Highlights

High-order asymptotic expansions of Gaussian quadrature rules with classical and generalised weight functions

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- Gaussian quadrature rules can be computed efficiently from asymptotic expansions

- Generalized weight functions include all classical orthogonal polynomials as special cases

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Abstract

Gaussian quadrature rules are a classical tool for the numerical approximation of integrals with smooth integrands and positive weight functions. We derive and explicitly list asymptotic expressions for the points and weights of Gaussian quadrature rules for three general classes of positive weight functions: analytic functions on a bounded interval with algebraic singularities at the endpoints, analytic weight functions on the halfline with exponential decay at infinity and an algebraic singularity at the finite endpoint, and analytic functions on the real line with exponential decay in both directions at infinity. The results include the Gaussian rules of classical orthogonal polynomials (Legendre, Jacobi, Laguerre and Hermite) as special cases. Explicit expressions for these cases are included in the appendix. We present experiments indicating the range of the number of points at which these expressions achieve high precision. We provide an algorithm that can compute arbitrarily many terms in these expansions for the classical cases, and many though not all terms for the generalized cases.

1. Introduction

The topic of this paper is a methodology to obtain high-order asymptotic expansions of nodes and weights of Gaussian quadrature rules. We derive results for generalizations of the classical Jacobi and Laguerre weights. The resulting explicit expressions for the classical Gauss-rules are listed in Appendix A, with more terms then appeared before in literature. The generalizations consist of smooth perturbations of the weight functions and hence to different Gauss rules. The proposed methodology is based on the asymptotic analysis of the corresponding orthogonal polynomials for large degree. It results in explicit formulae for the nodes and weights which can simply be evaluated one by one, leading to an efficient $\mathcal{O}(n)$ computational scheme for the construction of Gaussian quadrature. In this paper we focus mainly on the essential principles. Detailed computations, which are quite technical in nature, can be found in the PhD thesis of the first author [1].

Different expansions are presented for three distinct regions of the integration domain in which the asymptotic behaviour is different: near the endpoints (called a *hard edge* in the literature on asymptotic analysis), near infinity (*soft edge*) and in the interior of the domain (the *bulk*). An algorithm is presented to obtain an arbitrary number of terms. It is made available in two ways. The first version is written in symbolic software, in which parameters may be present (such as α and β) that are taken into account analytically. This is a very general approach, but the computation of very high order terms may become unwieldy. A second version is purely numerical and can be used for specific numeric choices of the parameters and without a further need for symbolic computations. This approach is more limited in scope but much more efficient. The implementation is accompanied by heuristical approaches to choose the appropriate asymptotic expansion and number of terms.

1.1. Overview of the paper

We give a brief motivation for our use of asymptotic expansions arising from Riemann–Hilbert problems of orthogonal polynomials in § 2. In order to understand the structure and derivation of the expansions, we recall the Riemann–Hilbert background of the orthogonal polynomials in § 3. We outline the procedure to turn expansions of polynomials into expansions of the nodes and weights in § 4. The explicit expressions themselves are listed in § 5, § 6 and § 7, with the classical cases listed in Appendix A. We illustrate the statements with numerical experiments throughout the paper.

2. Scope and motivation

The orthonormal polynomials associated with Gaussian quadrature satisfy the conditions

$$\int_a^b p_n(x)p_k(x)w(x)dx = \begin{cases} 1, & k = n, \\ 0, & k \neq n. \end{cases}$$

The nodes x_k are the zeros of p_n. The weights, sometimes referred to as Christoffel numbers, can be expressed in various ways (see [2, p. 35], [3, p. 323], [4, §1]). One useful expression is

$$w_k = \frac{-\gamma_{n+1}}{\gamma_n p'_n(x_k)p_{n+1}(x_k)} = \frac{\gamma_n}{\gamma_{n-1}p'_n(x_k)p_{n-1}(x_k)} > 0. \tag{2.1}$$

Here, γ_n is the leading order coefficient of the orthonormal polynomial p_n.

2.1. Computation of Gaussian quadrature

Historically, the main algorithm for the computation of Gaussian quadrature rules has been based on the eigenvalue decomposition of a tridiagonal matrix (the Jacobi matrix), which involves the coefficients of the three-term recurrence relation of the orthogonal polynomials [5]. The computation can be performed in $\mathcal{O}(n^2)$ operations using the Golub–Welsch algorithm [4]. Since then, algorithms with lower computational complexity for large n have started to appear for a number of cases, starting with the Glaser–Liu–Rokhlin algorithm [6] and including at least [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. These approaches are based on asymptotic analysis, solving Riemann–Hilbert problems, or rootfinding methods for non-oscillatory differential equations. A brief history of fast algorithms for the special case of Gauss–Legendre quadrature rules is described in [18].

2.2. Asymptotic expansions of orthogonal polynomials and numerical aspects

Asymptotic analysis of orthogonal polynomials has witnessed significant progress in the last two decades. Large-degree asymptotic expansions can be obtained from integral representations, differential equations satisfied by the polynomials (see [19] and references therein), or the associated Riemann–Hilbert problem (see in particular [20, 21] for Jacobi and Laguerre polynomials).

From a computational point of view, asymptotic expansions of orthogonal polynomials present two important advantages: they become increasingly accurate as the degree n increases, and their evaluation time is independent of n.[1] In this sense, they compare favourably to other methods for computing orthogonal polynomials. On the other hand, the terms in the expansion become increasingly involved with increasing order. Moreover, asymptotic expansions in general do not converge for a fixed value of n by adding terms. A typical recommendation is to truncate the

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[1]Strictly speaking, for this statement we assume that all functions involved can be evaluated in a time independent of n, which in some cases may require lookup tables [22].

expansion after the smallest term, after which the accuracy often deteriorates [2, p. 182-187]. Still, arbitrary accuracy can not be achieved and the availability of error bounds is, unfortunately, lagging behind the construction of the expansions themselves.

Gaussian quadrature rules can be computed from these expansions most simply by numerical rootfinding [8, 12]. The classical quadrature rules with n points may be computed in $\mathcal{O}(n)$ cost. However, asymptotic expansions can also be found directly for the nodes and weights themselves. For the classical Gauss rules, this was achieved in [7, 10, 11, 16, 17] and the thesis [1]. These expansions have $\mathcal{O}(n)$ cost as well, with a constant that is often (but not always) smaller than that of the rootfinding class of methods. Care has to be taken in the efficient evaluation of certain special functions and in various kinds of precomputations. Therefore, whether or not direct asymptotic expansions are faster is not a priori guaranteed and still depends on several implementation aspects, but they offer an appealing simplicity.

2.3. Riemann–Hilbert analysis

The asymptotic expansions in this paper are based on the Riemann–Hilbert (RH) problem of the associated orthogonal polynomials [23, 14]. Riemann–Hilbert problems provide a uniform framework in which to analyze orthogonal polynomials. Compared to alternative known approaches to asymptotic analysis, RH analysis is lengthy, but considerably more general. In particular the approach, once established, works the same for perturbations such as the multiplication of a weight function by an analytic function. Although the expansions derived in this paper have required considerable effort, the final outcome is simple to use in any language or tool simply by copying the expressions, and new expressions can be derived using the symbolic code, with little effort and without having to be familiar with the underlying mathematics. The main focus of the paper is to sketch the effort that remains, with sufficient technical detail to appreciate the scope and limitations of the approach.

2.4. Main results

We have previously described the numerical computation of arbitrary order asymptotic expansions for orthogonal polynomials for Jacobi–type weights [24] and Laguerre-type weights [25]. This was based on the asymptotic analysis of the associated RH problems by Kuijlaars, McLaughlin, Van Assche and Vanlessen in [20] for the Jacobi case and by Vanlessen in [21] for the Laguerre case. It resulted in high-order expansions in four different regions of the complex plane in which the polynomials exhibit different asymptotic behaviour. These expansions form the starting points for the expansions of the current paper of the associated Gaussian quadrature rules.

We consider integrals with weight functions for three canonical types of quadrature rules:

- The first are **modified Jacobi–type weight functions** of the form

$$w(x) = (1 - x)^\alpha (1 + x)^\beta h(x), \qquad x \in [-1, 1] \tag{2.2}$$

  where h is a strictly positive analytic function that represents the modification and $[a, b] = [-1, 1]$.

- The second are **modified Laguerre-type weight functions** of the form

$$w(x) = x^\alpha e^{-Q(x)}, \qquad x \in [0, \infty), \tag{2.3}$$

  where $[a, b) = [0, \infty)$ and the standard Laguerre case corresponds to $Q(x) = x$. Here, unlike in the Jacobi case, more general functions $Q(x)$ are restricted as detailed further on. Our results are most explicit for monomials $Q(x) = x^m$.

- Finally, we consider **modified Hermite–type weight functions**

$$w(x) = \mathrm{e}^{-Q(x^2)}, \qquad x \in (-\infty, \infty), \tag{2.4}$$

where $(a, b) = (-\infty, \infty)$ and where $Q(x) = x$ corresponds to the classical Gauss–Hermite case. The functions $Q(x)$ we consider are those of the Laguerre case.

The Hermite–type rules in the form above can be obtained from the Laguerre–type rules by a change of variables, hence we focus mostly on the first two cases. The substitution entails the specific values $\alpha = \pm 1/2$ which results in a simplification of the Laguerre expansions.

Numerical experiments indicate that the errors of the expansions decrease at the expected rate for increasing n, although no error bounds are provided in this paper. The expansions in the current paper hold in principle for any fixed value of α and β. However, when those parameters are large, accuracy of the expansions may decrease and any beneficial effect may require very high n. This is a known problem, and in that setting we recommend exploring different expansions, for example based on asymptotics with a varying weight (varying with n) as studied in [26, 27, 28, 29, 30].

