ON THE OPTIMAL LOCATION OF TRANSMISSION ZEROS IN LEAST SQUARES ARMA FILTERING

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On the optimal location of transmission zeros in least squares ARMA filtering

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Abstract

Recently, the Levinson-Schur algorithm, classical in AR-filtering, has been extended to cover the ARMA case [1-4]. The ARMA filter that was derived was optimal in a least squares sense, given the transmission zeros (TZ). This determines the denominator of the filter as a function of the numerator (TZ) using only second order information (autocorrelation coefficients). The resulting optimal least squares error is parametrized in these TZ and can be further reduced. This is a nonlinear programming problem in the TZ. This paper investigates the nature of this optimization problem and compares optimal and suboptimal solutions. Some simple examples give an illustration of practical implementation of these results. They reveal however the ill conditioning of the problem.

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1. Introduction

The Levinson algorithm is well known in the literature of linear filtering. It constructs an autoregressive (AR) predictive filter from the spectral information of the unknown filter. The approximation is optimal in a least squares sense. Recently a generalization of the Levinson-Schur algorithm was given, constructing optimal filters of ARMA type [1-4]. These are also optimal in least squares sense, but they are parametrized in the transmission zeros (TZ). In this note we give a criterion to find the optimal location of the TZ.

Section 2 will recall the results for the classical Levinson-Schur extracting algorithm. Section 3 gives the generalizations for the extraction of an arbitrary set of TZ and section 4 introduces the resulting optimization problem for the optimal location of the TZ. In section 5 some examples of previous theory are given, illustrating the ill conditioning of the problem.

2. The classical Szegö-Levinson-Schur extraction

Suppose $\sigma(z)$ is the unknown filter, and $\sigma_\star(z) = \overline{\sigma(1/\bar{z})}$ its para-hermitian conjugate (w.r.t. the unit circle T), then $\omega(z) = \sigma(z)\sigma_\star(z)$ is called its power spectrum. Suppose $\omega(z)$ is log-integrable and its geometric mean

$$G(\omega) = \exp\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\,\omega(e^{j\theta})\,d\theta\} = \sigma_\star(0)^2 > 0$$

then

$$\sigma(z) = \exp\{\frac{1}{4\pi}\int_{-\pi}^{\pi} \log\,\omega(e^{j\theta})\,\frac{e^{j\theta}+z}{e^{j\theta}-z}\,d\theta\}$$

is in H_2 and non-vanishing in $D = \{z \,|\, |z| < 1\}$, while $\sigma(0)$ is real and positive. The Fourierseries of the power spectrum $\omega(z) = \sum_{-\infty}^{\infty} R_k z^k$ has $R_k = \bar{R}_{-k}$: the autocorrelation coefficients.

$$\Omega(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1+ze^{j\theta}}{1-ze^{j\theta}}\,\omega(e^{j\theta})\,d\theta = R_0 + 2\sum_{1}^{\infty} R_k z^k$$

is a Carathéodory function, i.e. analytic in D and with positive real part there. $\Gamma_0(z) = (\Omega(z) - R_0)/(\Omega(z) + R_0)$ is a Schur function, i.e. analytic inside D and bounded by 1. On these results see e.g. [5]. The Schur algorithm constructs reflection coefficients ρ_k for $k = 0,1,2,\ldots$ as follows : [6]

for k = 0,1,2,... set

$$\begin{cases} \Gamma_k'(z) = \Gamma_k(z)/z \\ \rho_k = \Gamma_k'(0) = \lim_{z\to 0} \Gamma_k(z)/z \\ \Gamma_{k+1}(z) = (\Gamma_k'(z) - \rho_k)/(1 - \bar{\rho}_k \Gamma_k'(z)) \end{cases}$$

All $\Gamma_k(z)$ are Schur functions, vanishing at the origin and all ρ_k lay in D.

In an homogeneous form, this algorithm is reformulated as follows. Set $\Delta_1^0(z) = \Omega(z) - R_0$ and $\Delta_2^0(z) = \Omega(z) + R_0$, then $\Delta^0(z) = \frac{1}{2}[\Delta_1^0(z)\Delta_2^0(z)]$ is called an acceptable function because it is analytic in D and consists of the homogeneous coordinates of a Schur function viz. $\Gamma_0(z) = \Delta_1^0(z)/\Delta_2^0(z)$, is a Schur function. The outgoing version of the Levinson algorithm then is :

for k = 0,1,2,... set

$$\begin{cases} \Delta^{k'}(z) = \Delta^k(z) \begin{bmatrix} z^{-1} & 0 \\ 0 & 1 \end{bmatrix} \\ \rho_k = \Delta_1^{k'}(0)/\Delta_2^{k'}(0) \\ \Delta^{k+1}(z) = \Delta^{k'}(z) \dfrac{1}{\sqrt{1-|\rho_k|^2}} \begin{bmatrix} 1 & -\bar{\rho}_k \\ -\rho_k & 1 \end{bmatrix} \end{cases}$$

which is the same as

$$[\Delta_1^{k+1}(z) \;\; \Delta_2^{k+1}(z)] = [\Delta_1^k(z) \;\; \Delta_2^{(k)}(z)]\theta_k^{-1}$$

with $\theta_k = \dfrac{1}{\sqrt{1-|\rho_k|^2}} \begin{bmatrix} 1 & \bar{\rho}_k \\ \rho_k & 1 \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix}$.

All $\Delta^k(z)$ are acceptable and $\Delta_1^k(0) = 0$.

Set $\Theta_0 = R_0^{-1/2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\Theta_{n+1} = \theta_n \Theta_n$ for $n \geq 0$,

then Θ_n has the following structure [1]

$$\Theta_n = \frac{1}{2}\begin{bmatrix} \phi_n + \psi_n & \phi_n - \psi_n \\ \phi_n^\star - \psi_n^\star & \phi_n^\star + \psi_n^\star \end{bmatrix}$$

where the $\phi_n(z)$ are the Szegö orthonormal polynomials [7-9]

$$\text{i.e. : } \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi_k(e^{j\theta})\overline{\phi_\ell(e^{j\theta})}\omega(e^{j\theta})d\theta = \delta_{k\ell}$$

and ψ_n are the corresponding polynomials of the second kind [8]. The upper star indicates the reciprocal of a polynomial, e.g. $\phi_n^\star(z) = z^n \phi_{n\star}(z)$. This yields

$$\begin{bmatrix} \phi_{n+1}(z) \\ \phi_{n+1}^\star(z) \end{bmatrix} = \Theta_n \begin{bmatrix} \phi_n(z) \\ \phi_n^\star(z) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \psi_{n+1}(z) \\ -\psi_{n+1}^\star(z) \end{bmatrix} = \Theta_n \begin{bmatrix} \psi_n(z) \\ -\psi_n^\star(z) \end{bmatrix}$$

with $\phi_0 = \psi_0 = R_0^{-1/2}$. This is the incoming version of the Levinson algorithm, which is equivalent with the Szegö recursion for polynomials orthogonal on the unit circle and with a special case of the Trench-Zohar-Rissanen algorithm [10-12] for symmetric Toeplitz matrix inversion. $\tilde{\Omega}_n(z) = \psi_n^\star(z)/\phi_n^\star(z)$ is an order n+1 approximation to $\tilde{\Omega}(z) = \Omega(z)/R_0$ and $\tilde{\omega}_n(z) = \frac{1}{2}[\tilde{\Omega}_n(z) + \tilde{\Omega}_{n\star}(z)] = \dfrac{z^n}{\phi_n(z)\phi_n^\star(z)}$ is a Laurent-Padé approximation of order [0/n] for $\tilde{\omega}(z) = R_0^{-1}\,\omega(z)$ [13].

We also have a least squares approximation of $\sigma^{-1}(z)$ given by [8]:

$$S_n = \min_{p_n} \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\frac{1}{\sigma_\star(e^{j\theta})} - p_n(e^{j\theta})\right|^2 \omega(e^{j\theta})d\theta$$

where p_n is a polynomial of degree at most n. This is minimized for $p_n(z) = \sigma_\star(0)\kappa_n \phi_n^\star(z)$, $\kappa_n = \phi_n^\star(0)$ and the minimum is

$$S_n = 1 - \kappa_n^2 \sigma_\star(0)^2$$

where

$$\kappa_n^{-2} = R_0 \prod_{k=0}^{n-1} (1 - |\rho_k|^2)$$

with $\lim_{n\to\infty} \kappa_n = \sigma_\star(0)^{-1}$.

The approximation $(\kappa_n \phi_n^\star(z))^{-1}$ to $\sigma_\star(0)\sigma_\star(z)$ is stable because $\phi_n^\star(z)$ has all its zeros outside the unit disc.

