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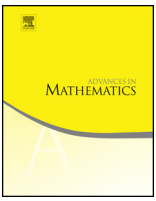

Asymptotics of matrix valued orthogonal polynomials on $[-1, 1]$

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ABSTRACT

We analyze the large degree asymptotic behavior of matrix valued orthogonal polynomials (MVOPs), with a weight that consists of a Jacobi scalar factor and a matrix part. Using the Riemann–Hilbert formulation for MVOPs and the Deift–Zhou method of steepest descent, we obtain asymptotic expansions for the MVOPs as the degree tends to infinity, in different regions of the complex plane (outside the interval of orthogonality, on the interval away from the endpoints and in neighborhoods of the endpoints), as well as for the matrix coefficients in the three-term recurrence relation for these MVOPs. The asymptotic analysis follows the work of Kuijlaars, McLaughlin, Van Assche and Vanlessen on scalar Jacobi-type orthogonal polynomials, but it also requires several different factorizations of the matrix part of the weight, in terms of eigenvalues/eigenvectors and using a matrix Szegő function. We illustrate the results with two main examples, MVOPs of Jacobi and Gegenbauer type, coming from group theory.

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1. Introduction and statement of results

1.1. Introduction

In this paper, we are interested in the large degree asymptotic behavior of matrix valued orthogonal polynomials (MVOPs), with orthogonality defined on $[-1,1]$. The weight matrix W on $[-1,1]$ is of size $r\times r$, and we take it of the form

$$W(x)=(1-x)^{\alpha}(1+x)^{\beta}H(x) \tag{1.1}$$

with $\alpha,\beta>-1$ and where the matrix valued function $H(x)$ satisfies the following:

Assumption 1.1.

(a) $H(x)$ is an $r\times r$ complex valued matrix for $x\in[-1,1]$,
(b) $H(x)$ is Hermitian positive definite for $x\in(-1,1)$,
(c) $H(x)$ is real analytic on $[-1,1]$,
(d) $H(-1)$ and $H(1)$ are not identically zero.

The real analyticity means that H has an analytic extension to a neighborhood of $[-1,1]$ in the complex plane that we will also denote by H. The requirement in (b) is that $H(x)$ is a Hermitian matrix, i.e., $H(x)=H(x)^*$, for every $x\in(-1,1)$, with positive eigenvalues. Then by real analyticity $H(-1)$ and $H(1)$ are Hermitian non-negative definite, but not necessarily positive definite, as some of the eigenvalues could vanish at ±1. However, not all eigenvalues can vanish because of the requirement in (d). In our examples the matrix valued function H is polynomial in x, and $H(\pm1)$ will be singular.

If $H(x)$ is a diagonal matrix for every $x\in[-1,1]$, then the MVOPs reduce to r usual scalar orthogonal polynomials with weight functions of the type $w(x)=(1-x)^{\alpha}(1+x)^{\beta}h(x)$, where $h(x)$ is analytic in a neighborhood of $[-1,1]$. Strong asymptotics for these kind of orthogonal polynomials was obtained with Riemann-Hilbert methods by Kuijlaars, McLaughlin, Van Assche and Vanlessen in [32], and the present paper can be viewed as a matrix valued extension of that work.

The monic MVOP P_n is defined by the property that for $m,n\geq 0$,

$$\int_{-1}^{1}P_n(x)W(x)P_m(x)^*dx=\delta_{n,m}\Gamma_n \tag{1.2}$$

with a positive definite matrix Γ_n, where $P_n(x)=x^nI_r+\cdots$ is a matrix valued polynomial of degree n whose leading coefficient is the identity matrix I_r. The integral in (1.2) is taken entrywise. Under Assumption 1.1, existence and uniqueness of the sequence $(P_n)_n$ is guaranteed.

Matrix orthogonal polynomials have appeared in many different contexts in the literature in the last years. Following classical ideas in the scalar case, Durán and Grünbaum in [18,19] studied MVOPs from the perspective of eigenfunctions of second order differential operators with matrix coefficients. This work has produced a large number of contributions in the literature, extending classical identities for scalar OPs to the matrix case. A general analysis of the matrix Bochner problem (the classification of $N \times N$ weight matrices whose associated MVOPs are eigenfunctions of a second order differential operator) has been recently addressed by Casper and Yakimov in [9], using techniques from noncommutative algebra.

From the point of view of group theory and representation theory, the study of matrix valued spherical functions has led to families of MVOPs associated to compact symmetric spaces. The first example of this connection is given by Grünbaum, Pacharoni and Tirao in [24] for the symmetric pair $(G, K) = (\mathrm{SU}(3), \mathrm{U}(2))$, see also [34,35,37]. Another approach was developed in [28,29] for the $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{diag})$, and later extended to a more general set-up in the context of the so-called multiplicity free pairs. In particular, [27] gives a detailed study of the Gegenbauer matrix valued orthogonal polynomials, which can be considered as matrix valued analogues of the Chebyshev polynomials, i.e., the spherical polynomials on $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{diag})$, better known as the characters on $\mathrm{SU}(2)$, see also [1] for the quantum group case.

The Riemann–Hilbert formulation for MVOPs appears in the works of Grünbaum, de la Iglesia and Martínez-Finkelshtein [23], and Cassatella-Contra and Mañas [10], as a generalization of the classical result of Fokas, Its and Kitaev [21]. This formulation has been used in several examples, like Hermite and Laguerre–type MVOPs in [7,8] or matrix biorthogonal polynomials in [5,6], in order to obtain algebraic and differential identities for MVOPs that can be seen as non-commutative analogues of well known identities in the theory of integrable systems, such as the Toda lattice equation or Painlevé equations.

Asymptotic results for MVOPs obtained from the Riemann–Hilbert formulation using the Deift–Zhou method [13] of steepest descent are much more scarce. In the last few years, MVOPs have appeared in the area of integrable probability, more precisely in the study of random tilings of plane figures. We mention the recent work by Duits and Kuijlaars [17] and Berggren and Duits [3] on periodic tilings of the Aztec diamond, as well as the papers by Charlier [11] and by Groot and Kuijlaars [22] on doubly periodic lozenge tilings of a hexagon. In these cases, an essential step in the asymptotic analysis is the connection between matrix orthogonality in the complex plane and scalar orthogonality on suitable curves in a Riemann surface.

Our results are strong asymptotic formulas for $P_n(z)$ as $n \to \infty$, for z in three regions in the complex plane, namely in the exterior region $\mathbb{C} \setminus [-1, 1]$, in the oscillatory region $(-1, 1)$ away from the endpoints, and near the endpoints. An important aspect of this work is the fact that we use different factorizations of the weight matrix for the asymptotic analysis: in the outer region and on the interval $(-1, 1)$, we use a matrix Szegő function D, which is obtained from a matrix spectral factorization of the weight on the unit circle; in neighborhoods of the endpoints, we use the spectral decomposi-

tion of $W(x)$, since the possible vanishing of the eigenvalues at $z = \pm 1$ is essential in the construction of the local parametrices. The same methodology allows us to include asymptotic expansions for the recurrence coefficients as well.

Throughout we assume that W is of the form (1.1) with H satisfying Assumption 1.1, and P_n is the degree n monic MVOP satisfying (1.2). We use A^T to denote the transpose of a matrix A and A^* for its Hermitian transpose. For a matrix valued function $A(z)$ defined for $z \in \mathbb{C} \setminus \Sigma$, where Σ is an oriented contour, we use $A_+(x)$ $(A_-(x))$ for the limits of $A(z)$ as $z \to x \in \Sigma$ from the $+$-side ($-$-side). The $+$-side ($-$-side) is on our left (right) as we follow Σ according to its orientation.

1.2. Factorizations of the weight matrix

Our asymptotic results rely on three factorizations of the weight matrix.

1.2.1. First factorization

The first one is the familiar spectral decomposition of $H(x)$

$$H(x) = Q(x)\Lambda(x)Q(x)^*, \qquad x \in [-1, 1] \tag{1.3}$$

with a unitary matrix $Q(x)$ and a diagonal matrix

$$\Lambda(x) = \operatorname{diag}(\lambda_1(x), \dots, \lambda_r(x)) \tag{1.4}$$

containing the eigenvalues $\lambda_j(x)$, $j = 1, \dots, r$ of $H(x)$. The assumption that H is real analytic on $[-1, 1]$ has the following important consequence.

Lemma 1.2. *$Q(x)$ and $\Lambda(x)$ can (and will) be taken to be real analytic on $[-1, 1]$.*

Proof. This is a well-known theorem of Rellich, see [40] or [38, Theorem 1.4.4]. □

We choose $Q(x)$ and $\Lambda(x)$ as in Lemma 1.2, and we continue to use Q and Λ for their analytic continuations to a neighborhood of $[-1, 1]$ in the complex plane. Then each eigenvalue $\lambda_j(x)$, $j = 1, \dots, r$ is analytic in that same neighborhood of $[-1, 1]$, and it satisfies $\lambda_j(x) > 0$ for $x \in (-1, 1)$ because of Assumption 1.1 (b), but $\lambda_j(x)$ could be zero at $x = \pm 1$, since we do not assume positive definiteness of H at the endpoints.

Definition 1.3. We define for $j = 1, \dots, r$,

(a) n_j is the order of vanishing of $\lambda_j(x)$ at $x = 1$, where we put $n_j = 0$ if $\lambda_j(1) > 0$, and

$$\alpha_j = \alpha + n_j, \tag{1.5}$$

(b) m_j is the order of vanishing of $\lambda_j(x)$ at $x = -1$, where we put $m_j = 0$ if $\lambda_j(-1) > 0$, and

$$\beta_j = \beta + m_j. \tag{1.6}$$

Because of Assumption 1.1 (d) at least one of the numbers $n_1, \dots, n_r$ is equal to zero, and similarly for the m_j's. Thus we have

$$\min\{n_j \mid j = 1, \dots, r\} = \min\{m_j \mid j = 1, \dots, r\} = 0.$$

We emphasize that $\lambda_j(x)$, for $j = 1, \dots, r$ are the eigenvalues of $H(x)$, and so by (1.1) the eigenvalues of $W(x)$ are $(1-x)^\alpha(1+x)^\beta\lambda_j(x)$ for $j = 1, \dots, r$.

1.2.2. Second factorization

The second factorization of $W(x)$ is less familiar.

Proposition 1.4. *There exists an analytic matrix valued function $D : \mathbb{C} \setminus [-1, 1] \to \mathbb{C}^{r\times r}$ with boundary values $D_\pm$ on $(-1, 1)$ satisfying*

$$W(x) = D_-(x)D_-(x)^* = D_+(x)D_+(x)^*, \tag{1.7}$$

where $D(z)$ is invertible for every $z \in \mathbb{C} \setminus [-1, 1]$, and such that

$$D(\infty) = \lim_{z\to\infty} D(z) \tag{1.8}$$

exists and is invertible as well.

Proposition 1.4 follows from Lemma 3.2 below.

A similar factorization, but for weight matrices on the unit circle appeared in [3], in the study of correlation functions for determinantal processes involving infinite Toeplitz minors, which arise in random tilings of certain planar domains.

Remark 1.5. We consider $D(z)$ as a matrix valued Szegő function. It arises from a matrix spectral factorization of the weight matrix W. It is unique up to a constant unitary matrix. That is, if D satisfies the conditions of Proposition 1.4 and U is a unitary matrix, independent of z, then DU satisfies the conditions as well. Uniqueness of the matrix valued Szegő function is guaranteed if we require that $D(\infty)$ is a positive definite Hermitian matrix. We call this the normalized matrix valued Szegő function.

If $W(x)$ is real valued for $x \in (-1, 1)$, then the normalized matrix valued Szegő function D will satisfy the symmetry condition

$$D(\overline{z}) = \overline{D(z)}, \qquad z \in \mathbb{C} \setminus [-1, 1]. \tag{1.9}$$

In that case $D_-(x) = \overline{D_+(x)}$ and the factorization (1.7) can be alternatively written as

$$W(x) = D_-(x)D_+(x)^T = D_+(x)D_-(x)^T. \tag{1.10}$$

Also $D(\infty)$ is a positive definite real matrix in this case.

1.2.3. Third factorization

The third factorization is very much related to the spectral decomposition (1.3). We use modified eigenvalues

$$\widetilde{\lambda}_j = (-1)^{n_j}\lambda_j, \qquad j = 1, \dots, r, \tag{1.11}$$

and

$$\widetilde{\Lambda} = \operatorname{diag}(\widetilde{\lambda}_1, \dots, \widetilde{\lambda}_r). \tag{1.12}$$

Recall from Definition 1.3 that n_j denotes the order of vanishing of λ_j at $x = 1$. Thus $\widetilde{\lambda}_j(x) > 0$ for $x \in (1, 1+\delta)$ for some $\delta > 0$, and we use $\widetilde{\lambda}_j(x)^{1/2}$ to denote its positive square root. This has an analytic continuation to a neighborhood of $[-1, 1]$ with a branch cut along $(-\infty, 1]$ that we also denote by $\widetilde{\lambda}_j^{1/2}$. Then we define

$$\widetilde{\Lambda}^{1/2} = \operatorname{diag}\left(\widetilde{\lambda}_1^{1/2}, \dots, \widetilde{\lambda}_r^{1/2}\right), \tag{1.13}$$

and

$$V(z) = (z-1)^{\alpha/2}(z+1)^{\beta/2}Q(z)\widetilde{\Lambda}(z)^{1/2}, \tag{1.14}$$

which is defined and analytic with a branch cut along $(-\infty, 1]$. In particular it is defined and analytic in $D(1, \delta) \setminus (1-\delta, 1]$ for some $\delta > 0$.

We will use V for the local analysis around 1. Near -1 we have a similarly defined matrix valued function. We define

$$\widehat{\Lambda} = \operatorname{diag}\left(\widehat{\lambda}_1, \dots, \widehat{\lambda}_r\right), \quad \widehat{\lambda}_j = (-1)^{m_j}\lambda_j, \tag{1.15}$$

so that $\widehat{\lambda}_j(x) > 0$ for $x \in (-1-\delta, -1)$ for some $\delta > 0$. Then we define

$$\widehat{V}(z) = (1-z)^{\alpha/2}(-1-z)^{\beta/2}Q(z)\widehat{\Lambda}^{1/2}(z), \tag{1.16}$$

defined with a branch cut along $[-1, \infty)$.

The third factorization of W is as follows:

Lemma 1.6. *We have for $x \in (-1,1)$,*

$$\begin{aligned} W(x) &= V_-(x)V_-(x)^* = V_+(x)V_+(x)^* \\ &= \widehat{V}_-(x)\widehat{V}_-(x)^* = \widehat{V}_+(x)\widehat{V}_+(x)^*, \end{aligned} \tag{1.17}$$

where V and $\widehat{V}$ are defined by (1.14) and (1.16).

Proof. This follows by straightforward calculation from the definitions (1.14) and (1.16). See also Lemma 3.4 for details. □

Comparing (1.17) and (1.7) we see that V and $\widehat{V}$ share the same factorization property with the matrix valued Szegő function D. Actually $D^{\pm}(x)^{-1}V_{\pm}(x)$ and $D_{\pm}(x)^{-1}\widehat{V}_{\pm}(x)$ are unitary matrices for every $x \in (-1,1)$, see formula (3.46) below. For our asymptotic results we need their values at the endpoints.

Lemma 1.7. *The two limits*

$$U_1 = \lim_{z\to 1} D(z)^{-1}V(z), \qquad U_{-1} = \lim_{z\to -1} D(z)^{-1}\widehat{V}(z) \tag{1.18}$$

exist, and define unitary matrices U_1 and U_{-1}.

The proof of Lemma 1.7 is in Section 3.5.8.

1.3. Asymptotics in the exterior region

Throughout the paper, we need the conformal map

$$\varphi(z) = z + (z^2-1)^{1/2}, \qquad z \in \mathbb{C}\setminus[-1,1] \tag{1.19}$$

from $\mathbb{C}\setminus[-1,1]$ to the exterior of the unit circle. Our first result is the asymptotics of $P_n(z)$ as $n\to\infty$ for $z \in \mathbb{C}\setminus[-1,1]$. The main term in the asymptotic formula (1.20) is not new as it is known at least since [2], where it is proved under weaker assumptions as well, namely W is assumed to satisfy a matrix Szegő condition on $[-1,1]$ with a finite number of mass points outside $[-1,1]$). See also [30] for an infinite number of mass points.

Theorem 1.8. *Let W be the weight matrix (1.1) with H satisfying Assumption 1.1. Let D be the matrix Szegő function associated with W as in Proposition 1.4. Then as $n\to\infty$ the monic MVOP P_n has an asymptotic series expansion*

$$\frac{2^n P_n(z)}{\varphi(z)^n} \sim \frac{\varphi(z)^{1/2}}{\sqrt{2}(z^2-1)^{1/4}} D(\infty)\left[I_r + \sum_{k=1}^{\infty}\frac{\Pi_k(z)}{n^k}\right] D(z)^{-1}, \quad z\in\mathbb{C}\setminus[-1,1], \tag{1.20}$$

uniformly for z in compact subsets of $\mathbb{C} \setminus [-1,1]$, where each Π_k is an analytic function in $\mathbb{C} \setminus [-1,1]$. The first one is

$$\Pi_1(z) = -\frac{1}{8(\varphi(z)-1)} U_1 \operatorname{diag}\left(4\alpha_1^2 - 1, \dots, 4\alpha_r^2 - 1\right) U_1^{-1} + \frac{1}{8(\varphi(z)+1)} U_{-1} \operatorname{diag}\left(4\beta_1^2 - 1, \dots, 4\beta_r^2 - 1\right) U_{-1}^{-1}, \quad (1.21)$$

where U_1 and U_{-1} are as in (1.18) and the parameters α_j and β_j for $j = 1, \dots, r$ are given by (1.5) and (1.6).

The proof of Theorem 1.8 is in Section 4.1.

The leading term in (1.20) is known. The limit

$$\lim_{n\to\infty} \frac{2^n P_n(z)}{\varphi(z)^n} = \frac{\varphi(z)^{1/2}}{\sqrt{2}(z^2-1)^{1/4}} D(\infty) D(z)^{-1}, \quad z \in \mathbb{C} \setminus [-1,1], \quad (1.22)$$

can equivalently be written as

$$\lim_{n\to\infty} (2z)^n P_n\left(\frac{z+z^{-1}}{2}\right) = \frac{1}{(1-z^2)^{1/2}} D(\infty) D\left(\frac{z+z^{-1}}{2}\right)^{-1} \qquad |z| < 1,$$

which corresponds to the asymptotics stated in [2, Theorem 2] and [30]. The analogous result for MVOP on the unit circle dates back to [41] and [15].

The existence of a full asymptotic expansion is new, as well as the explicit form (1.21) of the first subleading term.

Remark 1.9. The expression (1.21) simplifies if $n_j = m_j = 0$ for every $j = 1, \dots, r$, since in that case the two diagonal matrices in (1.21) are multiples of the identity matrix. Then (1.21) reduces to

$$\Pi_1(z) = \left(-\frac{4\alpha^2 - 1}{8(\varphi(z)-1)} + \frac{4\beta^2 - 1}{8(\varphi(z)+1)}\right) I_r,$$

which is consistent with the formula given in [32, formula (1.13)] for the scalar case.

1.4. Asymptotics on the interval $(-1,1)$

The MVOP P_n has oscillatory behavior on the interval $(-1,1)$. Theorem 1.10 should be compared with Theorem 2 (f) in [2], where the boundary values are given in L^2 sense, while our asymptotic formula (1.24) holds uniformly on compact subsets of $(-1,1)$.

Theorem 1.10. *With the same assumptions as in Theorem 1.8, we have uniformly for x in compact subsets of $(-1,1)$,*

$$2^n P_n(x) = \frac{1}{\sqrt{2}\sqrt[4]{1-x^2}} D(\infty) \times \left(e^{i(n+\frac{1}{2})\arccos(x) - \frac{\pi i}{4}} D_+(x)^{-1} + e^{-i(n+\frac{1}{2})\arccos(x) + \frac{\pi i}{4}} D_-(x)^{-1} \right) + \mathcal{O}(n^{-1}), \quad (1.23)$$

as $n \to \infty$.

In case $W(x)$ *is real symmetric for every* $x \in (-1,1)$*, then the MVOP* $P_n(x)$ *is real valued for real* x*. Then, if we use the normalized Szegő function as in Remark 1.5, we have uniformly for* x *in compact subsets of* $(-1,1)$*,*

$$2^n P_n(x) = \frac{\sqrt{2}}{(1-x^2)^{\frac{1}{4}}} D(\infty) \operatorname{Re}\left(e^{i(n+\frac{1}{2})\arccos x - \frac{\pi i}{4}} D_+(x)^{-1} \right) + \mathcal{O}(n^{-1}), \quad (1.24)$$

where the real part of the matrix is taken entrywise.

We prove Theorem 1.10 in Section 4.2.

1.5. *Asymptotics near the endpoints* $z = \pm 1$

Near the endpoints ± 1 we find asymptotic formulas in terms of Bessel functions. For the scalar case $r = 1$, the following is known (see Theorem 1.13 of [32]): there exists $\delta > 0$ such that for $x \in (1-\delta, 1)$ we have

$$P_n(x) = \frac{D(\infty)}{2^n \sqrt{W(x)}} \frac{\sqrt{n\pi \arccos x}}{(1-x^2)^{1/4}} \begin{pmatrix} \cos(\zeta(x)) & \sin(\zeta(x)) \end{pmatrix} \left(I_2 + \mathcal{O}(n^{-1}) \right) \times \begin{pmatrix} J_\alpha(n \arccos x) \\ J'_\alpha(n \arccos x) \end{pmatrix} \quad (1.25)$$

as $n \to \infty$, where J_α is the Bessel function of the first kind and order α and $\zeta(x)$ is a certain explicit function that depends on the weight W.

In the matrix valued generalization of (1.25), it turns out that Bessel functions of various orders appear. The orders of the Bessel functions in the asymptotics near 1 are determined by the parameters α_j introduced in (1.5).

We write

$$J_{\vec{\alpha}}(x) = \operatorname{diag}\left(J_{\alpha_1}(x), \dots, J_{\alpha_r}(x) \right) \quad (1.26)$$

for the diagonal matrix containing the Bessel function of orders $\alpha_1, \dots, \alpha_r$ on the diagonal, and similarly for $\left(J_{\vec{\alpha}}\right)'(x)$. We also use

$$A(z) = \frac{(z+1)^{1/2} + (z-1)^{1/2}}{\sqrt{2}} D(z)^{-1} V(z), \quad (1.27)$$

with the principal branch of the square roots and V is defined in (1.14).

Theorem 1.11. *We make the same assumptions as in Theorem 1.8, and we let Q and A be given by (1.3) and (1.27). Then there exists $\delta > 0$ such that for $x \in (1-\delta, 1)$,*

$$P_n(x)\sqrt{W(x)} = \frac{\sqrt{\pi n \arccos x}}{2^n(1-x^2)^{1/4}} D(\infty) \begin{pmatrix} \frac{A_+(x)+A_-(x)}{2} & \frac{A_+(x)-A_-(x)}{2i} \end{pmatrix} \left(I_{2r} + \mathcal{O}(n^{-1})\right) \times \begin{pmatrix} J_{\vec{\alpha}}(n\arccos x) \\ (J_{\vec{\alpha}})'(n\arccos x) \end{pmatrix} Q(x)^*, \quad (1.28)$$

with $J_{\vec{\alpha}}$ as in (1.26).

From (1.27) we obtain

$$A_\pm(x) = e^{\pm i \arccos x} D_\pm^{-1} V_\pm(x)$$

and $A_\pm(x)$ turn out to be unitary matrices for $x \in (-1,1)$, see (3.46) below. The limit

$$U_1 = \lim_{z\to 1} A(z) = \lim_{z\to 1} D(z)^{-1} V(z) \quad (1.29)$$

exists and is also a unitary matrix. It agrees with (1.18).

