\exists R. (5:\mathbb{N}) \sim_0^R (\text{true} : \text{Bool})$<br>$\forall R. \text{if}_X \sim_0^{\text{Bool} \rightarrow R \rightarrow R \rightarrow R} \text{if}_Y$ | $((\lambda X.X) \text{Bool} : \mathcal{U}_0) \sim_0^{\mathcal{U}_0} (\text{Bool} : \mathcal{U}_0)$<br>$([] : \text{List}_4 \kappa) \sim_0^{\text{List}_4 \kappa} ([] : \text{List}_6 \kappa)$<br>$\dots$   | $((\lambda \xi.\xi) \kappa : \mathcal{U}_1) \sim_0^{\mathcal{U}_1} (\kappa : \mathcal{U}_1)$<br>$([] : \text{List}_4 \mathcal{A}) \sim_0^{\text{List}_4 \mathcal{A}} ([] : \text{List}_6 \mathcal{A})$<br>$\dots$   |
| 1-related               | n/a  | $((A : \mathcal{U}_0) \sim_1^{\mathcal{U}_0} (B : \mathcal{U}_0)) := \text{Rel}(A, B)$<br>$\mathbb{N} := \text{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \sim_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$<br>$\text{List}_* A : \text{List}_4 A \sim_1^{\mathcal{U}_0} \text{List}_6 A$<br>$R : (G : \text{Grp}) \sim_1^{\text{Grp}} (H : \text{Grp})$<br>$R : (G : \text{Grp}) \sim_1^V (M : \text{Mon})$ | $((\kappa : \mathcal{U}_1) \sim_1^{\mathcal{U}_1} (\lambda : \mathcal{U}_1)) := \text{Rel}(\kappa, \lambda)^{* \rightarrow *}$<br>$\mathcal{U}_0 : (\mathcal{U}_0 : \mathcal{U}_1) \sim_1^{\mathcal{U}_1} (\mathcal{U}_0 : \mathcal{U}_1)$<br>$\text{List}_* \kappa : \text{List}_4 \kappa \sim_1^{\mathcal{U}_1} \text{List}_6 \kappa$<br>$p : (\alpha : \text{Cat}) \sim_1^{\text{Cat}} (\beta : \text{Cat})$ |
| 2-related               | n/a  | n/a  | $V : (\text{Grp} : \mathcal{U}_1) \sim_2^{\mathcal{U}_1} (\text{Mon} : \mathcal{U}_1)$  |

$((\lambda X.X) \text{Bool} : \mathcal{U}_0) \sim_0^{\mathcal{U}_0} (\text{Bool} : \mathcal{U}_0)$   
because

$$(\lambda X.X) \text{Bool} \equiv \text{Bool}$$

|                         | Value-level objects<br>$a : A : \mathcal{U}_0$ can be   | Type-level objects<br>$A : \kappa : \mathcal{U}_1$ can be   | Kind-level objects<br>$\kappa : \mathcal{A} : \mathcal{U}_2$ can be   |
|-------------------------|---|---|---|
| 0-related<br>(het. eq.) | $(2+5:\mathbb{N}) \sim_0^{\mathbb{N}} (7:\mathbb{N})$<br>$([] : \text{List}_4 A) \sim_0^{\text{List}\bullet A} ([] : \text{List}_6 A)$<br>$\exists R. (5:\mathbb{N}) \sim_0^R (\text{true} : \text{Bool})$<br>$\forall R. if_X \sim_0^{\text{Bool} \rightarrow R \rightarrow R \rightarrow R} if_Y$ | $((\lambda X.X) \text{Bool} : \mathcal{U}_0) \sim_0^{\mathcal{U}_0} (\text{Bool} : \mathcal{U}_0)$<br>$([] : \text{List}_4 \kappa) \sim_0^{\text{List}\bullet \kappa} ([] : \text{List}_6 \kappa)$<br>...   | $((\lambda \xi.\xi) \kappa : \mathcal{U}_1) \sim_0^{\mathcal{U}_1} (\kappa : \mathcal{U}_1)$<br>$([] : \text{List}_4 \mathcal{A}) \sim_0^{\text{List}\bullet \mathcal{A}} ([] : \text{List}_6 \mathcal{A})$<br>...  |
| 1-related               | n/a   | $((A : \mathcal{U}_0) \sim_1^{\mathcal{U}_0} (B : \mathcal{U}_0)) := \text{Rel}(A, B)$<br>$\mathbb{N} := \text{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \sim_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$<br>$\text{List}\bullet A : \text{List}_4 A \sim_1^{\mathcal{U}_0} \text{List}_6 A$<br>$R : (G : \text{Grp}) \sim_1^{\text{Grp}} (H : \text{Grp})$<br>$R : (G : \text{Grp}) \sim_1^V (M : \text{Mon})$ | $((\kappa : \mathcal{U}_1) \sim_1^{\mathcal{U}_1} (\lambda : \mathcal{U}_1)) := \text{Rel}(\kappa, \lambda)^{\{*\rightarrow *\}}$<br>$\mathcal{U}_0 : (\mathcal{U}_0 : \mathcal{U}_1) \sim_1^{\mathcal{U}_1} (\mathcal{U}_0 : \mathcal{U}_1)$<br>$\text{List}\bullet \kappa : \text{List}_4 \kappa \sim_1^{\mathcal{U}_1} \text{List}_6 \kappa$<br>$p : (\alpha : \text{Cat}) \sim_1^{\text{Cat}} (\beta : \text{Cat})$ |
| 2-related               | n/a   | n/a   | $V : (\text{Grp} : \mathcal{U}_1) \sim_2^{\mathcal{U}_1} (\text{Mon} : \mathcal{U}_1)$  |