3. The generalized Schur extractions

When projecting $(\sigma_\star(0)\sigma_\star(z))^{-1}$ on span$\{1,z,\ldots,z^n\}$ in the space $L^2(\omega)$, we allow only polynomial approximations. We could also allow rational approximations with preset poles outside the unit disc. (A polynomial of degree n is a rational function with an n-th order pole at infinity). Therefore we shall take the $L^2(\omega)$ projection of $(\sigma_\star(0)\sigma_\star(z))^{-1}$ onto $L_n = \text{span}\{U_0,U_1,U_2,\ldots,U_n\}$ where $\{U_k\}_0^n$ constitutes a basis for rational functions of degree n with poles $1/\bar{\alpha}_k$ $k = 1,2,\ldots$. We take for U_k the partial Blaschke products

$$U_0 = 1 \quad \text{and} \quad U_k = U_{k-1}\,\zeta_k \quad \text{with} \quad \zeta_k = \frac{z-\alpha_k}{1-\bar{\alpha}_k\, z}\,, \quad k = 1,2,\ldots$$

where α_k, $k = 1,2,3,\ldots$ are all in D. The algorithm of Schur is now extended to the Nevanlinna-Pick algorithm [14], which goes as follows : We set $\Gamma_0(z) = (\Omega(z)-R_0)/(\Omega(z)+R_0)$ as before and then iteratively

for $k = 0,1,2,\ldots$

$$\gamma_k = \begin{cases} \Gamma_k(\alpha_{k+1}) & \text{for} \quad \alpha_{k+1} \neq 0 \\ 0 & \text{for} \quad \alpha_{k+1} = 0 \end{cases}$$

$$\Gamma_k''(z) = (\Gamma_k(z)-\gamma_k)/(1-\bar{\gamma}_k\,\Gamma_k(z))$$

$$\Gamma_k'(z) = \Gamma_k''(z)/\zeta_{k+1}(z)$$

$$\rho_k = \Gamma_k'(0) = \begin{cases} \gamma_k/\alpha_{k+1} & \text{for} \quad \alpha_{k+1} \neq 0 \\ \lim\limits_{z\to 0} \Gamma_k(z) & \text{for} \quad \alpha_{k+1} = 0 \end{cases}$$

$$\Gamma_{k+1}(z) = (\Gamma_k'(z)-\rho_k)/(1-\bar{\rho}_k\,\Gamma_k'(z))$$

As in the classical case, all $\Gamma_k(z)$ are Schur functions, normalized such that $\Gamma_k(0) = 0$ and all γ_k and ρ_k are inside D. For computational purposes, it is more useful to have an homogeneous reformulation, in terms of acceptable functions. Suppose $\Delta^0(z)$ is as in the previous section then we have to do the following computations :

for k = 0,1,2,3,...

$$\gamma_k = \Delta_1^k(\alpha_{k+1})/\Delta_2^k(\alpha_{k+1}) \quad \text{if} \quad \alpha_{k+1} \neq 0, \text{ otherwise } \gamma_k = 0$$

$$\Delta^{k"}(z) = \Delta^k(z) \frac{1}{\sqrt{1-|\gamma_k|^2}} \begin{bmatrix} 1 & -\bar{\gamma}_k \\ -\gamma_k & 1 \end{bmatrix}$$

$$\Delta^{k'}(z) = \Delta^{k"}(z) \begin{bmatrix} \zeta_{k+1}^{-1}(z) & 0 \\ 0 & 1 \end{bmatrix}$$

$$\rho_k = \Delta_1^{k'}(0)/\Delta_2^k(0) \quad (= \gamma_k/\alpha_{k+1} \quad \text{if} \quad \alpha_{k+1} \neq 0)$$

$$\Delta^k(z) = \Delta^k(z) \frac{1}{\sqrt{1-|\rho_k|^2}} \begin{bmatrix} 1 & -\bar{\rho}_k \\ -\rho_k & 1 \end{bmatrix}$$

Again all $\Delta^k(z)$ are acceptable. One iteration can be summarized by

$$\Delta^{k+1}(z) = \Delta^k(z)\theta_k^{-1}$$

with $$\theta_k = \frac{1}{\sqrt{1-|\rho_k|^2}} \begin{bmatrix} 1 & \bar{\rho}_k \\ \rho_k & 1 \end{bmatrix} \begin{bmatrix} \zeta_{k+1}(z) & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{\sqrt{1-|\gamma_k|^2}} \begin{bmatrix} 1 & \bar{\gamma}_k \\ \gamma_k & 1 \end{bmatrix}$$

The structure of the Θ_n matrix, defined by the recursion

$$\Theta_0 = R_0^{-1/2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \Theta_{n+1} = \theta_n \Theta_n \quad \text{for } n \geq 0 \text{ is the following [1-4] :}$$

$$\Theta_n = \frac{1}{2} \begin{bmatrix} \tilde{k}_n^{\star}(z,0)+\tilde{\ell}_n^{\star}(z,0) & \tilde{k}_n^{\star}(z,0)-\tilde{\ell}_n^{\star}(z,0) \\ \tilde{k}_n(z,0)-\tilde{\ell}_n(z,0) & \tilde{k}_n(z,0)+\tilde{\ell}_n(z,0) \end{bmatrix}$$

The $\tilde{k}_n(x,y)$ are normalized reproducing kernels for L_n, i.e. $\tilde{k}_n(x,y) = k_n(x,y)/\sqrt{k_n(y,y)}$ with $k_n(x,y)$ the reproducing kernel for L_n [15]. If $\{\phi_s(z)\}_0^n$ are orthonormal functions in $L^2(\omega)$ for the spaces L_n, then the reproducing kernel is given by

$$k_n(x,y) = \sum_{s=0}^{n} \overline{\phi_s(y)}\ \phi_s(x).$$

The upper star generalizes the concept of reciprocal polynomial for a function in L_n, i.e.

$$f^\star(z) = U_n(z) f_\star(z) \qquad \forall f \in L_n.$$

We shall call this the reciprocal of the rational function f.
Remark that $k_n(z,\alpha_n) = \phi_n^\star(\alpha_n)\phi_n^\star(z)$, so that in case $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$ $\tilde{k}_n(z,0)$ is exactly $\phi_n^\star(z)$ with $\phi_n(z)$ the n-th Szegö polynomial. In the same way, $\tilde{\ell}_n(x,y) = \ell_n(x,y)/\sqrt{\ell_n(y,y)}$ where $\ell_n(x,y)$ generalizes the concept of second kind polynomial. It is therefore called the second kind function for $k_n(x,y)$. This may be defined as follows : $f^\#(z) \in L_n$ is the second kind function for $f(z) \in L_n$ if $f^\#(z)$ is an (Hermitian) interpolant for $\tilde{\Omega}(z)f(z)$ at the points $0,\alpha_1,\alpha_2,\ldots,\alpha_n$ on which L_n is constructed, and $\tilde{\Omega}(z)$ is related to the weight in $L^2(\omega)$ as before. This is equivalent with $f^\#(z)$ being the $L^2(d\theta)$ projection of $\tilde{\Omega}(z)f(z)$ onto L_n [16]. This means that $\tilde{\Omega}_n(z) = \tilde{\ell}_n(z,0)/\tilde{k}_n(z,0)$ approximates $\tilde{\Omega}(z)$ by interpolation at the points indicated and is a Carathéodory function itself. $\tilde{\omega}_n(z)\ \frac{1}{2}\ [\tilde{\Omega}_n(z)+\tilde{\Omega}_{n\star}(z)] = \dfrac{z^n}{\tilde{k}_n(z,0)\tilde{k}_n^\star(z,0)}$ is a kind of "Laurent-Hermite-interpolant" for $\tilde{\omega}(z) = R_0^{-1}\ \omega(z)$.
It can also be shown that $\ell_n(x,y)$ is the reproducing kernel for L_n in $L^2(w)$ where $w = \frac{1}{2}\ [W(z)+W_\star(z)]$ and $W(z) = \Omega(z)^{-1}$. The corresponding orthogonal functions ψ_s are direct generalizations of the Szegö polynomials of the second kind.

The previous discussion gives recursions for the reproducing kernels in their normalized form :

$$\begin{bmatrix} \tilde{k}_{n+1}^\star(z,0) \\ \tilde{k}_{n+1}(z,0) \end{bmatrix} = \theta_n \begin{bmatrix} \tilde{k}_n^\star(z,0) \\ \tilde{k}_n(z,0) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \tilde{\ell}_{n+1}^\star(z,0) \\ -\tilde{\ell}_{n+1}(z,0) \end{bmatrix} = \theta_n \begin{bmatrix} \tilde{\ell}_n^\star(z,0) \\ -\tilde{\ell}_n(z,0) \end{bmatrix}$$

with $\tilde{k}_0(z,0) = \tilde{\ell}_0(z,0) = R_0^{-1/2}$. This is the incoming version of the Nevanlinna-Pick problem. Remark that from this recursion one can not

directly derive a recursion for the orthonormal functions ϕ_n and ψ_n. Another remark is that the normalization at 0 (Denjoy-normalization) is arbitrary and one could choose any other point inside the unit disc instead of the origin [2,4].