If W is real symmetric, then $A_-(x) = \overline{A_+(x)}$, and A is a real orthogonal matrix. Then we may write

$$A_\pm(x) = U_1 e^{\pm i Z(x)} \quad (1.30)$$

with a Hermitian matrix valued function $Z(x)$ that varies analytically and $Z(x) \to O_r$ as $x \to 1-$. In fact we have $Z(x) = \mathcal{O}\left(\sqrt{1-x}\right)$ as $x \to 1-$. Then

$$\frac{A_+(x)+A_-(x)}{2} = U_1 \cos Z(x), \qquad \frac{A_+(x)-A_-(x)}{2i} = U_1 \sin Z(x)$$

and we obtain the following.

Corollary 1.12. *If $W(x)$ is real symmetric for every $x \in (-1,1)$, then (1.28) takes the form*

$$P_n(x)\sqrt{W(x)} = \frac{\sqrt{\pi n \arccos x}}{2^n(1-x^2)^{1/4}} D(\infty) U_1 \begin{pmatrix} \cos(Z(x)) & \sin(Z(x)) \end{pmatrix} \left(I_{2r} + \mathcal{O}(n^{-1})\right) \times \begin{pmatrix} J_{\vec{\alpha}}(n\arccos x) \\ (J_{\vec{\alpha}})'(n\arccos x) \end{pmatrix} Q(x)^T.$$

We obtain from Theorem 1.11 the Mehler-Heine asymptotics at $z = 1$.

Theorem 1.13. *Suppose the weight matrix W satisfies Assumption 1.1. Suppose λ_j, $j = 1, \dots, r$ be the eigenvalues of H as in (1.3), (1.4) and let*

$$c_j = 2^{-\alpha_j+\beta} \lim_{x\to 1} \frac{\lambda_j(x)}{(1-x)^{n_j}}, \qquad \text{for } j = 1, \dots, r, \tag{1.31}$$

where $\alpha_j = \alpha + n_j$ as in (1.5). Then we have the following Mehler–Heine asymptotics of the monic MVOP associated with W:

$$\lim_{n\to\infty} \frac{2^n}{\sqrt{n\pi}} P_n\left(\cos\frac{\theta}{n}\right) Q\left(\cos\frac{\theta}{n}\right) \operatorname{diag}\left(c_1^{1/2} n^{-\alpha_1}, \dots, c_r^{1/2} n^{-\alpha_r}\right) = D(\infty) U_1 \operatorname{diag}\left(\theta^{-\alpha_1} J_{\alpha_1}(\theta), \dots, \theta^{-\alpha_r} J_{\alpha_r}(\theta)\right) \tag{1.32}$$

with Q and U_1 given by (1.3) and (1.18) and D the matrix valued Szegő function.

The proofs of Theorems 1.11 and 1.13 are in Section 4.3.

Analogous results hold near -1, with Bessel functions of order $\beta_j = \beta + m_j$.

1.6. Asymptotics of recurrence coefficients

The monic MVOPs satisfy a three term recurrence relation:

$$xP_n(x) = P_{n+1}(x) + B_n P_n(x) + C_n P_{n-1}(x), \tag{1.33}$$

with initial values $P_{-1}(x) = 0_r$ and $P_0(x) = I_r$, see e.g. [12]. From the Riemann-Hilbert asymptotic analysis that we present in this paper, one can obtain large n asymptotics for the recurrence coefficients B_n and C_n, see also [32] for the scalar case.

Recall that α_j and β_j for $j = 1, \dots, r$ are defined in (1.5) and (1.6).

Theorem 1.14. *Suppose the weight matrix W satisfies the assumptions of Theorem 1.8. The recurrence coefficients B_n and C_n in (1.33) admit asymptotic expansions of the form*

$$B_n \sim \sum_{k=2}^{\infty} \frac{\mathcal{B}_k}{n^k}, \qquad C_n \sim \frac{1}{4} I_{2r} + \sum_{k=2}^{\infty} \frac{\mathcal{C}_k}{n^k}, \qquad n \to \infty, \tag{1.34}$$

with certain computable $r \times r$ matrices $\mathcal{B}_k$, $\mathcal{C}_k$, for $k = 2, 3, \dots$.

We have an explicit formula for $\mathcal{B}_2$,

$$\mathcal{B}_2 = -\frac{1}{16} D(\infty) U_1 \operatorname{diag}\left(4\alpha_1^2 - 1, \dots, 4\alpha_r^2 - 1\right) U_1^{-1} D(\infty)^{-1} + \frac{1}{16} D(\infty) U_{-1} \operatorname{diag}\left(4\beta_1^2 - 1, \dots, 4\beta_r^2 - 1\right) U_{-1}^{-1} D(\infty)^{-1}, \tag{1.35}$$

where U_1 and U_{-1} are given by (1.18), and D is the matrix valued Szegő function.

The proof is in Section 4.4. The matrices $\mathcal{B}_k$ and $\mathcal{C}_k$ in (1.34) are, in principle, explicitly computable in an iterative manner. However, the computations become very involved with increasing k, and we limit ourselves in Theorem 1.14 to the explicit form of $\mathcal{B}_2$.

If $\alpha_j = \alpha$ and $\beta_j = \beta$ for every j, then (1.35) simplifies to $\mathcal{B}_2 = \frac{\beta^2-\alpha^2}{4} I_r$, which is consistent with the formula given in [32, (1.30)] for the scalar case. In the scalar case more terms are given in [32, Theorem 1.10].

2. Two examples

In this section we discuss two examples that arise from the study of matrix valued orthogonal polynomials associated to compact symmetric pairs. We find it remarkable that in both examples the matrix Szegő function $D(z)$ can be computed explicitly.

2.1. A Jacobi weight

Our first example is a family of Jacobi-type matrix orthogonal polynomials which is connected with the matrix valued spherical functions associated to the compact symmetric pair $(\mathrm{SU}(n+1), \mathrm{SU}(n-1))$. This is the result of a series of papers, starting with [24] and later extended in [37,34,33,35]. The weight matrix is given in [34, Corollary 3.3 and Theorem 3.4].

Let $\alpha, \beta > -1$, $0 < k < \alpha + 1$ and $\ell \in \mathbb{N}_0$. We consider the $(\ell+1) \times (\ell+1)$ weight matrix

$$W(x) = (1-x)^\alpha (1+x)^\beta H(x), \quad H(x) = \Psi(x) T \Psi(x)^T, \qquad x \in [-1,1], \tag{2.1}$$

where $\Psi(x)$ is upper triangular and T is a constant diagonal matrix. Explicitly, we have

$$T_{j,j} = \binom{\ell+k-1-j}{\ell-j} \binom{\alpha-k+j}{j}, \qquad j = 0, \dots, \ell,$$

and

$$\Psi(x)_{i,j} = \binom{j}{i} 2^{-\frac{\ell-j}{2}-i} (1+x)^{\ell-\frac{j}{2}} (1-x)^i, \qquad 0 \le i \le j \le \ell. \tag{2.2}$$

We note that the orthogonality interval in [34] is $[0,1]$, so in (2.1) we have made a change of variables to $[-1,1]$ in order to match with the setup in Assumption 1.1 of the present paper. We have also interchanged the exponents α and β in order to be consistent with standard notation for Jacobi polynomials, that we also follow in this paper, and we take as $\Psi(x)$ the transpose of the corresponding matrix from [34].

The matrix part H of the weight (2.1) has the factorized form

$$\begin{aligned} H(x) = {} & \operatorname{diag}\left(1, 1-x, \dots, (1-x)^\ell\right) R \\ & \times \operatorname{diag}\left((1+x)^\ell, (1+x)^{\ell-1}, \dots, 1+x, 1\right) R^T \operatorname{diag}\left(1, 1-x, \dots, (1-x)^\ell\right) \end{aligned} \tag{2.3}$$

with a constant upper triangular matrix R containing the entries

$$R_{i,j} = \binom{j}{i} 2^{-\frac{\ell-j}{2}-i} T_{j,j}^{1/2}, \qquad 0 \le i \le j \le \ell.$$

Thus the entries of $H(x)$ are polynomial in x.

For any choice of invertible upper triangular matrix R, we can compute the matrix Szegő function for H explicitly, and this will allow us to make the asymptotic results explicit for this class of examples.

Proposition 2.1. *Let R be any invertible upper triangular matrix. Then the matrix Szegő function D_H for the matrix weight (2.3) is equal to*

$$\begin{aligned} D_H(z) &= \operatorname{diag}\left(1, 1-z, \dots, (1-z)^\ell\right) R \\ &\quad \times \operatorname{diag}\left((z+1)^{\frac{\ell}{2}}\varphi(z)^{-\frac{\ell}{2}}, (z+1)^{\frac{\ell-1}{2}}\varphi(z)^{-\frac{\ell+1}{2}}, \dots, \varphi(z)^{-\ell}\right), \quad z \in \mathbb{C} \setminus [-1,1] \end{aligned} \tag{2.4}$$

with principal branches of the fractional powers, where we recall that φ is the conformal map (1.19). The matrix Szegő function for W given by (2.1) is

$$D(z) = \frac{(z+1)^{\frac{\beta}{2}}(z-1)^{\frac{\alpha}{2}}}{\varphi(z)^{\frac{\alpha+\beta}{2}}} D_H(z), \qquad z \in \mathbb{C} \setminus [-1,1], \tag{2.5}$$

with D_H given by (2.4).

Proof. Let $D_H(z)$ be defined by (2.4). We show that it satisfies the requirements for the matrix Szegő function of H.

The diagonal entries in the last factor on the right-hand side of (2.4) are

$$(z+1)^{\frac{\ell-j}{2}}\varphi(z)^{-\frac{\ell+j}{2}} = \left(\frac{(z+1)^{1/2}}{\varphi(z)^{1/2}}\right)^{\ell-j} \varphi(z)^{-j}, \qquad j = 0, \dots, \ell.$$

These entries are analytic in $\mathbb{C} \setminus [-1,1]$, since $\frac{(z+1)^{1/2}}{\varphi(z)^{1/2}}$, which may be initially defined for $z \in \mathbb{C} \setminus (-\infty, 1]$, has an analytic continuation across $(-\infty, -1)$. Hence D_H is analytic in $\mathbb{C} \setminus [-1,1]$.

For $z \in \mathbb{C} \setminus [-1,1]$ the factors on the right-hand side of (2.4) are invertible matrices, and therefore $D_H(z)$ is invertible for $z \in \mathbb{C} \setminus [-1,1]$. As $z \to \infty$, we have $\frac{z+1}{\varphi(z)} \to \frac{1}{2}$ and $(z-1)^i\varphi(z)^{-j} \to \frac{1}{2}$ if $i = j$ and $(z-1)^i\varphi(z)^{-j} \to 0$ if $i < j$. Since R is upper triangular, it then follows that $D_H(z)$ tends to a diagonal matrix with nonzero diagonal entries $(-1)^j 2^{-\frac{\ell+j}{2}} R_{j,j}$, $j = 0, \dots, \ell$. Hence $D_H(\infty)$ exists and is invertible as well.

Finally, the identity $H(x) = D_{H-}(x)D_{H+}(x)^T = D_{H+}(x)D_{H-}(x)^T$ for $x \in (-1,1)$ is immediate from (2.3) and (2.4) and the fact that $\varphi_+(x)\varphi_-(x) = 1$ for $x \in (-1,1)$.

The formula (2.5) for the matrix Szegő function for W follows from (2.4), and the fact that the scalar prefactor in (2.5) is the Szegő function for the standard Jacobi weight $(1-x)^\alpha(1+x)^\beta$. □

Next we work out the details of the different asymptotic expansions in the case of a 2×2 matrix valued weight, which corresponds to $\ell = 1$ in (2.1). Up to an inessential scalar factor $\frac{k}{p}$, we have

$$W(x) = (1-x)^\alpha(1+x)^\beta H(x), \qquad H(x) = \frac{1}{4}\begin{pmatrix} 4+2p+2px & 2(1-x) \\ 2(1-x) & (1-x)^2 \end{pmatrix}, \tag{2.6}$$

with $p = k(\alpha+1-k)^{-1} > 0$, and $H(x)$ depends on the parameter p only.

Corollary 2.2. *The monic MVOP P_n associated with the weight matrix (2.6) has the following asymptotic behavior as $n\to\infty$:*

(a) *For $z \in \mathbb{C}\setminus[-1,1]$,*

$$\frac{2^n P_n(z)}{\varphi(z)^n} = F^{\text{outer}}(z)\left(I_2 + \mathcal{O}(n^{-1})\right), \tag{2.7}$$

where

$$F^{\text{outer}}(z) = \left(\frac{\varphi(z)}{2}\right)^{\frac{\alpha+\beta+1}{2}} \frac{1}{2(z-1)^{\frac{2\alpha+3}{4}}(z+1)^{\frac{2\beta+3}{4}}(1-\varphi(z))} \times \begin{pmatrix} (\varphi(z)-1)^2 & 4\varphi(z) \\ 0 & \varphi(z)+1 \end{pmatrix}. \tag{2.8}$$

(b) *For $x\in(-1,1)$,*

$$2^n P_n(x) = F^{\text{inner}}(x) + \mathcal{O}(n^{-1}), \qquad x\in(-1,1), \tag{2.9}$$

where

$$F^{\text{inner}}(x) = \frac{2^{-\frac{\alpha+\beta}{2}}}{(1-x)^{\frac{2\alpha+3}{4}}(1+x)^{\frac{2\beta+3}{4}}} \times \begin{pmatrix} (1-x)\cos(\gamma(x)) & -2\cos(\gamma(x)) \\ 0 & -\frac{1}{\sqrt{2}}\sqrt{1+x}\cos\left(\gamma(x)+\frac{\theta(x)}{2}\right) \end{pmatrix}, \tag{2.10}$$

with $\theta(x) = \arccos x$ and

$$\gamma(x) = \left(n+1+\frac{\alpha+\beta}{2}\right)\arccos x - \frac{\alpha\pi}{2} - \frac{\pi}{4}. \tag{2.11}$$

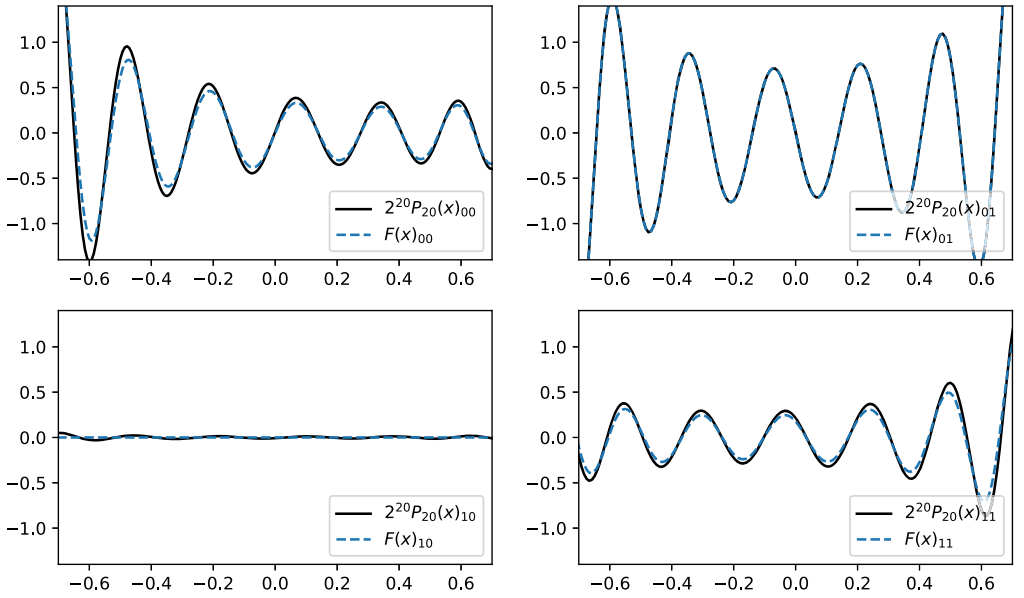

Fig. 1. Plot of the entries of the scaled Jacobi MVOP $2^n P_n(x)$ for $n = 20$ and $(\alpha, \beta, k) = (1, 2, 1)$ (solid line). The approximation $F^{\text{inner}}(x)$ (dashed line) is given in formula (2.10) of Corollary 2.2. We observe that the $(2, 1)$ entry is of order $\mathcal{O}(n^{-1})$, since the $(2, 1)$ entry of the leading term in (2.8) is 0.

(c) *Mehler–Heine asymptotics near $z = 1$:*

$$\lim_{n\to\infty} \frac{2^n}{\sqrt{n\pi}} P_n\left(\cos\frac{\theta}{n}\right) Q\left(\cos\frac{\theta}{n}\right) \operatorname{diag}\left(c_1^{1/2} n^{-\alpha}, c_2^{1/2} n^{-\alpha-2}\right)$$
$$= \frac{2^{-\frac{\alpha+\beta}{2}-2}}{\sqrt{1+p}} \begin{pmatrix} 2p & 2\sqrt{p} \\ -1 & \sqrt{p} \end{pmatrix} \begin{pmatrix} \theta^{-\alpha} J_\alpha(\theta) & 0 \\ 0 & \theta^{-\alpha-2} J_{\alpha+2}(\theta) \end{pmatrix},$$

where the constants are

$$c_1 = 2^{-\alpha+\beta} \frac{1+p}{p}, \qquad c_2 = 2^{-\alpha-4+\beta} \frac{p}{1+p}.$$

See Fig. 1 for a plot of the four entries of $2^n P_n(x)$ on the interval $[-1, 1]$ for the value $n = 20$, together with a plot of the entries of the approximation (2.10).

Proof. The matrix Szegő function for the 2×2 weight W from (2.6) is

$$D(z) = \frac{(z-1)^{\frac{\alpha}{2}} (z+1)^{\frac{\beta}{2}}}{\varphi(z)^{\frac{\alpha+\beta}{2}}} D_H(z), \qquad D_H(z) = \frac{1}{4} \begin{pmatrix} 2\sqrt{p}(1+\varphi(z)^{-1}) & 4\varphi(z)^{-1} \\ 0 & -(1-\varphi(z)^{-1})^2 \end{pmatrix}. \tag{2.12}$$

This follows from (2.4) with $\ell = 1$: after removing the scalar factor $\sqrt{k}/\sqrt{p}$, we obtain

$$D_H(z) = \frac{1}{4}\begin{pmatrix} 2\sqrt{2p}\sqrt{z+1}\varphi(z)^{-1/2} & 4\varphi(z)^{-1} \\ 0 & 2(1-z)\varphi(z)^{-1} \end{pmatrix}.$$

This leads to (2.12) because of the two identities $\sqrt{2}\sqrt{z+1} = \varphi(z)^{1/2} + \varphi(z)^{-1/2}$ and $2z = \varphi(z) + \varphi(z)^{-1}$. From (2.12), we obtain the limit behavior

$$D(\infty) = 2^{-\frac{\alpha+\beta}{2}} D_H(\infty) = 2^{-\frac{\alpha+\beta}{2}-2}\begin{pmatrix} 2\sqrt{p} & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.13}$$

With this information, we can apply Theorem 1.8 to obtain the outer asymptotics of P_n as stated in part (a).

For the inner asymptotics, we note that the weight $W(x)$ given by (2.6) is real symmetric on $[-1,1]$, so by (1.24) we only need $D_+(x)^{-1}$. Write $\varphi_\pm(x) = e^{\pm i\theta(x)}$ with $\theta(x) = \arccos(x)$. From (2.12), we obtain

$$D_+(x) = e^{-\frac{i}{2}((\alpha+\beta)\theta(x)-\alpha\pi)}(1-x)^{\frac{\alpha}{2}}(1+x)^{\frac{\beta}{2}}\begin{pmatrix} \sqrt{\frac{p}{2}}\sqrt{1+x}e^{-i\theta(x)/2} & e^{-i\theta(x)} \\ 0 & \frac{1}{2}(1-x)e^{-i\theta(x)} \end{pmatrix},$$

and therefore

$$D_+(x)^{-1} = \frac{\sqrt{2}\, e^{\frac{i}{2}((\alpha+\beta+1)\theta(x)-\alpha\pi)}}{\sqrt{p}(1+x)^{\frac{1+\beta}{2}}(1-x)^{1+\frac{\alpha}{2}}}\begin{pmatrix} 1-x & -2 \\ 0 & \sqrt{2p}\sqrt{1+x}e^{i\theta(x)/2} \end{pmatrix}.$$

Hence

$$\begin{aligned}&\operatorname{Re}\left(e^{i(n+\frac{1}{2})\theta(x)-\frac{\pi i}{4}}D_+(x)^{-1}\right)\\ &\quad= \frac{\sqrt{2}}{\sqrt{p}(1+x)^{\frac{1+\beta}{2}}(1-x)^{1+\frac{\alpha}{2}}}\begin{pmatrix} (1-x)\cos(\gamma(x)) & -2\cos(\gamma(x)) \\ 0 & \sqrt{2p}\sqrt{1+x}\cos\left(\gamma(x)+\frac{\theta(x)}{2}\right) \end{pmatrix},\end{aligned}$$

where the phase function $\gamma(x)$ is given by (2.11). Also $D(\infty) = 2^{-\frac{\alpha+\beta}{2}} D_H(\infty)$ and $D_H(\infty)$ is given by (2.13). Using this in the formula (1.24) of Theorem 1.10, we obtain the result of part (b).

The Mehler–Heine asymptotics of part (c) follows from Theorem 1.13. The eigenvalues $\lambda_{1,2}$ of the matrix $H(x)$ in (2.6) can be computed explicitly, and as $x \to \pm 1$, they behave as follows:

$$\begin{aligned} \lambda_1(x) &= 1 + p + \mathcal{O}(x-1), & \lambda_1(x) &= 2 + \mathcal{O}(x+1), \\ \lambda_2(x) &= \frac{p}{4(1+p)}(x-1)^2 + \mathcal{O}((x-1)^3), & \lambda_2(x) &= \frac{p}{4}(x+1) + \mathcal{O}((x+1)^2). \end{aligned} \tag{2.14}$$

Therefore, the exponents are $\alpha_1 = \alpha$ and $\alpha_2 = \alpha + 2$, and the constants from (1.31) are

$$c_1 = 2^{-\alpha_1+\beta} \lim_{x\to 1} \frac{\lambda_1(x)}{(1-x)^{n_1}} = 2^{-\alpha+\beta} \lim_{x\to 1} \lambda_1(x) = 2^{-\alpha+\beta} \frac{1+p}{p},$$
$$c_2 = 2^{-\alpha_2+\beta} \lim_{x\to 1} \frac{\lambda_2(x)}{(1-x)^{n_2}} = 2^{-\alpha-2+\beta} \lim_{x\to 1} \frac{\lambda_2(x)}{(1-x)^2} = 2^{-\alpha-4+\beta} \frac{p}{1+p}.$$

We can also calculate the matrix U_1 using the explicit expressions for $D_H(z)$ using (2.12), as well as $V(z)$:

$$V(z) = (z-1)^{\alpha/2}(z+1)^{\beta/2} Q(z) \tilde{\Lambda}(z)^{1/2},$$

where $\tilde{\Lambda}(z)^{1/2} = \operatorname{diag}(\lambda_1^{1/2}(z), \lambda_2^{1/2}(z))$ and we use the normalized matrix of eigenvectors:

$$Q(z) = \begin{pmatrix} \frac{1}{\sqrt{1+\rho_2(z)^2}} & -\frac{1}{\sqrt{1+\rho_1(z)^2}} \\ -\frac{\rho_2(z)}{\sqrt{1+\rho_2(z)^2}} & \frac{\rho_1(z)}{\sqrt{1+\rho_1(z)^2}} \end{pmatrix}, \qquad \rho_{1,2}(z) = \frac{2\lambda_{1,2}(z) - 2 - p - pz}{z-1}.$$

Using this information, we can calculate the matrix U_1 given by (1.18) explicitly:

$$U_1 = \lim_{z\to 1} D^{-1}(z) V(z) = \frac{1}{\sqrt{1+p}} \begin{pmatrix} \sqrt{p} & 1 \\ 1 & -\sqrt{p} \end{pmatrix}. \tag{2.15}$$

Then, we combine it with $D(\infty)$ in (2.13), and the right hand side of the Mehler–Heine asymptotic formula (1.32) becomes

$$D(\infty) A \operatorname{diag}\left(\theta^{-\alpha_1} J_{\alpha_1}(\theta), \theta^{-\alpha_2} J_{\alpha_2}(\theta)\right)$$
$$= \frac{2^{-\frac{\alpha+\beta}{2}-2}}{\sqrt{1+p}} \begin{pmatrix} 2p & 2\sqrt{p} \\ -1 & \sqrt{p} \end{pmatrix} \begin{pmatrix} \theta^{-\alpha} J_\alpha(\theta) & 0 \\ 0 & \theta^{-\alpha-2} J_{\alpha+2}(\theta) \end{pmatrix},$$

which completes the proof of part (c). □

By a zero of a matrix valued polynomial P_n, one commonly means a zero of the determinant of P_n. If P_n is a matrix valued orthogonal polynomial with respect to an a.e. positive definite weight matrix on $[-1,1]$, then it is known that all zeros of P_n are in $[-1,1]$. The multiplicity is at most r if r is the size of P_n, see [20, Theorem 1.1].