We have implemented an algorithm to compute asymptotic expansions for nodes and weights with arbitrary many terms in the symbolic software package SAGE. By making all steps of the algorithm explicit, i.e. not requiring symbolic treatment, the algorithm can also be implemented in a more efficient non-symbolic language. The resulting expressions themselves have to be computed only once, and can be implemented in any language. The figures in this paper have been generated using code in JULIA.[2] Some limitations are present in the case of modified Laguerre, as the expansions involve several integrals that have to be evaluated numerically for specific numerical choices of the parameters. This renders a generic listing of expansions rather difficult.

3. Preliminaries

We briefly recall the background of the asymptotic expansions for Jacobi–type and Laguerre–type orthogonal polynomials as derived in [24, 25], see also [1]. In particular, we define the quantities that arise from the modified weight functions (2.2)–(2.3) and that appear in the final expansions: the constants c_k and d_k, the phase functions $\lambda_n(z)$ and $\zeta_n(z)$ and, in case of Laguerre–type weights, the constant β_n. These quantities depend on the analytic modifier $h(x)$ in (2.2) or on the function $Q(x)$ in (2.3) and (2.4).

3.1. Riemann-Hilbert analysis

The explicit expansions in [24, 25] are derived from the Deift-Zhou nonlinear steepest descent analysis [32] of a Riemann–Hilbert problem associated with orthogonal polynomials [33, 23]. The asymptotic analysis for Jacobi-type weight functions of the form (2.2) was carried out in [20], while the description for Laguerre-type weight functions (2.3) is in [21].

The Riemann-Hilbert problem has a 2×2 complex matrix-valued solution. The main result in [24, 25] has been the explicit asymptotic approximations to a matrix-valued function $R(z)$, defined in the complex plane. For large n, this matrix is the identity matrix plus an asymptotic series,

$$R(z) \sim I + \sum_{k=1}^{\infty} \frac{R_k(z)}{n^k}, \qquad n \to \infty.$$

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[2]The implementation is available as a Julia package `AsymptoticGaussianQuadrature.jl` from the GitHub repository `https://github.com/daanhb/AsymptoticGaussianQuadrature.jl`. That repository also contains the SAGE symbolic notebooks. We note that several results of this paper have also been incorporated in the Julia package `FastGaussQuadrature.jl` and in Chebfun [31].
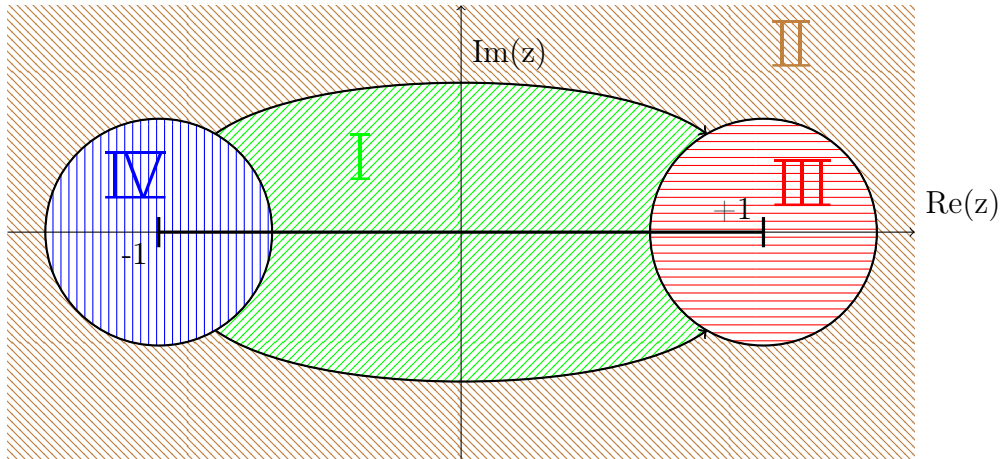
Figure 1: Regions of the complex plane in which the Jacobi-type polynomials have different asymptotic expansions: the lens (I), the outer region (II) and the right and left disks (III and IV).

The expansions of the polynomials are formulated in terms of $R(z)$ and, hence, additional terms $R_k(z)$ give rise to additional terms in the expansions of the polynomials. This happens via an explicit and invertible sequence of transformations.

The asymptotic analysis is different in different regions of the complex plane. The regions are illustrated for the case of Jacobi-type polynomials in Figure 1. There are four types of asymptotic expansions of the orthogonal polynomials $p_n(z)$: (I) inner asymptotics for z in the so-called 'lens' around the interval, (II) outer asymptotics valid for z away from the lens and the disks around the endpoints, (III) boundary asymptotics valid for z near the rightmost endpoint $z = +1$ and (IV) boundary asymptotics near the left endpoint $z = -1$. The matrix-valued function $R(z)$ has different expressions in these regions, which we label $R^{\text{outer}}(z)$ away from the disks, and $R^{\text{right}}(z)$ and $R^{\text{left}}(z)$ in the right and left disks respectively. Only the expansions in regions (I), (III) and (IV) contain zeros of the polynomials and they give rise to expansions for the bulk of the zeros in $(-1, 1)$, and for the extreme zeros near ± 1.

The figure with four regions is qualitatively similar for Laguerre-type and Hermite-type polynomials, albeit with different endpoint regions. We elaborate on the differences further on.

Mathematically, the regions shown in Figure 1 are of arbitrary size. Depending on how they are chosen, any given point $z \in \mathbb{C}$ can in principle belong to several regions. As a result, the expansions for the different regions have overlapping regions in which they are valid. In terms of implementation, the choice of the expansion is relevant as the accuracy can differ. We formulate heuristic choices of the expansions in later sections of this paper. We experimentally observed that, when two expansions apply at the same point, the differences in the corresponding relative errors for large n are fairly modest, in most cases not exceeding a factor of 2. The exception is when one expansion approaches the boundaries of its domain of validity: for example, the expansion for the outer region is not accurate near the interval. Within a radius of about 0.2 around an endpoint, the asymptotic expansion for that disk is orders of magnitude more accurate than the three other ones.

3.2. Jacobi-type polynomials

The asymptotic expansion for the orthonormal Jacobi-type polynomials valid in region (I) in the complex plane, as formulated in [24], is[3]

$$p_n(z) = \frac{2^{\frac{1}{2}-n}\,\gamma_n}{\sqrt{w(z)}\,(1-z^2)^{1/4}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}^T R^{\text{outer}}(z) \begin{pmatrix} D_\infty & 0 \\ 0 & -\frac{i}{D_\infty} \end{pmatrix} \begin{pmatrix} \cos\left(\lambda_n(z) + \frac{1}{2}\arccos z\right) \\ \cos\left(\lambda_n(z) - \frac{1}{2}\arccos z\right) \end{pmatrix}. \tag{3.1}$$

Here, γ_n is the leading order coefficient (with z^n) of the polynomial and D_∞ is a constant,

$$D_\infty = 2^{-\frac{\alpha+\beta}{2}} \exp\left(\frac{1}{4\pi i} \oint \frac{\log h(\zeta)}{(\zeta^2-1)^{1/2}} d\zeta\right),$$

where the contour encircles the interval $[-1,1]$ and $h(z)$ is the additional function that appears in the Jacobi-type weight function (2.2). The constant D_∞ is the limit for $z \to \infty$ of the Szëgo function $D(z)$ of the orthogonal polynomial. Note that for the classical Jacobi case, where $h(x) = 1$, we have $D_\infty = 2^{-\frac{\alpha+\beta}{2}}$. The leading order term of (3.1) corresponds to substituting $R^{\text{outer}}(z)$ by I, the 2×2 identity matrix.

The trigonometric behaviour of the polynomials in this region is determined by the phase function

$$\lambda_n(z) = \left(n + \frac{\alpha+\beta}{2}\right)\arccos z - \frac{\pi}{4} - \frac{\alpha\pi}{2} + \frac{(1-z^2)^{\frac{1}{2}}}{4\pi i} \oint \frac{\log h(\zeta)}{(\zeta^2-1)^{1/2}} \frac{d\zeta}{\zeta-z}. \tag{3.2}$$

The phase depends on the $h(x)$ function via a contour integral as well, which in this case should encircle the point z. This contour integral is a function of z and can be expanded around $z = \pm 1$, for example using the power series $\sum_{n=0}^\infty c_n(z-1)^n$ and $\sum_{n=0}^\infty d_n(z+1)^n$. These series have coefficients given by

$$c_k = \frac{1}{2\pi i} \oint_\gamma \frac{\log h(\zeta)}{(\zeta^2-1)^{1/2}} \frac{d\zeta}{(\zeta-1)^{k+1}}, \tag{3.3}$$

$$d_k = \frac{1}{2\pi i} \oint_\gamma \frac{\log h(\zeta)}{(\zeta^2-1)^{1/2}} \frac{d\zeta}{(\zeta+1)^{k+1}},$$

for $k \geq 0$. The coefficients enter the asymptotic expansions of the polynomials, and of the points and weights in this paper. We refer to [24] for comments on their computation.