Most comfortably also the $L^2(\omega)$ projection theory goes through for this generalization. Indeed : $\sigma_\star^{-1}(z)$ is projected onto L_n by looking for the minimum

$$S_n = \min_{f_n \in L_n} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{1}{\sigma_\star(e^{j\theta})} - f_n(e^{j\theta}) \right|^2 \omega(e^{j\theta}) d\theta .$$

The minimum is obtained for $f_n = k_n(z,0)\sigma_\star(0)$ and the residual norm squared is

$$S_n = 1 - k_n(0,0)\sigma_\star(0)^2$$

and

$$k_n(0,0) = R_0^{-1} \prod_{k=0}^{n-1} (1-|\gamma_k|^2)/(1-|\rho_k|^2) .$$

Stability of the approximation is guaranteed if all ρ_k and γ_k remain inside D.

This concludes the parallel treatment of classical and generalized Schur-Nevanlinna-Pick algorithms. In the next section we shall investigate how the transmission zeros $\alpha_1, \alpha_2, \alpha_3, \ldots$ have to be chosen to be optimal.

4. The optimal location of the transmission zeros

It follows from the previous section that the least squares error depends upon the choice of the transmission zeros α_k upon which the spaces L_n are built. There is a nested dependence through ρ_k and γ_k because

$$\rho_n = \rho_n(\alpha_1, \ldots, \alpha_{n+1}) \quad \text{and} \quad \gamma_n = \gamma_n(\alpha_1, \ldots, \alpha_{n+1}), \quad n \geqslant 0.$$

The squared residual is minimal if the $\alpha_1, \alpha_2, \ldots$ are chosen such that they are optimal for

$$E_n = \min_{\substack{\alpha_k \in D \\ k=1,2,\ldots,n}} \prod_{s=0}^{n-1} (1-|\rho_s(\alpha_1, \ldots, \alpha_{s+1})|^2)/(1-|\gamma_s(\alpha_1, \ldots, \alpha_{s+1})|^2)$$

This is a constrained optimization problem which can be solved by an optimization routine not requiring derivatives. The constraints are not active at the optimum because $\alpha_{p+1} \in T$ is a degenerate situation with $(1-|\rho_p|^2)/(1-|\gamma_p|^2) = 1$, not giving a reduction of the residual error. Thus the restrictions $\alpha_k \in D$, $k = 1,2,\ldots$ are rather an aid in bounding the steplength for the optimizing routine, than an additional complication.

It is an important drawback however that the optimization problem has to be solved for all $\{\alpha_k\}_1^{n+1}$ simultaneously. The recursive procedure fixing $\alpha_1,\alpha_2,\alpha_3,\ldots$ successively by solving for $k = 1,2,\ldots$

$$\min_{\alpha_k \in D} \; S_k(\alpha_1^0,\alpha_2^0,\ldots,\alpha_{k-1}^0,\alpha_k)$$

with $S_k = \prod_{s=0}^{k-1} (1-|\rho_s|^2)/(1-|\gamma_s|^2)$

yielding α_k^0 as the optimizer is not optimal. This would be the same as finding α_k, $k \geqslant 1$ from

$$\min_{\alpha_k \in D} \; (1-|\rho_{k-1}(\alpha_1^0,\ldots,\alpha_{k-1}^0,\alpha_k)|^2)/(1-|\gamma_k(\alpha_1^0,\ldots,\alpha_{k-1}^0,\alpha_k)|^2)$$

Only the first step then would give the best approximation, e.g. at least as good as the approximation resulting from one ordinary Levinson extraction. After two or more steps however nothing guarantees that ordinary Levinson extractions, or even arbitrary chosen transmission zeros do not give a better result.

Another possibility to exploit the recursive structure is to minimise S_k over all $\alpha_1,\ldots,\alpha_k$ simultaneously, with some standard library routine, taking for $\alpha_1 \ldots \alpha_{k-1}$ as starting values the optimal values for the minimization of S_{k-1} and for α_k, e.g. 0. This is not a good procedure either because e.g. the optimal value of α_1 in $\min_{\alpha_1} S_1(\alpha_1)$ can deviate quite a bit from the optimal value of α_i in $\min_{\alpha_1,\alpha_2} S_2(\alpha_1,\alpha_2)$. See e.g. example 3 in the next section.

It is however true that if $\sigma_\star(z)$ is rational of degree n, then if $S_m(\alpha_1,\ldots,\alpha_m)$ is minimized for $m \geqslant n$, obviously the exact transmission zeros are found. This follows from the projection properties given in the previous section.

5. Examples and numerical considerations

In order to obtain an idea about the nature of the optimization problem described in previous sections we give some numerical computations on very simple examples.

First of all we restrict ourselves to real autocorrelation sequences and filters $\sigma(z)$ having finitely many poles and zeros. This implies that transmission zeros are either real or appear in complex conjugate pairs.

In example 1 we suppose there is only one transmission zero, so $\sigma_\star(z) = (z-\alpha_1)/(z-\beta_1)$, then after one extraction the minimum of $S_1(\alpha) = (1-|\rho_0(\alpha)|^2)/(1-|\gamma_0(\alpha)|^2)$ must yield the correct value of α. In fig. 1 we took $\alpha_1 = 0.5$ and $\beta_1 = 0$. The contour lines of $S_1(\alpha)$ for $\alpha \in D$ are given. $S_1(\alpha) = 1$ for $\alpha \in T$ as it will always be in the following examples. Here the minimal value is obtained for $\alpha = 0.5$.

If there is more than one transmission zero, then the optimization of $S_1(\alpha)$ will not give the correct transmission zero. E.g. take $\sigma_\star(z) = \Pi_1^2(z-\alpha_i)/(z-\beta_i)$.
In example 2, $\alpha_1 = -\alpha_2 = 0.5$ and $\beta_1 = \beta_2 = 0$. The minima of $S_1(\alpha)$ are found at locations different from α_i (fig. 2). The optimal values are shifted towards the unit circle.

The optimization of $S_1(\alpha)$ can give a quite misleading idea about the location of the transmission zero as is illustrated in example 3 where $\alpha_1 = 0.5$, $\alpha_2 = 0.$ and $\beta_1 = \bar{\beta}_2 = 0.5 \exp(j\pi/4)$. Fig. 3 shows a minimum close to -0.5 which is the opposite of the exact α_1. This has however an intuitive explanation. The approximation is least squares on the unit circle. The two poles work together to overgrow the effect of the zero on the unit circle in the 1st and 4th quadrant. So if only one zero is allowed for the approximation, it will obviously be found in $(-1,0)$, dominating the 2nd and 3rd quadrant.

Now let us suppose that $\sigma_\star(z)$ is of arbitrary degree n but has only one transmission zero different from 0. It may have a multiplicity $m \leq n$. It can only be real, so it has to be in $(-1,1)$. The algorithm to find this zero exactly is the following. First perform a number of ordinary Levinson extractions. At least n-m, but there may be more. On the resulting acceptable function, a number of generalized extractions are done at α, at least m. Fig. 10 gives a fortran subroutine implementing such a generalized extraction, except that in this case all complex computations may be replaced by real ones. If the function

$$S_M(\alpha) = \prod_{0}^{M-1} (1-|\rho_k(\alpha)|^2)/(1-|\gamma_k(\alpha)|^2)$$

(with $M \geq m$ and ρ_k and γ_k resulting from the generalized extractions) is minimized with respect to $\alpha \in (-1,1)$, then this yields the exact location of the transmission zero.

As an example take $\sigma_\star(z) = (z-0.5)^2/z^2$. No ordinary Levinson extractions are needed. Fig. 4 gives a plot of $S_2(\alpha)$ for $\alpha \in (-1,1)$, which has a global minimum at $\alpha = 0.5$ as expected.

The next two figures are related to the filter $\sigma_\star(z) = \prod_1^4 (z-\alpha_i)/(z-\beta_i)$ with $\alpha_1 = \alpha_2 = 0.5$, $\alpha_3 = \alpha_4 = 0$, $\beta_1 = \beta_2 = -0.5$, $\beta_3 = \bar{\beta}_4 = 0.5 \exp(j\pi/4)$. In fig. 5 $S_M(\alpha)$ is plotted for two generalized extractions done on the acceptable function obtained after two ordinary Levinson extractions. Fig. 6 is obtained after 3 generalized extractions preceded by 4 Levinson extractions.