From (2.10) one gets an asymptotic formula for $\det P_n(x)$, $x \in (-1,1)$ as $n \to \infty$. From the determinant of the matrix part of (2.10), we conclude that to leading order the zeros of P_n come from the solutions of $\cos(\gamma(x)) = 0$ and $\cos(\gamma(x) + \frac{\theta(x)}{2}) = 0$. That is

$$x = \cos\left(\frac{\frac{\alpha}{2}\pi + \frac{3\pi}{4} + k\pi}{n+1+\frac{\alpha+\beta}{2}}\right), \qquad k \in \mathbb{Z} \tag{2.16}$$

and

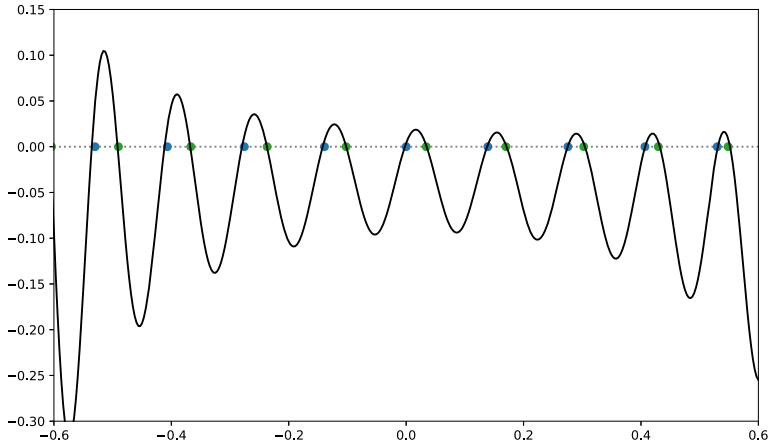

Fig. 2. Plot of the determinant of the scaled Jacobi MVOP $2^{20}P_{20}$ for $(\alpha,\beta,k)=(1,2,1)$ (solid line). The blue dots are solutions of (2.16) and the green dots are solutions of (2.17). (For interpretation of the colors in the figure, the reader is referred to the web version of this article.)

$$x=\cos\left(\frac{\frac{\alpha}{2}\pi+\frac{3\pi}{4}+k\pi}{n+\frac{3}{2}+\frac{\alpha+\beta}{2}}\right),\qquad k\in\mathbb{Z}. \tag{2.17}$$

The zeros come in two groups, see Fig. 2.

Remark 2.3. Using formula (2.3), in this example we can actually calculate the order of vanishing of the eigenvalues at $z=\pm1$ for general ℓ: $\alpha_k=\alpha+2k-2$, $\beta_k=\beta+k-1$ for $k=0,\dots,\ell$.

Regarding the recurrence coefficients, we have the following result:

Corollary 2.4. *The recurrence coefficients for the Jacobi weight (2.6) have the following asymptotic behavior as $n\to\infty$: $B_n=\frac{\mathcal{B}_2}{n^2}+\mathcal{O}(n^{-3})$ with*

$$\mathcal{B}_2=\frac{1}{4}\begin{pmatrix}(\beta+1)^2-\alpha^2 & 0\\ 0 & \beta^2-(\alpha+2)^2\end{pmatrix}-\frac{\alpha+1}{1+p}\begin{pmatrix}1 & 2p\\ \frac{1}{2} & -1\end{pmatrix} \tag{2.18}$$

and $C_n=\frac{1}{4}I_2+\mathcal{O}(n^{-2})$.

Proof. In view of Theorem 1.14 we only need to verify the expression (2.18). We have the matrix U_1 from (2.15), and we can similarly compute

$$U_{-1}=\lim_{z\to-1}D^{-1}(z)\widehat{V}(z)=\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}.$$

Therefore, we obtain from this and (2.13)

$$D(\infty)U_1=\frac{2^{-\frac{\alpha+\beta}{2}-2}}{\sqrt{1+p}}\begin{pmatrix}2p & -2\sqrt{p}\\ -1 & -\sqrt{p}\end{pmatrix},\qquad D(\infty)U_{-1}=2^{-\frac{\alpha+\beta}{2}-2}\begin{pmatrix}0 & -2\sqrt{p}\\ 1 & 0\end{pmatrix}.$$

From (2.14) we have $\alpha_1 = \alpha$, $\alpha_2 = \alpha + 2$, $\beta_1 = \beta$ and $\beta_2 = \beta + 1$. Using all this in (1.35), we obtain the expression (2.18) for $\mathcal{B}_2$. □

Remark 2.5. The previous result is consistent with the explicit recurrence coefficients for matrix Jacobi polynomials. However these coefficients have rather complicated expressions. They can be calculated using the approach of shift operators given in [27] and by extensive use of Maple. For reasons of space, we omit the expression of B_n. The coefficient C_n has a closed factorized form given by

$$C_n = \frac{4n(\alpha+n+1)}{(k+n)(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)} \begin{pmatrix} 1 & 0 \\ \frac{\alpha-k+1}{\alpha+n+\beta-k+2} & 1 \end{pmatrix} \times \begin{pmatrix} \frac{(\alpha+\beta+n+1)(\alpha+\beta+n-k+2)(\beta+n)(k+n-1)}{(\alpha+\beta+2n+1)(\alpha+\beta+2n)(\alpha+\beta+n+1-k)} & -\frac{k}{(k+n)} \\ 0 & \frac{(\alpha+\beta+n+2)(\alpha+\beta+n+1-k)(\beta+n+1)(k+n+1)}{(\alpha+\beta+n-k+2)(\alpha+\beta+2n+2)(\alpha+\beta+2n+3)} \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ -\frac{\alpha-k+1}{\alpha+n+1+\beta-k} & 1 \end{pmatrix}.$$

The coefficient C_n is written in terms of the parameter k, which is related to p as in (2.6). It indeed satisfies $C_n = \frac{1}{4} I_2 + \mathcal{O}(n^{-2})$ as $n \to \infty$.

2.2. A Gegenbauer weight

The second example is a family of matrix valued Gegenbauer-type polynomials, introduced in [27] and is a one parameter extension of [28,29]. Let K be the constant matrix

$$K_{i,j} = K_i(j, 1/2, 2\ell), \qquad i, j \in \{0, \dots, 2\ell\}, \tag{2.19}$$

where $K_n(x, p, N)$ is the Krawtchouk polynomial, see e.g. [26] or [16, §18.19]. For $\ell \in \frac{1}{2}\mathbb{N}_0$ and $\nu > 0$, we consider $(2\ell+1) \times (2\ell+1)$ weight matrix

$$W(x) = (1-x^2)^{\nu-\frac{1}{2}} H(x), \qquad H(x) = \Psi(x) T \Psi(x)^*, \qquad x \in (-1, 1), \tag{2.20}$$

where $\Psi(x) = K \Upsilon(x) K$ and $\Upsilon(x)$, T are diagonal matrices with entries

$$\Upsilon(x)_{j,j} = e^{\frac{j\pi i}{2}} \binom{2\ell}{j} (1-x)^{\frac{j}{2}} (1+x)^{\ell-\frac{j}{2}},$$

$$T_{j,j} = 2^{-6\ell-1} \frac{(2\nu+2\ell)_{2\ell+1}}{(\nu+\frac{1}{2})_{2\ell}} \sum_{j=0}^{2\ell} \binom{2\ell}{j} \frac{(\nu)_j}{(\nu+2\ell-j)_j}.$$

This factorization for the weight matrix is taken from [36, Theorem 3.1]. The weight W coincides with that in [27, Definition 2.1] for $\nu > 0$ and with that in [28] for $\nu = 1$. The

matrix H is a matrix polynomial in x. This follows from [27] or from a direct computation using the above expressions. Note that the even diagonal entries of Υ are real and the odd diagonal entries are purely imaginary. The fact that H has polynomial entries with real coefficients relies on particular properties of the Krawtchouk polynomials in the entries of the matrix K and the matrix Υ, see [36, Corollary 3.7, Remark 3.8].

Let $\xi^{\frac{j}{2}}(z)$ be the function

$$\xi^{\frac{j}{2}}(z) = \left(\frac{z-1}{z+1}\right)^{\frac{j}{2}}, \qquad j = 0, \dots, 2\ell,$$

with principal branches of the fractional powers, so that for $x \in (-1,1)$, we have

$$\xi_{\pm}^{\frac{j}{2}}(x) = e^{\pm \frac{j\pi i}{2}} \left(\frac{1-x}{1+x}\right)^{\frac{j}{2}},$$

where $\pm$ indicates boundary values from the left (right) of the interval $(-1,1)$. The matrix H has the factorized form:

$$H(x) = (1+x)^{2\ell} K \operatorname{diag}\left(1, \xi_+^{\frac{1}{2}}(x), \dots, \xi_+^{\ell}(x)\right) RR^T \operatorname{diag}\left(1, \xi_-^{\frac{1}{2}}(x), \dots, \xi_-^{\ell}(x)\right) K^T, \tag{2.21}$$

where

$$R = \operatorname{diag}\left(\binom{2\ell}{0}, (-1)\binom{2\ell}{1}, \dots, (-1)^{2\ell}\binom{2\ell}{2\ell}\right) KT^{\frac{1}{2}}. \tag{2.22}$$

Proposition 2.6. *Let R be an invertible matrix. Then the matrix Szegő function D_H for the matrix weight (2.21) is*

$$D_H(z) = \frac{(1+z)^{\ell}}{\varphi(z)^{\ell}} K \operatorname{diag}\left(1, \xi(z)^{\frac{1}{2}}, \dots, \xi(z)^{\ell}\right) R, \tag{2.23}$$

for $z \in \mathbb{C} \setminus [-1,1]$ with principal branches of the fractional powers.

The matrix Szegő function for the weight W is

$$D(z) = \frac{(z^2-1)^{\frac{\nu}{2}-\frac{1}{4}}}{\varphi(z)^{\nu-\frac{1}{2}}} D_H(z).$$

Proof. The entries $\xi^{\frac{j}{2}}(z)$ of the diagonal matrix in $D_H(z)$ are analytic in $\mathbb{C} \setminus [-1,1]$, so $D_H(z)$ is an analytic function on $\mathbb{C} \setminus [-1,1]$. On the other hand, the factor $(1+z)^{\ell}\varphi(z)^{-\ell}$ is analytic in $\mathbb{C} \setminus [-1,1]$ for $\ell \in \mathbb{N}_0$ and has an analytic continuation across $(-\infty,-1)$ for fractional values of $\ell \in \frac{1}{2}\mathbb{N}_0$. Therefore D_H is analytic in $\mathbb{C} \setminus [-1,1]$.

Moreover, since the factors on the right hand side of (2.23) are invertible, $D_H(z)$ is invertible for $z \in \mathbb{C} \setminus [-1,1]$. As $z \to \infty$ we have that $\xi(z)^{\frac{j}{2}} \to 1$ and $(z+1)^{\ell}\varphi(z)^{-\ell} \to 2^{-\ell}$. Therefore the limit $D_H(\infty)$ exists and is invertible.

For $x \in (-1,1)$, by taking $\pm$ boundary values we get $H(x) = D_{H,+}(x)D_{H,-}(x)^T$, which coincides with (2.21).

The matrix Szegő function for the weight W follows from (2.23) and (2.20) by using that the scalar prefactor in (2.23) is the Szegő function for the Gegenbauer weight $(1-x^2)^{\nu-\frac{1}{2}}$. □

Remark 2.7. As in the previous example, any choice of invertible matrix R gives the matrix Szegő function for $H(z)$ explicitly in (2.21). However, in order for $H(z)$ to be real valued on $(-1,1)$, we need the specific matrix R in (2.22).

The weight W (2.20) is an instance of a reducible weight matrix. More precisely, for any $n \in \mathbb{N}$ let I_n be the $n \times n$ identity matrix and let J_n be the $n \times n$ matrix

$$J_n = \sum_{i=0}^{n-1} E_{i,n-1-i},$$

where $E_{i,j}$ indicates the matrix with all 0 entries, except for a 1 in the position (i,j), and let Y be given by

$$Y = \frac{1}{\sqrt{2}} \begin{pmatrix} I_{\ell+\frac{1}{2}} & J_{\ell+\frac{1}{2}} \\ -J_{\ell+\frac{1}{2}} & I_{\ell+\frac{1}{2}} \end{pmatrix}, \quad \ell \in \mathbb{N}_0 + \frac{1}{2}, \qquad Y = \frac{1}{\sqrt{2}} \begin{pmatrix} I_\ell & 0 & J_\ell \\ 0 & \sqrt{2} & 0 \\ -J_\ell & 0 & I_\ell \end{pmatrix}, \quad \ell \in \mathbb{N}_0. \tag{2.24}$$

We note that Y is orthogonal, i.e. $YY^T = I_{2\ell+1}$. The weight matrix $W(x)$ satisfies:

$$YW(x)Y^T = \begin{pmatrix} W_2(x) & 0 \\ 0 & W_1(x) \end{pmatrix}, \tag{2.25}$$

where W_1 and W_2 have a strictly lower dimension, see for instance [28, Theorem 6.5] and [27, Proposition 2.6]. Given (2.25), it is straightforward to check from the orthogonality property that, if $\widetilde{P}_n(x)$ are monic MVOPs with respect to the original weight $W(x)$, then $Y\widetilde{P}_n(x)Y^T$ are monic MVOPs orthogonal with respect to the weight in block form $\begin{pmatrix} W_2(x) & 0 \\ 0 & W_1(x) \end{pmatrix}$. Moreover, using the uniqueness property of the family of monic MVOPs, see also [28, Corollary 5.6], we have

$$Y\widetilde{P}_n(x)Y^T = \begin{pmatrix} P_n(x) & 0 \\ 0 & Q_n(x) \end{pmatrix},$$

where $P_n(x)$ and $Q_n(x)$ are monic MVOPs with respect to the blocks $W_2(x)$ and $W_1(x)$ respectively.

For $\ell = \frac{1}{2}$, the weight W decomposes into two 1×1 blocks and therefore reduces to a scalar situation. The first nontrivial example is for $\ell = 1$, where the weight W

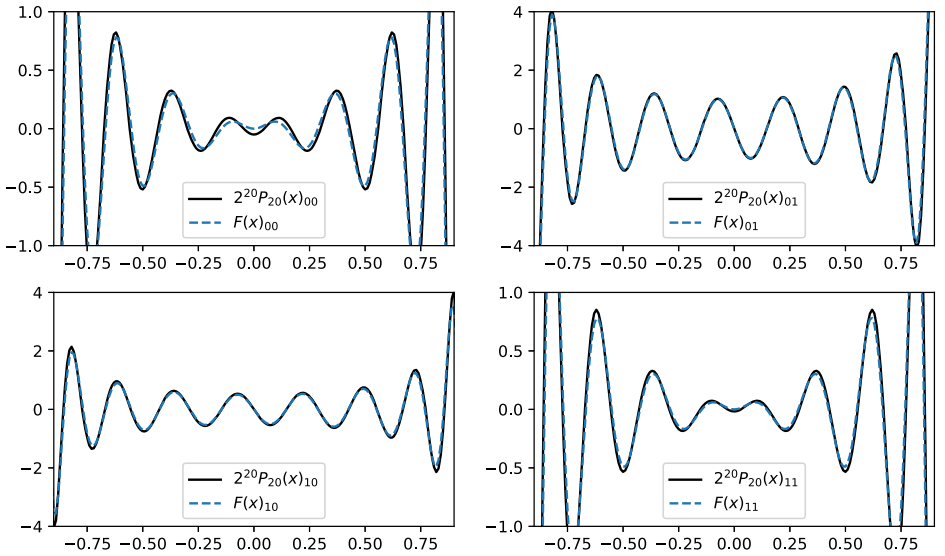

Fig. 3. Plot of the entries of the scaled Gegenbauer MVOP $2^n P_n(x)$ for $n = 20$ and $\nu = \frac{1}{2}$ (solid line). The approximation $F^{\text{inner}}(x)$ (dashed line) is given in formula (2.28) of Corollary 2.8.

decomposes into an irreducible 2×2 block W_2 and a 1×1 block W_1. This irreducible 2×2 block W_2 is, up to the scalar factor $\frac{2\nu+1}{2+\nu}$, the following:

$$W_2(x) = (1-x^2)^{\nu-1/2} \begin{pmatrix} 2(\nu+1)x^2 + 2\nu & (2\nu+1)\sqrt{2}x \\ (2\nu+1)\sqrt{2}x & \nu x^2 + \nu + 1 \end{pmatrix}, \qquad -1 < x < 1, \tag{2.26}$$

with $\nu > 0$.

Corollary 2.8. *The monic MVOP P_n associated with the weight matrix (2.26) has the following asymptotic behavior as $n \to \infty$:*

(a) *For $z \in \mathbb{C} \setminus [-1, 1]$,*

$$\frac{2^n P_n(z)}{\varphi(z)^n} = F^{\text{outer}}(z) \left(I_2 + \mathcal{O}(n^{-1})\right), \qquad n \to \infty,$$

where

$$F^{\text{outer}}(z) = \frac{\varphi(z)^{\nu+1}}{2^{\nu+2}(z^2-1)^{\frac{\nu}{2}+1}} \begin{pmatrix} 2z & -2\sqrt{2} \\ -\sqrt{2} & 2z \end{pmatrix}. \tag{2.27}$$

(b) *For $x \in (-1, 1)$,*

$$2^n P_n(x) = F^{\text{inner}}(x) + \mathcal{O}(n^{-1}), \qquad n \to \infty,$$

where (see Fig. 3)

$$F^{\text{inner}}(x) = \frac{2^{-\nu-1}}{(1-x^2)^{\frac{\nu}{2}+1}} \cos\left((n+\nu+1)\theta(x) - \frac{\nu\pi}{2}\right) \begin{pmatrix} 2x & -2\sqrt{2} \\ -\sqrt{2} & 2x \end{pmatrix}, \tag{2.28}$$

and $\theta(x) = \arccos x$.

(c) *Mehler–Heine asymptotics near* $z = 1$*:*

$$\lim_{n\to\infty} \frac{2^n}{\sqrt{n\pi}} P_n\left(\cos\frac{\theta}{n}\right) Q\left(\cos\frac{\theta}{n}\right) \operatorname{diag}\left(c_1^{1/2} n^{-\nu+\frac{1}{2}}, c_2^{1/2} n^{-\nu-\frac{3}{2}}\right)$$
$$= \frac{2^{-\nu-\frac{1}{2}}}{\sqrt{1+2\nu}} \begin{pmatrix} \sqrt{2}(\nu+1) & -2\sqrt{\nu(\nu+1)} \\ \nu & \sqrt{\nu(\nu+1)} \end{pmatrix} \begin{pmatrix} \theta^{-\nu+\frac{1}{2}} J_{\nu-\frac{1}{2}}(\theta) & 0 \\ 0 & \theta^{-\nu-\frac{3}{2}} J_{\nu+\frac{3}{2}}(\theta) \end{pmatrix},$$

where the constants are $c_1 = 3(1+2\nu)$ *and* $c_2 = \frac{2\nu(1+\nu)}{3(1+2\nu)}$.

Proof. The matrix Szegő function for $W_2(x)$ in (2.26) is

$$D(z) = \frac{(z^2-1)^{\nu/2-1/4}}{\varphi(z)^{\nu-1/2}} \frac{1}{\varphi(z)} \begin{pmatrix} \sqrt{2(\nu+1)}z & \sqrt{2\nu} \\ \sqrt{\nu+1} & \sqrt{\nu}z \end{pmatrix}, \tag{2.29}$$

and as a consequence

$$D_2(\infty) = 2^{-\nu-\frac{1}{2}} \begin{pmatrix} \sqrt{2(\nu+1)} & 0 \\ 0 & \sqrt{\nu} \end{pmatrix}. \tag{2.30}$$

This follows from conjugating the matrix Szegő function from Proposition 2.6 for $\ell = 1$ with Y coming from (2.24). Thus, we obtain up to the factor $(2\nu+1)^{\frac{1}{2}}(2+\nu)^{-\frac{1}{2}}$:

$$Y D(z) Y^T = \begin{pmatrix} D_2(z) & 0 \\ 0 & D_1(z) \end{pmatrix},$$

where $D_2(z)$ is given by (2.29), using the fact that $\varphi(z) + \varphi(z)^{-1} = 2z$. From this and the block decomposition of the weight W in (2.25), we verify that $D_2(z)$ is the matrix Szegő function for $W_2(z)$. Then, the matrix $D_2(\infty)$ is obtained directly by taking the limit of (2.29) as $z \to \infty$.

With this information, direct calculation gives

$$D_2(\infty) D_2^{-1}(z) = \frac{\varphi(z)^{\nu+\frac{1}{2}}}{2^{\nu+\frac{3}{2}}(z^2-1)^{\frac{\nu}{2}+\frac{3}{4}}} \begin{pmatrix} 2z & -2\sqrt{2} \\ -\sqrt{2} & 2z \end{pmatrix}, \tag{2.31}$$

and application of Theorem 1.8 gives the result from part (a).

For $x \in (-1,1)$, we have from (2.29) the boundary value

$$e^{i(n+\frac{1}{2})\theta - \frac{\pi i}{4}} D_{2+}(x)^{-1} = \frac{(1-x^2)^{-\frac{\nu}{2}-\frac{3}{4}}}{\sqrt{2\nu(\nu+1)}} e^{i(n+\nu+1)\theta(x) - \frac{\nu\pi}{2}} \begin{pmatrix} \sqrt{\nu}x & -\sqrt{2\nu} \\ -\sqrt{\nu+1} & \sqrt{2(\nu+1)}x \end{pmatrix}.$$

Then, from this, the fact that $W(x)$ is real symmetric on $[-1,1]$ and Theorem 1.10 we have the inner asymptotics for $x \in (-1,1)$,

$$\begin{aligned} 2^n P_n(x) &= \frac{\sqrt{2}}{(1-x^2)^{\frac{1}{4}}} D_2(\infty) \operatorname{Re}\left[e^{i(n+\frac{1}{2})\theta(x) - \frac{\pi i}{4}} D_{2+}(x)^{-1} \right] + \mathcal{O}(n^{-1}) \\ &= \frac{2^{-\nu}}{\sqrt{2\nu(\nu+1)}(1-x^2)^{\frac{\nu}{2}+1}} \cos\left((n+\nu+1)\theta - \frac{\nu\pi}{2} \right) \\ &\times \begin{pmatrix} \sqrt{2(\nu+1)} & 0 \\ 0 & \sqrt{\nu} \end{pmatrix} \begin{pmatrix} \sqrt{\nu}x & -\sqrt{2\nu} \\ -\sqrt{\nu+1} & \sqrt{2(\nu+1)}x \end{pmatrix} + \mathcal{O}(n^{-1}) \\ &= \frac{2^{-\nu-1}}{(1-x^2)^{\frac{\nu}{2}+1}} \cos\left((n+\nu+1)\theta - \frac{\nu\pi}{2} \right) \begin{pmatrix} 2x & -2\sqrt{2} \\ -\sqrt{2} & 2x \end{pmatrix} + \mathcal{O}(n^{-1}), \end{aligned} \tag{2.32}$$

where we have used (2.30). This proves part (b).