Around the endpoints the polynomials exhibit behaviour that relates to the Bessel function. Each endpoint ± 1 is independent of n and for this reason is sometimes referred to as a 'hard edge'. In a disk around the right endpoint $z = 1$ the expression given in [24] is

$$p_n(z) = \gamma_n \frac{2^{-n}\sqrt{n\pi \arccos z}}{\sqrt{w(z)}\,(1-z^2)^{1/4}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}^T R^{\text{right}}(z) \begin{pmatrix} D_\infty & 0 \\ 0 & -\frac{i}{D_\infty} \end{pmatrix} \times \tag{3.4}$$

$$\begin{pmatrix} \cos\left(\zeta(z) + \frac{1}{2}\arccos z\right) J_\alpha(n\arccos z) + \sin\left(\zeta(z) + \frac{1}{2}\arccos z\right) J'_\alpha(n\arccos z) \\ \cos\left(\zeta(z) - \frac{1}{2}\arccos z\right) J_\alpha(n\arccos z) + \sin\left(\zeta(z) - \frac{1}{2}\arccos z\right) J'_\alpha(n\arccos z) \end{pmatrix}.$$

This expression may seem to be undefined near $z = 1$; however, the singularities cancel analytically with other terms and the problem can be avoided using series expansions around the endpoints. Here, the phase function $\zeta(z)$ is given by

$$\zeta(z) = \frac{\alpha+\beta}{2}\arccos z + \frac{(1-z^2)^{1/2}}{4\pi i} \oint_\gamma \frac{\log h(\zeta)}{(\zeta^2-1)^{1/2}} \frac{d\zeta}{\zeta-z},$$

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[3]We note that the formulation in [24] is for the monic orthogonal polynomials, which differs from the formulation here in the leading order coefficient γ_n. The latter can be expanded asymptotically too, see [24, (26)].

where the contour integral is similar as before. The expansion for the left disk is entirely similar due to the symmetry relation $P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x)$ [34, Tab. 18.6.1]. The endpoint expansions remain structurally similar even when $h(x)$ is not symmetric, but in that more general case the symmetry involves substituting $h(x)$ for $h(-x)$.

Both expansions have the form of a prefactor consisting of a normalization factor and an envelope function, multiplied by an oscillatory part consisting of $R(z)$ and oscillatory factors. Note that the function h appears in the phase of the oscillations, via a contour integral; in $R(z)$, via c_k and d_k; and in the prefactor, via the weight function $w(z)$. The prefactor does not affect the points x_k, but it does affect the weights.

3.3. Laguerre-type polynomials

The structure of the expansions for Laguerre-type polynomials is very similar to that of Jacobi-type polynomials. We do not repeat the expressions here, but refer to [25, 1]. We do recall the context and the expressions that are necessary for understanding and implementing the expansions of this paper.

Recall the generalised Laguerre weight function (2.3). We consider three cases for the function $Q(x)$:

- a monomial: $Q(x) = q_m x^m + q_0$,

- a general polynomial: $Q(x) = \sum_{k=0}^m q_k x^k$,

- a general function $Q(x)$ analytic on the integration interval.

The function $Q(x)$ should be such that all moments $\int_0^\infty x^\alpha x^k e^{-Q(x)} dx$ are finite, which implies at least that

$$\lim_{x \to \infty} Q(x) = +\infty.$$

For polynomial Q, it is sufficient that the leading order coefficient q_m is positive. Here, as we shall see the expansions depend on certain contour integrals involving Q which have to be evaluated by other means, and we present only limited results.

The left endpoint $x = 0$ of the Laguerre-type polynomials is very similar to the left endpoint of Jacobi-type polynomials: it is a hard edge that gives rise to expansions in terms of the Bessel function and its derivative.

The behaviour is quite different for $x \to \infty$, a so-called *soft edge*. The zeros of the Laguerre-type polynomials accumulate on a finite interval $(0, 1)$ after an n-dependent rescaling of the variable

$$x = \beta_n z,$$

where β_n is the *Mhaskar-Rakhmanov-Saff (MRS) number* (see [25, §3]). The local asymptotic behaviour near the soft edge is given in terms of the Airy function and its derivative. This leads to higher order poles in the derivations and thus also to longer formulae for the $R_k(z)$.

For classical Laguerre polynomials with $\alpha = 0$, the MRS number is $\beta_n = 4n$. Indeed, the largest zero of the Laguerre polynomial of degree n scales approximately as $4n$. For general monomial functions $Q(x) = q_m x^m$, we have

$$\beta_n = n^{1/m} \left(\frac{m \, q_m \, A_m}{2} \right)^{-\frac{1}{m}},$$

with

$$A_k = 4^{-k} \binom{2k}{k}. \tag{3.5}$$

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In the more general polynomial case, β_n itself has an expansion in powers of $n^{-\frac{1}{m}}$. For general Q, the MRS number is such that

$$2\pi n = \int_0^{\beta_n} Q'(x)\sqrt{\frac{x}{\beta_n - x}}\,\mathrm{d}x. \tag{3.6}$$

A full expansion of β_n can not be derived at this level of generality. However, equation (3.6) can be solved numerically, such that for each n a value of β_n can be computed. These values appear in the expansions. We refer to [25, §3] for more considerations on the computation and expansion of the MRS numbers.

In the asymptotic expansions of Laguerre-type orthogonal polynomials, the generalisation of the weight function $Q(x)$ appears in the argument of the Airy and Bessel functions, but not in the trigonometric functions that are multiplied by them, while it was the other way around in (3.4) for the Jacobi case. This is due to choices made in the derivation of the Riemann-Hilbert analysis.

Finally, the modification $Q(x)$ results in c_k and d_k coefficients that are analogous to (3.3). For the Laguerre-type case, they are

$$c_k = \frac{\beta_n}{2\pi i n}\oint \frac{\sqrt{y}Q'(\beta_n y)}{\sqrt{y-1}y^{k+1}}\,\mathrm{d}y, \tag{3.7}$$

$$d_k = \frac{\beta_n}{2\pi i n}\oint \frac{\sqrt{y}Q'(\beta_n y)}{\sqrt{y-1}(1-y)^{k+1}}\,\mathrm{d}y, \tag{3.8}$$

where the contours should enclose the interval $[0,1]$. These coefficients, too, appear in the asymptotic expansions for the nodes and weights.

4. Methodology for the asymptotic expansions of x_k and w_k

The process of inverting an expansion in order to produce expansions for its zeros is, in principle, well known. However, in practice, the computations involved are lengthy and laborious. Here, rather than detailing all computations, we illustrate the general principles. This allows to describe the structure of the expansions themselves, as well as the scope and limitations of our implementation.

We proceed formally. Consider an asymptotic expansion of the form

$$u(n,x) \sim \sum_{k=0}^{\infty} u_k(x;n)\,n^{-\mu_k}, \tag{4.1}$$

where $\{\mu_k\}_{k=0}^{\infty}$ is a strictly increasing sequence. Though the functions u_k may depend on n, it is assumed they do not grow with n, such that the truncation of the right hand side in (4.1) after T terms has an error that decays like $\mathcal{O}(n^{-\mu_{T+1}})$.

4.1. Leading order term

To leading order, the roots of $u(n,x)$ are determined by the roots of the first term $u_0(x;n)$ in (4.1). This implies that we need to find all solutions to the equation

$$u_0(x;n) = 0. \tag{4.2}$$

We enumerate the roots as t_k with $k = 1, 2, \ldots$, such that $t_k < t_{k+1}$.

The nature of this equation, and the difficulty of finding its solution, depends on the asymptotic expansion and on the special functions it contains. The leading order terms of the expansions in this paper include trigonometric functions (for the bulk regions), Bessel functions (for a hard edge) or

Airy functions (for a soft edge). That is the reason why the roots of the Bessel function, for example, appear in the expansions for the nodes x_k near the endpoints in the Jacobi-type case.

The more general modified weight functions also lead to more complicated leading order terms. This is a factor that significantly contributes to the difficulty of providing explicit asymptotic expansions for these more general cases. Explicit expressions appearing further on in this paper that specialize (4.2) include (4.4) for the bulk region in the case of Laguerre polynomials, and (6.1) and (6.5) for the bulk region for Jacobi polynomials with and without the additional function $h(x)$ in the weight.

4.2. Subsequent terms

We formally consider a full expansion of the zeros x_k of the form

$$x_k \sim \sum_{l=0}^{\infty} a_{k,l}\, n^{-\nu_l}, \tag{4.3}$$

where $\{\nu_l\}_{l=0}^{\infty}$ is a (suitably chosen) strictly increasing sequence. One subsitutes (4.3) into (4.1) and then proceeds by expanding all functions u_k around x_k. This results again in an asymptotic expansion in inverse powers of n. Next, the terms of this expansion are equated to zero one by one.

Since the functions u_k are known, the unknowns are the $a_{k,l}$ coefficients. These can be computed one by one, starting with $l = 0$. Once the value of $a_{k,l}$ has been found, it can be substituted back into the expansion. Equating the expression multiplying a higher power of n to zero yields a linear equation for the next coefficient $a_{k,l+1}$. In contrast to the first term to find t_k, for which (4.2) has to be solved, the computation of subsequent terms is a linear problem.

The above represents the general methodology. For Jacobi, in case of the lens, we use the notation

$$x_k = t_k + \sum_{l=1}^{\infty} z_{1,l+1} n^{-l},$$

and we solve recursively for the values of $z_{1,l+1}$. The expression is similar for Laguerre in the lens,

$$\frac{x_k}{\beta_n} = t_k + \sum_{l=1}^{\infty} z_{1,l+1} n^{-l},$$

where β_n is the MRS number. In case of the left boundary, we use

$$x_k = -1 + \frac{t_k}{n^2} + \sum_{l=1}^{\infty} z_{1,l+1} n^{-2-l}$$

for Jacobi and

$$\frac{x_k}{\beta_n} = \frac{t_k}{n^2} + \sum_{l=1}^{\infty} z_{1,l+1} n^{-2-l}$$

for Laguerre, where again β_n is the MRS number.