Figures 4-6 illustrate in a clear way the awful ill conditioning of the problem. Although the originating $\sigma_\star(z)$ are nicely behaving and of a rather simple structure it is numerically almost impossible to find the optimum accurately. In terms of the least squares error it is of little importance, within certain bounds, where the transmission zero is chosen. However a relatively significant improvement can be obtained over the ordinary Levinson extraction where α is taken as 0.

Taking a closer look, the other figures will also reveal a more or less ill conditioning of the problem. This problem is well known in least squares rational approximations [17] and is as difficult as exponential curve fitting.

Example 7 has a somewhat nicer conditioning as illustrated in fig. 7. These are the contour lines for $S_1(\alpha)$, $\alpha \in D$ obtained by one generalized extraction at α on the acceptable function generated by two ordinary extractions at $\alpha = 0$ for the data associated with $\sigma_\star(z) = \prod_1^3 (z-\alpha_i)/(z-\beta_i)$ where $\alpha_1 = 0.5$, $\alpha_2 = \alpha_3 = 0$ and $\beta_1 = -0.5$, $\beta_2 = \bar{\beta}_3 = 0.5 \exp(j\pi/4)$.

Finally we consider the problem of extracting one conjugate pair of transmission zeros, with eventually a higher multiplicity, appart from the transmission zeros at the origin. In this case a number of pairs of extractions at α and $\bar{\alpha}$ are performed with the routine extra of fig. 10, after the transmission zeros at the origin are taken out by ordinary Levinson extraction. If $\alpha = r \exp(j\theta)$ then an optimization of the resulting $S_M(\alpha) = S_M(r,\theta)$ has to be done for $r \in (0,1)$ and $\theta \in (0,\pi)$. Fig. 8 gives such contour lines for $S_M(r,\theta)$ generated for $\sigma_\star(z) = (z-\alpha)(z-\bar{\alpha})/(z+0.5)^2$ with $\alpha = 0.5 \exp(j\pi/4)$

obtained after one pair of complex conjugate extractions. The minimum is at the expected place, but again remark the ill conditioning. Fig. 9 gives similar contour lines for $\sigma_\star(z) = \Pi_1^3 (z-\alpha_i)/(z-\beta_i)$ with $\alpha_1 = \bar{\alpha}_2 = 0.5 \exp(j\pi/4)$, $\alpha_3 = 0$ and $\beta_1 = \beta_2 = -0.5$, $\beta_3 = 0.8$. One transmission zero at 0 is taken out, followed by a pair of conjugate extractions. Again the minimum is found at the expected location.

6. Conclusion

The Nevanlinna-Pick algorithm is resumed together with a review of related approximation results. The straightforward generalizations of the classical Levinson-Schur algorithm is illustrated. Especially an $L^2(\omega)$ projection theory is maintained, but now on subspaces L_n of rational functions with preset poles. These poles turn out to be the transmission zeros of the filter. The least squares error is parametrized in the chosen transmission zeros and can thus be further reduced if we let those zeros vary over the unit disc. The corresponding optimization problem is however very ill conditioned as it was illustrated by some simple examples. This suggests the idea that with only a rough idea about the location of the transmission zeros, a low order ARMA approximation can be constructed which will result in a least squares error which is as low as for a high order AR filter, even if the estimation of the transmission zero is very poor, but within a certain region.

7. References

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APPENDIX

In this appendix, the figures are resumed. They all relate to a given filter of the form

$$\sigma_*(z) = \Pi_1^n (z-\alpha_i)/(z-\beta_i).$$

n, α_i, and β_i are indicated for each figure.

A plot is given of the objective function

$$S_M(\alpha) = \Pi_1^M (1-|\rho_i(\alpha)|^2)/(1-|\gamma_i(\alpha)|^2),$$

obtained with M generalized extractions performed on the acceptable function resulting from L ordinary Levinson extractions at $\alpha=0$.

In fig. 1-3 and 7 , α ranges over the unit disc. In fig. 4-6, $\alpha \in [-1,1]$, while in fig. 8 and 9, $\alpha = r \exp(j\theta)$ with $r \in [0,1]$ and $\theta \in [0,\pi]$.

For fig. 1-3 and 7-9 contour lines are plotted for $S_M(\alpha) = c_i$. On the unit circle is $S_M(\alpha) = 1$, the contour constants c_i are indicated for each figure.

Poles are shown by a cross and zeros by a circle.

$n=1$, $\alpha_1=0.5$, $\beta_1=0.0$

$c_i=0.81(0.02)0.99,1.0$

min=0.8002, L=0, M=1.

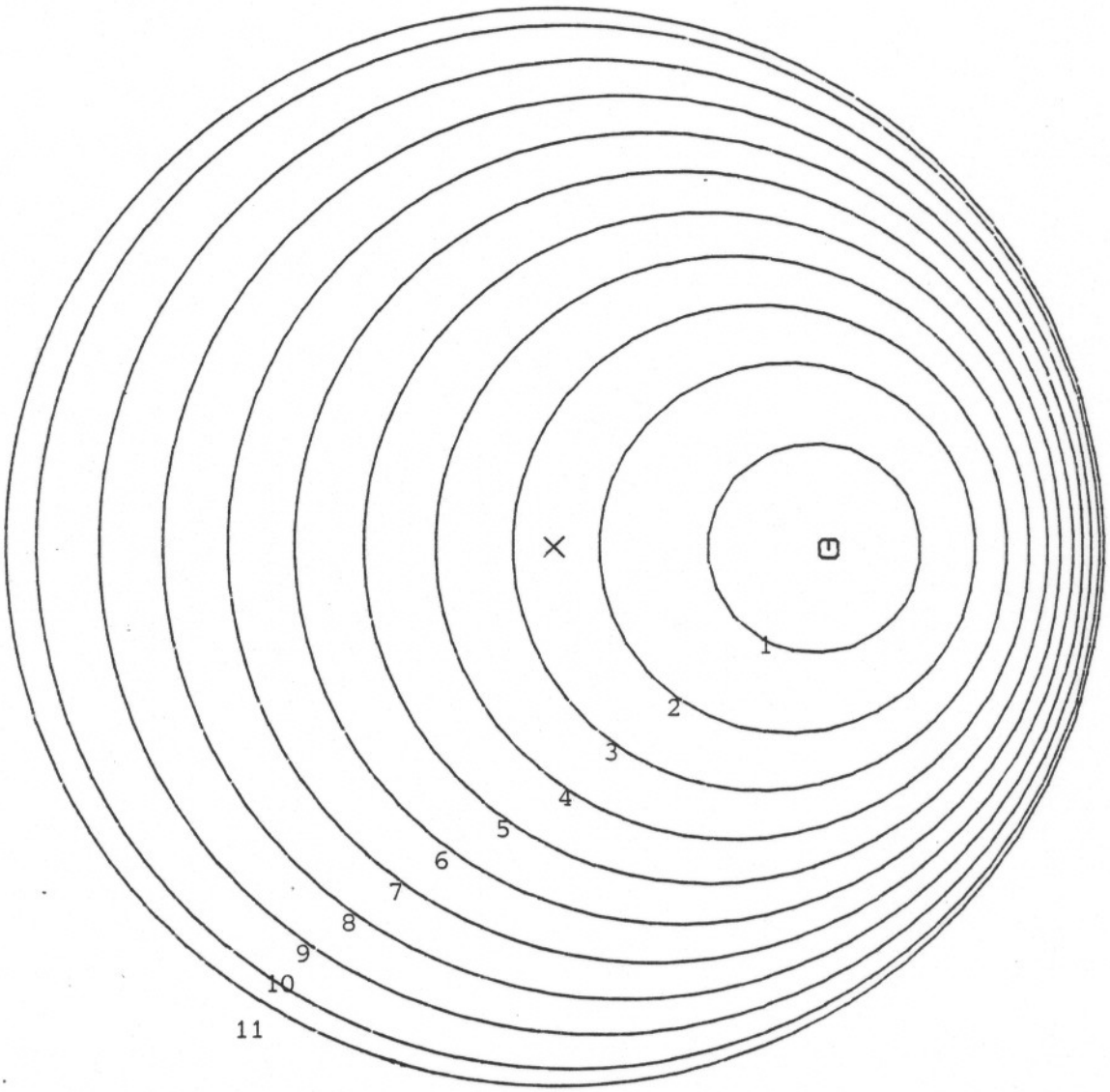


fig. 1

$n=2,\ \alpha_1=-\alpha_2=0.5,\ \beta_1=\beta_2=0.0$

$c_i = 0.982, 0.9825(0.0025)1.0$

min=0.98151, L=0, M=1.