$([] : \text{List}_4 \kappa) \sim_0^{\text{List}\bullet \kappa} ([] : \text{List}_6 \kappa)$

|                         | Value-level objects<br>$a : A : \mathcal{U}_0$ can be  | Type-level objects<br>$A : \kappa : \mathcal{U}_1$ can be  | Kind-level objects<br>$\kappa : \mathcal{A} : \mathcal{U}_2$ can be  |
|-------------------------|--|--|--|
| 0-related<br>(het. eq.) | $(2+5:\mathbb{N}) \sim_0^{\mathbb{N}} (7:\mathbb{N})$<br>$([] : \text{List}_4 A) \sim_0^{\text{List} \bullet A} ([] : \text{List}_6 A)$<br>$\exists R. (5:\mathbb{N}) \sim_0^R (\text{true} : \text{Bool})$<br>$\forall R. \text{if}_X \sim_0^{\text{Bool} \rightarrow R \rightarrow R \rightarrow R} \text{if}_Y$ | $((\lambda X.X) \text{Bool} : \mathcal{U}_0) \sim_0^{\mathcal{U}_0} (\text{Bool} : \mathcal{U}_0)$<br>$([] : \text{List}_4 \kappa) \sim_0^{\text{List} \bullet \kappa} ([] : \text{List}_6 \kappa)$<br>...   | $((\lambda \xi.\xi) \kappa : \mathcal{U}_1) \sim_0^{\mathcal{U}_1} (\kappa : \mathcal{U}_1)$<br>$([] : \text{List}_4 \mathcal{A}) \sim_0^{\text{List} \bullet \mathcal{A}} ([] : \text{List}_6 \mathcal{A})$<br>...  |
| 1-related               | n/a  | $((A : \mathcal{U}_0) \sim_1^{\mathcal{U}_0} (B : \mathcal{U}_0)) := \text{Rel}(A, B)$<br>$\mathbb{N} := \text{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \sim_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$<br>$\text{List} \bullet A : \text{List}_4 A \sim_1^{\mathcal{U}_0} \text{List}_6 A$<br>$R : (G : \text{Grp}) \sim_1^{\text{Grp}} (H : \text{Grp})$<br>$R : (G : \text{Grp}) \sim_1^V (M : \text{Mon})$ | $((\kappa : \mathcal{U}_1) \sim_1^{\mathcal{U}_1} (\lambda : \mathcal{U}_1)) := \text{Rel}(\kappa, \lambda)^{\{\bullet \rightarrow \bullet\}}$<br>$\mathcal{U}_0 : (\mathcal{U}_0 : \mathcal{U}_1) \sim_1^{\mathcal{U}_1} (\mathcal{U}_0 : \mathcal{U}_1)$<br>$\text{List} \bullet \kappa : \text{List}_4 \kappa \sim_1^{\mathcal{U}_1} \text{List}_6 \kappa$<br>$\rho : (\alpha : \text{Cat}) \sim_1^{\text{Cat}} (\beta : \text{Cat})$ |
| 2-related               | n/a  | n/a  | $V : (\text{Grp} : \mathcal{U}_1) \sim_2^{\mathcal{U}_1} (\text{Mon} : \mathcal{U}_1)$   |

$$\left( (A : \mathcal{U}_0) \sim_1^{\mathcal{U}_0} (B : \mathcal{U}_0) \right) := \text{Rel}(A, B)$$