4.3. The quadrature weights

The quadrature nodes are the roots of the orthogonal polynomials and are estimated asymptotically using the procedure described above. For the quadrature weights, we recall formula (2.1). There are several alternative expressions for the weights, yet in the end they should all lead to the same

expansion. One advantage of formula (2.1) is that it is generally applicable, whereas some other known expressions may be shorter but specific to the classical polynomials.

The simplest approach is to explicitly evaluate the right hand side of (2.1) using the asymptotic expansion of the orthogonal polynomials. The derivative p'_n can be evaluated by differentiating an expansion for p_n term by term. The shifted polynomial p_{n-1} has an expansion in powers of $(n-1)$ rather than n, but it can also readily be evaluated. Still, we have proceeded to derive explicit expansions for the weights directly.

Thus, we substitute the previously computed expansion (4.3) of the node into (2.1) and re-expand the entire formula. We take into account the following considerations:

- We obtain an asymptotic expansion of p_{n-1} by re-expanding $(n-1)^{-1}$ in inverse powers of n.

- An expansion for p'_n is obtained by differentiating p_n term by term. However, here, some efficiency can be gained. The expansions typically have the form $p_n(x) \sim E_n(x)\varepsilon(x)$, where $E_n(x)$ is a pre-factor without roots and $\varepsilon(x)$ is the term that vanishes at the roots of p_n. Using the fact that x_k is a root of p_n, we find that

$$p'_n(x_k) = E'_n(x_k)\varepsilon(x_k) + E_n(x_k)\varepsilon'(x_k) = E_n(x_k)\varepsilon'(x).$$

  Thus, we can avoid having to compute the (expansions of the) derivatives of the prefactor E_n.

- The weight function evaluated at the node, $w(x_k)$, is a common factor in all terms and is factored out once for the final expression.

4.4. Explicit expansions and explicit convolutions

The steps that lead to the expansions of the nodes and weights involve a great number of expansions and re-expansions. Many of these take the form of convolutions: this occurs each time an expansion is applied to a variable that is itself an expansion, or whenever two expansions are multiplied.

In principle, this process is greatly simplified with the aid of computer algebra software. However, any use of such software prohibits an implementation of the derivations of the expansions in a non-symbolic language, though of course the final result can be copied and implemented in any language. Still, even then, the length of the expressions leads to a substantial practical limitation on the number of terms that can be computed.

An alternative is to restrict ourselves to cases where computations can be done analytically. If succesful, this has the advantage of leading to exact expressions in analytic form, for which possibly arbitrary many terms can be generated on the fly. Yet, this strongly restricts the scope of the methodology to cases where analytical derivations are feasible. This rules out any of the modifications of the more general weight functions under consideration.

For these reasons, we have upheld the following two principles in our implementation:

- Each expansion of a special function (such as $\sqrt{1+z}$ around $z = 0$ or $\frac{1}{n+1}$ in terms of n^{-k}) is derived analytically and the resulting formulas are implemented manually. (Hence, we do not rely on the capability of the software to produce series expansions.)

- Each convolution is programmed explicitly. Here, too, we do not rely on the algebraic software to recombine, e.g., products of power series into a single power series.

These principles make the implementation much more efficient, compared to relying on the symbolic manipulations of a software package.

Having said that, we do compute expansions that contain parameters such as, e.g., α and β for the Jacobi-case, α for the Laguerre case, and the c_n and d_n coefficients given by (3.3) in the case of modified weight functions. If these parameters do not have numeric values, we do currently use the capabilities of a symbolic software package (SAGE) to manipulate the resulting expressions. When translated into an imperative programming language, all the parameters should have numerical values.

4.5. Transcendental equation for the bulk in the Laguerre case

For the classical Laguerre case, or Laguerre-type case with $Q(x) = x$, equation (4.2) for the leading order term of the nodes becomes the transcendental equation [35]

$$2\arccos(\sqrt{t}) - 2\sqrt{t - t^2} - \pi p = 0 \tag{4.4}$$

with

$$p = \frac{4n - 4k + 3}{4n + 2\alpha + 2}. \tag{4.5}$$

We use an iterative Newton procedure, with starting values

$$t_k \approx \frac{\pi^2}{16}(p - 1)^2.$$

This approximation arises from the series expansion of the left hand side of (4.4) near $t = 0$, which yields

$$p\pi = \pi - 4\sqrt{t} + \frac{2}{3}t^{3/2} + \mathcal{O}(t^{5/2}).$$

Heuristically, we have observed that 6 iterations are sufficient for double precision accuracy for all k in the bulk region.

The exact form of the transcendental equation results from choices in the Riemann-Hilbert analysis, and could appear differently by selecting another function for splitting the lens. However, the rootfinding problem is numerically straightforward, as for each value of k there is a unique simple root in $(0, 1)$. This is shown in the following lemma.

Lemma 4.1. *The left hand side of equation (4.4) has a single simple root in the interval $(0, 1)$ for each k.*

Proof. We denote (4.4) as $F(t) = 0$. The derivative is

$$F'(t) = -(t - t^2)^{-1/2} - \frac{1 - 2t}{\sqrt{t - t^2}} = -2\sqrt{\frac{1 - t}{t}},$$

which is strictly negative, so $F(t)$ is monotonically decreasing on $(0, 1)$. The limiting values of F at the endpoints are the maximum

$$F(0) = 2\arccos(0) - 2\sqrt{0} - p\pi = \pi (1 - p),$$

with p as in (4.5), and the minimum

$$F(1) = -p\pi = \frac{4n - 4k + 3}{4n + 2\alpha + 2}\pi.$$

Since $k \in [1, n]$, the numerator $4n - 4k + 3$ lies in $[3, 4n - 1]$. Moreover, since $\alpha > -1$ the denominator $4n + 2\alpha + 2$ is larger than $4n$. As a result, the maximum is always strictly positive and the minimum is strictly negative, hence there is a single simple zero for $t \in (0, 1)$. $\qquad\square$

5. Gauss–Laguerre rules

5.1. Standard associated Gauss–Laguerre

The weight function in the standard case of associated Gauss–Laguerre is $w(x) = x^\alpha e^{-x}$. We have computed asymptotic expansions in terms of inverse powers of n. However, the formulae are shorter in terms of inverse powers of $(4n + 2\alpha + 2)$ and for that reason they are presented that way in this paper. Explicit expressions are listed in Appendix A.

The higher order terms of the expansions contain large integer coefficients with alternating signs, which may form a possible source of cancellation errors. Still, all terms multiply an inverse power of n where the exponent decreases as those integers increase, and therefore any cancellation error will decay accordingly with increasing n. Note that this problem is not specific to asymptotic expansions for Gauss–Laguerre quadrature, it is a rather general problem with expansions.

5.2. Heuristical choices

In existing implementations of our approach in [31, 36] we switch to asymptotics for $n \geq 128$, since that yields accurate results to machine precision in double precision floating point arithmetic. For smaller values of n, we suggest to use the Golub-Welsch algorithm. If higher accuracy is required, the lower bound for n should be increased and high-precision arithmetic may be necessary. We note that small weights may be computed with some relative accuracy using asymptotic expansions or iterative methods, but typically not using Golub-Welsch. Also, large values of α may lead to lower accuracy. Accurate results are guaranteed only when α remains bounded as n increases.

The simplest heuristic procedure is to sum terms of the asymptotic expansion as long as they decay, and to truncate the sum once the terms start to grow. Experiments indicate this works well. However, in the absence of known error estimates to provide mathematical justification of the truncation, we also embark on a more in depth discussion of truncation of the expansions in different regimes.

First, we limit the number of nodes to compute based on the size of the weights, which may underflow in floating point precision. Indeed, the weight w_k behaves as $w(x_k) = x_k^\alpha e^{-Q(x_k)}$. From (4.4) we find, after some calculations, that underflow at a threshold ϵ happens in the monomial case $Q(x) = q_m x^m$ when

$$k < \frac{1}{4} - \frac{\alpha}{2} + \frac{(2n + \alpha + 1)}{\pi(2m - 1)} \left(\frac{-mA_m \log \epsilon}{2n} \right)^{\frac{1}{2m}},$$

where A_m is given by (3.5). We simplify this expression for $\epsilon = 10^{-308}$ to the heuristical choice

$$k \leq \min(n, \exp[1.05 e^{\frac{1}{m}}] n^{1 - \frac{1}{2m}}).$$

This agrees with the $\mathcal{O}(\sqrt{n})$ subsampling complexity employed in [37] for standard Hermite. In the standard Laguerre case, our heuristic becomes

$$k \leq \min(n, 17\sqrt{n}).$$

A second heuristic concerns the number of terms T. Note that the leading order terms for the hard edge and the bulk in § 5.1 only give approximately double precision for $n > 10^8$. One can not simply use all available terms for all n, as the asymptotic expansion diverges for fixed n and increasing T. Yet, since we adopt a minimal value of n, divergence is less of an issue for a certain predetermined accuracy. On the other hand, T may actually decrease as n increases, because fewer terms may be required to reach machine precision. More specifically, when n squares, the number
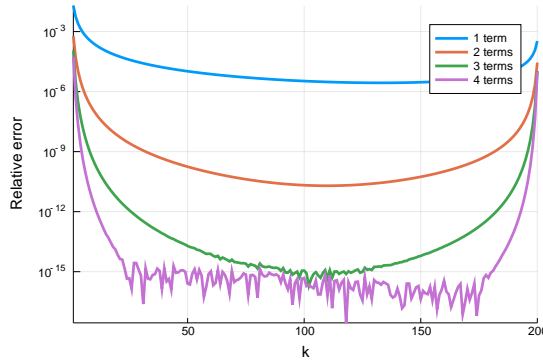
Figure 2: Relative error of the nodes in the bulk (A.2) for $n = 200$ and $w(x) = x^{0.7}e^{-x}$ on $[0, \infty)$. The accuracy of the expression for the bulk deteriorates near the endpoints, hence the need for special boundary asymptotics.