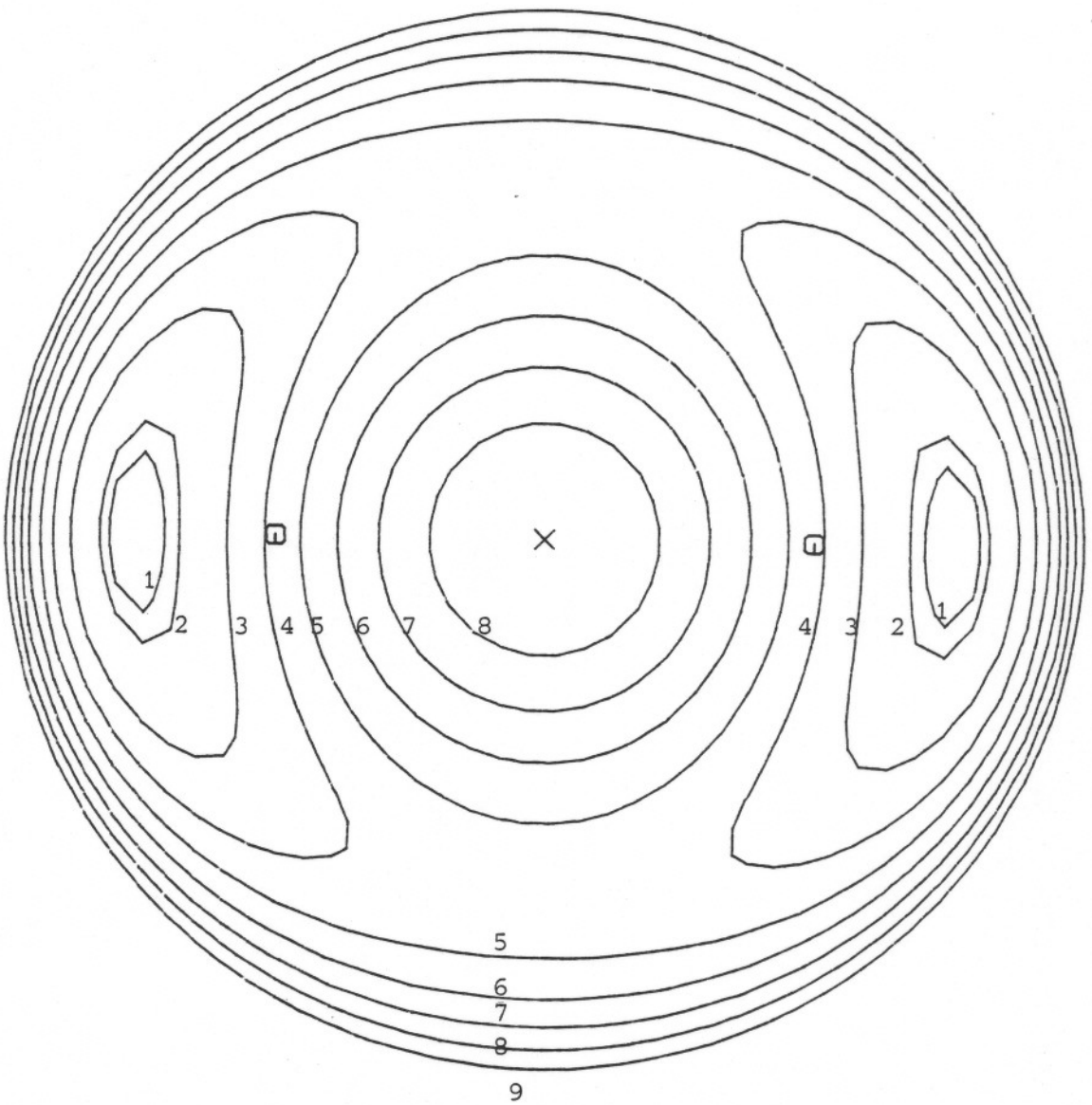


fig. 2

$n=2, \alpha_1=0.5, \alpha_2=0.0, \beta_1=\bar{\beta}_2=0.5 \exp(j\pi/4)$

$c_i=0.95(0.005)1.0$

min=0.94982, L=0, M=1

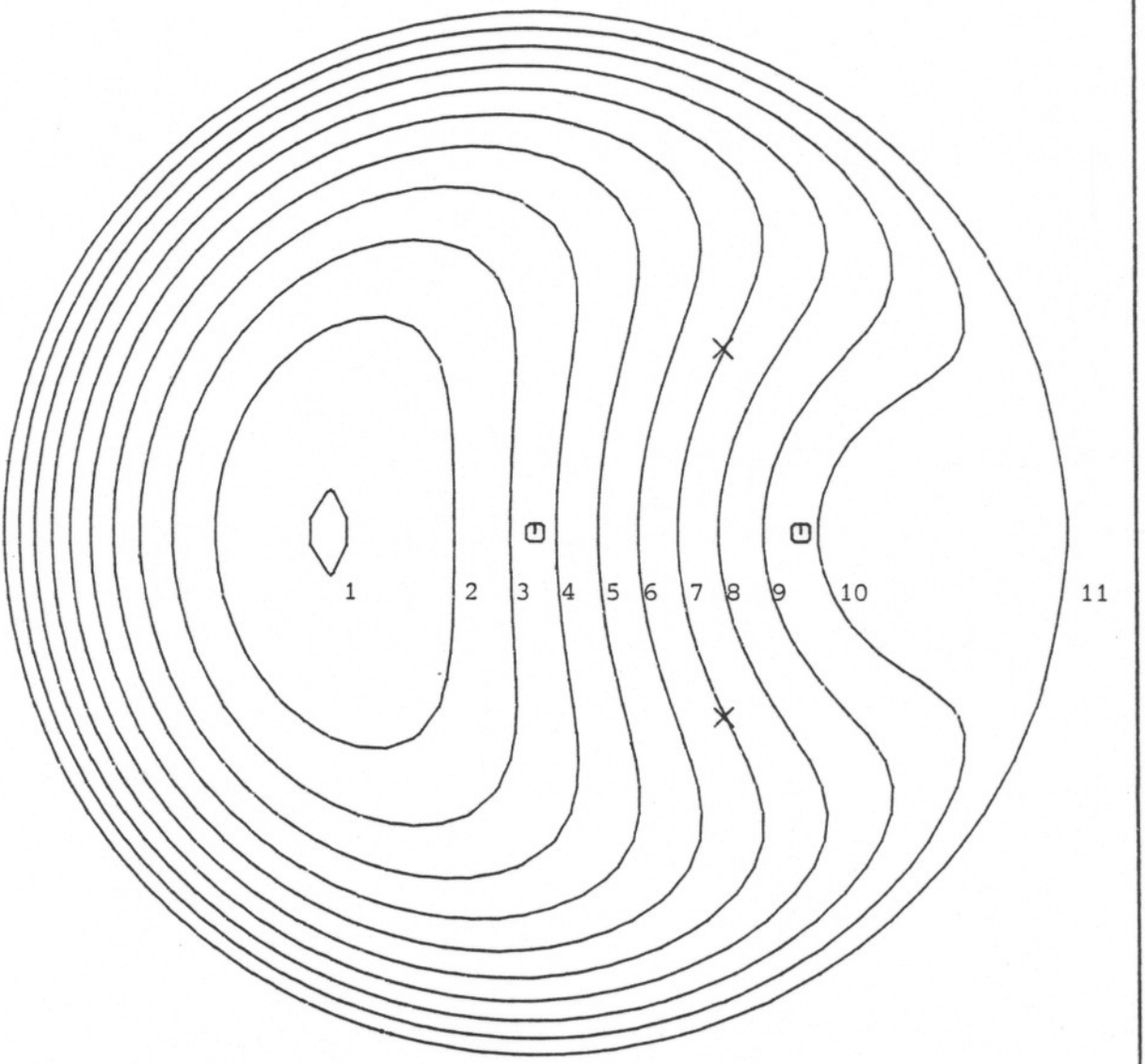


fig. 3.