$(a : A) \sim_i^R (b : B)$  is always w.r.t.  $R : (A : \mathcal{U}_n) \sim_{i+1}^{\mathcal{U}_n} (B : \mathcal{U}_n)$

|                         | Value-level objects<br>$a : A : \mathcal{U}_0$ can be   | Type-level objects<br>$A : \kappa : \mathcal{U}_1$ can be   | Kind-level objects<br>$\kappa : \mathcal{A} : \mathcal{U}_2$ can be  |
|-------------------------|---|---|--|
| 0-related<br>(het. eq.) | $(2+5:\mathbb{N}) \sim_0^{\mathbb{N}} (7:\mathbb{N})$<br>$([] : \text{List}_4 A) \sim_0^{\text{List}\bullet A} ([] : \text{List}_6 A)$<br>$\exists R. (5:\mathbb{N}) \sim_0^R (\text{true} : \text{Bool})$<br>$\forall R. \text{if}_X \sim_0^{\text{Bool} \rightarrow R \rightarrow R \rightarrow R} \text{if}_Y$ | $((\lambda X.X) \text{Bool} : \mathcal{U}_0) \sim_0^{\mathcal{U}_0} (\text{Bool} : \mathcal{U}_0)$<br>$([] : \text{List}_4 \kappa) \sim_0^{\text{List}\bullet \kappa} ([] : \text{List}_6 \kappa)$<br>...   | $((\lambda \xi.\xi) \kappa : \mathcal{U}_1) \sim_0^{\mathcal{U}_1} (\kappa : \mathcal{U}_1)$<br>$([] : \text{List}_4 \mathcal{A}) \sim_0^{\text{List}\bullet \mathcal{A}} ([] : \text{List}_6 \mathcal{A})$<br>...   |
| 1-related               | n/a   | $((A : \mathcal{U}_0) \sim_1^{\mathcal{U}_0} (B : \mathcal{U}_0)) := \text{Rel}(A, B)$<br>$\mathbb{N} := \text{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \sim_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$<br>$\text{List}\bullet A : \text{List}_4 A \sim_1^{\mathcal{U}_0} \text{List}_6 A$<br>$R : (G : \text{Grp}) \sim_1^{\text{Grp}} (H : \text{Grp})$<br>$R : (G : \text{Grp}) \sim_1^V (M : \text{Mon})$ | $((\kappa : \mathcal{U}_1) \sim_1^{\mathcal{U}_1} (\lambda : \mathcal{U}_1)) := \text{Rel}(\kappa, \lambda)^{\{\bullet \rightarrow \bullet\}}$<br>$\mathcal{U}_0 : (\mathcal{U}_0 : \mathcal{U}_1) \sim_1^{\mathcal{U}_1} (\mathcal{U}_0 : \mathcal{U}_1)$<br>$\text{List}\bullet \kappa : \text{List}_4 \kappa \sim_1^{\mathcal{U}_1} \text{List}_6 \kappa$<br>$p : (\alpha : \text{Cat}) \sim_1^{\text{Cat}} (\beta : \text{Cat})$ |
| 2-related               | n/a   | n/a   | $V : (\text{Grp} : \mathcal{U}_1) \sim_2^{\mathcal{U}_1} (\text{Mon} : \mathcal{U}_1)$   |

$$\mathbb{N} := \text{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \sim_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$$