of terms can halve. Thus, T depends inversely logarithmically on n. In order to obtain about 10^{-16} relative error at $n = 128$, we need eight terms. This leads us to our heuristical choice

$$T = \lceil 34/\log(n) \rceil. \tag{5.1}$$

A third heuristic involves the choice of the region, i.e., which expansion to use – recall the domains shown in Figure 1. The sizes of the disks near the endpoints are flexible, as mentioned before. We transition between expansions at the values k^{left} and k^{right}. The optimal values of the transition points depend on T, n, α and, more generally, on the weight function $w(x)$. At least for the standard associated Laguerre weight, the dependency on the parameters appears mild in comparison to the dependency on n. Experiments suggest the following scaling:

$$k^{\text{left}} = \lceil \sqrt{n} \rceil, \qquad \text{and} \qquad k^{\text{right}} = \lfloor 0.9n \rfloor. \tag{5.2}$$

This corresponds approximately to the points $x_{k^{\text{left}}} \approx \frac{\pi^2}{4}$ and $x_{k^{\text{right}}} \approx 3.6n$.

5.3. Accuracy

We validate the heuristics of § 5.2 by comparing to a reference solution computed using the recurrence relation in high-precision arithmetic. In Figure 2, we show the relative error for the nodes in the asymptotic expansion (A.2) for the bulk region. The relative error reaches machine precision accuracy in the bulk when using four terms, which corresponds to $\mathcal{O}(n^{-8})$ error. The nodes quickly become less accurate near the left and right endpoints, which is the reason why our heuristics in § 5.2 switch to another asymptotic expansion there.

In Figure 3, we show the error of the explicit expansions of the weights (A.1), (A.3) and (A.4) up to relative order $\mathcal{O}(n^{-2})$, $\mathcal{O}(n^{-4})$, $\mathcal{O}(n^{-6})$ and $\mathcal{O}(n^{-8})$ combined with the heuristic (5.2) at $n = 200$. As expected, increasing the number of terms in those expansions decreases the error.

The expansions of the weights match closely when we switch from the left disk to the bulk near $k^{\text{left}} = 15$. A steep jump in the relative error is seen near $k^{\text{right}} = 180$ when we switch to Airy asymptotics. This is because the expansion of the weight in the latter region (A.4) is only accurate up to $\mathcal{O}(n^{-2/3})$. However, the first weight in this regime has size $w_{180} \approx 9 \times 10^{-222}$, hence the absolute error is very small. It is an interesting property of asymptotic expansions that such a small weight can still be computed with some relative accuracy. This property is seen also with some iterative methods in which individual points and weights can be computed, but not with the Golub-Welsch algorithm as the latter computes several points and weights at once – hence their relative difference in size can not be too large.
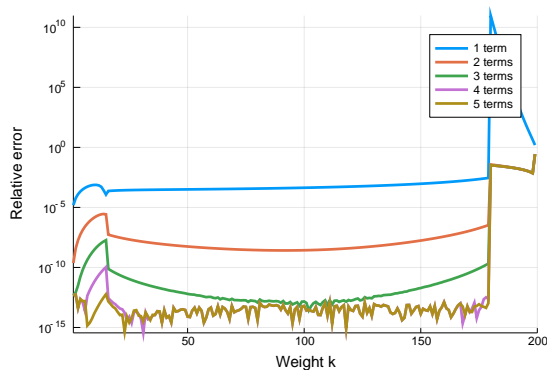
Figure 3: Relative error of the weights for $n = 200$ and $w(x) = x^{0.7}\mathrm{e}^{-x}$ on $[0, \infty)$. At $k = 0.9n = 180$ we switch to Airy asymptotics with fewer terms and seemingly less accuracy (visible as the spike in the figure). Yet, because $w_{180} < 10^{-220}$ in this case, the absolute error is still extremely small.

5.4. Modified Gauss–Laguerre

We do not pursue all expansions in the full generality of the modification of the weight function (2.3), i.e., for all possible analytic functions $Q(x)$. In [25] we considered asymptotic expansions for the polynomials when $Q(x)$ is a monomial, a general polynomial of degree m, or a more general analytic function on the positive halfline for which the orthogonal polynomials exist. In principle, these expansions could be used to generate asymptotic expansions for the corresponding Gaussian quadrature rules. However, the level of complexity of the computation increases.

We briefly elaborate on some of the issues. First, the procedure for general polynomial $Q(x)$ yields fractional powers of n, due to expanding the MRS number β_n in fractional powers of n as shown in [25, §3]. This increases the complexity of obtaining full asymptotic expansions quite substantially, as the computation of not one but several terms is required to increase the order by $1/n$.

Next, general functions $Q(x)$ require the calculation of contour integrals. Recall that a weight of the form $w(x) = x^{\alpha}\mathrm{e}^{-Q(x)}$ leads to the coefficients c_n and d_n that were given by contour integrals (3.7)–(3.8). These contour integrals have to be computed, either analytically or numerically, since the coefficients appear in the expansions. One advantage of the monomial case $Q(x) = x^m$ is that the contour integrals (3.7)–(3.8) can be evaluated analytically for all m. Other examples where these contour integrals can be found analytically are given in [25, §3.4] and [1, (2.19)]. Similar contour integrals also appear in the formula for the phase function in the leading order term of the expansion of the orthogonal polynomial. Expressions are given in [25, §6], and they are the analogue of (3.2) for the Jacobi case.

Finally, since we have to numerically solve for the roots of the leading order term, recall (4.2), with the leading order term for general $Q(x)$ itself involves requiring the evaluation of contour integrals, producing expansions may be quite time-consuming.

For these reasons, the expansions for modified Laguerre that we list explicitly in this paper are somewhat limited. Still, for a specific choice of weight function, the code may be used to produce expansions.

Hard edge. The expansion of Laguerre-type nodes near the left endpoint is

$$x_k = \frac{16 j_{\alpha,k}^2 \beta_n}{d_0^2 (4n + 2\alpha + 2)} \left(1 + \frac{4[\alpha + 1][d_0 - 2]}{d_0 (4n + 2\alpha + 2)} \right. \tag{5.3}$$

$$+ \frac{4}{3 c_0^2 d_0^3 (4n + 2\alpha + 2)^2} \left[9(\alpha^2 + 2\alpha + 1) c_0^2 d_0^3 + 2(22\alpha^2 + 36\alpha + 17) c_0^2 d_0 - (4\alpha^2 - 1) c_0^2 d_1 \right.$$
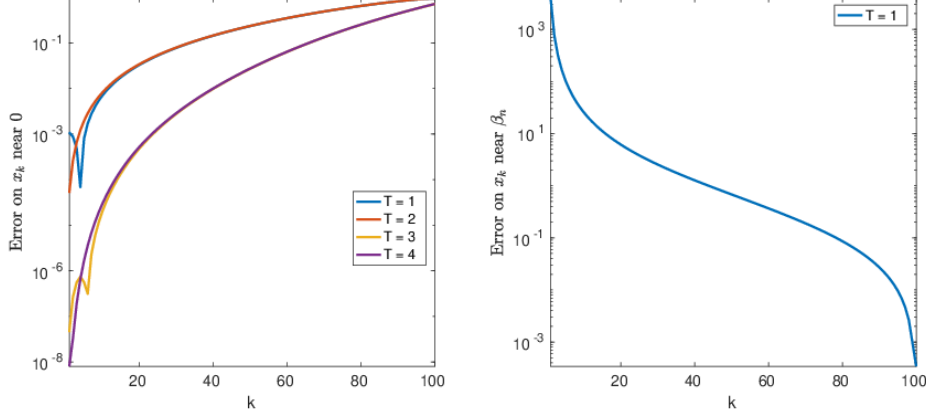
14

Figure 4: Relative error of the explicit expansions of the nodes (5.3) near the left endpoint (left panel) and (5.4) near the soft edge (right panel) for the weight function $w(x) = \mathrm{e}^{-\mathrm{e}^x}$ on $[0, \infty)$ and $n = 100$.

$$
- 3(12[\alpha^2 + 2\alpha + 1]c_0^2 - 2[2\alpha^2 + 4\alpha + 1]c_0 + c_1)d_0^2 + 4(c_0^2 d_0 - 2c_0^2 d_1)j_{\alpha,k}^2 \Big]
$$

$$
+ \frac{16}{3c_0^4 d_0^4 (4n + 2\alpha + 2)^3} \Big[6(\alpha^3 + 3\alpha^2 + 3\alpha + 1)c_0^4 d_0^4 - 3(28\alpha^3 + 60\alpha^2 + 45\alpha + 13)c_0^4 d_0
$$

$$
- 3(12\{\alpha^3 + 3\alpha^2 + 3\alpha + 1\}c_0^4 - 4(2\alpha^3 + 6\alpha^2 + 5\alpha + 1)c_0^3
$$

$$
+ 2(2\alpha^3 + 6\alpha^2 + 3\alpha - 1)c_0^2 - 6(\alpha + 1)c_1^2 + 5(\alpha + 1)c_0 c_2 + \{2(\alpha + 1)c_0^2 + (4\alpha^3 + 12\alpha^2 + 5\alpha - 3)c_0\}c_1)d_0^3
$$

$$
+ \big(4[22\alpha^3 + 58\alpha^2 + 53\alpha + 17]c_0^4 - 3[16\alpha^3 + 40\alpha^2 + 29\alpha + 5]c_0^3 + 9[\alpha + 1]c_0^2 c_1 \big)d_0^2
$$

$$
+ 8\big([\alpha + 1]c_0^4 d_0^2 - 3(\alpha + 1)c_0^4 d_0 - [2(\alpha + 1)c_0^4 d_0 - 5(\alpha + 1)c_0^4]d_1\big)j_{\alpha,k}^2
$$

$$
- \big(2[4\alpha^3 + 4\alpha^2 - \alpha - 1]c_0^4 d_0 - 5[4\alpha^3 + 4\alpha^2 - \alpha - 1]c_0^4\big)d_1 \Big] + \mathcal{O}(n^{-4}).
$$