$n=2,\ \alpha_1=\alpha_2=0.5,\ \beta_1=\beta_2=0.0$
min=0.48497, L=0, M=2

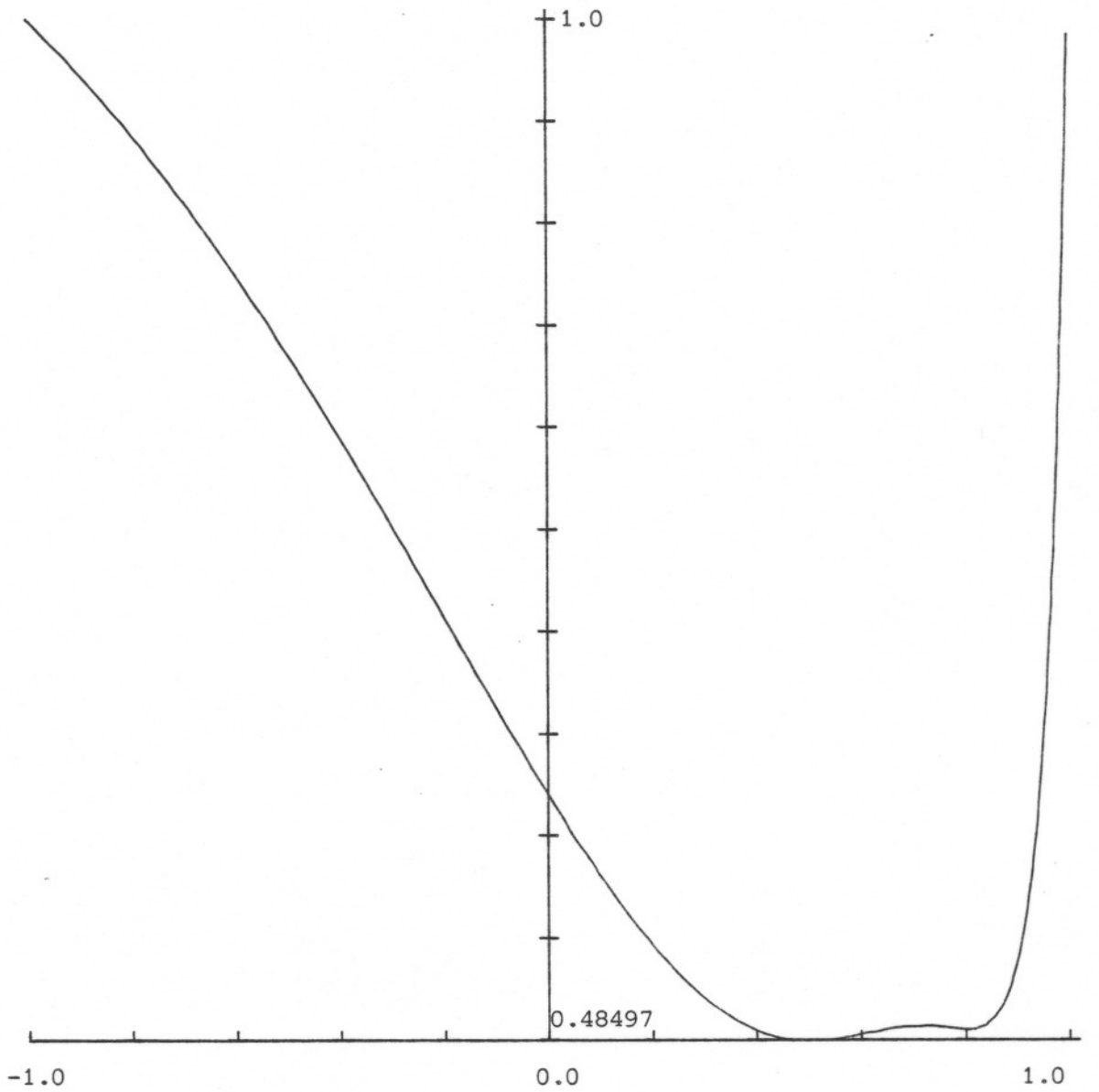


fig. 4.

$n=4$, $\alpha_1=\alpha_2=0.5$, $\alpha_3=\alpha_4=0.0$
$\beta_1=\bar{\beta}_2=0.5\ \exp(j\pi/4)$, $\beta_3=\beta_4=-0.5$
min=0.97507, L=4, M=3

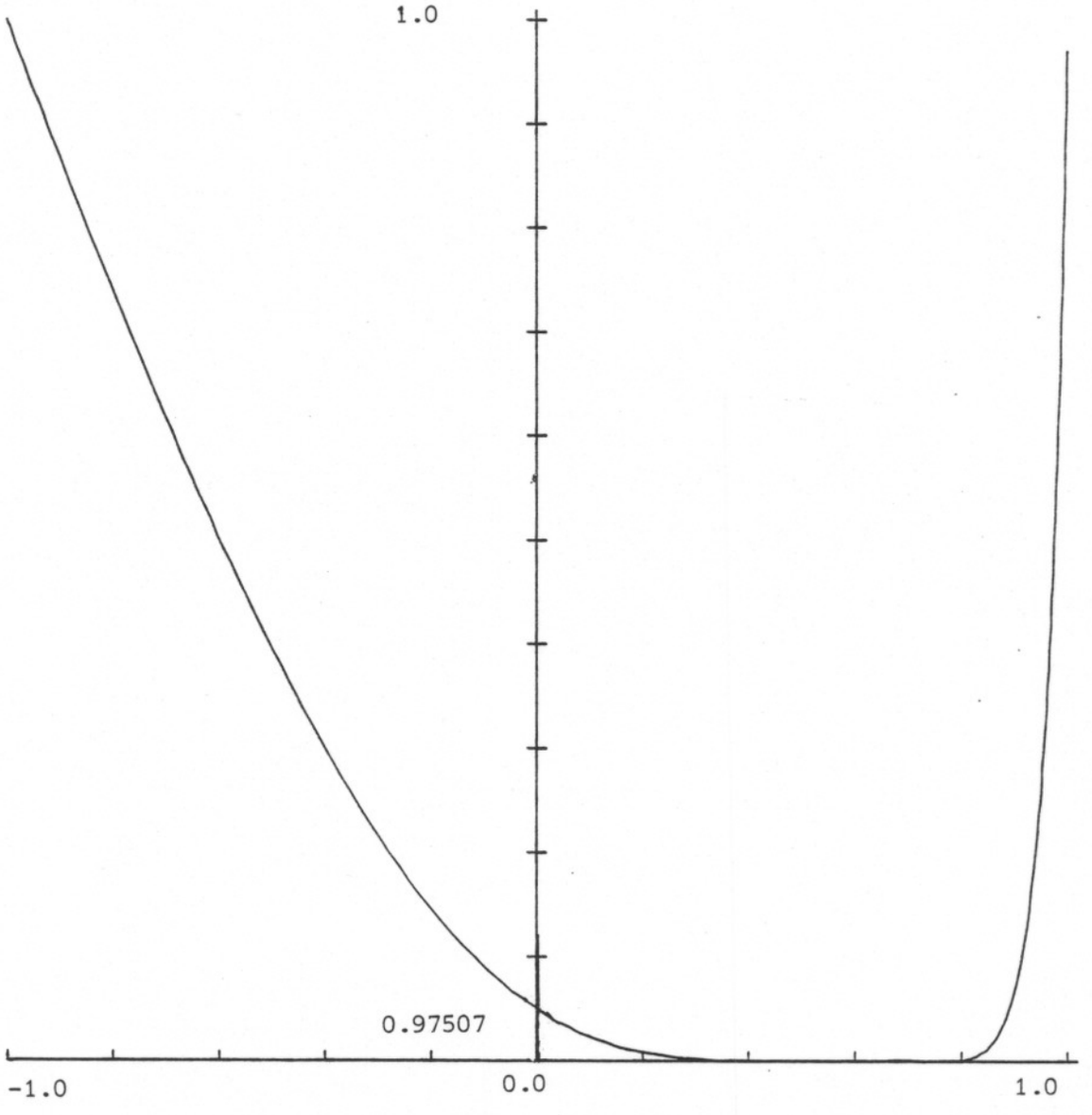


fig. 6.