|                         | Value-level objects<br>$a : A : \mathcal{U}_0$ can be  | Type-level objects<br>$A : \kappa : \mathcal{U}_1$ can be  | Kind-level objects<br>$\kappa : \mathcal{A} : \mathcal{U}_2$ can be   |
|-------------------------|--|--|---|
| 0-related<br>(het. eq.) | $(2+5:\mathbb{N}) \sim_0^{\mathbb{N}} (7:\mathbb{N})$<br>$([] : \text{List}_4 A) \sim_0^{\text{List}_\bullet A} ([] : \text{List}_6 A)$<br>$\exists R. (5:\mathbb{N}) \sim_0^R (\text{true} : \text{Bool})$<br>$\forall R. \text{if}_X \sim_0^{\text{Bool} \rightarrow R \rightarrow R \rightarrow R} \text{if}_Y$ | $((\lambda X.X) \text{Bool} : \mathcal{U}_0) \sim_0^{\mathcal{U}_0} (\text{Bool} : \mathcal{U}_0)$<br>$([] : \text{List}_4 \kappa) \sim_0^{\text{List}_\bullet \kappa} ([] : \text{List}_6 \kappa)$<br>...   | $((\lambda \xi.\xi) \kappa : \mathcal{U}_1) \sim_0^{\mathcal{U}_1} (\kappa : \mathcal{U}_1)$<br>$([] : \text{List}_4 \mathcal{A}) \sim_0^{\text{List}_\bullet \mathcal{A}} ([] : \text{List}_6 \mathcal{A})$<br>...   |
| 1-related               | n/a  | $((A : \mathcal{U}_0) \sim_1^{\mathcal{U}_0} (B : \mathcal{U}_0)) := \text{Rel}(A, B)$<br>$\mathbb{N} := \text{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \sim_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$<br>$\text{List}_\bullet A : \text{List}_4 A \sim_1^{\mathcal{U}_0} \text{List}_6 A$<br>$R : (G : \text{Grp}) \sim_1^{\text{Grp}} (H : \text{Grp})$<br>$R : (G : \text{Grp}) \sim_1^V (M : \text{Mon})$ | $((\kappa : \mathcal{U}_1) \sim_1^{\mathcal{U}_1} (\lambda : \mathcal{U}_1)) := \text{Rel}(\kappa, \lambda)^{\{\bullet \rightarrow \bullet\}}$<br>$\mathcal{U}_0 : (\mathcal{U}_0 : \mathcal{U}_1) \sim_1^{\mathcal{U}_1} (\mathcal{U}_0 : \mathcal{U}_1)$<br>$\text{List}_\bullet \kappa : \text{List}_4 \kappa \sim_1^{\mathcal{U}_1} \text{List}_6 \kappa$<br>$p : (\alpha : \text{Cat}) \sim_1^{\text{Cat}} (\beta : \text{Cat})$ |
| 2-related               | n/a  | n/a  | $V : (\text{Grp} : \mathcal{U}_1) \sim_2^{\mathcal{U}_1} (\text{Mon} : \mathcal{U}_1)$  |

$\text{List}_\bullet A : (\text{List}_4 A : \mathcal{U}_0) \sim_1^{\mathcal{U}_0} (\text{List}_6 A : \mathcal{U}_0)$

|                         | Value-level objects<br>$a : A : \mathcal{U}_0$ can be  | Type-level objects<br>$A : \kappa : \mathcal{U}_1$ can be  | Kind-level objects<br>$\kappa : \mathcal{A} : \mathcal{U}_2$ can be  |
|-------------------------|--|--|--|
| 0-related<br>(het. eq.) | $(2+5:\mathbb{N}) \sim_0^{\mathbb{N}} (7:\mathbb{N})$<br>$([] : \text{List}_4 A) \sim_0^{\text{List}_4 A} ([] : \text{List}_6 A)$<br>$\exists R. (5:\mathbb{N}) \sim_0^R (\text{true} : \text{Bool})$<br>$\forall R. \text{if}_X \sim_0^{\text{Bool} \rightarrow R \rightarrow R \rightarrow R} \text{if}_Y$ | $((\lambda X.X) \text{Bool} : \mathcal{U}_0) \sim_0^{\mathcal{U}_0} (\text{Bool} : \mathcal{U}_0)$<br>$([] : \text{List}_4 \kappa) \sim_0^{\text{List}_4 \kappa} ([] : \text{List}_6 \kappa)$<br>...   | $((\lambda \xi.\xi) \kappa : \mathcal{U}_1) \sim_0^{\mathcal{U}_1} (\kappa : \mathcal{U}_1)$<br>$([] : \text{List}_4 \mathcal{A}) \sim_0^{\text{List}_4 \mathcal{A}} ([] : \text{List}_6 \mathcal{A})$<br>...  |
| 1-related               | n/a  | $((A : \mathcal{U}_0) \sim_1^{\mathcal{U}_0} (B : \mathcal{U}_0)) := \text{Rel}(A, B)$<br>$\mathbb{N} := \text{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \sim_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$<br>$\text{List}_\bullet A : \text{List}_4 A \sim_1^{\mathcal{U}_0} \text{List}_6 A$<br>$R : (G : \text{Grp}) \sim_1^{\text{Grp}} (H : \text{Grp})$<br>$R : (G : \text{Grp}) \sim_1^V (M : \text{Mon})$ | $((\kappa : \mathcal{U}_1) \sim_1^{\mathcal{U}_1} (\lambda : \mathcal{U}_1)) := \text{Rel}(\kappa, \lambda)^{\{\bullet \rightarrow \bullet\}}$<br>$\mathcal{U}_0 : (\mathcal{U}_0 : \mathcal{U}_1) \sim_1^{\mathcal{U}_1} (\mathcal{U}_0 : \mathcal{U}_1)$<br>$\text{List}_\bullet \kappa : \text{List}_4 \kappa \sim_1^{\mathcal{U}_1} \text{List}_6 \kappa$<br>$\rho : (\alpha : \text{Cat}) \sim_1^{\text{Cat}} (\beta : \text{Cat})$ |
| 2-related               | n/a  | n/a  | $V : (\text{Grp} : \mathcal{U}_1) \sim_2^{\mathcal{U}_1} (\text{Mon} : \mathcal{U}_1)$   |