Bulk. Based on the asymptotic expansions in [25, §3.4], equation (4.2) for the leading order specializes to

$$
\pi(4k - 4n - 3) + (\alpha + 1)2\arccos(2t - 1) + \frac{n}{4i}\int_1^t \frac{\sqrt{y-1}}{\sqrt{y}}\left[\frac{1}{2\pi i}\oint_{\Gamma_y}\frac{\beta_n\sqrt{x}Q'(x)\mathrm{d}x}{n\sqrt{x-1}(x-y)}\right]\mathrm{d}y = 0.
$$

The contour Γ_y should enclose the interval $[0, 1]$ and the point y.

Unfortunately, this expression does not simplify even in the monomial case.

Soft edge. We only provide the leading order term near the soft edge, as the corresponding weights underflow even for moderate n. We have

$$
x_k \sim \beta_n\left[1 + \left(\frac{2}{nc_0}\right)^{2/3}a_{n-k+1} + o(n^{-2/3})\right] \tag{5.4}
$$

and

$$
w_k^{-1} \sim 2\beta_n^{n-1}\beta_{n-1}^{1-n-\alpha-1}\exp\left[nl_n/2 - (n-1)l_{n-1}/2\right]\left(\frac{\beta_n - \beta_{n-1}}{\beta_{n-1}} + \left(\frac{2}{nc_0}\right)^{2/3}a_{n-k+1}\right)^{-1/4}
$$

$$
\exp[Q(x_k)]Ai'(a_{n-k+1})^2\,(c_0/2)^{3/2}\,n^{1/2}a_{n-k+1}^{1/4}.
$$

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Here, recall that a_m are the roots of the Airy function. The constant l_n is given for general Laguerre-type weight functions in [25, §3.2] and also involves an integral to be evaluated numerically.

In Figure 4, we can see that the expansions near the hard and the soft edge give good approximations. Moreover, increasing the number of terms T near the hard edge decreases the error, with results shown using $n = 100$ for the nodes of a Gauss–Laguerre-type quadrature rule with respect to the doubly-exponential weight function

$$w(x) = \exp(-e^x).$$

In that case we have
$$\beta_n \sim \log(n) - \log(\log(8\pi n^2))/2 + \log(8\pi)/2.$$

6. Gauss–Jacobi rules

6.1. Standard Gauss–Jacobi

The polynomials exhibit Bessel-like behaviour near both hard edges at $+1$ and -1. Similar to the Laguerre case, the expansions were computed in terms of inverse powers of n, but the results are presented in shorter form using inverse powers of $(2n + \alpha + \beta + 1)$ (in part following the notation of [38]). The expansion for the nodes x_k near $x = -1$ are given in (A.5) up to $\mathcal{O}(n^{-12})$. The corresponding weights w_k are given up to relative order $\mathcal{O}(n^{-8})$, relative to the size of $w(x_k)$, in (A.6). Near the right endpoint at $x = +1$, we can simply interchange α and β in these expressions, and multiply x_{n+1-k} by -1 for $k = 1, \ldots, n$.

For the nodes and weights in the bulk, we first have to find the roots of the leading order term of the polynomials. In the modified Jacobi case the generic equation (4.2) specializes to (6.5) further on. In the current classical case, there is no contour integral and the roots are obtained with the explicit expression

$$t_k = \cos\left(\pi \frac{4n - 4k + 2\alpha + 3}{4n + 2\alpha + 2\beta + 2}\right). \tag{6.1}$$

The expansion of the nodes up to $\mathcal{O}(n^{-10})$ and weights up to (relative order) $\mathcal{O}(n^{-8})$ are given in (A.7) and (A.8) respectively.

Sometimes the change of variables $x = \cos\theta$ is applied and expansions are obtained in terms of θ [39]. This can also improve numerical accuracy near ± 1 in other approaches [8]. We observed no numerical issues near the endpoints in our code, but do not rule out that for very large values of n the change of variables $x = \cos\theta$ may indeed be beneficial.

6.2. Heuristical choices

We are faced with similar choices as in the Laguerre case in § 5.2, namely the choice of the number of terms T in the expansion and the choice of the expansion to use. Due to the difference in nature with Gauss-Laguerre and because we have fewer terms available in the current case, we adjust our assumption for n: we suggest to use the recurrence relation up to about $n \leq 300$ for double precision accuracy. As in the case of Laguerre, accurate results can be guaranteed only if α and β remain bounded as n increases. Since there is no underflow associated with the weights of Gauss–Jacobi at moderate n, we compute all nodes and weights.

Based on our aim of achieving machine precision in double precision accuracy, we heuristically choose
$$T = \lceil 50/\log(n)\rceil.$$

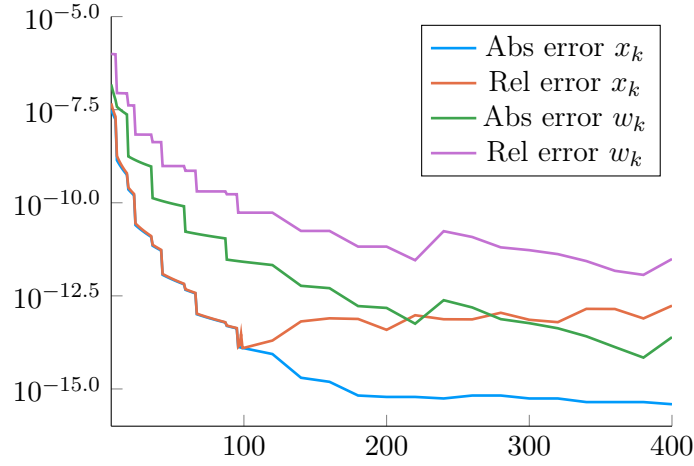This is slightly higher than the corresponding choice (5.1) in the Laguerre case.

Figure 5: Maximal absolute and relative error of the points and weights of all Jacobi quadrature rules for n up to 400 for the weight function $w(x) = (1-x)^{0.42}(1+x)^{-1/\sqrt{5}}$.

For the switch between the asymptotic regimes, we have settled on the choices

$$k^{\text{left}} = \lceil \sqrt{n} \rceil, \qquad \text{and} \qquad k^{\text{right}} = n - \lceil \sqrt{n} \rceil. \tag{6.2}$$

This corresponds to switching approximately at the points $x_{k^{\text{left}}} = \frac{\pi^2}{2n} - 1$ and $x_{k^{\text{right}}} = 1 - \frac{\pi^2}{2n}$.

6.3. Accuracy

In Figure 5, we show the maximal absolute or relative error over all nodes or weights at each integer value of n between 1 and 400. The asymptotic expansions are compared to a computation of the corresponding Gaussian quadrature rule using the recurrence relation in higher precision arithmetic. This allows to assess the accuracy to all digits.

6.4. Modified Gauss–Jacobi

For the sake of brevity, we only give the first few terms and indicate how many were computed in our implementation. Recall that these expressions are symbolic because they contain the parameters α and β. When numeric values for α and β are supplied, more terms can be readily computed with modest effort. In contrast to the case of modified Laguerre, the results here are quite complete.

Hard edge at $x = -1$. The expansions near the left endpoint $x = -1$ are

$$x_k \sim -1 + \frac{2j_{\beta,k}^2}{(2n+\alpha+\beta+1-d_0)^2} + \frac{-2j_{\beta,k}^2}{3(2n+\alpha+\beta+1-d_0)^4} \left[j_{\beta,k}^2 - 3\alpha^2 - \beta^2 + 1 \right] \tag{6.3}$$

$$+ \frac{-j_{\beta,k}^2}{6(2n+\alpha+\beta+1-d_0)^5} \left[16(d_0 - 3d_1)j_{\beta,k}^4 + 3(4\alpha^2 - 1)c_0 + (12\alpha^2 + 8\beta^2 - 5)d_0 - 6(4\beta^2 - 1)d_1 \right]$$

$$+ \dots + \mathcal{O}(n^{-9}).$$

The corresponding weights are asymptotically

$$\frac{w_k}{w(x_k)} \sim \frac{8}{J_{\beta-1}^2(j_{\beta,k})[2n+\alpha+\beta+1-d_0]^2} \tag{6.4}$$

$$+ \frac{8}{3J_{\beta-1}^2(j_{\beta,k})[2n+\alpha+\beta+1-d_0]^4} \left[3\alpha^2 + \beta^2 - 1 - 2j_{\beta,k}^2 \right]$$

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$$-\frac{2\left[32(d_0 - 3d_1)j_{\beta,k}^2 + 3(4\alpha^2 - 1)c_0 + (12\alpha^2 + 8\beta^2 - 5)d_0 - 6(4\beta^2 - 1)d_1\right]}{3J_{\beta-1}^2(j_{\beta,k})[2n + \alpha + \beta + 1 - d_0]^5}$$

$$+ \ldots + \mathcal{O}(n^{-8}).$$

Hard edge at $x = +1$. Near the right endpoint at $x = +1$, we can simply interchange α and β in the expressions above, and multiply x_{n+1-k} by -1 for $k = 1, \ldots, n$. This implies that one has to substitute $h(-x)$ for $h(x)$ when recomputing the expansions coefficients c_n and d_n.