The eigenvalues $\lambda_{1,2}$ of the matrix part of (2.26) are explicit, and as $x \to 1$ they satisfy

$$\begin{aligned} \lambda_1(x) &= 3(1+2\nu) + 2(2+3\nu)(x-1) + \mathcal{O}((x-1)^2), \\ \lambda_2(x) &= \frac{8\nu(1+\nu)}{3(1+2\nu)}(x-1)^2 + \mathcal{O}((x-1)^3), \end{aligned} \tag{2.33}$$

so $n_1 = 0$ and $n_2 = 2$, and the exponents are

$$\alpha_1 = \nu - \tfrac{1}{2}, \qquad \alpha_2 = \nu - \tfrac{1}{2} + 2 = \nu + \tfrac{3}{2}.$$

The constants in this example are

$$\begin{aligned} c_1 &= 2^{-\alpha_1+\beta} \lim_{x\to 1} \frac{\lambda_1(x)}{(1-x)^{n_1}} = \lim_{x\to 1} \lambda_1(x) = 3(1+2\nu), \\ c_2 &= 2^{-\alpha_2+\beta} \lim_{x\to 1} \frac{\lambda_2(x)}{(1-x)^{n_2}} = 2^{-2} \lim_{x\to 1} \frac{\lambda_2(x)}{(1-x)^2} = \frac{2\nu(1+\nu)}{3(1+2\nu)}. \end{aligned}$$

We can also calculate the matrix U_1 using the explicit expressions for $D_2(z)$ in (2.29), as well as $V(z)$:

$$V(z) = (z^2-1)^{\frac{\nu}{2}-\frac{1}{4}} Q(z) \tilde{\Lambda}(z)^{1/2},$$

where $\tilde{\Lambda}(z)^{1/2} = \operatorname{diag}(\lambda_1(z)^{1/2}, \lambda_2(z)^{1/2})$ and we use the normalized matrix eigenvectors

$$Q(z) = \begin{pmatrix} \frac{1}{\sqrt{1+\rho_2(z)^2}} & -\frac{1}{\sqrt{1+\rho_1(z)^2}} \\ -\frac{\rho_2(z)}{\sqrt{1+\rho_2(z)^2}} & \frac{\rho_1(z)}{\sqrt{1+\rho_1(z)^2}} \end{pmatrix}, \qquad \rho_{1,2}(z) = \frac{\lambda_{1,2}(z) - (\nu z^2 + \nu + 1)}{\sqrt{2}(1+2\nu)z}. \tag{2.34}$$

From this and (2.29), we calculate

$$U_1 = \lim_{z\to 1} D_2^{-1}(z) V(z) = \frac{1}{\sqrt{1+2\nu}} \begin{pmatrix} \sqrt{\nu+1} & -\sqrt{\nu} \\ \sqrt{\nu} & \sqrt{\nu+1} \end{pmatrix}. \tag{2.35}$$

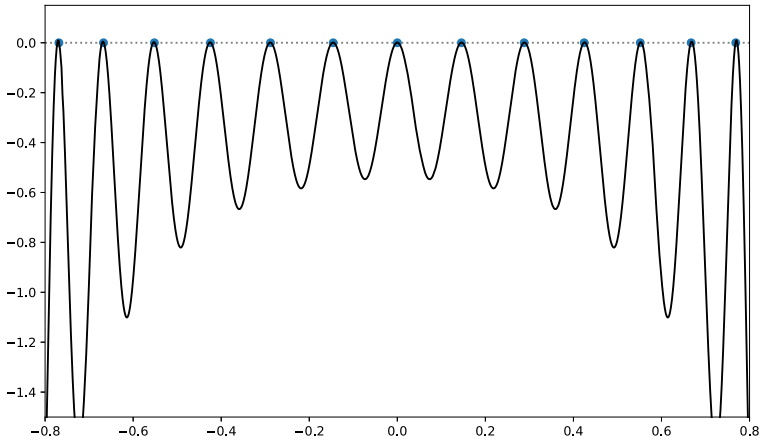

Fig. 4. Plot of the determinant of scaled Gegenbauer MVOPs $2^{20}P_{20}$ for $\nu = \frac{1}{2}$ (solid line). The blue dots are zeros of (2.36). (For interpretation of the colors in the figure, the reader is referred to the web version of this article.)

We combine this with $D_2(\infty)$ from (2.30), to obtain

$$D_2(\infty)U_1 = \frac{2^{-\nu-\frac{1}{2}}}{\sqrt{1+2\nu}} \begin{pmatrix} \sqrt{2(\nu+1)} & 0 \\ 0 & \sqrt{\nu} \end{pmatrix} \begin{pmatrix} \sqrt{\nu+1} & -\sqrt{\nu} \\ \sqrt{\nu} & \sqrt{\nu+1} \end{pmatrix} = \frac{2^{-\nu-\frac{1}{2}}}{\sqrt{1+2\nu}} \begin{pmatrix} \sqrt{2}(\nu+1) & -2\sqrt{\nu(\nu+1)} \\ \nu & \sqrt{\nu(\nu+1)} \end{pmatrix},$$

which we use in the right hand side of the Mehler–Heine asymptotic formula (1.32) to obtain part (c). □

From part (b) of Corollary 2.8, we have $\det\left(2^n P_n(x)\right) = \det F^{\text{inner}}(x) + \mathcal{O}(n^{-1})$ for $x \in (-1,1)$, with

$$\det F^{\text{inner}}(x) = -2^{-2\nu}(1-x^2)^{-\nu-\frac{1}{2}} \cos^2\left((n+\nu+1)\arccos(x) - \frac{\nu\pi}{2}\right). \tag{2.36}$$

Thus $\det F^{\text{inner}}(x)$ has double zeros on the interval $(-1,1)$, which gives asymptotic information about the zeros of the Gegenbauer MVOPs as $n \to \infty$. For large n, the zeros of $\det(2^n P_n(x))$ come in close pairs. However, for finite n, the zeros are still simple. See Fig. 4 for a plot of $\det(2^n P_n)$ with $n = 20$ and $\nu = \frac{1}{2}$.

Regarding the recurrence coefficients, we have the following result.

Corollary 2.9. *The recurrence coefficients for the Gegenbauer weight* (2.26) *have the following asymptotic behavior as* $n \to \infty$*:*

$$B_n = \frac{1}{n^2} \begin{pmatrix} 0 & \sqrt{2}(1+\nu) \\ \frac{\nu}{\sqrt{2}} & 0 \end{pmatrix} + \mathcal{O}(n^{-3}), \qquad C_n = \frac{1}{4} I_2 + \mathcal{O}(n^{-2}). \tag{2.37}$$

Proof. We have the matrix U_1 from (2.35), and we can compute in a similar way

$$U_{-1} = \lim_{z\to -1} D_2^{-1}(z)\widehat{V}(z) = \frac{1}{\sqrt{1+2\nu}} \begin{pmatrix} -\sqrt{\nu+1} & -\sqrt{\nu} \\ \sqrt{\nu} & -\sqrt{\nu+1} \end{pmatrix}.$$

Therefore, we obtain by (2.30)

$$D_2(\infty)U_1 = \frac{2^{-\nu-\frac{1}{2}}}{\sqrt{1+2\nu}} \begin{pmatrix} \sqrt{2}(\nu+1) & \sqrt{2\nu(\nu+1)} \\ \nu & \sqrt{\nu(\nu+1)} \end{pmatrix},$$
$$D_2(\infty)U_{-1} = \frac{2^{-\nu-\frac{1}{2}}}{\sqrt{1+2\nu}} \begin{pmatrix} -\sqrt{2}(\nu+1) & -\sqrt{2\nu(\nu+1)} \\ \nu & -\sqrt{\nu(\nu+1)} \end{pmatrix}.$$

Finally, the exponents of the eigenvalues at $z=1$ are $\alpha_1 = \nu - \frac{1}{2}$ and $\alpha_2 = \nu + \frac{3}{2}$, while at $z=-1$ they are $\beta_1 = \nu - \frac{1}{2}$, and $\beta_2 = \nu + \frac{3}{2}$. Collecting all this in (1.35), we obtain the coefficient $\mathcal{B}_2$ in the expansion of B_n and (2.37) follows from Theorem 1.14. □

Remark 2.10. The previous result is consistent with the explicit recurrence coefficients for matrix Gegenbauer polynomials given in [27, Proposition 3.3]. If we denote by $\widetilde{P}_n(x)$ the MVOPs with respect to the 3×3 Gegenbauer weight, and if $\widetilde{B}_n$ and $\widetilde{C}_n$ are the recurrence coefficients for $\widetilde{P}_n(x)$, then $Y\widetilde{B}_nY^T$ and $Y\widetilde{C}_nY^T$ give block diagonal matrices that contain the recurrence coefficients for the MVOPs $P_n(x)$ and $Q_n(x)$.

In the case given by the 2×2 block in (2.26), we start with 3×3 matrix Gegenbauer polynomials, we have $\ell = 1$ and recurrence coefficients given explicitly by

$$\widetilde{B}_n = \begin{pmatrix} 0 & \frac{\nu+1}{(n+\nu+1)(n+\nu+2)} & 0 \\ \frac{\nu}{(n+\nu)(n+\nu+1)} & 0 & \frac{\nu}{2(n+\nu)(n+\nu+1)} \\ 0 & \frac{\nu+1}{(n+\nu+1)(n+\nu+2)} & 0 \end{pmatrix}$$

and

$$\widetilde{C}_n = \frac{n(n+2\nu+1)}{4(n+\nu)(n+\nu+1)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{(n+\nu-1)(n+\nu+2)}{(n+\nu)(n+\nu+1)} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then, we conjugate with the matrix Y in (2.24), with $\ell = 1$:

$$Y\widetilde{B}_nY^T = \frac{1}{(n+\nu+1)(n+\nu+2)} \begin{pmatrix} 0 & \sqrt{2}(\nu+1) & 0 \\ \frac{\nu}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$= \frac{1}{n^2} \begin{pmatrix} 0 & \sqrt{2}(\nu+1) & 0 \\ \frac{\nu}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \mathcal{O}(n^{-3}),$$

$$Y\widetilde{C}_nY^T = \frac{n(n+2\nu+1)}{4(n+\nu)(n+\nu+1)}\begin{pmatrix}1 & 0 & 0\\ 0 & \frac{(n+\nu-1)(n+\nu+2)}{(n+\nu)(n+\nu+1)} & 0\\ 0 & 0 & 1\end{pmatrix}$$
$$= \frac{1}{4}I_3 + \frac{1}{4n^2}\begin{pmatrix}-\nu(\nu+1) & 0 & 0\\ 0 & -2-\nu(\nu+1) & 0\\ 0 & 0 & -\nu(\nu+1)\end{pmatrix} + \mathcal{O}(n^{-3}).$$

We see that the 2×2 upper blocks indeed agree with the terms up to $\mathcal{O}(n^{-2})$ that we obtain in the asymptotic expansions for B_n and C_n in (2.37).

3. RH steepest descent analysis

We use the Riemann-Hilbert (RH) problem for MVOP to prove the theorems stated in Section 1. The steepest descent analysis of RH problems originates with the work of Deift and Zhou [13] and was applied to orthogonal polynomials in [4,14] and in many different contexts in subsequent works. We follow in particular [32], where the RH method is applied to Jacobi-type OPs in $[-1,1]$, see also [31].

3.1. RH problem

The matrix valued orthogonal polynomial P_n is characterized by a RH problem of size $2r\times 2r$, see [10,23]. It is the upper left block in the solution of the following $2r\times 2r$ matrix valued RH problem: we seek $Y:\mathbb{C}\setminus[-1,1]\to\mathbb{C}^{2r\times 2r}$ such that

1. $Y:\mathbb{C}\setminus[-1,1]\to\mathbb{C}^{2r\times 2r}$ is analytic.
2. For $-1<x<1$, with this segment oriented from left to right, the matrix Y admits boundary values $Y_\pm(x)=\lim_{\varepsilon\to 0+}Y(x\pm i\varepsilon)$, which are related by

$$Y_+(x) = Y_-(x)\begin{pmatrix}I_r & W(x)\\ 0_r & I_r\end{pmatrix}, \qquad -1<x<1. \tag{3.1}$$

3. As $z\to\infty$, we have the asymptotic behavior

$$Y(z) = \left(I_{2r}+\mathcal{O}(z^{-1})\right)\begin{pmatrix}z^nI_r & 0_r\\ 0_r & z^{-n}I_r\end{pmatrix} \quad \text{as } z\to\infty. \tag{3.2}$$

4. To ensure a unique solution we also need to specify endpoint conditions at ± 1. As in [23], we have the following endpoint behavior (by $r\times r$ blocks):

$$Y(z) = \begin{pmatrix}\mathcal{O}(1) & \mathcal{O}(h_\alpha(z))\\ \mathcal{O}(1) & \mathcal{O}(h_\alpha(z))\end{pmatrix}, \quad \text{as } z\to 1, \tag{3.3}$$

since we assume that the matrix part of the weight is not identically 0 at $z=\pm 1$. Here

$$h_\alpha(z) = \begin{cases} |z-1|^\alpha, & -1 < \alpha < 0, \\ \log(|z-1|), & \alpha = 0, \\ 1, & \alpha > 0, \end{cases} \tag{3.4}$$

and a similar behavior holds as $z \to -1$, with β instead of α in (3.3), and $z+1$ instead of $z-1$ in (3.4).

We have

$$P_n(z) = \begin{pmatrix} I_r & 0_r \end{pmatrix} Y(z) \begin{pmatrix} I_r \\ 0_r \end{pmatrix}, \tag{3.5}$$

that is, P_n is the left upper $r \times r$ block of the solution Y to the RH problem. The right upper $r \times r$ block is given by the Cauchy transform

$$C(P_n W)(z) := \frac{1}{2\pi i} \int_{-1}^{1} \frac{P_n(s) W(s)}{s - z} ds, \qquad z \in \mathbb{C} \setminus [-1, 1],$$

which has the jump $C(P_n W)_+ = C(P_n W)_- + P_n W$ on $(-1, 1)$. The lower half of the matrix Y is constructed in a similar way out of the degree $n-1$ MVOP,

$$Y(z) = \begin{pmatrix} P_n(z) & C(P_n W)(z) \\ -2\pi i \Gamma_{n-1}^{-1} P_{n-1}(z) & -2\pi i \Gamma_{n-1}^{-1} C(P_{n-1} W)(z) \end{pmatrix}, \tag{3.6}$$

where $\Gamma_{n-1} = \displaystyle\int_{-1}^{1} P_{n-1}(x) W(x) P_{n-1}(x)^* dx$ as in (1.2), see [10,23] for details.

3.2. First transformation

We use the conformal map $\varphi(x)$ given by (1.19) in the first transformation $Y \mapsto T$, which is given by

$$T = \begin{pmatrix} 2^n I_r & 0_r \\ 0_r & 2^{-n} I_r \end{pmatrix} Y \begin{pmatrix} \varphi^{-n} I_r & 0_r \\ 0_r & \varphi^n I_r \end{pmatrix}. \tag{3.7}$$

Using the properties $\varphi(z) = 2z + \mathcal{O}(z^{-1})$ as $z \to \infty$ and $\varphi_+(x)\varphi_-(x) = 1$ for $x \in (-1, 1)$, we obtain that T solves the following RH problem:

1. $T : \mathbb{C} \setminus [-1, 1] \to \mathbb{C}^{2r \times 2r}$ is analytic.
2. On $(-1, 1)$ we have the jump

$$T_+ = T_- \begin{pmatrix} \varphi_-^{-2n} I_r & W \\ 0_r & \varphi_+^{-2n} I_r \end{pmatrix}. \tag{3.8}$$

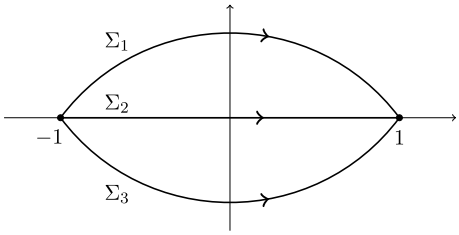

Fig. 5. Lens for the $T \mapsto S$ transformation. $\Sigma_S = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ is the jump contour in the RH problem for S.

3. As $z \to \infty$ the matrix T is normalized at infinity:

$$T(z) = I_{2r} + \mathcal{O}(z^{-1}) \quad \text{as } z \to \infty. \tag{3.9}$$

4. As $z \to \pm 1$, the matrix T has the same endpoint behavior as Y has.

3.3. Second transformation

The jump matrix on $(-1, 1)$ can be factorized as

$$\begin{pmatrix} I_r & 0_r \\ \varphi_-^{-2n} W^{-1} & I_r \end{pmatrix} \begin{pmatrix} 0_r & W \\ -W^{-1} & 0_r \end{pmatrix} \begin{pmatrix} I_r & 0_r \\ \varphi_+^{-2n} W^{-1} & I_r \end{pmatrix}.$$

This leads to the second transformation where we open a lens around $[-1, 1]$ as in Fig. 5, and we define

$$S = T \times \begin{cases} \begin{pmatrix} I_r & 0_r \\ -\varphi^{-2n} W^{-1} & I_r \end{pmatrix}, & \text{in upper part of the lens,} \\ \begin{pmatrix} I_r & 0_r \\ \varphi^{-2n} W^{-1} & I_r \end{pmatrix}, & \text{in lower part of the lens,} \end{cases} \tag{3.10}$$

and

$$S = T \qquad \text{outside of the lens.} \tag{3.11}$$

In the definition of S we use the analytic continuation of the weight matrix into the complex plane with branch cuts along $(-\infty, -1]$ and $[1, \infty)$. That is

$$W(z) = (1 - z)^{\alpha} (1 + z)^{\beta} H(z).$$

We also make sure that the lens is inside the region where H is analytic and invertible.

Then S is defined and analytic in $\mathbb{C} \setminus \Sigma_S$ where $\Sigma_S = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ consists of the interval $[-1, 1]$ together with the upper and lower lip of the lens, see Fig. 5. This matrix S satisfies the following RH problem:

1. $S(z)$ is analytic in $\mathbb{C} \setminus \Sigma_S$.
2. For $z \in \Sigma_S$, we have the following jumps:

$$S_+ = S_- \times \begin{cases} \begin{pmatrix} 0_r & W \\ -W^{-1} & 0_r \end{pmatrix} & \text{on } (-1,1), \\ \begin{pmatrix} I_r & 0_r \\ \varphi^{-2n} W^{-1} & I_r \end{pmatrix} & \text{on } \Sigma_1 \text{ and } \Sigma_3. \end{cases} \tag{3.12}$$

3. As $z \to \infty$, the matrix S has the asymptotic behavior

$$S(z) = I_{2r} + \mathcal{O}(z^{-1}) \quad \text{as } z \to \infty. \tag{3.13}$$

4. Outside the lens, S has the same endpoint conditions as T has. Inside the lens, the local behavior follows from the jump relations (3.12).

3.4. Global parametrix and proof of Proposition 1.4

The jump matrix on the lips of the lens in (3.12) tends to the identity matrix as $n \to \infty$, because $\varphi(z)$ maps $\mathbb{C} \setminus [-1,1]$ onto the exterior of the unit circle, and therefore $|\varphi(z)| > 1$ for $z \in \mathbb{C} \setminus [-1,1]$. We ignore these jumps and look for a global parametrix M satisfying the following RH problem:

1. $M(z)$ is analytic in $\mathbb{C} \setminus [-1,1]$.
2. On $(-1,1)$ we have the jump relation

$$M_+ = M_- \begin{pmatrix} 0_r & W \\ -W^{-1} & 0_r \end{pmatrix}. \tag{3.14}$$

3. As $z \to \infty$, the matrix M has the asymptotic behavior

$$M(z) = I_{2r} + \mathcal{O}(z^{-1}). \tag{3.15}$$

The case $W = I_r$ can be readily solved. The solution is

$$\begin{aligned} M_0(z) &= \frac{1}{2} \begin{pmatrix} I_r & iI_r \\ iI_r & I_r \end{pmatrix} \begin{pmatrix} \gamma(z) I_r & 0_r \\ 0_r & \gamma(z)^{-1} I_r \end{pmatrix} \begin{pmatrix} I_r & -iI_r \\ -iI_r & I_r \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} (\gamma(z) + \gamma(z)^{-1}) I_r & -i(\gamma(z) - \gamma(z)^{-1}) I_r \\ i(\gamma(z) - \gamma(z)^{-1}) I_r & (\gamma(z) + \gamma(z)^{-1}) I_r \end{pmatrix}, \quad \gamma(z) = \left(\frac{z-1}{z+1} \right)^{1/4}. \end{aligned} \tag{3.16}$$

The solution for general W requires the matrix Szegő function D from Proposition 1.4.

In Lemma 3.1 and throughout the paper, we use the notation $D(\infty)^{-*}$, where for an invertible matrix X we put $X^{-*} := \left(X^{-1}\right)^* = (X^*)^{-1}$.

Lemma 3.1. *Let D be the matrix Szegő function for W. Then M defined by*

$$M(z) = \begin{pmatrix} D(\infty) & 0_r \\ 0_r & D(\infty)^{-*} \end{pmatrix} M_0(z) \begin{pmatrix} D(z)^{-1} & 0_r \\ 0_r & D(\overline{z})^{*} \end{pmatrix} \tag{3.17}$$

satisfies the above RH problem for M.

Proof. Note that $z \mapsto D(\overline{z})^* = \left(\overline{D(\overline{z})}\right)^T$ is analytic for $z \in \mathbb{C} \setminus [-1,1]$, as it involves two anti-holomorphic conjugations, and therefore M is analytic. The asymptotic behavior (3.15) is satisfied because $M_0(z) \to I_{2r}$ and $D(z) \to D(\infty)$ as $z \to \infty$. It remains to check the jump (3.14).

For $x \in (-1,1)$, we have by (3.17) and the jump property $M_{0,+} = M_{0,-} \begin{pmatrix} 0_r & I_r \\ -I_r & 0_r \end{pmatrix}$ of M_0 that

$$\begin{aligned} M_-(x)^{-1} M_+(x) &= \begin{pmatrix} D_-(x) & 0_r \\ 0_r & D_+(x)^{-*} \end{pmatrix} \begin{pmatrix} 0_r & I_r \\ -I_r & 0_r \end{pmatrix} \begin{pmatrix} D_+(x)^{-1} & 0_r \\ 0_r & D_-(x)^{*} \end{pmatrix} \\ &= \begin{pmatrix} 0_r & D_-(x) D_-(x)^* \\ -\left(D_+(x) D_+(x)^*\right)^{-1} & 0_r \end{pmatrix} \end{aligned}$$

and this is $\begin{pmatrix} 0_r & W(x) \\ -W(x)^{-1} & 0_r \end{pmatrix}$ because of the defining property (1.7) of D. □

We still have to show existence of the matrix Szegő function, thereby proving Proposition 1.4.