$$\begin{array}{c}
 (\text{Grp} : \text{Grp}) \sim_1^{\text{Grp}} (\text{Grp} : \text{Grp}) \\
 \cong \\
 (\underline{R} : \underline{G} \sim_1^{\mathcal{U}_0} \underline{H}) \times (e_G \sim_0^R e_H) \times (*_G \sim_0^{R \rightarrow R \rightarrow R} *_H)
 \end{array}$$

|                         | Value-level objects<br>$a : A : \mathcal{U}_0$ can be  | Type-level objects<br>$A : \kappa : \mathcal{U}_1$ can be  | Kind-level objects<br>$\kappa : \mathcal{A} : \mathcal{U}_2$ can be  |
|-------------------------|--|--|--|
| 0-related<br>(het. eq.) | $(2+5:\mathbb{N}) \sim_0^{\mathbb{N}} (7:\mathbb{N})$<br>$([] : \text{List}_4 A) \sim_0^{\text{List}_4 A} ([] : \text{List}_6 A)$<br>$\exists R. (5:\mathbb{N}) \sim_0^R (\text{true} : \text{Bool})$<br>$\forall R. \text{if}_X \sim_0^{\text{Bool} \rightarrow R \rightarrow R \rightarrow R} \text{if}_Y$ | $((\lambda X.X) \text{Bool} : \mathcal{U}_0) \sim_0^{\mathcal{U}_0} (\text{Bool} : \mathcal{U}_0)$<br>$([] : \text{List}_4 \kappa) \sim_0^{\text{List}_4 \kappa} ([] : \text{List}_6 \kappa)$<br>...   | $((\lambda \xi.\xi) \kappa : \mathcal{U}_1) \sim_0^{\mathcal{U}_1} (\kappa : \mathcal{U}_1)$<br>$([] : \text{List}_4 \mathcal{A}) \sim_0^{\text{List}_4 \mathcal{A}} ([] : \text{List}_6 \mathcal{A})$<br>...  |
| 1-related               | n/a  | $((A : \mathcal{U}_0) \sim_1^{\mathcal{U}_0} (B : \mathcal{U}_0)) := \text{Rel}(A, B)$<br>$\mathbb{N} := \text{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \sim_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$<br>$\text{List}_\bullet A : \text{List}_4 A \sim_1^{\mathcal{U}_0} \text{List}_6 A$<br>$R : (G : \text{Grp}) \sim_1^{\text{Grp}} (H : \text{Grp})$<br>$R : (G : \text{Grp}) \sim_1^V (M : \text{Mon})$ | $((\kappa : \mathcal{U}_1) \sim_1^{\mathcal{U}_1} (\lambda : \mathcal{U}_1)) := \text{Rel}(\kappa, \lambda)^{\{\bullet \rightarrow \bullet\}}$<br>$\mathcal{U}_0 : (\mathcal{U}_0 : \mathcal{U}_1) \sim_1^{\mathcal{U}_1} (\mathcal{U}_0 : \mathcal{U}_1)$<br>$\text{List}_\bullet \kappa : \text{List}_4 \kappa \sim_1^{\mathcal{U}_1} \text{List}_6 \kappa$<br>$\rho : (\alpha : \text{Cat}) \sim_1^{\text{Cat}} (\beta : \text{Cat})$ |
| 2-related               | n/a  | n/a  | $V : (\text{Grp} : \mathcal{U}_1) \sim_2^{\mathcal{U}_1} (\text{Mon} : \mathcal{U}_1)$   |