Bulk. For the expansions in the bulk, equation (4.2) for the parameter t_k becomes

$$\pi\frac{4k + 2\alpha + 3}{4n + 2\alpha + 2\beta + 2} = \arccos(t_k) + \frac{\sqrt{1 - t_k^2}}{2n + \alpha + \beta + 1}\frac{1}{2\pi i}\oint_\gamma \frac{\log h(\zeta)\mathrm{d}\zeta}{\sqrt{\zeta^2 - 1}(\zeta - t_k)}. \tag{6.5}$$

This corresponds to the expression in [38], with a minor modification of the denominator and the addition of a contour integral for our generalised case.

One needs to compute the series expansion coefficients of the latter:

$$\frac{1}{2\pi i}\oint_\gamma \frac{\log h(\zeta)\mathrm{d}\zeta}{\sqrt{\zeta^2 - 1}(\zeta - z)} \sim \sum_{i=0}^{\infty} h_i(z - t_k)^i.$$

In the absence of an analytical approach for the evaluation of these integrals, they have to be computed numerically for each k. Examples of cases where they can be evaluated analytically are given in [24, §6.1].

Still, in terms of the coefficients h_i, the expansion for the nodes in the bulk region is, up to $\mathcal{O}(n^{-5})$:

$$x_k \sim t_k + \frac{2\alpha^2 - 2\beta^2 + (2\alpha^2 + 2\beta^2 - 1)t_k}{2[2n + \alpha + \beta + 1 + h_0]^2} - \frac{1}{4[2n + \alpha + \beta + 1 + h_0]^3}\left(2(2\alpha^2 + 2\beta^2 - 1)h_1 t_k^3 \right. \tag{6.6}$$

$$+ 2\left[(2\alpha^2 + 2\beta^2 - 1)h_0 + 2(\alpha^2 - \beta^2)h_1\right]t_k^2 + (4\alpha^2 - 1)c_0 + (4\beta^2 - 1)d_0 + 8(\alpha^2 - \beta^2)h_0 - 4(\alpha^2 - \beta^2)h_1$$

$$+ \left.\left[(4\alpha^2 - 1)c_0 - (4\beta^2 - 1)d_0 + 4(3\alpha^2 + \beta^2 - 1)h_0 - 2(2\alpha^2 + 2\beta^2 - 1)h_1\right]t_k\right) + \ldots + \mathcal{O}(n^{-5}).$$

The corresponding weights are, up to $\mathcal{O}(n^{-4})$ relative error (relative to $w(x_k)$):

$$\frac{w_k}{w(x_k)} \sim \frac{\pi\sqrt{1 - t_k^2}}{2n + \alpha + \beta + 1}\left[2 - \frac{2h_1(1 - t_k^2) - 2h_0 t_k}{2n + \alpha + \beta + 1} + \frac{1}{(2n + \alpha + \beta + 1)^2}\left(2h_1^2 t_k^4 + 4h_0 h_1 t_k^3 - 4h_0 h_1 t_k\right.\right.$$

$$+ 2(h_0^2 - 2h_1^2)t_k^2 + 2\alpha^2 + 2\beta^2 + 2h_1^2 - 1\left.\left.\right) + \ldots + \mathcal{O}(n^{-4})\right]. \tag{6.7}$$

We illustrate the accuracy of these expansions throughout the domain $[-1, 1]$ in Figure 6 for $n = 200$ for the weight function

$$w(x) = (1 - x)^{1/\sqrt{3}}(1 + x)^{-1/\pi}\mathrm{e}^x.$$

The results show that the expansions improve with T and that their accuracy is very high in the regions where they are valid.
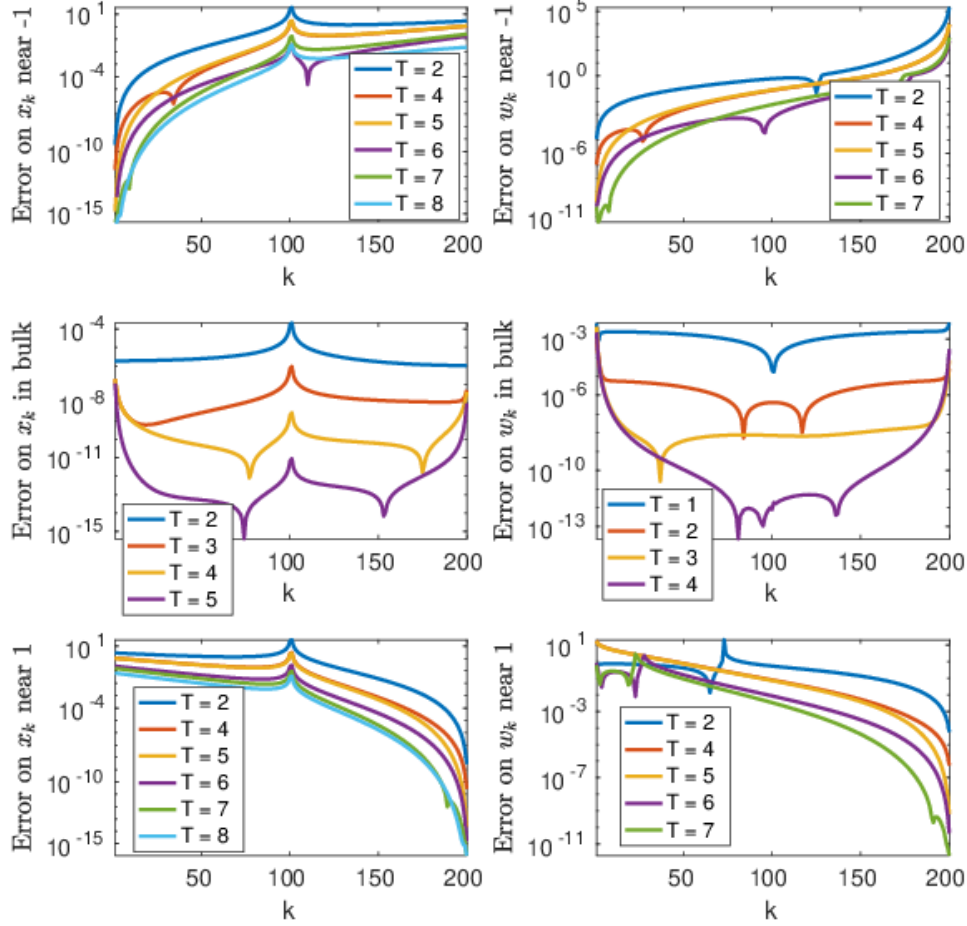
Figure 6: Relative error of the explicit expansions of the nodes (6.3) and weights (6.4) near the left endpoint (top), and the nodes (6.6) and weights (6.7) in the bulk (middle) for the weight function $w(x) = (1-x)^{1/\sqrt{3}}(1+x)^{-1/\pi}e^x$ at $n = 200$ for a varying number of terms T. The expressions for the left endpoint are reused for those near the right endpoint (bottom) by interchanging α and β and computing c_n and d_n with $h(x) = e^{-x}$.

7. Gauss–Hermite rules

The standard Gauss–Hermite polynomials are related to associated Gauss–Laguerre polynomials with $\alpha = \mp 1/2$ [34, 18.7.19-20]. Written in terms of the orthonormal polynomials, we have the relations

$$p_{2n}^{(H)}(x) = p_n^{(L,-\frac{1}{2})}(x^2), \qquad p_{2n+1}^{(H)}(x) = x p_n^{(L,\frac{1}{2})}(x^2).$$

This generalises to other functions $Q(x)$ with $\alpha = \mp 1/2$ as was shown in [25, §7.2]. Using the change of variables $x = t^2$, we have

$$\int_0^\infty x^j\, p_n(x)\, x^{-\frac{1}{2}}e^{-Q(x)}\, \mathrm{d}x = 2\int_0^\infty t^{2j}\, p_n(t^2)e^{-Q(t^2)}\, \mathrm{d}t = \int_{-\infty}^\infty t^{2j}\, p_n(t^2)e^{-Q(t^2)}\, \mathrm{d}t = 0, \quad j = 0, \ldots, 2n-1.$$

$$(7.1)$$

Since the odd moments of $p_n(t^2)$ vanish by symmetry, it follows that $p_n(t^2)$ is the Hermite-type polynomial of degree $2n$ with respect to the weight function $e^{-Q(t^2)}$ on the real line.

Similarly, for the odd-degree polynomials we have

$$\int_0^\infty x^j \, p_n(x) \, x^{\frac{1}{2}} \mathrm{e}^{-Q(x)} \, \mathrm{d}x = \int_{-\infty}^\infty t^{2j+1} \, t \, p_n(t^2) \mathrm{e}^{-Q(t^2)} \, \mathrm{d}t = 0, \quad j = 0, \dots, 2n-1.$$

In this case, the even moments vanish due to the odd symmetry of $p_n(t^2)$.

It remains to determine the connection between the corresponding Gaussian quadrature nodes and weights.