$n=4$, $\alpha_1=\alpha_2=0.5, \alpha_3=\alpha_4=0.0$,
$\beta_1=\bar{\beta}_2=0.5\ \exp(j\pi/4)$, $\beta_3=\beta_4=-0.5$
min=0.90398, L=2, M=2

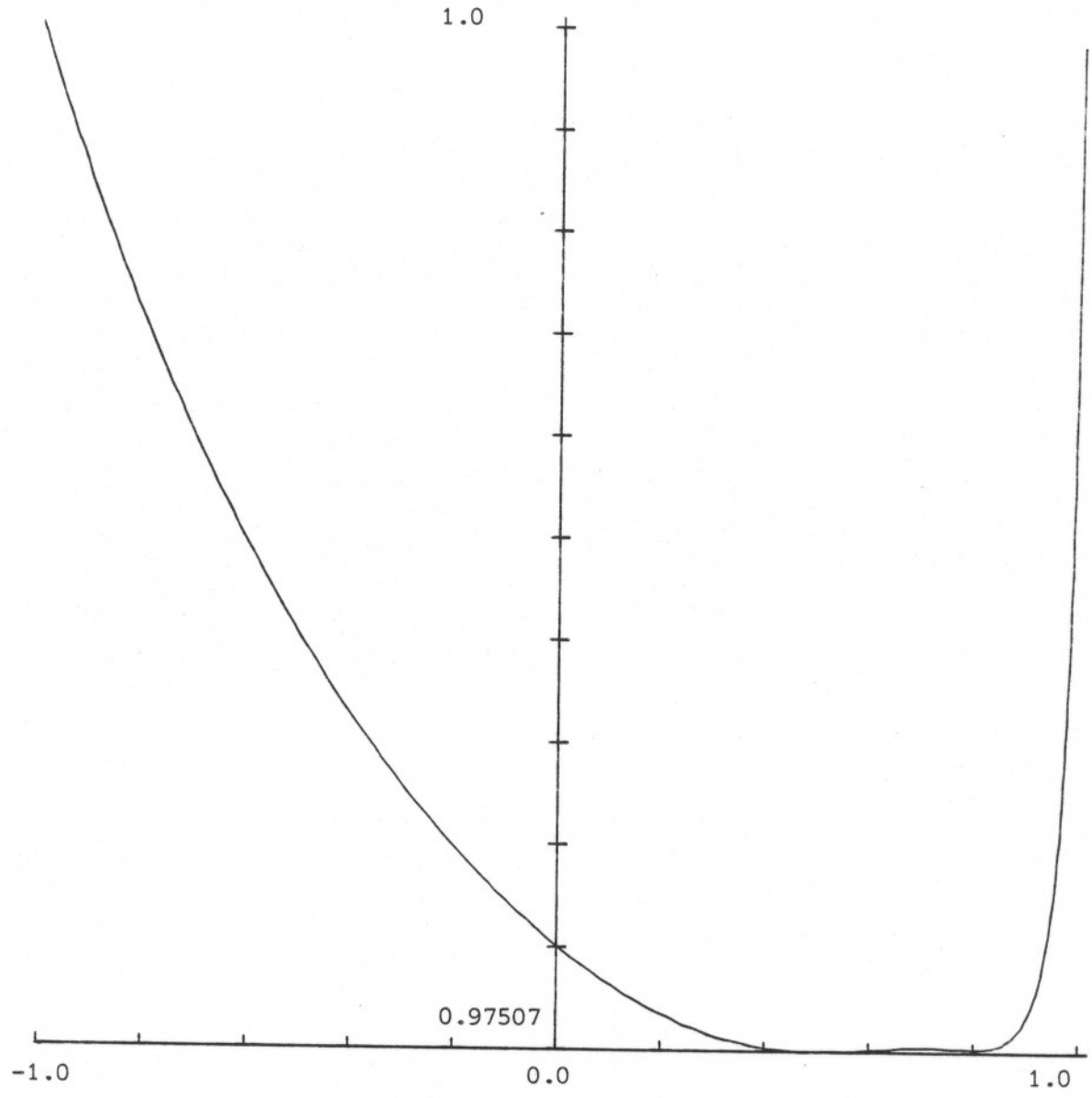


fig. 5.

$n=3,\ \alpha_1=\alpha_2=0.0,\ \alpha_3=0.5$

$\beta_1=\bar{\beta}_2=0.5\ \exp(j\pi/4),\ \beta_3=-0.5,$

$c_i = 0.98(0.0025)1.0$

min=0.97944, L=2, M=1

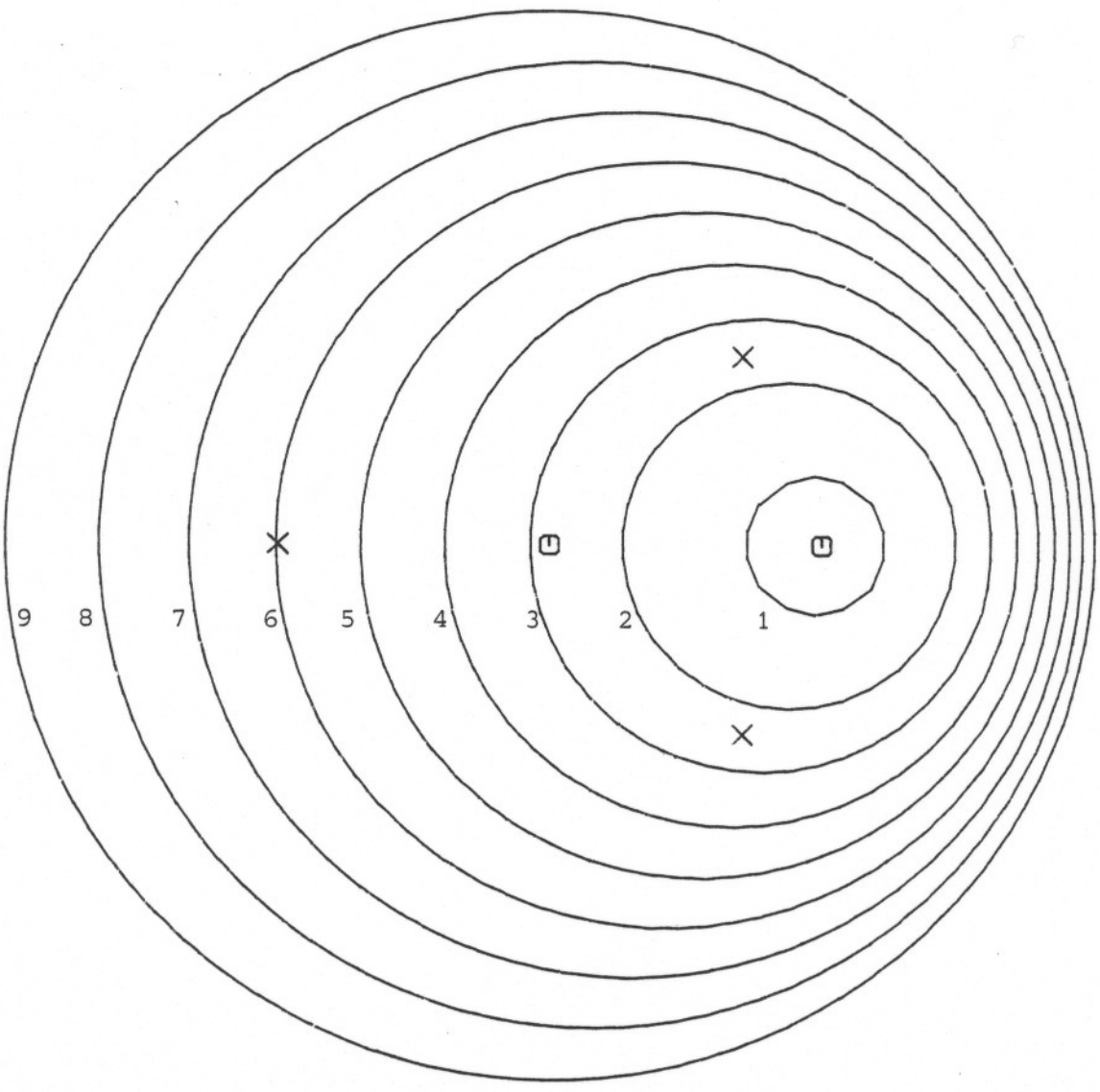


fig. 7.