We find $D(z)$ through a matrix valued factorization theorem that we pose on the unit circle by means of the conformal map φ, whose inverse is the rational function

$$\varphi^{-1}(z) = \frac{z + z^{-1}}{2}.$$

Then $W\left(\frac{z+z^{-1}}{2}\right)$, $|z| = 1$, is a matrix valued function on the unit circle that is Hermitian positive definite except possibly at $z = \pm 1$. A classical result of Wiener and Masani [39, Theorem 7.13], and Helson and Lowdenslager [25, Theorem 9] states that a factorization

$$W\left(\frac{z + z^{-1}}{2}\right) = G(z) G(z)^*, \qquad |z| = 1, \tag{3.18}$$

exists where G is analytic and invertible in the interior $|z| < 1$. In addition, G is unique if we specify that $G(0)$ is Hermitian positive definite, but we do not insist on the uniqueness here. The factorization (3.18) is valid under the matrix Szegő condition

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log \det W(\cos\theta)\, d\theta > -\infty, \tag{3.19}$$

which is certainly satisfied in the present situation. In the general setting the identity (3.18) holds a.e. on the unit circle, but in our setting it is valid everywhere, except possibly at ± 1.

Lemma 3.2. *Let G solve the matrix factorization problem (3.18). Then the matrix Szegő function is given by*

$$D(z) = G\left(\frac{1}{\varphi(z)}\right), \qquad z \in \mathbb{C} \setminus [-1,1]. \tag{3.20}$$

Proof. Since $|\varphi(z)| > 1$ for every $z \in \mathbb{C} \setminus [-1,1]$, we have that (3.20) is well-defined and analytic for $|z| < 1$. Also $D(z)$ is invertible for every $z \in \mathbb{C} \setminus [-1,1]$, and $D(\infty) = G(0)$ is also invertible, due to the corresponding property of G.

To check the property (1.7) we let $x \in (-1,1)$ and write $z = \varphi_+(x)$. Then $|z| = 1$, $\overline{z} = 1/z = \varphi_-(x)$ and $x = \frac{z+z^{-1}}{2}$. From (3.20) we get $D_-(x) = G\left(\frac{1}{\varphi_-(x)}\right) = G(z)$ and thus by (3.18)

$$D_-(x)D_-(x)^* = W\left(\frac{z+z^{-1}}{2}\right) = W(x),$$

which is the first identity in (1.7). The second identity follows in similar fashion. We have $D_+(x) = G\left(\frac{1}{\varphi_+(x)}\right) = G(\overline{z})$ and by (3.18)

$$D_+(x)D_+(x)^* = W\left(\frac{\overline{z}+\overline{z}^{-1}}{2}\right) = W\left(\frac{z^{-1}+z}{2}\right) = W(x). \quad \square$$

Remark 3.3. If D is chosen such that $D(\infty)$ is Hermitian positive definite, then D satisfies

$$D(\overline{z}) = D(z)^*, \qquad z \in \mathbb{C} \setminus [-1,1].$$

The identity $D_+(x)D_+(x)^*$ shows that $|D_+(x)| = \sqrt{W(x)}$ in the sense of the polar decomposition

$$D_+(x) = \sqrt{W(x)}U^*(x), \qquad -1 < x < 1, \tag{3.21}$$

where U is a unitary matrix, and $\sqrt{W(x)}$ denotes the positive definite square root of the Hermitian positive definite matrix. Thus

$$\sqrt{W(x)} = D_+(x)U(x), \qquad -1 < x < 1.$$

3.5. Local parametrix around $z = 1$

3.5.1. Statement

We fix a disk $D(1,\delta) = \{z \in \mathbb{C} \mid |z-1| < \delta\}$, around 1, with radius $\delta > 0$ sufficiently small. The local parametrix P should satisfy the following RH problem:

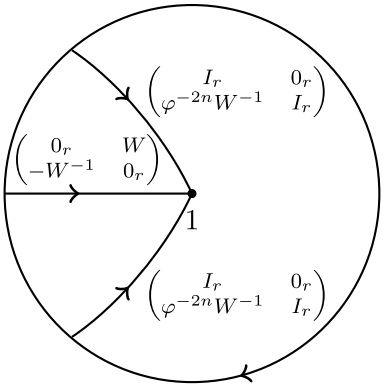

Fig. 6. Contours and jumps in the RH problem for $P(z)$.

1. $P(z)$ is analytic for $z \in D(1,\delta) \setminus \Sigma_S$.
2. For $z \in D(1,\delta) \cap \Sigma_S$, the matrix $P(z)$ should have the same jumps as S in this disk, see also Fig. 6:

$$P_+ = P_- \times \begin{cases} \begin{pmatrix} 0_r & W \\ -W^{-1} & 0_r \end{pmatrix} & \text{on } (1-\delta, 1), \\ \begin{pmatrix} I_r & 0_r \\ \varphi^{-2n}W^{-1} & I_r \end{pmatrix} & \text{on the lips of the lens inside the disk.} \end{cases} \tag{3.22}$$

3. As $n \to \infty$, uniformly for $z \in \partial D(1,\delta) \setminus \Sigma_S$, we have the matching condition

$$P(z)M^{-1}(z) = I_{2r} + \mathcal{O}(n^{-1}). \tag{3.23}$$

4. As $z \to 1$, $P(z)$ has the same behavior as $S(z)$ in the sense that $S(z)P^{-1}(z)$ remains bounded as $z \to 1$.

3.5.2. Properties of V

Recall that V is defined in (1.14) in terms of the modified eigenvalues (1.11). We need the following properties.

Lemma 3.4. *V is analytic in $D(1,\delta) \setminus [1-\delta, 1]$ and it satisfies (1.17), and*

$$W(z) = V(z)\,\mathrm{diag}\left(e^{-\alpha_1\pi i}, \dots, e^{-\alpha_r\pi i}\right) V(\overline{z})^* \quad \textit{in upper half plane within } D(1,\delta), \tag{3.24}$$

$$W(z) = V(z)\,\mathrm{diag}\left(e^{\alpha_1\pi i}, \dots, e^{\alpha_r\pi i}\right) V(\overline{z})^* \quad \textit{in lower half plane within } D(1,\delta). \tag{3.25}$$

Proof. The analyticity is clear from (1.14). For $x \in (1-\delta, 1)$, we have by (1.14)

$$V_\pm(x) = e^{\pm\alpha\pi i/2}(1-x)^{\alpha/2}(1+x)^{\beta/2}Q(x)\widetilde{\Lambda}_\pm(x)^{1/2}, \tag{3.26}$$

and,

$$V_\pm(x)^* = e^{\mp\alpha\pi i/2}(1-x)^{\alpha/2}(1+x)^{\beta/2}\overline{\widetilde{\Lambda}_\pm(x)^{1/2}}Q(x)^*. \tag{3.27}$$

We obtain (1.17) from (3.26) and (3.27) because of (1.1), (1.3) and the property

$$\widetilde{\Lambda}_\pm^{1/2}\overline{\widetilde{\Lambda}_\pm^{1/2}} = \Lambda.$$

The latter identity holds, since for each $j = 1, \dots, r$ and $x \in (1-\delta, 1]$, we have

$$\widetilde{\lambda}_{j,\pm}(x)^{1/2}\overline{\widetilde{\lambda}_{j,\pm}(x)^{1/2}} = \left|\widetilde{\lambda}_j(x)\right| = \lambda_j(x),$$

see (1.11).

To obtain (3.24), we first note that an analytic function f on $D(1,\delta)\setminus[1-\delta,1]$ that is real valued on $(1, 1+\delta)$ has the symmetry $\overline{f(\overline{z})} = f(z)$. This applies to the functions $z \mapsto \widetilde{\lambda}_j(z)$, and to $z \mapsto (z-1)^{\alpha/2}$, $z \mapsto (z+1)^{\beta/2}$. Therefore we obtain from (1.14)

$$V(\overline{z})^* = (z-1)^{\alpha/2}(z+1)^{\beta/2}\widetilde{\Lambda}(z)^{1/2}Q(\overline{z})^*.$$

Then by this and (1.14) the right-hand side of (3.24) is

$$\begin{aligned} &V(z)\operatorname{diag}\left(e^{-\alpha_1\pi i}, \dots, e^{-\alpha_r\pi i}\right)V(\overline{z})^* \\ &\qquad = (z-1)^{\alpha}(z+1)^{\beta}Q(z)\operatorname{diag}\left(e^{-\alpha_1\pi i}\widetilde{\lambda}_1(z), \dots, e^{-\alpha_r\pi i}\widetilde{\lambda}_r(z)\right)Q(\overline{z})^*. \end{aligned}$$

Then using (1.5) and (1.11), wet get for $z \in D(1,\delta)$ with $\operatorname{Im} z > 0$,

$$\begin{aligned} &V(z)\operatorname{diag}\left(e^{-\alpha_1\pi i}, \dots, e^{-\alpha_r\pi i}\right)V(\overline{z})^* \\ &\quad = e^{-\alpha\pi i}(z-1)^{\alpha}(1+z)^{\beta}Q(z)\operatorname{diag}\left((-1)^{n_1}\widetilde{\lambda}_1(z), \dots, (-1)^{n_r}\widetilde{\lambda}_r(z)\right)Q(\overline{z})^* \\ &\qquad = (1-z)^{\alpha}(1+z)^{\beta}Q(z)\Lambda(z)Q(\overline{z})^*. \end{aligned} \tag{3.28}$$

We finally note that $Q(z)\Lambda(z)Q(\overline{z})^*$ is analytic and agrees with $H(z)$ for $z \in (-1,1)$ because of (1.3), and so it is equal to the analytic continuation of H into the complex plane. Thus (3.28) is equal to the analytic continuation of W into the upper half-plane by (1.1). This proves (3.24).

The proof of (3.25) is similar. The only difference is that $e^{\alpha\pi i}(z-1)^{\alpha} = (1-z)^{\alpha}$ for z in the lower half plane. □

3.5.3. Reduction to constant jumps

Having V we seek P in the form

$$P(z) = E_n(z)P^{(1)}(z) \begin{pmatrix} \varphi(z)^{-n}V^{-1}(z) & 0_r \\ 0_r & \varphi(z)^n V(\overline{z})^* \end{pmatrix} \tag{3.29}$$

with an analytic prefactor $E_n(z)$, and an unknown $P^{(1)}(z)$ that are yet to be determined. The properties of V will guarantee that $P^{(1)}$ needs to have piecewise constant jumps.

Lemma 3.5. *Suppose that P is defined by (3.29) with an analytic prefactor and invertible E_n. Then P satisfies the jump conditions (3.22) if and only if $P^{(1)}$ satisfies*

$$P_+^{(1)} = P_-^{(1)} \times \begin{cases} \begin{pmatrix} 0_r & I_r \\ -I_r & 0_r \end{pmatrix} & \text{on } (1-\delta, 1), \\ \begin{pmatrix} I_r & 0_r \\ e^{\vec{\alpha}\pi i} & I_r \end{pmatrix} & \text{on the upper lip of the lens inside the disk,} \\ \begin{pmatrix} I_r & 0_r \\ e^{-\vec{\alpha}\pi i} & I_r \end{pmatrix} & \text{on the lower lip of the lens inside the disk,} \end{cases} \tag{3.30}$$

where $\vec{\alpha} = \operatorname{diag}(\alpha_1, \dots, \alpha_r)$ and $e^{\pm\vec{\alpha}\pi i} = \operatorname{diag}\left(e^{\pm\alpha_1\pi i}, \dots, e^{\pm\alpha_r\pi i}\right)$.

Proof. This is a straightforward calculation. The prefactor E_n does not influence the jumps. Then it follows from (3.29) that

$$\begin{aligned} &\left(P_-^{(1)}(z)\right)^{-1} P_+^{(1)}(z) \\ &\quad = \begin{pmatrix} \varphi(z)^{-n}V^{-1}(z) & 0_r \\ 0_r & \varphi(z)^n V(\overline{z})^* \end{pmatrix}_- P_-^{-1}(z)P_+(z) \begin{pmatrix} \varphi(z)^n V(z) & 0_r \\ 0_r & \varphi(z)^{-n} V(\overline{z})^{-*} \end{pmatrix}_+ \end{aligned}$$

For $z = x \in (1-\delta, 1)$ this gives us, because of (3.22) and the fact that $\varphi_+(x)\varphi_-(x) = 1$,

$$\begin{aligned} \left(P_-^{(1)}(x)\right)^{-1} P_+^{(1)}(x) &= \begin{pmatrix} \varphi_-(x)^{-n}V_-^{-1}(x) & 0_r \\ 0_r & \varphi_-(x)^n V_+(x)^* \end{pmatrix} \\ &\quad \times \begin{pmatrix} 0_r & W(x) \\ -W^{-1} & 0_r \end{pmatrix} \begin{pmatrix} \varphi_+(x)^n V_+(x) & 0_r \\ 0_r & \varphi_+(x)^{-n} V_-(x)^{-*} \end{pmatrix} \\ &= \begin{pmatrix} 0_r & V_-^{-1}(x)W(x)V_-(x)^{-*} \\ -V_+(x)^* W(x)^{-1} V_+(x) & 0_r \end{pmatrix}. \end{aligned}$$

Because of (1.17), this reduces to $\begin{pmatrix} 0_r & I_r \\ -I_r & 0_r \end{pmatrix}$ which gives us the required jump (3.30) on $(1-\delta, 1)$. The jumps (3.30) on the lips of the lens follow in a similar way, where we now use the identities (3.24) and (3.25).

Thus the jump conditions (3.22) imply those in (3.30). It is easy to revert the arguments to show that the jumps (3.30) imply (3.22), which proves the lemma. □

3.5.4. Bessel functions

To construct $P^{(1)}(z)$, we use the standard size 2×2 local parametrix with (modified) Bessel functions from the paper [32], see formulas (6.23)–(6.25) therein, that we reproduce here for ease of reference. It involves the modified Bessel functions I_α and K_α of order α, as well as the two Hankel functions $H_\alpha^{(1)}$ and $H_\alpha^{(2)}$ of order α.

$$\Psi_\alpha(\zeta) = \begin{cases} \begin{pmatrix} I_\alpha(2\zeta^{1/2}) & \frac{i}{\pi}K_\alpha(2\zeta^{1/2}) \\ 2\pi i\zeta^{1/2}I_\alpha'(2\zeta^{1/2}) & -2\zeta^{1/2}K_\alpha'(2\zeta^{1/2}) \end{pmatrix}, & |\arg\zeta| < \frac{2\pi}{3}, \\ \begin{pmatrix} \frac{1}{2}H_\alpha^{(1)}(2(-\zeta)^{1/2}) & \frac{1}{2}H_\alpha^{(2)}(2(-\zeta)^{1/2}) \\ \pi\zeta^{1/2}\left(H_\alpha^{(1)}\right)'(2(-\zeta)^{1/2}) & \pi\zeta^{1/2}\left(H_\alpha^{(2)}\right)'(2(-\zeta)^{1/2}) \end{pmatrix} \\ \qquad\qquad \times \begin{pmatrix} e^{\frac{1}{2}\alpha\pi i} & 0 \\ 0 & e^{-\frac{1}{2}\alpha\pi i} \end{pmatrix}, & \frac{2\pi}{3} < \arg\zeta < \pi, \\ \begin{pmatrix} \frac{1}{2}H_\alpha^{(2)}(2(-\zeta)^{1/2}) & -\frac{1}{2}H_\alpha^{(1)}(2(-\zeta)^{1/2}) \\ -\pi\zeta^{1/2}\left(H_\alpha^{(2)}\right)'(2(-\zeta)^{1/2}) & \pi\zeta^{1/2}\left(H_\alpha^{(1)}\right)'(2(-\zeta)^{1/2}) \end{pmatrix} \\ \qquad\qquad \times \begin{pmatrix} e^{-\frac{1}{2}\alpha\pi i} & 0 \\ 0 & e^{\frac{1}{2}\alpha\pi i} \end{pmatrix}, & -\pi < \arg\zeta < -\frac{2\pi}{3}. \end{cases} \tag{3.31}$$

We evaluate Ψ_α at $\zeta = n^2 f(z)$, where

$$f(z) = \frac{1}{4}\left(\log\varphi(z)\right)^2 \tag{3.32}$$

is a conformal map from $D(1,\delta)$ to a neighborhood of $\zeta = f(1) = 0$. We may (and do) assume that the lens is opened in such a way that $\arg f(z) = \pm 2\pi/3$ for z on the lips of the lens within $D(1,\delta)$. Then

$$\Psi_\alpha(n^2 f(z))_+ = \Psi_\alpha(n^2 f(z))_- \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{on } (1-\delta, 1), \\ \begin{pmatrix} 1 & 0 \\ e^{\pm\alpha\pi i} & 0 \end{pmatrix}, & \text{on lips of the lens.} \end{cases} \tag{3.33}$$

We use Ψ_α in block form with parameters $\alpha_1, \dots, \alpha_r$. We also need the permutation matrix Π_r of size $2r \times 2r$ with

$$(\Pi_r)_{2j-1,j} = 1, \quad (\Pi_r)_{2j,j+r} = 1, \qquad \text{for } j = 1, \dots, r, \tag{3.34}$$

while $(\Pi_r)_{j,k} = 0$ otherwise. Thus for example,

$$\Pi_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \Pi_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Lemma 3.6. *We define*

$$P^{(1)}(z) = \Pi_r^{-1} \operatorname{diag}\left(\Psi_{\alpha_1}(n^2 f(z)), \dots, \Psi_{\alpha_r}(n^2 f(z))\right) \Pi_r, \tag{3.35}$$

where diag(...) *denotes a block diagonal matrix of size* $2r \times 2r$ *with blocks of size* 2×2. *Then* $P^{(1)}$ *satisfies the jump properties* (3.30).

Proof. The permutation matrix Π_r has the following property, which can be readily checked from (3.34). Given 2×2 matrices $A_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}$ for $j = 1, \dots, r$, one has

$$\Pi_r^{-1} \operatorname{diag}(A_1, A_2, \dots, A_r) \Pi_r = \begin{pmatrix} \operatorname{diag}(\vec{a}) & \operatorname{diag}(\vec{b}) \\ \operatorname{diag}(\vec{c}) & \operatorname{diag}(\vec{d}) \end{pmatrix} \tag{3.36}$$

where $\vec{a} = (a_1, \dots, a_r)$ and so on.

The jumps (3.30) follow by direct calculation from the definition (3.35), the jumps (3.33) of Ψ_α and the property (3.36). □

Remark 3.7. The reader may note that the jump conditions in (3.30) remain the same in case one or more of the α_j's is shifted by an even integer. Therefore a construction of $P^{(1)}$ with such shifted parameters would also satisfy the jump conditions. Then we could go on and construct E_n as below (E_n does not depend on the parameters α_j) and define P by (3.29). However, we have to use the modified Bessel functions with exact orders $\alpha_1, \dots, \alpha_r$ in order to be able to match P with S in the sense that $S(z)P(z)^{-1}$ should remain bounded near $z = 1$.

The matching will be done in section 3.6 below. There we will use that $V(z)^{-1}$ and $V(\overline{z})^*$ have a certain behavior as $z \to 1$ with exponents $\pm\alpha_1/2, \dots, \pm\alpha_r/2$, see (3.50). These exponents agree with the exponents coming from the Bessel parametrix at the origin, see (3.48), but only if we use the modified Bessel functions of orders $\alpha_1, \dots, \alpha_r$.

3.5.5. Definition of E_n

With $P^{(1)}$ given by (3.35), we let P be as in (3.29) with an analytic prefactor E_n that is still to be determined. Then P will have the correct jumps from (3.22) and we choose E_n in such a way that it also satisfies the matching condition (3.23) on the circle $|z - 1| = \delta$.

The leading term in the asymptotic behavior of $\Psi_\alpha(n^2 f(z))$ as $n \to \infty$ (with fixed $z \in \partial D(1, \delta)$) is given in formula (6.29) of [32]. It does not depend on α. Thus for every $j = 1, \dots, r$ we have

$$\Psi_{\alpha_j}(n^2 f(z)) = \begin{pmatrix} (2\pi n)^{-1/2} f(z)^{-1/4} & 0 \\ 0 & (2\pi n)^{1/2} f(z)^{1/4} \end{pmatrix} \times \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} + \mathcal{O}(n^{-1}) \right) \begin{pmatrix} e^{2nf(z)^{1/2}} & 0 \\ 0 & e^{-2nf(z)^{-1/2}} \end{pmatrix}.$$

By (3.35), (3.36), and (3.32) this leads to (we use principal branches of the fractional powers)

$$P^{(1)}(z) = \begin{pmatrix} (2\pi n)^{-1/2} f(z)^{-1/4} I_r & 0_r \\ 0_r & (2\pi n)^{1/2} f(z)^{1/4} I_r \end{pmatrix} \times \left(\frac{1}{\sqrt{2}} \begin{pmatrix} I_r & iI_r \\ iI_r & I_r \end{pmatrix} + \mathcal{O}(n^{-1}) \right) \begin{pmatrix} \varphi(z)^n I_r & 0_r \\ 0_r & \varphi(z)^{-n} I_r \end{pmatrix} \tag{3.37}$$

as $n \to \infty$.

To obtain (3.23), we ignore the $\mathcal{O}(n^{-1})$ term and define E_n in view of (3.29) and (3.37) by

$$E_n(z) = M(z) \begin{pmatrix} V(z) & 0_r \\ 0_r & V(\overline{z})^{-*} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} I_r & -iI_r \\ -iI_r & I_r \end{pmatrix} \times \begin{pmatrix} (2\pi n)^{1/2} f(z)^{1/4} I_r & 0_r \\ 0_r & (2\pi n)^{-1/2} f(z)^{-1/4} I_r \end{pmatrix}. \tag{3.38}$$

Then the matching condition (3.23) can be readily verified from (3.29) and (3.38).

The definition (3.38) shows that $E_n(z)$ is analytic in $D(1,\delta) \setminus (1-\delta, 1]$. We need that it has analytic extension to $D(1,\delta)$, and this is what we are going to prove in the next subsection. Once we have that, we will have completed the construction of the local parametrix at 1.

3.5.6. Analyticity of E_n across $(1-\delta, 1)$

The analyticity of E_n across $(1-\delta, 1)$ follows from the following lemma.

Lemma 3.8. *We have $E_{n,+}(x) = E_{n,-}(x)$ for $x \in (1-\delta, 1)$.*

Proof. Let $x \in (1-\delta, 1)$. We first note that by (3.14) and (1.17)

$$\begin{aligned} &\begin{pmatrix} V_-(x)^{-1} & 0_r \\ 0_r & V_+(x)^* \end{pmatrix} M_-^{-1}(x) M_+(x) \begin{pmatrix} V_+(x) & 0_r \\ 0_r & V_-(x)^{-*} \end{pmatrix} \\ &\quad = \begin{pmatrix} V_-(x)^{-1} & 0_r \\ 0_r & V_+(x)^* \end{pmatrix} \begin{pmatrix} 0_r & W(x) \\ -W(x)^{-1} & 0_r \end{pmatrix} \begin{pmatrix} V_+(x) & 0_r \\ 0_r & V_-(x)^{-*} \end{pmatrix} \\ &\quad = \begin{pmatrix} 0_r & V_-(x)^{-1} W(x) V_-(x)^{-*} \\ -V_+(x)^* W(x)^{-1} V_+(x) & 0_r \end{pmatrix} = \begin{pmatrix} 0_r & I_r \\ -I_r & 0_r \end{pmatrix}. \end{aligned}$$

This gives by (3.38)

$$E_{n,-}^{-1}(x)E_{n,+}(x) = \begin{pmatrix} (2\pi n)^{-1/2} f_-(x)^{-1/4} I_r & 0_r \\ 0_r & (2\pi n)^{1/2} f_-(x)^{1/4} I_r \end{pmatrix} \times \frac{1}{\sqrt{2}} \begin{pmatrix} I_r & iI_r \\ iI_r & I_r \end{pmatrix} \begin{pmatrix} 0_r & I_r \\ -I_r & 0_r \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} I_r & -iI_r \\ -iI_r & I_r \end{pmatrix} \times \begin{pmatrix} (2\pi n)^{1/2} f_+(x)^{1/4} I_r & 0_r \\ 0_r & (2\pi n)^{-1/2} f_+(x)^{-1/4} I_r \end{pmatrix}.$$

The product of the three matrices in the middle line is equal to $\begin{pmatrix} -iI_r & 0_r \\ 0_r & iI_r \end{pmatrix}$, and we get

$$E_{n,-}^{-1}(x)E_{n,+}(x) = \begin{pmatrix} -if_-^{-1/4}(x) f_+^{1/4}(x) I_r & 0_r \\ 0_r & if_-^{1/4}(x) f_+^{-1/4}(x) I_r \end{pmatrix} = I_{2r}.$$

The final identity holds since $f(x) < 0$ for $x \in (1-\delta, 1)$ and due to the choice of principal branch of the fourth root, one has $f_+^{1/4}(x) = if_-^{1/4}(x)$ for $x \in (1-\delta, 1)$. □

From (3.8) it follows that E_n has analytic continuation across $(1-\delta, 1)$ by Morera's theorem. Thus E_n is analytic in the punctured disk $D(1,\delta) \setminus \{1\}$.