$$\begin{aligned}
 & (G : \text{Grp}) \sim_1^V (M : \text{Mon}) \\
 & \quad := \\
 & (\underline{R} : \underline{G} \sim_1^{\mathcal{U}_0} \underline{M}) \times (e_G \sim_0^R e_M) \times (*_G \sim_0^{R \rightarrow R \rightarrow R} *_M)
 \end{aligned}$$

|                         | Value-level objects<br>$a : A : \mathcal{U}_0$ can be  | Type-level objects<br>$A : \kappa : \mathcal{U}_1$ can be  | Kind-level objects<br>$\kappa : \mathcal{A} : \mathcal{U}_2$ can be  |
|-------------------------|--|--|--|
| 0-related<br>(het. eq.) | $(2+5:\mathbb{N}) \sim_0^{\mathbb{N}} (7:\mathbb{N})$<br>$([] : \text{List}_4 A) \sim_0^{\text{List}_4 A} ([] : \text{List}_6 A)$<br>$\exists R. (5:\mathbb{N}) \sim_0^R (\text{true} : \text{Bool})$<br>$\forall R. \text{if}_X \sim_0^{\text{Bool} \rightarrow R \rightarrow R \rightarrow R} \text{if}_Y$ | $((\lambda X.X) \text{Bool} : \mathcal{U}_0) \sim_0^{\mathcal{U}_0} (\text{Bool} : \mathcal{U}_0)$<br>$([] : \text{List}_4 \kappa) \sim_0^{\text{List}_4 \kappa} ([] : \text{List}_6 \kappa)$<br>...   | $((\lambda \xi.\xi) \kappa : \mathcal{U}_1) \sim_0^{\mathcal{U}_1} (\kappa : \mathcal{U}_1)$<br>$([] : \text{List}_4 \mathcal{A}) \sim_0^{\text{List}_4 \mathcal{A}} ([] : \text{List}_6 \mathcal{A})$<br>...  |
| 1-related               | n/a  | $((A : \mathcal{U}_0) \sim_1^{\mathcal{U}_0} (B : \mathcal{U}_0)) := \text{Rel}(A, B)$<br>$\mathbb{N} := \text{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \sim_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$<br>$\text{List}_\bullet A : \text{List}_4 A \sim_1^{\mathcal{U}_0} \text{List}_6 A$<br>$R : (G : \text{Grp}) \sim_1^{\text{Grp}} (H : \text{Grp})$<br>$R : (G : \text{Grp}) \sim_1^V (M : \text{Mon})$ | $((\kappa : \mathcal{U}_1) \sim_1^{\mathcal{U}_1} (\lambda : \mathcal{U}_1)) := \text{Rel}(\kappa, \lambda)^{\{\bullet \rightarrow \bullet\}}$<br>$\mathcal{U}_0 : (\mathcal{U}_0 : \mathcal{U}_1) \sim_1^{\mathcal{U}_1} (\mathcal{U}_0 : \mathcal{U}_1)$<br>$\text{List}_\bullet \kappa : \text{List}_4 \kappa \sim_1^{\mathcal{U}_1} \text{List}_6 \kappa$<br>$\rho : (\alpha : \text{Cat}) \sim_1^{\text{Cat}} (\beta : \text{Cat})$ |
| 2-related               | n/a  | n/a  | $V : (\text{Grp} : \mathcal{U}_1) \sim_2^{\mathcal{U}_1} (\text{Mon} : \mathcal{U}_1)$   |

$$V : (\text{Grp} : \mathcal{U}_1) \sim_2^{\mathcal{U}_1} (\text{Mon} : \mathcal{U}_1)$$