$n=2,\ \alpha_1=\bar{\alpha}_2=0.5\ \exp(j\pi/4),\ \beta_1=\beta_2=-0.5$

$c_i=$ 0.104(0.001)0.11(0.002)0.12(0.01)0.14
0.14(0.02)0.2,0.3(0.2)0.9,1.0

min=0.10378, L=0, M=2

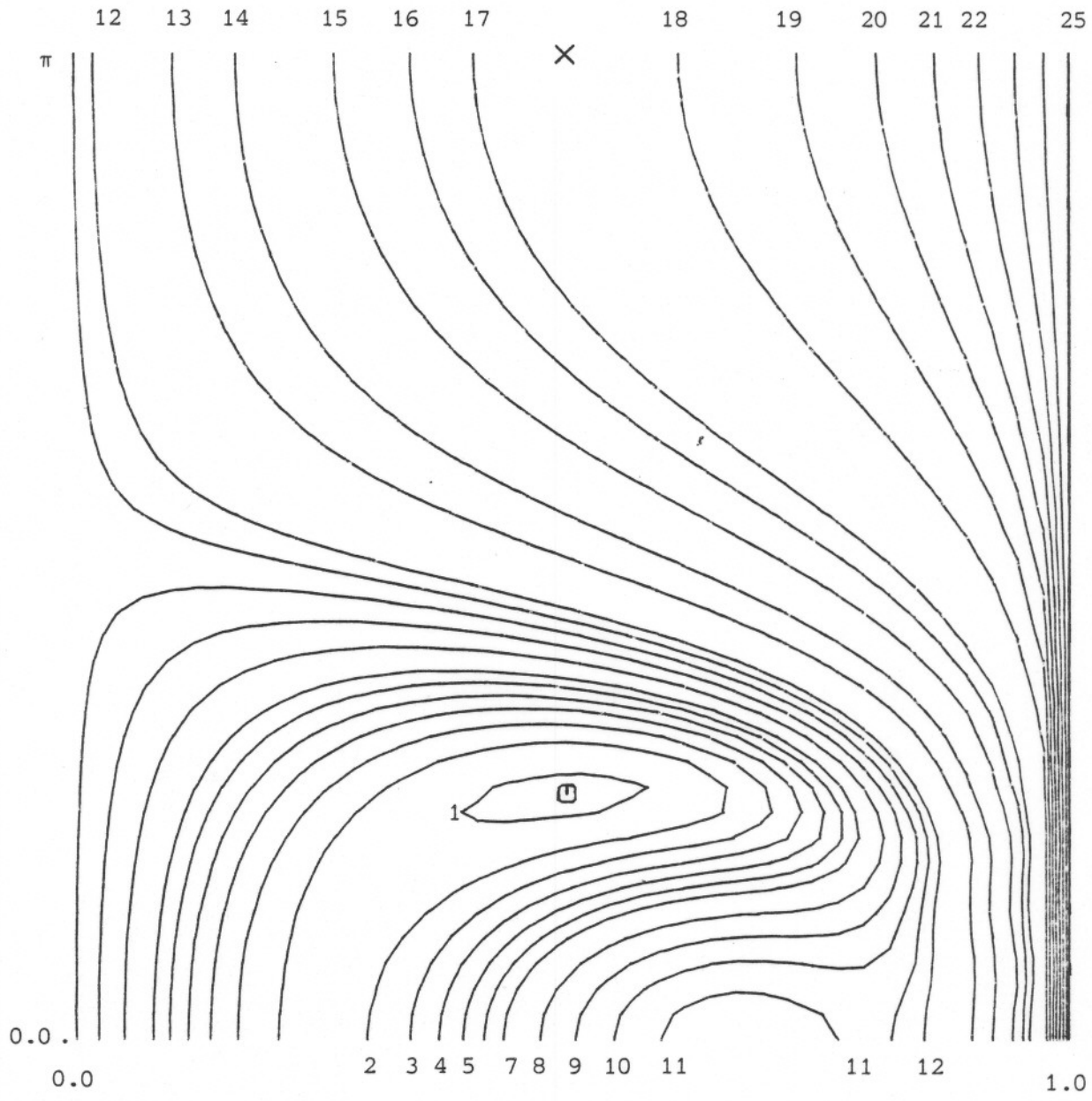


fig. 8.

$n=3,\ \alpha_1=\bar{\alpha}_2=0.5\ \exp(j\pi/4),\alpha_3=0.0$

$\beta_1=\beta_2=-0.5,\ \beta_3=0.8$

$c_i=0.814,0.82,0.85,0.9,0.95,0.97,$
$0.98,0.983,0.99,1.0$

min=0.81361, L=1, M=2

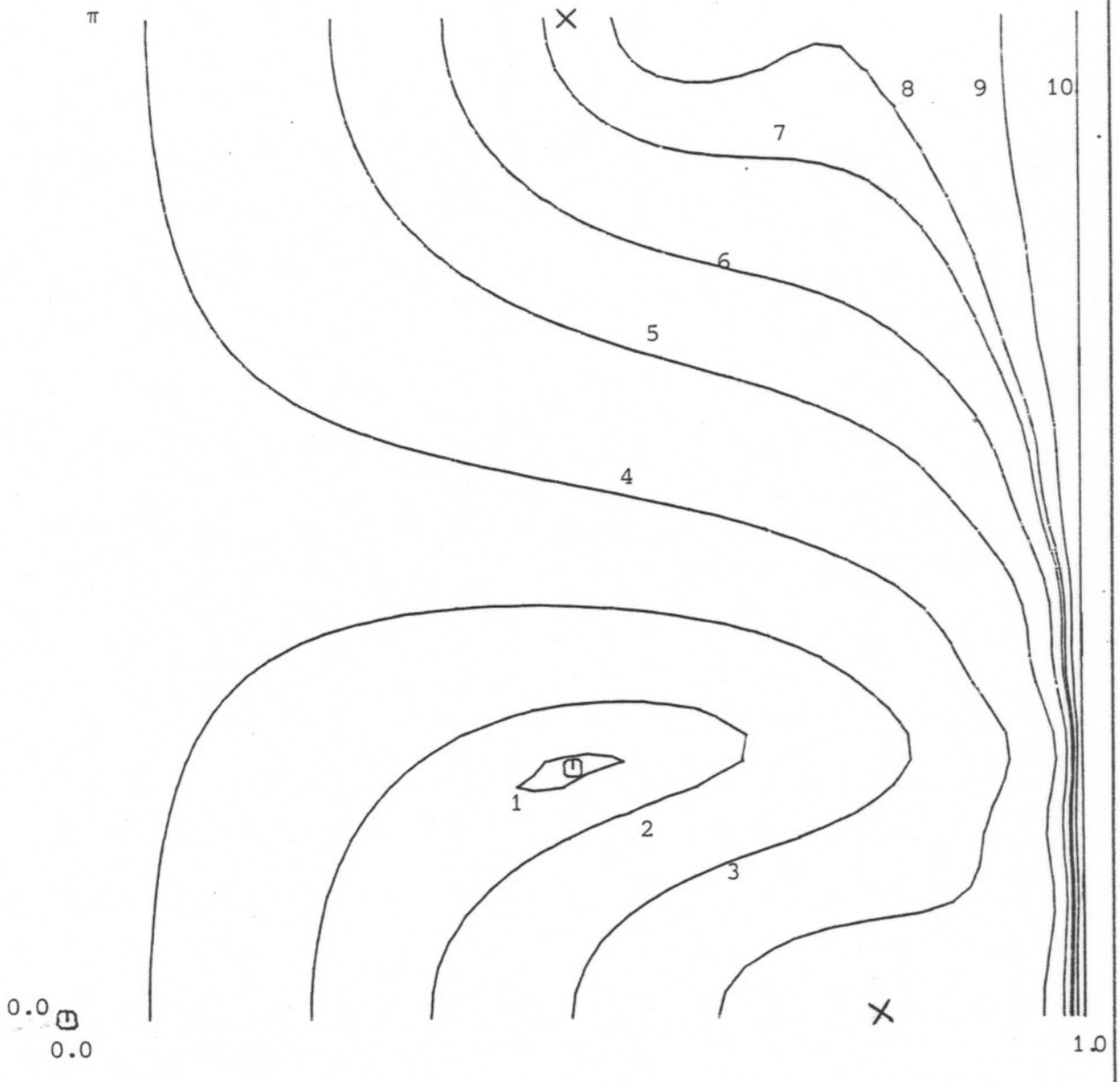


fig. 9.

extra.f

```
      subroutine extra(nd,d1,d2,alfa,gamma,ro)
c Schur extraction of transmission zero at alfa to generate
c the reflection coefficients gamma and ro.
c d1(i),d2(i),i=1,2,...,nd are acceptable function coefficients.
      complex d1(nd),d2(nd),alfa,gamma,ro,t,n,dh,qh,qk
      complex cgamma,cro,calfa
      nd1=nd-1
      if(cabs(alfa).lt.1.e-5) goto 1
      calfa=conjg(alfa)
      t=d1(nd)
      n=d2(nd)
      do 2 i=1,nd1
            n=n*alfa+d2(nd-i)
2           t=t*alfa+d1(nd-i)
      gamma=t/n
      cgamma=conjg(gamma)
      do 3 i=1,nd
            dh=d1(i)-gamma*d2(i)
            d2(i)=d1(i)*cgamma-d2(i)
3           d1(i)=dh
      qh=cmplx(0.,0.)
      do 4 i=1,nd
            qk=(d1(i)+qh)/alfa
            d1(i)=qk-calfa*qh
4           qh=qk
      ro=gamma/alfa
      go to 6
1     do 8 i=1,nd1
8           d1(i)=d1(i+1)
      d1(nd)=cmplx(0.,0.)
      ro=d1(1)/d2(1)
      gamma=cmplx(0.,0.)
      cgamma=gamma
6     cro=conjg(ro)
      do 5 i=1,nd
            dh=d1(i)*cro-d2(i)
            d1(i)=d1(i)-ro*d2(i)
5           d2(i)=dh
      return
      end
```

fig. 10.