3.5.7. Removable singularity

It remains to show that the isolated singularity at 1 is removable.

Lemma 3.9. *The isolated singularity of E_n at 1 is removable.*

Proof. From (3.38) it is clear that the n-dependence of E_n is only in the last factor in the right hand side of (3.38), and we have in view of (3.16), (3.17) and (3.38)

$$E_n(z) = \frac{1}{2} \begin{pmatrix} D(\infty) & 0_r \\ 0_r & D(\infty)^{-*} \end{pmatrix} \begin{pmatrix} I_r & iI_r \\ iI_r & I_r \end{pmatrix} \times \left[(\pi n)^{1/2} E^{(1)}(z) \begin{pmatrix} I_r & 0_r \end{pmatrix} + \frac{1}{2} (\pi n)^{-1/2} E^{(2)}(z) \begin{pmatrix} 0_r & I_r \end{pmatrix} \right] \tag{3.39}$$

with $E^{(j)}(z)$, $j = 1, 2$, of size $2r \times r$ and independent of n, namely

$$\begin{aligned} E^{(1)}(z) &= \begin{pmatrix} \gamma(z) I_r & 0_r \\ 0_r & \gamma(z)^{-1} I_r \end{pmatrix} \begin{pmatrix} I_r & -iI_r \\ -iI_r & I_r \end{pmatrix} \begin{pmatrix} D(z)^{-1} V(z) & 0_r \\ 0_r & D(\overline{z})^* V(\overline{z})^{-*} \end{pmatrix} \\ &\quad \times \begin{pmatrix} I_r & -iI_r \\ -iI_r & I_r \end{pmatrix} \begin{pmatrix} f(z)^{1/4} I_r \\ 0_r \end{pmatrix} \\ &= \begin{pmatrix} \gamma(z) f(z)^{1/4} (D(z)^{-1} V(z) - D(\overline{z})^* V(\overline{z})^{-*}) \\ \gamma(z)^{-1} f(z)^{1/4} \left(-iD(z)^{-1} V(z) - iD(\overline{z})^* V(\overline{z})^{-*} \right) \end{pmatrix}, \end{aligned} \tag{3.40}$$

and similarly

$$E^{(2)}(z) = \begin{pmatrix} \gamma(z) f(z)^{-1/4}(-iD(z)^{-1}V(z) - iD(\overline{z})^{*}V(\overline{z})^{-*}) \\ \gamma(z)^{-1} f(z)^{-1/4}\left(-D(z)^{-1}V(z) + D(\overline{z})^{*}V(\overline{z})^{-*}\right) \end{pmatrix} \tag{3.41}$$

Both $E^{(1)}$ and $E^{(2)}$ are analytic in the punctured disk $D(1,\delta)\setminus\{1\}$. Since both $\gamma(z)$ and $f(z)^{1/4}$ behave like $\approx (z-1)^{1/4}$ as $z\to 1$ (see their definitions in (3.16) and (3.32)), we conclude that both

$$\Omega_1(z) := D(z)^{-1}V(z) + D(\overline{z})^{*}V(\overline{z})^{-*}, \tag{3.42}$$

and

$$\Omega_2(z) := (z-1)^{-1/2}\left(D(z)^{-1}V(z) - D(\overline{z})^{*}V(\overline{z})^{-*}\right) \tag{3.43}$$

are analytic in $D(1,\delta)\setminus\{1\}$. It suffices to prove that both Ω_1 and Ω_2 have removable singularities at $z=1$.

We first show that there cannot be an essential singularity. From the definition (1.14) it is clear that

$$V(z) = \mathcal{O}\left((z-1)^{\alpha_{\min}/2}\right), \quad V^{-1}(z) = \mathcal{O}\left((z-1)^{-\alpha_{\max}/2}\right), \tag{3.44}$$

where $\alpha_{\min} = \min(\alpha_1,\dots,\alpha_r)$ and $\alpha_{\max} = \max(\alpha_1,\dots,\alpha_r)$. The same estimates

$$D(z) = \mathcal{O}\left((z-1)^{\alpha_{\min}/2}\right), \quad D^{-1}(z) = \mathcal{O}\left((z-1)^{-\alpha_{\max}/2}\right), \tag{3.45}$$

hold for $D(z)$ and $D^{-1}(z)$. To see this we argue that the Jacobi prefactor $(1-x)^{\alpha}(1+x)^{\beta}$ has the scalar Szegő function $\frac{(z-1)^{\alpha/2}(z+1)^{\beta/2}}{\varphi(z)^{(\alpha+\beta)/2}}$ and

$$D(z) = \frac{(z-1)^{\alpha/2}(z+1)^{\beta/2}}{\varphi(z)^{(\alpha+\beta)/2}} D_H(z),$$

where $D_H(z)$ is the matrix Szegő function for H. Since H is bounded and analytic at $z=1$, also D_H is bounded at $z=1$. It thus follows that $D(z) = \mathcal{O}\left((z-1)^{\alpha/2}\right)$ as $z\to 1$, which is the first statement of (3.45), since $\alpha_{\min} = \alpha$. The second statement follows in a similar fashion since $z\mapsto D(\overline{z})^{-*}$ is the matrix Szegő function for W^{-1} and the eigenvalues of W^{-1} have exponents $-\alpha_1,\dots,-\alpha_r$ at $z=1$. From (3.44) and (3.45) we see that both (3.42) and (3.43) have the behavior $\mathcal{O}((z-1)^{-p})$ as $z\to 1$ for some $p\geq 0$, which implies that the isolated singularity at $z=1$ cannot be an essential singularity. It can be at most a pole of order $\leq p$.

To exclude the possibility of a pole we consider $x\in(-1,1)$. From (1.1), (3.21) and (1.3) we have

$$D_+^{-1}(x) = (1-x)^{-\alpha/2}(1+x)^{-\beta/2}U(x)\left(Q(x)\Lambda(x)Q(x)^*\right)^{-1/2}$$
$$= (1-x)^{-\alpha/2}(1+x)^{-\beta/2}U(x)Q(x)\Lambda(x)^{-1/2}Q(x)^*,$$

where $U(x)$ and $Q(x)$ are unitary. Then by (1.14), (1.11) and (1.4)

$$D_+^{-1}(x)V_+(x) = e^{\alpha\pi i/2}U(x)Q(x)\Lambda(x)^{-1/2}\widetilde{\Lambda}_+(x)^{1/2}$$
$$= U(x)Q(x)\operatorname{diag}\left(e^{\alpha_1\pi i/2},\dots,e^{\alpha_r\pi i/2}\right), \quad -1<x<1, \tag{3.46}$$

where we also used (1.5).

The three factors on the right-hand side of (3.46) are unitary matrices that remain bounded as $x\to 1-$. Thus (3.46) remains bounded as $x\to 1-$. The same reasoning applies to $D_-^{-1}(x)V_-(x)$ and to their Hermitian transposes. Because of (3.42) we then have that $\Omega_1(x)$ remains bounded as $x\to 1-$, while by (3.43) we have that $\Omega_2(x) = \mathcal{O}\left((x-1)^{-1/2}\right)$ as $x\to 1-$, and both behaviors exclude the possibility of a pole at 1. Thus Ω_1 and Ω_2 have removable singularities at 1, and this completes the proof. □

3.5.8. Proof of Lemma 1.7

From the proof of Lemma 3.9 we also obtain the existence of the limit defining U_1 as claimed in Lemma 1.7.

Proof of Lemma 1.7. In the proof of Lemma 3.9 we established that Ω_1 and Ω_2 are analytic in a neighborhood of 1, where Ω_1 and Ω_2 are defined by (3.42) and (3.43). From these definitions we see that

$$D(z)^{-1}V(z) = \frac{1}{2}\left(\Omega_1(z) + (z-1)^{1/2}\Omega_2(z)\right),$$

and therefore the limit defining U_1 in (1.18) exists and $U_1 = \frac{1}{2}\Omega_1(1)$. For $z = x \in (1-\delta, 1)$ we have by (3.46) that $D_+^{-1}(x)V_+(x)$ is unitary, and the unitarity is preserved in the limit $x\to 1-$. Thus U_1 is unitary.

The statements for U_{-1} in Lemma 1.7 follow similarly. □

3.6. $S(z)P(z)^{-1}$ remains bounded near $z=1$

We finally check the last item in the RH problem for P.

Lemma 3.10. *$S(z)P(z)^{-1}$ remains bounded near $z=1$.*

Proof. We first note that by [32, Remark 7.1] we have $\det\Psi_\alpha(\zeta) = 1$ for every ζ where it is defined. Then by (3.35) also $P^{(1)}(z) = 1$ for every $z\in D(1,\delta)\setminus\Sigma_S$. Since $\det V(z) = \overline{\det V(\overline{z})}$ we then also get that E_n defined by (3.38) has determinant 1 (as also M has determinant 1). Hence by (3.29) also

$$\det P(z) = 1, \text{ for } z \in D(1,\delta) \setminus \Sigma_S, \tag{3.47}$$

and in particular the inverse $P(z)^{-1}$ exists.

Next, because S and P have the same jumps inside $D(1,\delta)$, the product $S(z)P(z)^{-1}$ has analytic continuation to $D(1,\delta)\setminus\{1\}$ with an isolated singularity at $z=1$. We have to show that the isolated singularity is removable.

By construction both S and P can have at most power like singularities, say $S(z) = \mathcal{O}\left((z-1)^{-p}\right)$ and $P(z) = \mathcal{O}\left((z-1)^{-p}\right)$ as $z \to 0$, for some $p \geq 0$. Then also $P^{-1}(z) = \mathcal{O}\left((z-1)^{-p}\right)$ since $\det P = 1$, and $SP^{-1}(z) = \mathcal{O}\left((z-1)^{-2p}\right)$ which implies that SP^{-1} does not have an essential singularity at 1.

The behavior of Ψ_α near 0 is given by formulas (6.19)–(6.21) in [32]. For $\alpha \neq 0$, we have

$$\Psi_\alpha(\zeta) = \begin{pmatrix} \mathcal{O}(|\zeta|^{\alpha/2}) & \mathcal{O}(|\zeta|^{-|\alpha|/2}) \\ \mathcal{O}(|\zeta|^{\alpha/2}) & \mathcal{O}(|\zeta|^{-|\alpha|/2}) \end{pmatrix} \tag{3.48}$$

as $\zeta \to 0$ with $|\arg\zeta| < 2\pi/3$. Then from (3.35) and (3.36) we get that,

$$P^{(1)}(z) = \begin{pmatrix} \operatorname{diag}\left(\mathcal{O}(|z-1|^{\alpha_1/2}),\dots,\mathcal{O}(|z-1|^{\alpha_r/2})\right) & \operatorname{diag}\left(\mathcal{O}(|z-1|^{-|\alpha_1|/2}),\dots,\mathcal{O}(|z-1|^{-|\alpha_r|/2})\right) \\ \operatorname{diag}\left(\mathcal{O}(|z-1|^{\alpha_1/2}),\dots,\mathcal{O}(|z-1|^{\alpha_r/2})\right) & \operatorname{diag}\left(\mathcal{O}(|z-1|^{-|\alpha_1|/2}),\dots,\mathcal{O}(|z-1|^{-|\alpha_r|/2})\right) \end{pmatrix} \tag{3.49}$$

as $z \to 1$ outside the lens. From (1.14) we have

$$\begin{aligned} V(z)^{-1} &= \operatorname{diag}\left(\mathcal{O}(|z-1|^{-\alpha_1/2}),\dots,\mathcal{O}(|z-1|^{-\alpha_r/2})\right) Q(z)^{-1}, \\ V(\overline{z})^{*} &= \operatorname{diag}\left(\mathcal{O}(|z-1|^{\alpha_1/2}),\dots,\mathcal{O}(|z-1|^{\alpha_r/2})\right) Q(\overline{z})^{*}. \end{aligned} \tag{3.50}$$

We use (3.49) and (3.50) in (3.29) and we note that $E_n(z)$ and $\varphi(z)^{\pm}$ remain bounded as $z \to 1$. Then the definition (3.29) of $P(z)$ tells us that

$$P(z) = \begin{cases} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(|z-1|^{\alpha}) \\ \mathcal{O}(1) & \mathcal{O}(|z-1|^{\alpha}) \end{pmatrix}, & \text{if } -1 < \alpha < 0, \\ \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix}, & \text{if } \alpha > 0, \end{cases}$$

as $z \to 1$ from outside the lens, where we recall that $\alpha = \min(\alpha_1,\dots,\alpha_r)$, and all $\alpha_j - \alpha$ are non-negative integers. For $\alpha = 0$, logarithmic terms appear in (3.48), and then the above reasoning leads to

$$P(z) = \begin{pmatrix} \mathcal{O}(\log|z-1|) & \mathcal{O}(\log|z-1|) \\ \mathcal{O}(\log|z-1|) & \mathcal{O}(\log|z-1|) \end{pmatrix}, \quad \text{if } \alpha = 0,$$

as $z \to 1$ from outside the lens. Because of (3.47) we obtain from the above that

$$P^{-1}(z) = \begin{cases} \begin{pmatrix} \mathcal{O}(|z-1|^\alpha) & \mathcal{O}(|z-1|^\alpha) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix}, & \text{if } -1 < \alpha < 0, \\ \begin{pmatrix} \mathcal{O}(\log|z-1|) & \mathcal{O}(\log|z-1|) \\ \mathcal{O}(\log|z-1|) & \mathcal{O}(\log|z-1|) \end{pmatrix}, & \text{if } \alpha = 0, \\ \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix}, & \text{if } \alpha > 0, \end{cases} \tag{3.51}$$

as $z \to 1$ from outside the lens.

From item 4. in the RH problem for S we get that $S(z)$ behaves in the same way as $Y(z)$ when $z \to 1$ from outside the lens. That is,

$$S(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(h_\alpha(z)) \\ \mathcal{O}(1) & \mathcal{O}(h_\alpha(z)) \end{pmatrix}, \tag{3.52}$$

with h_α as in (3.4). Then by (3.51) and (3.52) it follows that

$$S(z)P^{-1}(z) = \begin{pmatrix} \mathcal{O}(|z-1|^\alpha) & \mathcal{O}(|z-1|^\alpha) \\ \mathcal{O}(|z-1|^\alpha) & \mathcal{O}(|z-1|^\alpha) \end{pmatrix}, \quad \text{if } -1 < \alpha < 0,$$

as $z \to 1$ from outside the lens. This behavior shows that $S(z)P^{-1}(z)$ cannot have a pole at $z = 1$, since $\alpha > -1$. If $\alpha > 0$ then (3.51) and (3.52) give us that $SP^{-1} = \mathcal{O}(1)$ and again there is no pole. If $\alpha = 0$ then SP^{-1} has a potential logarithmic behavior, but again it is not enough for a pole.

We already excluded the possibility of an essential singulariy and thus SP^{-1} has a removable singularity at $z = 1$. The lemma follows. □

3.7. Local parametrix around $z = -1$

The local parametrix $\widetilde{P}$ around -1 is constructed in a similar way. It satisfies the following RH problem:

1. $\widetilde{P}(z)$ is analytic for $z \in D(-1, \delta) \setminus \Sigma_S$.
2. For $z \in D(-1, \delta) \cap \Sigma_S$, the matrix $\widetilde{P}(z)$ should have the same jumps as $S(z)$ in this disk, see also Fig. 7:

$$\widetilde{P}_+ = \widetilde{P}_- \times \begin{cases} \begin{pmatrix} 0_r & W \\ -W^{-1} & 0_r \end{pmatrix} & \text{on } (1-\delta, 1), \\ \begin{pmatrix} I_r & 0_r \\ \varphi^{-2n} W^{-1} & I_r \end{pmatrix} & \text{on the lips of the lens inside the disk.} \end{cases} \tag{3.53}$$

3. As $n \to \infty$, uniformly for $z \in \partial D(-1, \delta) \setminus \Sigma_S$, we have the matching condition

$$\widetilde{P}(z) M^{-1}(z) = I_{2r} + \mathcal{O}(n^{-1}). \tag{3.54}$$

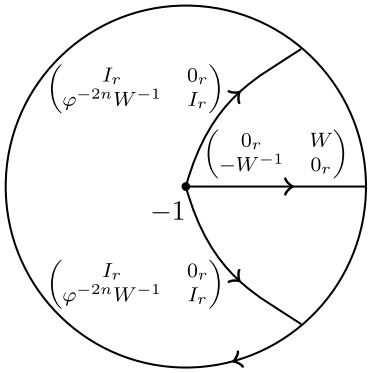

Fig. 7. Contours and jumps in the RH problem for $\widetilde{P}(z)$.

4. $S(z)\left(\widetilde{P}(z)\right)^{-1}$ remains bounded as $z \to -1$.

The local parametrix takes the form

$$\widetilde{P}(z) = \widetilde{E}_n(z)\widetilde{P}^{(1)}(z)\begin{pmatrix} \widetilde{\varphi}(z)^{-n}\widetilde{V}^{-1}(z) & 0_r \\ 0_r & \widetilde{\varphi}(z)^n\widetilde{V}(\overline{z})^* \end{pmatrix} \tag{3.55}$$

which is similar to (3.29). All quantities with a tilde are slight modifications of their non-tilded counterparts. We use $\widetilde{\varphi}(z) = \varphi(-z)$, and we note that changing $\varphi(z) \to \widetilde{\varphi}(z)$ does not alter the jumps for $\widetilde{P}(z)$ in (3.53). Let m_j be the order of vanishing of λ_j at -1, and put

$$\beta_j = \beta + m_j.$$

Then, with appropriate branches of the square roots,

$$\widetilde{V}(z) = (1-z)^{\alpha/2}(-1-z)^{\beta/2}Q(z)\operatorname{diag}\left((-1)^{m_1}\lambda_1(z),\dots,(-1)^{m_r}\lambda_r(z)\right)^{1/2}$$

for $z \in D(-1,\delta)\setminus[-1,-1+\delta]$. $\widetilde{P}^{(1)}$ is built out of the 2×2 Bessel parametrix (3.31), but now with parameters $\beta_1,\dots,\beta_r$, namely, similar to (3.35),

$$\widetilde{P}^{(1)}(z) = \Pi_r^{-1}\operatorname{diag}\left(\sigma_3\Psi_{\beta_1}(n^2\tilde{f}(z))\sigma_3,\dots,\sigma_3\Psi_{\beta_r}(n^2\tilde{f}(z))\sigma_3\right)\Pi_r,$$

with $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\widetilde{f}(z) = \frac{1}{4}\left(\log\widetilde{\varphi}(z)\right)^2$. The analytic prefactor takes the form

$$\widetilde{E}_n(z) = M(z)\begin{pmatrix} \widetilde{V}(z) & 0_r \\ 0_r & \widetilde{V}(\overline{z})^{-*} \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} I_r & iI_r \\ iI_r & I_r \end{pmatrix} \times \begin{pmatrix} (2\pi n)^{1/2}\tilde{f}(z)^{1/4}I_r & 0_r \\ 0_r & (2\pi n)^{-1/2}\tilde{f}(z)^{-1/4}I_r \end{pmatrix} \tag{3.56}$$

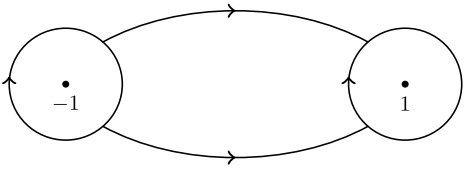

Fig. 8. System of contours Σ_R in the RH problem for $R(z)$.

which is analogous to (3.38). The items in the RH problem for $\widetilde{P}$ then follow in the same way as we proved them for P. We do not give any more details.

3.8. Final transformation

The final transformation $S \mapsto R$ is

$$R(z) = \begin{cases} S(z)M^{-1}(z), & z \in \mathbb{C} \setminus \left(\overline{D(1,\delta)} \cup \overline{D(-1,\delta)} \cup \Sigma_S \right), \\ S(z)P^{-1}(z), & z \in D(1,\delta) \setminus \Sigma_S, \\ S(z)\widetilde{P}^{-1}(z), & z \in D(-1,\delta) \setminus \Sigma_S. \end{cases} \tag{3.57}$$

Then R is defined and analytic in $\mathbb{C} \setminus (\Sigma_S \cup \partial D(1,\delta) \cup \partial D(-1,\delta))$ with analytic continuation across $(-1,1)$ and on the parts of Σ_S inside the disks. This follows immediately from the fact that the jumps of M and S agree on $(-1,1)$, the jumps of $P^{(1)}$ and S agree on $\Sigma_S \cap D(1,\delta)$, and the jumps of $\widetilde{P}^{(1)}$ and S agree on $\Sigma_S \cap D(-1,\delta)$. The isolated singularities at ± 1 are removable, since R remains bounded near the endpoints, as follows from item 4. in the RH problems for P and $\widetilde{P}$. Therefore $R(z)$ satisfies the following RH problem on the oriented contour Σ_R shown in Fig. 8:

1. $R(z)$ is analytic in $\mathbb{C} \setminus \Sigma_R$.
2. For $z \in \Sigma_R$, the matrix has the following jumps:

$$R_+(z) = R_-(z) \begin{cases} M(z) \begin{pmatrix} I_r & 0_r \\ \varphi^{-2n}(z) W^{-1}(z) & I_r \end{pmatrix} M(z)^{-1}, & z \text{ on the lips of the lens outside of the disks,} \\ P(z)M(z)^{-1}, & z \in \partial D(1,\delta), \\ \widetilde{P}(z)M(z)^{-1}, & z \in \partial D(-1,\delta). \end{cases}$$

3. As $z \to \infty$, we have the asymptotic behavior $R(z) = I_{2r} + \mathcal{O}(z^{-1})$.

Since $M(z)$ and $W(z)$ are independent of n, and $|\varphi(z)| > 1$ on the lips of lens outside the disks, we can verify that $R_+ = R_-(I_{2r} + \mathcal{O}(n^{-1}))$ on the two circles and $R_+ = R_-(I_{2r} + \mathcal{O}(e^{-cn}))$ (with $c > 0$) on the lips of the lens outside the disks. The conclusion of the steepest descent analysis then is that

$$R(z) = I_{2r} + \mathcal{O}\left(\frac{1}{n(1+|z|)}\right) \quad \text{as } n \to \infty, \tag{3.58}$$

uniformly for $z \in \mathbb{C} \setminus \Sigma_R$.

3.9. Asymptotic expansion of R

The large n behavior (3.58) will suffice for the proof of main term in Theorem 1.8 and for the proof of Theorems 1.10–1.13 that deal with the asymptotic behavior of the MVOP. For the large n behavior of the recurrence coefficients as stated in Theorem 1.14 we need more information on R. In fact it will be true that R has a full asymptotic expansion

$$R(z) \sim I_{2r} + \sum_{k=1}^{\infty} \frac{R_k(z)}{n^k} \tag{3.59}$$

as $n \to \infty$, that is uniform for $z \in \mathbb{C} \setminus (\partial D(1,\delta) \cup \partial D(-1,\delta))$. Furthermore the expansion has a double asymptotic property

$$\left\| R(z) - I_{2r} - \sum_{k=1}^{\ell} \frac{R_k(z)}{n^k} \right\| \leq \frac{C_\ell}{|z| n^{\ell+1}}, \quad C_\ell > 0,$$

for $\ell \geq 1$ and $|z| > 2$. This is analogous to [14, Theorem 7.10] or [32, Lemma 8.3], and the proof is similar. The matrix valued functions $R_k(z)$ are meromorphic with poles in ± 1 only and $R_k(z) = \mathcal{O}(z^{-1})$ as $z \to \infty$.

The asymptotic expansion of R follows from an expansion of the jump matrices of R on the two circles $\partial D(\pm 1, \delta)$. We write

$$\Delta(z) = \begin{cases} P(z)M(z)^{-1} - I_{2r} & \text{for } z \in \partial D(1,\delta) \\ \widetilde{P}(z)M(z)^{-1} - I_{2r} & \text{for } z \in \partial D(-1,\delta). \end{cases}$$

Then Δ also depends on n (which is suppressed in the notation) and Δ has an asymptotic expansion

$$\Delta(z) = \sum_{k=1}^{\infty} \frac{\Delta_k(z)}{n^k}, \tag{3.60}$$

as $n \to \infty$, with

$$\Delta_k(z) = \frac{1}{2^k g(z)^k} M(z) \begin{pmatrix} V(z) & 0_r \\ 0_r & V(\overline{z})^{-*} \end{pmatrix} \Pi_r^{-1} \Psi_k \Pi_r \begin{pmatrix} V^{-1}(z) & 0_r \\ 0_r & V(\overline{z})^{*} \end{pmatrix} M^{-1}(z)$$

with

$$g(z) = \begin{cases} \log \varphi(z), & \text{on } \partial D(1,\delta), \\ \log \tilde{\varphi}(z), & \text{on } \partial D(-1,\delta), \end{cases}$$

and Ψ_k is a piecewise constant matrix

$$\Psi_k = \begin{cases} \operatorname{diag}\left((\alpha_j, k-1) \begin{pmatrix} \frac{(-1)^k}{k}\left(\alpha_j^2 + \frac{k}{2} - \frac{1}{4}\right) & -(k-\frac{1}{2})i \\ (-1)^k (k-\frac{1}{2})i & \frac{1}{k}\left(\alpha_j^2 + \frac{k}{2} - \frac{1}{4}\right) \end{pmatrix} \right), & \text{on } \partial D(1,\delta), \\ \operatorname{diag}\left((\beta_j, k-1) \begin{pmatrix} \frac{(-1)^k}{k}\left(\beta_j^2 + \frac{k}{2} - \frac{1}{4}\right) & (k-\frac{1}{2})i \\ (-1)^{k+1} (k-\frac{1}{2})i & \frac{1}{k}\left(\beta_j^2 + \frac{k}{2} - \frac{1}{4}\right) \end{pmatrix} \right), & \text{on } \partial D(-1,\delta). \end{cases}$$

Thus Ψ_k is a block diagonal matrix with 2×2 blocks as j is varying from 1 to r. The numbers $(\alpha_j, k-1)$ and $(\beta_j, k-1)$ come from asymptotic expansions of Bessel functions. In general we have $(\nu, 0) = 1$ and

$$(\nu, k) = \frac{(4\nu^2 - 1)(4\nu^2 - 9) \cdots (4\nu^2 - (2k-1)^2)}{4^k k!}, \qquad k \geq 1. \tag{3.61}$$

The analogue of Lemma 8.2 in [32] holds. That is, for some $\delta_0 > \delta$, we have that Δ_k has an analytic continuation to $(D(1,\delta_0) \setminus \{1\}) \cup D(-1,\delta_0) \setminus \{-1\})$ with poles of order $\leq \left[\frac{k+1}{2}\right]$ at $z = 1$ and $z = -1$.

The matrix valued functions $R_k(z)$, for $k \geq 1$, are obtained from additive RH problems arising from the relation $R_+(z) = R_-(z)(I_{2r} + \Delta(z))$ for $z \in \partial D(1,\delta) \cup \partial D(-1,\delta)$ together with (3.59) and (3.60). The first one is

$$R_{1+}(z) = R_{1-}(z) + \Delta_1(z), \qquad z \in \partial D(1,\delta) \cup D(-1,\delta), \tag{3.62}$$

with $R_1(z) = \mathcal{O}(z^{-1})$ as $z \to \infty$. Since $\Delta_1(z)$ has simple poles at $z = \pm 1$, we write

$$\Delta_1(z) = \frac{A^{(1)}}{z-1} + \mathcal{O}(1), \quad z \to 1, \qquad \Delta_1(z) = \frac{B^{(1)}}{z+1} + \mathcal{O}(1), \quad z \to -1, \tag{3.63}$$

for some constant matrices $A^{(1)}$ and $B^{(1)}$. Then the solution of the additive RH problem for $R_1(z)$ is given by

$$R_1(z) = \begin{cases} \dfrac{A^{(1)}}{z-1} + \dfrac{B^{(1)}}{z+1}, & z \in \mathbb{C} \setminus \left(\overline{D(1,\delta)} \cup \overline{D(-1,\delta)} \right), \\ \dfrac{A^{(1)}}{z-1} + \dfrac{B^{(1)}}{z+1} - \Delta_1(z), & z \in D(1,\delta) \cup D(-1,\delta). \end{cases} \tag{3.64}$$

From the previous formulas, we have for $z \in D(1,\delta) \setminus \{1\}$,

$$\Delta_1(z) = \frac{1}{2g(z)} M(z) \begin{pmatrix} V(z) & 0_r \\ 0_r & V(\overline{z})^{-*} \end{pmatrix} \Pi_r^{-1} \operatorname{diag}\begin{pmatrix} -(\alpha_j^2 + \frac{1}{4}) & -\frac{1}{2}i \\ -\frac{1}{2}i & \alpha_j^2 + \frac{1}{4} \end{pmatrix} \Pi_r$$

$$\times \begin{pmatrix} V^{-1}(z) & 0_r \\ 0_r & V(\overline{z})^* \end{pmatrix} M^{-1}(z),$$

for $z \in \partial D(1,\delta)$. We note $\lim_{z\to 1} \dfrac{(z-1)^{1/2}}{g(z)} = \dfrac{1}{\sqrt{2}}$ and using (3.17), the explicit form of $M_0(z)$ and the local expansions of the matrices $D(z)^{-1}V(z)$ and $D(\overline{z})^*V(\overline{z})^{-*}$, where we recall that U_1 is unitary,

$$\lim_{z\to 1}(z-1)^{1/4} M(z) \begin{pmatrix} V(z) & 0_r \\ 0_r & V(\overline{z})^{-*} \end{pmatrix} = \frac{1}{2^{3/4}} \begin{pmatrix} D(\infty)U_1 & 0_r \\ 0_r & D(\infty)^{-*}U_1 \end{pmatrix} \begin{pmatrix} I_r & iI_r \\ -iI_r & I_r \end{pmatrix}$$

$$\lim_{z\to 1}(z-1)^{1/4} \begin{pmatrix} V^{-1}(z) & 0_r \\ 0_r & V(\overline{z})^* \end{pmatrix} M^{-1}(z)$$

$$= \frac{1}{2^{3/4}} \begin{pmatrix} I_r & -iI_r \\ iI_r & I_r \end{pmatrix} \begin{pmatrix} U_1^{-1}D(\infty)^{-1} & 0_r \\ 0_r & U_1^{-1}D(\infty)^* \end{pmatrix}.$$

Then we can calculate the residue $A^{(1)}$ in (3.63),

$$A^{(1)} = \lim_{z\to 1}(z-1)\Delta_1(z)$$

$$= \frac{1}{8} \begin{pmatrix} D(\infty)U_1 & 0_r \\ 0_r & D(\infty)^{-*}U_1 \end{pmatrix} \begin{pmatrix} I_r & iI_r \\ -iI_r & I_r \end{pmatrix} \Pi_r^{-1} \operatorname{diag} \begin{pmatrix} -(\alpha_j^2 + \frac{1}{4}) & -\frac{1}{2}i \\ -\frac{1}{2}i & \alpha_j^2 + \frac{1}{4} \end{pmatrix} \Pi_r$$

$$\times \begin{pmatrix} I_r & -iI_r \\ iI_r & I_r \end{pmatrix} \begin{pmatrix} U_1^{-1}D(\infty)^{-1} & 0_r \\ 0_r & U_1^{-1}D(\infty)^* \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} D(\infty)U_1 & 0_r \\ 0_r & D(\infty)^{-*}U_1 \end{pmatrix} \Pi_r^{-1} \operatorname{diag} \left((\alpha_j^2 - \frac{1}{4}) \begin{pmatrix} -1 & i \\ i & 1 \end{pmatrix} \right) \Pi_r$$

$$\times \begin{pmatrix} U_1^{-1}D(\infty)^{-1} & 0_r \\ 0_r & U_1^{-1}D(\infty)^* \end{pmatrix}. \quad (3.65)$$

Similarly,

$$B^{(1)} = \lim_{z\to -1}(z+1)\Delta_1(z)$$

$$= \frac{1}{4} \begin{pmatrix} D(\infty)U_{-1} & 0_r \\ 0_r & D(\infty)^{-*}U_{-1} \end{pmatrix} \Pi_r^{-1} \operatorname{diag} \left((\beta_j^2 - \frac{1}{4}) \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \right) \Pi_r$$

$$\times \begin{pmatrix} U_{-1}^{-1}D(\infty)^{-1} & 0_r \\ 0_r & U_{-1}^{-1}D(\infty)^* \end{pmatrix}. \quad (3.66)$$

4. Proofs of the theorems

4.1. Proof of Theorem 1.8

Proof. Let U be an open neighborhood of $[-1,1]$ in the complex plane. We may assume that the lens around $[-1,1]$ and the disks $D(\pm 1,\delta)$ are fully contained in U. Then for

$z \in \mathbb{C} \setminus U$, we have by (3.5), (3.7), and (3.11),

$$P_n(z) = \begin{pmatrix} I_r & 0_r \end{pmatrix} T(z) \begin{pmatrix} I_r \\ 0_r \end{pmatrix} \left(\frac{\varphi(z)}{2}\right)^n = \begin{pmatrix} I_r & 0_r \end{pmatrix} S(z) \begin{pmatrix} I_r \\ 0_r \end{pmatrix} \left(\frac{\varphi(z)}{2}\right)^n. \tag{4.1}$$

Then $S = RM$ by (3.57), and using (3.17), we obtain

$$\begin{aligned} \frac{2^n P_n(z)}{\varphi(z)^n} &= \begin{pmatrix} I_r & 0_r \end{pmatrix} R(z) \begin{pmatrix} D(\infty) & 0_r \\ 0_r & D(\infty)^{-*} \end{pmatrix} M_0(z) \begin{pmatrix} D(z)^{-1} & 0_r \\ 0_r & D(\overline{z})^{*} \end{pmatrix} \begin{pmatrix} I_r \\ 0_r \end{pmatrix} \\ &= \begin{pmatrix} R_{11}(z)D(\infty) & R_{12}(z)D(\infty)^{-*} \end{pmatrix} \begin{pmatrix} M_{0,11}(z) \\ M_{0,21}(z) \end{pmatrix} D(z)^{-1} \end{aligned}$$

where $R_{11} = \begin{pmatrix} I_r & 0_r \end{pmatrix} R \begin{pmatrix} I_r \\ 0_r \end{pmatrix}$ and $R_{12} = \begin{pmatrix} I_r & 0_r \end{pmatrix} R \begin{pmatrix} 0_r \\ I_r \end{pmatrix}$ denote $r \times r$ submatrices of R and similarly for $M_{0,11}$ and $M_{0,21}$. Because of (3.16) these latter matrices are multiples of the identity matrix, and we obtain

$$\begin{aligned} \frac{2^n P_n(z)}{\varphi(z)^n} = &\left[\frac{1}{2}\left(\gamma(z) + \gamma(z)^{-1}\right) R_{11}(z)D(\infty) - \frac{1}{2i}\left(\gamma(z) - \gamma(z)^{-1}\right) R_{12}(z)D(\infty)^{-*}\right] \\ &\times D(z)^{-1} \end{aligned}$$

with γ as in (3.16). Using (3.59) we conclude that

$$\frac{2^n P_n(z)}{\varphi(z)^n} \sim \sum_{k=0}^{\infty} \frac{\widetilde{\Pi}_k(z)}{n^k} \tag{4.2}$$

has a full asymptotic expansion in inverse powers of n, with analytic matrix valued functions $\widetilde{\Pi}_k$.

From

$$R(z) = I_{2r} + \frac{R_1(z)}{n} + \mathcal{O}(n^{-2})$$

uniformly for $z \in \mathbb{C} \setminus U$, we obtain

$$\begin{aligned} \frac{2^n P_n(z)}{\varphi(z)^n} = &\left[\frac{1}{2}\left(\gamma(z) + \gamma(z)^{-1}\right)\left(I_r + \frac{(R_1(z))_{11}}{n} + \mathcal{O}(n^{-2})\right) D(\infty)\right. \\ &\left. -\frac{1}{2i}\left(\gamma(z) - \gamma(z)^{-1}\right)\left(\frac{(R_1(z))_{12}}{n} + \mathcal{O}(n^{-2})\right) D(\infty)^{-*}\right] D(z)^{-1}. \end{aligned} \tag{4.3}$$

Recall that $D(\infty)$ and $D(z)^{-1}$ are invertible matrices that are independent of n. Then we arrive at the leading term in the expansion (4.2)

$$\widetilde{\Pi}_0(z) = \lim_{n\to\infty} \frac{2^n P_n(z)}{\varphi(z)^n} = \frac{1}{2}\left(\gamma(z) + \gamma(z)^{-1}\right) D(\infty) D(z)^{-1}, \tag{4.4}$$

which is (1.22) by simple rewriting of the scalar prefactor.

Next, we see from (4.3) that the n^{-1} term in (4.2) has the coefficient

$$\widetilde{\Pi}_1(z) = \left[\frac{1}{2}\left(\gamma(z)+\gamma(z)^{-1}\right)(R_1(z))_{11}D(\infty) - \frac{1}{2i}\left(\gamma(z)-\gamma(z)^{-1}\right)(R_1(z))_{12}D(\infty)^{-*}\right] \times D(z)^{-1}. \tag{4.5}$$

We use (3.64) and (3.65), (3.66) and the property (3.36) of the permutation matrix Π_r to conclude

$$\begin{aligned}(R_1(z))_{11} &= \frac{(A^{(1)})_{11}}{z-1} + \frac{(B^{(1)})_{11}}{z+1} \\ &= -\frac{1}{4(z-1)}D(\infty)U_1 \operatorname{diag}\left(\alpha_1^2-\frac{1}{4},\dots,\alpha_r^2-\frac{1}{4}\right)U_1^{-1}D(\infty)^{-1} \\ &\quad + \frac{1}{4(z+1)}D(\infty)U_{-1}\operatorname{diag}\left(\beta_1^2-\frac{1}{4},\dots,\beta_r^2-\frac{1}{4}\right)U_{-1}^{-1}D(\infty)^{-1}\end{aligned} \tag{4.6}$$

and similarly

$$\begin{aligned}(R_1(z))_{12} &= \frac{i}{4(z-1)}D(\infty)U_1 \operatorname{diag}\left(\alpha_1^2-\frac{1}{4},\dots,\alpha_r^2-\frac{1}{4}\right)U_1^{-1}D(\infty)^{*} \\ &\quad + \frac{i}{4(z+1)}D(\infty)U_{-1}\operatorname{diag}\left(\beta_1^2-\frac{1}{4},\dots,\beta_r^2-\frac{1}{4}\right)U_{-1}^{-1}D(\infty)^{*}\end{aligned} \tag{4.7}$$

Inserting (4.6) and (4.7) into (4.5), we obtain

$$\widetilde{\Pi}_1(z) = \frac{1}{2}\left(\gamma(z)+\gamma(z)^{-1}\right)D(\infty)\Pi_1(z)D(z)^{-1}$$

with

$$\begin{aligned}\Pi_1(z) = &-\frac{1}{4(z-1)}U_1\operatorname{diag}\left(\alpha_1^2-\frac{1}{4},\dots,\alpha_r^2-\frac{1}{4}\right)U_1^{-1} \\ &+\frac{1}{4(z+1)}U_{-1}\operatorname{diag}\left(\beta_1^2-\frac{1}{4},\dots,\beta_r^2-\frac{1}{4}\right)U_{-1}^{-1} \\ &-\frac{\gamma(z)-\gamma(z)^{-1}}{\gamma(z)+\gamma(z)^{-1}}\left[\frac{1}{4(z-1)}U_1\operatorname{diag}\left(\alpha_1^2-\frac{1}{4},\dots,\alpha_r^2-\frac{1}{4}\right)U_1^{-1}\right. \\ &\left.+\frac{1}{4(z+1)}U_{-1}\operatorname{diag}\left(\beta_1^2-\frac{1}{4},\dots,\beta_r^2-\frac{1}{4}\right)U_{-1}^{-1}\right].\end{aligned}$$

This leads to (1.21) since

$$\frac{1}{z-1}\left[1+\frac{\gamma(z)-\gamma(z)^{-1}}{\gamma(z)+\gamma(z)^{-1}}\right]=\frac{2}{\varphi(z)-1},$$
$$\frac{1}{z+1}\left[1-\frac{\gamma(z)-\gamma(z)^{-1}}{\gamma(z)+\gamma(z)^{-1}}\right]=\frac{2}{\varphi(z)+1},$$

as can be verified by direct calculation. □

4.2. Proof of Theorem 1.10

Proof. We have by (3.5) and (3.7)

$$P_n(z)=\begin{pmatrix} I_r & 0_r\end{pmatrix} Y(z)\begin{pmatrix} I_r\\ 0_r\end{pmatrix}=\begin{pmatrix} I_r & 0_r\end{pmatrix} T(z)\begin{pmatrix} I_r\\ 0_r\end{pmatrix}\left(\frac{\varphi(z)}{2}\right)^n.$$

For z in the upper part of the lens outside of the disks $D(\pm 1,\delta)$, we then have by (3.10) and (3.57)

$$2^n P_n(z)=\begin{pmatrix} I_r & 0_r\end{pmatrix} R(z)M(z)\begin{pmatrix} \varphi(z)^n I_r\\ \varphi(z)^{-n}W(z)^{-1}\end{pmatrix}.$$

We take the limit $z\to x\in(-1+\delta,1-\delta)$ and split the previous formula into two terms. Then we use (3.14) to obtain

$$\begin{aligned}2^n P_n(x)&=\begin{pmatrix} I_r & 0_r\end{pmatrix} R(x)M_+(x)\begin{pmatrix} I_r\\ 0_r\end{pmatrix}\varphi_+(x)^n\\ &\quad+\begin{pmatrix} I_r & 0_r\end{pmatrix} R(x)M_-(x)\begin{pmatrix} 0_r & W(x)\\ -W(x)^{-1} & 0_r\end{pmatrix}\begin{pmatrix} 0_2\\ \varphi_+(x)^{-n}W(x)^{-1}\end{pmatrix}\\ &=\begin{pmatrix} I_r & 0_r\end{pmatrix} R(x)\left(M_+(x)\varphi_+(x)^n+M_-(x)\varphi_-(x)^n\right)\begin{pmatrix} I_r\\ 0_r\end{pmatrix},\end{aligned}$$

where we also used $\varphi_+(x)\varphi_-(x)=1$.

Note that $\varphi_\pm(x)=e^{\pm i\theta(x)}$, with $\theta(x)=\arccos(x)$. Then by the above and (3.58) to obtain

$$2^n P_n(x)=\begin{pmatrix} I_r & 0_r\end{pmatrix} M_+(x)\begin{pmatrix} I_r\\ 0_r\end{pmatrix}e^{in\theta(x)}+\begin{pmatrix} I_r & 0_r\end{pmatrix} M_-(x)\begin{pmatrix} I_r\\ 0_r\end{pmatrix}e^{-in\theta(x)}+\mathcal{O}(n^{-1})$$

as $n\to\infty$. Using (3.17) and (3.16) and noting that

$$\frac{1}{2}\left(\gamma_\pm(x)+\gamma_\pm^{-1}(x)\right)=\frac{1}{\sqrt{2}\sqrt[4]{1-x^2}}e^{\pm\frac{i}{2}\theta(x)\mp\frac{\pi i}{4}},\qquad -1<x<1,$$

we arrive at (1.23), with a $\mathcal{O}(n^{-1})$ term that is uniform for $x\in(-1+\delta,1-\delta)$.

If the weight matrix W is real symmetric, then $P_n(x)$ is real valued for real x. Then the normalized Szegő function has the symmetry (1.9) which implies $D_-(x) = \overline{D_+(x)}$ for $-1 < x < 1$. Hence the two terms within parentheses in (1.23) are each other's complex conjugates, and (1.24) follows from (1.23).

Since $\delta > 0$ can be taken arbitrarily small, the asymptotic formulas (1.23) and (1.24) are valid uniformly for x in any compact subset of $(-1, 1)$. □

4.3. Proofs of Theorems 1.11 and 1.13

Proof of Theorem 1.11. Let $x \in (1-\delta, 1)$ in the upper part of the lens. Then, starting from (3.5) and following the transformations (3.7), (3.10), (3.57), we have

$$\begin{aligned} P_n(x) = (I_r \quad 0_r)\, Y_+(x) \begin{pmatrix} I_r \\ 0_r \end{pmatrix} &= 2^{-n} (I_r \quad 0_r)\, T_+(x) \begin{pmatrix} I_r \\ 0_r \end{pmatrix} \varphi_+(x)^n \\ &= 2^{-n} (I_r \quad 0_r)\, S_+(x) \begin{pmatrix} I_r \\ \varphi_+(x)^{-2n} W^{-1}(x) \end{pmatrix} \varphi_+(x)^n \\ &= 2^{-n} (I_r \quad 0_r)\, R(x) P_+(x) \begin{pmatrix} \varphi_+(x)^n \\ \varphi_+(x)^{-n} W^{-1}(x) \end{pmatrix}. \end{aligned}$$

Inserting the formula (3.29) for the local parametrix P, and using $W = V_- V_-^*$ from (1.17) we get

$$2^n P_n(x) = (I_r \quad 0_r)\, R(x) E_n(x) P_+^{(1)}(x) \begin{pmatrix} V_+^{-1}(x) \\ V_-^{-1}(x) \end{pmatrix}.$$

The definition (1.14) of V and the factorization (1.3) gives that

$$V_\pm(x) = \sqrt{W(x)} Q(x) e^{\pm \vec{\alpha} \pi i/2},$$

with $\vec{\alpha} = \operatorname{diag}(\alpha_1, \dots, \alpha_r)$, where

$$\sqrt{W(x)} = (1-x)^{\alpha/2} (1+x)^{\beta/2} Q(x) \operatorname{diag}\left(\lambda_1(x)^{1/2}, \dots, \lambda_r(x)^{1/2}\right) Q(x)^*$$

is the positive square root of $W(x)$. Thus

$$2^n P_n(x) \sqrt{W(x)} = (I_r \quad 0_r)\, R(x) E_n(x) P_+^{(1)}(x) \begin{pmatrix} e^{-\frac{1}{2}\vec{\alpha}\pi i} \\ e^{\frac{1}{2}\vec{\alpha}\pi i} \end{pmatrix} Q(x)^*. \tag{4.8}$$

Recall that $P^{(1)}$ is given by (3.35) in terms of the Bessel parametrices Ψ_{α_j} for $j = 1, \dots, r$. Given $x \in (1-\delta, 1)$ we have from (3.32) that $f(x) = -\arccos(x)^2 < 0$ and by (3.31), see also the second line in (3.31),

$$\Psi_{\alpha,+}(n^2 f(x))$$
$$= \begin{pmatrix} \frac{1}{2}H_\alpha^{(1)}(2n\sqrt{-f(x)}) & \frac{1}{2}H_\alpha^{(2)}(2n\sqrt{-f(x)}) \\ \pi i n\sqrt{-f(x)}\left(H_\alpha^{(1)}\right)'(2n\sqrt{-f(x)}) & \pi i n\sqrt{-f(x)}\left(H_\alpha^{(2)}\right)'(2n\sqrt{-f(x)}) \end{pmatrix} \times \begin{pmatrix} e^{\frac{1}{2}\alpha\pi i} & 0 \\ 0 & e^{-\frac{1}{2}\alpha\pi i} \end{pmatrix}, \quad (4.9)$$

where $H_\alpha^{(1)}$ and $H_\alpha^{(2)}$ are the Hankel functions of order α, and $\left(H_\alpha^{(1)}\right)'$ and $\left(H_\alpha^{(2)}\right)'$ are their derivatives. We use parameters $\alpha_1, \dots, \alpha_r$ and the short hand notation

$$H_{\vec{\alpha}}^{(j)}(\xi) = \operatorname{diag}\left(H_{\alpha_1}^{(j)}(\xi), \dots, H_{\alpha_r}^{(j)}(\xi)\right), \qquad j = 1, 2,$$

and similarly for $\left(H_{\vec{\alpha}}^{(j)}\right)'$. Thus by (3.35) and (3.31)

$$P_+^{(1)}(x) = \begin{pmatrix} \frac{1}{2}H_{\vec{\alpha}}^{(1)}(2n\sqrt{-f(x)}) & \frac{1}{2}H_{\vec{\alpha}}^{(2)}(2n\sqrt{-f(x)}) \\ \pi i n\sqrt{-f(x)}\left(H_{\vec{\alpha}}^{(1)}\right)'(2n\sqrt{-f(x)}) & \pi i n\sqrt{-f(x)}\left(H_{\vec{\alpha}}^{(2)}\right)'(2n\sqrt{-f(x)}) \end{pmatrix} \times \begin{pmatrix} e^{\frac{1}{2}\vec{\alpha}\pi i} & 0_r \\ 0_r & e^{-\frac{1}{2}\vec{\alpha}\pi i} \end{pmatrix}.$$

Hence, because of relation $J_\alpha = \frac{1}{2}\left(H_\alpha^{(1)} + H_\alpha^{(2)}\right)$ between the Bessel function of the first kind and the Hankel functions, we obtain

$$P_+^{(1)}(x)\begin{pmatrix} e^{-\frac{1}{2}\vec{\alpha}\pi i} \\ e^{\frac{1}{2}\vec{\alpha}\pi i} \end{pmatrix} = \begin{pmatrix} I_r & 0_r \\ 0_r & 2\pi i n I_r \end{pmatrix} \begin{pmatrix} J_{\vec{\alpha}}\left(2n\sqrt{-f(x)}\right) \\ \sqrt{-f(x)}\,(J_{\vec{\alpha}})'\left(2n\sqrt{-f(x)}\right) \end{pmatrix} \quad (4.10)$$

with $J_{\vec{\alpha}} = \operatorname{diag}(J_{\alpha_1}, \dots, J_{\alpha_r})$ and similarly for $(J_{\vec{\alpha}})'$. Using this in (4.8), we obtain

$$P_n(x)\sqrt{W(x)} = 2^{-n}\begin{pmatrix} I_r & 0_r \end{pmatrix} R(x) E_n(x) \begin{pmatrix} I_r & 0_r \\ 0_r & 2\pi i n I_r \end{pmatrix} \times \begin{pmatrix} J_{\vec{\alpha}}\left(2n\sqrt{-f(x)}\right) \\ \sqrt{-f(x)}\,(J_{\vec{\alpha}})'\left(2n\sqrt{-f(x)}\right) \end{pmatrix} Q(x)^*. \quad (4.11)$$

Next we focus on the product $E_n(x)\begin{pmatrix} I_r & 0_r \\ 0_r & 2\pi i n I_r \end{pmatrix}$. By (3.39) it is equal to

$$E_n(x)\begin{pmatrix} I_r & 0_r \\ 0_r & 2\pi i n I_r \end{pmatrix} = \frac{\sqrt{\pi n}}{2}\begin{pmatrix} D(\infty) & 0_r \\ 0_r & D(\infty)^{-*} \end{pmatrix}\begin{pmatrix} I_r & iI_r \\ iI_r & I_r \end{pmatrix}\begin{pmatrix} E^{(1)}(x) & iE^{(2)}(x) \end{pmatrix}. \quad (4.12)$$

The n-dependence appears only in the prefactor $\frac{\sqrt{\pi n}}{2}$. Since $R(x) = I_{2r} + \mathcal{O}(n^{-1})$ by (3.58), we obtain from (4.11) and (4.12) that

$$2^n P_n(x)\sqrt{W(x)} = \sqrt{\pi n} 2D(\infty) \begin{pmatrix} I_r & iI_r \end{pmatrix} \begin{pmatrix} E^{(1)}(x) & iE^{(2)}(x) \end{pmatrix} \left(I_{2r} + \mathcal{O}(n^{-1})\right) \times \begin{pmatrix} J_{\vec{\alpha}}(2n\sqrt{-f(x)}) \\ \sqrt{-f(x)}\,(J_{\vec{\alpha}})'(2n\sqrt{-f(x)}) \end{pmatrix} Q(x)^*. \quad (4.13)$$

Note that $E^{(1)}$ and $E^{(2)}$ are matrices of size $2r \times r$ that are explicitly given in (3.40) and (3.41). They are both analytic in the disk $D(1,\delta)$ around $z = 1$, as was shown in the proof of Lemma 3.9. For $z \in D(1,\delta)$ we may readily verify from (3.40) and (3.41), and from the formula (3.16) for $\gamma(z)$ that

$$\begin{pmatrix} I_r & iI_r \end{pmatrix} \begin{pmatrix} E^{(1)}(z) & iE^{(2)}(z) \end{pmatrix} = \frac{f(z)^{1/4}}{(z^2-1)^{1/4}} \times \begin{pmatrix} \left((z+1)^{1/2} + (z-1)^{1/2}\right) D(z)^{-1} V(z) & \left((z+1)^{1/2} - (z-1)^{1/2}\right) D(\overline{z})^* V(\overline{z})^{-*} \end{pmatrix} \times \begin{pmatrix} I_r & f(z)^{-1/2} I_r \\ I_r & -f(z)^{-1/2} I_r \end{pmatrix}. \quad (4.14)$$

We use this in (4.13) for $z = x \in (1-\delta, 1)$ with $+$ boundary values, where we note the identities

$$\frac{f_+(x)^{1/4}}{(x^2-1)_+^{1/4}} = \frac{\sqrt{\arccos(x)}}{\sqrt{2}(1-x^2)^{1/4}}, \qquad f_+(x)^{-1/2} = -i(\sqrt{-f(x)})^{-1}$$

and $(x-1)_+^{1/2} = i\sqrt{1-x}$ to obtain

$$P_n(x)\sqrt{W(x)} = \frac{\sqrt{\pi n \arccos x}}{2^{n+1}(1-x^2)^{1/4}} D(\infty) \times \begin{pmatrix} \frac{\sqrt{1+x}+i\sqrt{1-x}}{\sqrt{2}} D_+(x)^{-1} V_+(x) & \frac{\sqrt{1+x}-i\sqrt{1-x}}{\sqrt{2}} D_-(x)^* V_-(x)^{-*} \end{pmatrix} \times \begin{pmatrix} I_r & -iI_r \\ I_r & iI_r \end{pmatrix} \left(I_{2r} + \mathcal{O}(n^{-1})\right) \begin{pmatrix} J_{\vec{\alpha}}(2n\sqrt{-f(x)}) \\ (J_{\vec{\alpha}})'(2n\sqrt{-f(x)}) \end{pmatrix} Q(x)^*, \quad (4.15)$$

which proves (1.28) in view of the definition (1.27), and the fact that $D_-(x)D_-(x)^* = V_-(x)V_-(x)^*$ by (1.17), which can be rewritten as

$$D_-(x)^* V_-(x)^{-*} = D_-(x)^{-1} V_-(x). \quad \square$$

Proof of Theorem 1.13. We use $x = \cos\frac{\theta}{n}$ in (1.28). As $n \to \infty$, we then have $x \to 1-$, $A_+(x) \to A(1) = U_1$, $A_-(x) \to U_1$ and $\frac{\arccos x}{(1-x^2)^{1/4}} \to 1$. Because of (1.28) we then have

$$\lim_{n\to\infty} \frac{2^n}{\sqrt{\pi n}} P_n(x)\sqrt{W(x)} Q(x) = D(\infty) U_1 J_{\vec{\alpha}}(\theta) \qquad \text{with } x = \cos\frac{\theta}{n}. \quad (4.16)$$

We write

$$\sqrt{W(x)}Q(x) = Q(x)\Lambda^{1/2}(x)(1-x)^{\alpha/2}(1+x)^{\beta/2}, \tag{4.17}$$

where we used (1.1) and (1.3). For each $j = 1, \dots, r$ we have

$$\frac{\lambda_j(x)(1-x)^{\alpha}(1+x)^{\beta}}{(1-x)^{\alpha_j}} \to c_j 2^{\alpha_j} \quad \text{as } x \to 1-,$$

because of (1.31). Since $1 - x = \frac{\theta^2}{2n^2} + \mathcal{O}(n^{-4})$ as $n \to \infty$, when $x = \cos\frac{\theta}{n}$, we get

$$n^{2\alpha_j}\lambda_j(x)(1-x)^{\alpha}(1+x)^{\beta} \to c_j\theta^{2\alpha_j}$$

as $n \to \infty$. It follows that

$$\lim_{n\to\infty} \Lambda^{1/2}(x)(1-x)^{\alpha/2}(1+x)^{\beta/2}\operatorname{diag}\left(c_1^{-1/2}n^{\alpha_1}, \dots, c_r^{-1/2}n^{\alpha_r}\right) = \operatorname{diag}\left(\theta^{\alpha_1}, \dots, \theta^{\alpha_r}\right), \tag{4.18}$$

again with $x = \cos\frac{\theta}{n}$. Combining (4.16), (4.17) and (4.18), we obtain (1.32). □

4.4. Proof of Theorem 1.14

The recurrence coefficients can be obtained from the solution of the RH problem for Y. To emphasize the dependence on n, we write $Y^{(n)}$, and similarly for other matrices that depend on n. For a matrix X of size $2r \times 2r$ we use X_{ij} for $i, j = 1, 2$ to denote its submatrices of size $r \times r$, that is,

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}.$$

We start with explicit formulas for the recurrence coefficients B_n and C_n in terms of the matrices $R^{(n)}$ and $R^{(n+1)}$.

Lemma 4.1. *We have*

$$B_n = \lim_{z\to\infty} z\left(R_{11}^{(n)}(z) - R_{11}^{(n+1)}(z)\right) \tag{4.19}$$

and

$$C_n = \lim_{z\to\infty}\left(\frac{i}{2}D(\infty)D(\infty)^* + zR_{12}^{(n)}(z)\right)\left(-\frac{i}{2}D(\infty)^{-*}D(\infty)^{-1} + zR_{21}^{(n)}(z)\right). \tag{4.20}$$

Proof. The matrix valued function

$$U^{(n)}(z) = Y^{(n+1)}(z)Y^{(n)}(z)^{-1} \tag{4.21}$$

is entire, since the jump matrix for $Y^{(n)}$ is independent of n, and it is equal to

$$\begin{aligned} U^{(n)}(z) &= \begin{pmatrix} zI_r & 0_r \\ 0_r & 0_r \end{pmatrix} + Y_1^{(n+1)} \begin{pmatrix} I_r & 0_r \\ 0_r & 0_r \end{pmatrix} - \begin{pmatrix} I_r & 0_r \\ 0_r & 0_r \end{pmatrix} Y_1^{(n)} \\ &= \begin{pmatrix} zI_r + \left(Y_1^{(n+1)}\right)_{11} - \left(Y_1^{(n)}\right)_{11} & -\left(Y_1^{(n)}\right)_{12} \\ \left(Y_1^{(n+1)}\right)_{21} & 0_r \end{pmatrix}, \end{aligned} \tag{4.22}$$

where $Y_1^{(n)}$ denotes the residue matrix in

$$Y^{(n)}(z) \begin{pmatrix} z^{-n}I_r & 0_r \\ 0_r & z^n I_r \end{pmatrix} = I_{2r} + \frac{Y_1^{(n)}}{z} + \mathcal{O}(z^{-2}) \tag{4.23}$$

as $z \to \infty$ with a constant matrix $Y_1^{(n)}$. Then by [23, Theorem 2.16]

$$B_n = zI_r - U_{11}^{(n)}(z) = \left(Y_1^{(n)}\right)_{11} - \left(Y_1^{(n+1)}\right)_{11}. \tag{4.24}$$

$$C_n = \left(Y_1^{(n)}\right)_{12} \left(Y_1^{(n)}\right)_{21} \tag{4.25}$$

The transformations $Y \mapsto T \mapsto S$ in (3.7) and (3.11), and the definition (4.21) show that for z outside of the lens

$$\begin{aligned} U^{(n)}(z) = &\begin{pmatrix} 2^{-n-1}I_r & 0_r \\ 0_r & 2^{n+1}I_r \end{pmatrix} S^{(n+1)}(z) \begin{pmatrix} \varphi(z)I_r & 0_r \\ 0_r & \varphi(z)^{-1}I_r \end{pmatrix} S^{(n)}(z)^{-1} \\ &\times \begin{pmatrix} 2^n I_r & 0_r \\ 0_r & 2^{-n}I_r \end{pmatrix}. \end{aligned}$$

All factors in this product remain bounded as $z \to \infty$, except for the diagonal matrix with $\varphi(z)$. Since $\varphi(z) = 2z + \mathcal{O}(z^{-1})$ as $z \to \infty$ we obtain

$$\begin{aligned} U^{(n)}(z) = 2z &\begin{pmatrix} 2^{-n-1}I_r & 0_r \\ 0_r & 2^{n+1}I_r \end{pmatrix} S^{(n+1)}(z) \begin{pmatrix} I_r & 0_r \\ 0_r & 0_r \end{pmatrix} S^{(n)}(z)^{-1} \begin{pmatrix} 2^n I_r & 0_r \\ 0_r & 2^{-n}I_r \end{pmatrix} \\ &+ \mathcal{O}(z^{-1}) \end{aligned} \tag{4.26}$$

as $z \to \infty$.

We use the final transformation $S^{(n)} = R^{(n)}M$, see (3.57), where we note that

$$\begin{aligned} &M(z) \begin{pmatrix} I_r & 0_r \\ 0_r & 0_r \end{pmatrix} M(z)^{-1} \\ &\quad = \begin{pmatrix} I_r & 0_r \\ 0_r & 0_r \end{pmatrix} + \frac{1}{2iz} \begin{pmatrix} 0_r & D(\infty)D(\infty)^* \\ D(\infty)^{-*}D(\infty)^{-1} & 0_r \end{pmatrix} + \mathcal{O}(z^{-2}) \quad \text{as } z \to \infty, \end{aligned} \tag{4.27}$$

which follows from direct calculation from (3.16) and (3.17). Using $S^{(n)} = R^{(n)}M$ and (4.27) in (4.26) we arrive at (where we also use $R^{(n+1)}(z) = I_{2r}+\mathcal{O}(z^{-1})$ and $R^{(n)}(z)^{-1} = I_{2r} + \mathcal{O}(z^{-1})$)

$$U^{(n)}(z) = 2z \begin{pmatrix} 2^{-n-1}I_r & 0_r \\ 0_r & 2^{n+1}I_r \end{pmatrix} R^{(n+1)}(z) \begin{pmatrix} I_r & 0_r \\ 0_r & 0_r \end{pmatrix} R^{(n)}(z)^{-1} \begin{pmatrix} 2^n I_r & 0_r \\ 0_r & 2^{-n}I_r \end{pmatrix} + \frac{1}{2i} \begin{pmatrix} 0_r & 4^{-n}D(\infty)D(\infty)^* \\ 4^{n+1}D(\infty)^{-*}D(\infty)^{-1} & 0_r \end{pmatrix} + \mathcal{O}(z^{-1}) \quad (4.28)$$

as $z \to \infty$.

Using (4.28) in the first identity of (4.24) we note that the term with $D(\infty)$ does not contribute to B_n and we get

$$B_n = \lim_{z\to\infty} \left(zI_r - zR_{11}^{(n+1)}(z) \left(R^{(n)}(z)^{-1} \right)_{11} \right) \quad (4.29)$$

We use the Laurent expansion $R^{(n)}(z) = I_{2r}+\dfrac{R_1^{(n)}}{z}+\mathcal{O}(z^{-2})$, where we note $R^{(n)}(z)^{-1} = I_{2r} - \dfrac{R_1^{(n)}}{z} + \mathcal{O}(z^{-2})$, and we obtain from (4.29) that

$$B_n = \lim_{z\to\infty} \left(z \left(R_1^{(n)}(z) - R_1^{(n+1)}(z) \right)_{11} \right)$$

which is the same as (4.19).

To obtain the formula (4.20) for C_n we go back to (4.25). The transformations $Y \mapsto T \mapsto S = RM$ then show that

$$Y_{12}^{(n)}(z) = 2^{-n} \left[R_{11}^{(n)}(z)M_{12}(z) + R_{12}^{(n)}(z)M_{22}(z) \right] \varphi(z)^{-n}$$
$$Y_{21}^{(n)}(z) = 2^{n} \left[R_{21}^{(n)}(z)M_{11}(z) + R_{22}^{(n)}(z)M_{21}(z) \right] \varphi(z)^{n}$$

and by (4.23), since $\varphi(z) = 2z + \mathcal{O}(z^{-1})$ as $z \to \infty$,

$$\begin{aligned} \left(Y_1^{(n)} \right)_{12} &= \frac{1}{4^n} \lim_{z\to\infty} z \left[R_{11}^{(n)}(z)M_{12}(z) + R_{12}^{(n)}(z)M_{22}(z) \right], \\ \left(Y_1^{(n)} \right)_{21} &= 4^n \lim_{z\to\infty} z \left[R_{21}^{(n)}(z)M_{11}(z) + R_{22}^{(n)}(z)M_{21}(z) \right]. \end{aligned} \quad (4.30)$$

Since $R^{(n)}(z) \to I_{2r}$ and $M(z) \to I_{2r}$ as $z \to \infty$ with

$$\lim_{z\to\infty} zM_{12}(z) = \frac{i}{2}D(\infty)D(\infty)^*,$$
$$\lim_{z\to\infty} zM_{21}(z) = -\frac{i}{2}D(\infty)^{-*}D(\infty)^{-1},$$

see the formulas (3.16) and (3.17) for M, we obtain (4.20) from (4.25) and (4.30). □

We can now turn to the proof of Theorem 1.14.

Proof of Theorem 1.14. Due to the asymptotic expansion (3.59) for $R = R^{(n)}$, and the formulas (4.19) and (4.20) we see that B_n and C_n have an asymptotic expansion in inverse powers of n.

Because of (3.59) we have

$$z\left(R^{(n)}(z) - R^{(n+1)}(z)\right) \sim \sum_{k=1}^{\infty} \left(\frac{1}{n^k} - \frac{1}{(n+1)^k}\right) R_k(z)$$

and the kth term in this series is $\mathcal{O}(n^{-k-1})$ as $k \to \infty$, uniformly for z in a neighborhood of ∞. Keeping only the first term we have

$$z(R^{(n)}(z) - R^{(n+1)}(z)) = \frac{R_1(z)}{n^2} + \mathcal{O}(n^{-3}).$$

From the explicit form (3.64) we then obtain

$$\lim_{z\to\infty} z\left(R^{(n)}(z) - R^{(n+1)}(z)\right) = \frac{A^{(1)} + B^{(1)}}{n^2} + \mathcal{O}(n^{-3})$$

as $n \to \infty$. We thus conclude from (4.19)

$$B_n = \frac{A_{11}^{(1)} + B_{11}^{(1)}}{n^2} + \mathcal{O}(n^{-3}).$$

The explicit formulas (3.65) and (3.66) for $A^{(1)}$ and $B^{(1)}$ then lead to the formula (1.35) for $\mathcal{B}_2 = A_{11}^{(1)} + B_{11}^{(1)}$.

Now we turn to the recurrence coefficient C_n. From (4.20) with $\lim_{z\to\infty} zR_{12}^{(n)}(z) = \mathcal{O}(n^{-1})$, $\lim_{z\to\infty} zR_{21}^{(n)}(z) = \mathcal{O}(n^{-1})$, we find

$$C_n = \frac{1}{4}I_r - \frac{i}{2}\left(\lim_{z\to\infty} zR_{12}^{(n)}(z)\right) D(\infty)^{-*}D(\infty)^{-1} + \frac{i}{2}D(\infty)D(\infty)^{*}\left(\lim_{z\to\infty} zR_{21}^{(n)}(z)\right) + \mathcal{O}(n^{-2}) \quad (4.31)$$

as $n \to \infty$. We have the leading term $\frac{1}{4}I_r$ for C_n and to prove (1.34) it remains to show that the n^{-1} term in the expansion for C_n vanishes.

By (3.59) and (4.19) we have

$$\lim_{z\to\infty} zR_{12}^{(n)}(z) = \frac{1}{n}\lim_{z\to\infty} (zR_1(z))_{12} + \mathcal{O}(n^{-2}) = \frac{1}{n}\left(A_{12}^{(1)} + B_{12}^{(1)}\right) + \mathcal{O}(n^{-2}),$$

and similarly for $R_{21}^{(n)}$. Thus by (4.31), $C_n = \frac{1}{4}I_r + \frac{\mathcal{C}_1}{n} + \mathcal{O}(n^{-2})$ as $n \to \infty$ with

$$\mathcal{C}_1 = -\frac{i}{2}\left(A_{12}^{(1)} + B_{12}^{(1)}\right) D(\infty)^{-*} D(\infty)^{-1} + \frac{i}{2} D(\infty) D(\infty)^{*} \left(A_{21}^{(1)} + B_{21}^{(1)}\right). \quad (4.32)$$

The formulas (3.65) and (3.66) can be written as

$$\begin{pmatrix} D(\infty)^{-1} & 0_r \\ 0_r & D(\infty)^{*} \end{pmatrix} A^{(1)} \begin{pmatrix} D(\infty) & 0_r \\ 0_r & D(\infty)^{-*} \end{pmatrix} = \begin{pmatrix} -X & iX \\ iX & X \end{pmatrix}, \qquad X = \frac{1}{4} U_1 \operatorname{diag}\left(\alpha_j^2 - \frac{1}{4}\right) U_1^{-1}, \quad (4.33)$$

and

$$\begin{pmatrix} D(\infty)^{-1} & 0_r \\ 0_r & D(\infty)^{*} \end{pmatrix} B^{(1)} \begin{pmatrix} D(\infty) & 0_r \\ 0_r & D(\infty)^{-*} \end{pmatrix} = \begin{pmatrix} \widetilde{X} & i\widetilde{X} \\ i\widetilde{X} & -\widetilde{X} \end{pmatrix}, \qquad \widetilde{X} = \frac{1}{4} U_{-1} \operatorname{diag}\left(\beta_j^2 - \frac{1}{4}\right) U_{-1}^{-1}. \quad (4.34)$$

We observe that in both these formulas the off-diagonal blocks agree, which means that

$$D(\infty)^{-1} A_{12}^{(1)} D(\infty)^{-*} = D(\infty)^{*} A_{21}^{(1)} D(\infty),$$
$$D(\infty)^{-1} B_{12}^{(1)} D(\infty)^{-*} = D(\infty)^{*} B_{21}^{(1)} D(\infty).$$

These two identities and (4.32) show that indeed $\mathcal{C}_1 = 0_r$ and the proof is complete. □

Declaration of competing interest

The authors have no competing interests to declare that are relevant to the content of this article.

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