Lemma 7.2. *Let $x_k^{(L,-\frac{1}{2})}$ and $w_k^{(L,-\frac{1}{2})}$, $k = 1, \dots, n$, be the nodes and weights for the modified Gauss–Laguerre rule with weight function $x^{-\frac{1}{2}}\mathrm{e}^{-Q(x)}$. Then the nodes and weights of the modified Gauss–Hermite rule with weight function $\mathrm{e}^{-Q(x^2)}$ on $(-\infty, \infty)$ are, with $k = 1, \dots, n$,*

$$x_{n+k}^{(H)} = \sqrt{x_k^{(L,-\frac{1}{2})}}, \quad w_{n+k}^{(H)} = \frac{w_k^{(L,-\frac{1}{2})}}{2}, \quad x_{n+1-k}^{(H)} = -\sqrt{x_k^{(L,-\frac{1}{2})}}, \quad w_{n+1-k}^{(H)} = \frac{w_k^{(L,-\frac{1}{2})}}{2}.$$

Proof. The result for the nodes follows immediately from the change of variables in (7.1). The results for the weights follow by applying the same change of variables to the moment conditions of the Laguerre-type quadrature rule. Since

$$\int_0^\infty x^j x^{-\frac{1}{2}} \mathrm{e}^{-Q(x)} \, \mathrm{d}x = \sum_{k=1}^n w_k^{(L,-\frac{1}{2})} \left(x_k^{(L,-\frac{1}{2})} \right)^j, \qquad j = 0, \dots, 2n-1,$$

we have

$$\int_{-\infty}^\infty t^{2j} \mathrm{e}^{-Q(t^2)} \, \mathrm{d}t = \sum_{k=1}^n w_k^{(L,-\frac{1}{2})} \left(x_k^{(L,-\frac{1}{2})} \right)^j = \sum_{k=1}^{2n} w_k^{(H)} \left(x_k^{(H)} \right)^{2j}, \qquad j = 0, \dots, 2n-1.$$

The moment conditions for the odd monomials t^{2j+1}, $j = 0, \dots, 2n-1$, follow from the odd symmetry. $\qquad\square$

The result for Gauss–Hermite type rules with an odd number of points is similar, with some complications only for the extra node at the origin.

Lemma 7.3. *The nodes and weights of a $2n+1$ point modified Gauss–Hermite quadrature rule are given in terms of the modified Gauss–Laguerre quadrature rule with $\alpha = +\frac{1}{2}$ as, with $k = 1, \dots, n$,*

$$x_{n+1+k}^{(H)} = \sqrt{x_k^{(L,\frac{1}{2})}}, \qquad w_{n+1+k}^{(H)} = \frac{w_k^{(L,\frac{1}{2})}}{2x_k^{(L,\frac{1}{2})}},$$

$$x_{n+1}^{(H)} = 0, \qquad w_{n+1}^{(H)} = \int_{-\infty}^\infty \mathrm{e}^{-Q(t^2)} \, \mathrm{d}t - \sum_{k=1}^n \left[w_{n+1+k}^{(H)} + w_{n+1-k}^{(H)} \right],$$

$$x_{n+1-k}^{(H)} = -\sqrt{x_k^{(L,1/2)}}, \qquad w_{n+1-k}^{(H)} = \frac{w_k^{(L,\frac{1}{2})}}{2x_k^{(L,\frac{1}{2})}}.$$

Proof. The moment conditions for the modified Laguerre rule are

$$\int_0^\infty x^j x^{\frac{1}{2}} \mathrm{e}^{-Q(x)} \, \mathrm{d}x = \int_{-\infty}^\infty t^{2j+2} \, \mathrm{e}^{-Q(t^2)} \, \mathrm{d}t = \sum_{k=1}^n w_k^{(L,\frac{1}{2})} \left(x_k^{(L,\frac{1}{2})} \right)^j, \qquad j = 0, \dots, 2n-1.$$

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The modified Gauss–Hermite rule should satisfy

$$\int_{-\infty}^{\infty} t^m \, e^{-Q(t^2)} \, dt = \sum_{k=1}^{2n+1} w_k^{(H)} \left(x_k^{(H)} \right)^m, \qquad m = 0, \ldots, 4n+1.$$

The odd moments are satisfied by symmetry. The even moments are satisfied for $m = 2j + 2 > 0$ using the exactness conditions for Laguerre above, noting that $\left(x_{n+1}^{(H)} \right)^m = 0$ since $m > 0$. The moment condition for $m = 0$ is satisfied precisly by the weight formulated in the lemma. $\qquad\square$

There is no known general formula for the middle weight $w_{n+1}^{(H)}$. However, in case of the standard Gauss–Hermite rule once can find from [40, §3.5 & 18.6] that

$$w_{n+1}^{(H)} = \frac{\pi \Gamma(n+1)}{(2n+1)\Gamma(n+\frac{1}{2})}.$$

For large n, the ratio of Gamma functions can be approximated much like in [12, §3.2.3] as

$$\frac{\Gamma(n+1)}{\sqrt{n}\Gamma(n+1/2)} \sim 1 + \frac{1}{8n} + \frac{1}{128n^2} - \frac{5}{1024n^3} - \frac{21}{32768n^4} + \frac{399}{262144n^5} + \frac{869}{4194304n^6}$$
$$- \frac{39325}{33554432n^7} - \frac{334477}{2147483648n^8} + \frac{28717403}{17179869184n^9} + \frac{59697183}{274877906944n^{10}}, \quad n \to \infty.$$

In order to construct the Gauss-Hermite rule, one simply computes the Gauss-Laguerre rules from § 5.1 with approximately half the number of points and then exploits these relations.

The connection to Laguerre-type polynomials seemingly obviates the need to repeat the asymptotic analysis for Hermite-type polynomials, at least for even symmetric weight functions of the form $e^{-Q(x^2)}$. It may still be worthwhile to repeat the analysis for symmetric weight functions, since the Hermite-type polynomials do not have a hard edge at $x = 0$. Hence, the computation of their expansion is simpler and it becomes more tractable to compute more terms. For example, for symmetric weight functions and $\alpha = \pm 1/2$ the $U_{k,m}^{\text{left}}$ matrices in [21, 25] are zero. The analysis may also be useful for non-symmetric weight functions and for algebraic singularities near zero, i.e. $\alpha \neq \pm 1/2$.

Acknowledgements

Appendix A. Expansions for classical Gaussian quadrature rules

We include explicit expressions for the asymptotic expansions of nodes and weights of Gauss–Laguerre and Gauss–Jacobi quadrature rules. Expressions for Gauss–Legendre can be found in [7]. Earlier results on the classical rules include, among others, [41, 42, 43, 44, 45, 35, 46, 47, 48, 11, 10].

The expressions listed here are also implemented in the Julia package available from GitHub and called `AsymptoticGaussianQuadrature.jl`. The expressions are implemented in a manner that makes them easy to copy elsewhere.

Appendix A.1. Gauss–Laguerre

Asymptotic expansions for the nodes and weights of Gauss–Laguerre quadrature rules were described concurrently in [10, 1]. Here, we recall the explicit expansions as they are derived and presented in [1]. For brevity of notation we use the notation

$$\rho = 4n + 2\alpha + 2.$$

Following (5.2) we suggest to use the expressions for the hard edge up to $k = \lceil \sqrt{n} \rceil$ and the expressions for the bulk up to $k = \lfloor 0.9n \rfloor$.

Hard edge. Denote by $j_{\alpha,k}$ the k-th root of the Bessel function of order α. An explicit expansion for the nodes near the hard edge at $x = 0$ is:

$$
x_k = \frac{j_{\alpha,k}^2}{\rho} + \frac{j_{\alpha,k}^2 \left(j_{\alpha,k}^2 + 2\alpha^2 - 2 \right)}{3\rho^3} + \frac{j_{\alpha,k}^2 \left(11 j_{\alpha,k}^4 + 3 j_{\alpha,k}^2 (11\alpha^2 - 19) + 46\alpha^4 - 140\alpha^2 + 94 \right)}{45\rho^5}
$$

$$
+ \frac{j_{\alpha,k}^2}{3^4 35 \rho^7} \Big[657 j_{\alpha,k}^6 + 36 j_{\alpha,k}^4 (73\alpha^2 - 181) + 2 j_{\alpha,k}^2 (2459\alpha^4 - 10750\alpha^2 + 14051)
$$

$$
+ 4(1493\alpha^6 - 9303\alpha^4 + 19887\alpha^2 - 12077) \Big] + \frac{j_{\alpha,k}^2}{3^5 5^2 7 \rho^9} \Big[10644 j_{\alpha,k}^8 + 60 j_{\alpha,k}^6 (887\alpha^2 - 2879)
$$

$$
+ j_{\alpha,k}^4 (125671\alpha^4 - 729422\alpha^2 + 1456807)
$$

$$
+ 3 j_{\alpha,k}^2 (63299\alpha^6 - 507801\alpha^4 + 1678761\alpha^2 - 2201939)
$$

$$
+ 2(107959\alpha^8 - 1146220\alpha^6 + 5095482\alpha^4 - 10087180\alpha^2 + 6029959) \Big] + \mathcal{O}(n^{-11}).
$$

The inequality $x_k > j_{\alpha,k}^2 (4n + 2\alpha + 2)^{-1}$ [34, (18.16.10)] already provides an $\mathcal{O}(n^{-2})$ relative error and shows that the remainder has to be positive. The first two terms are exactly those found by Tricomi [45, (37)], where the error bound was reported to be $\mathcal{O}(n^{-4})$, while [46] reported a stricter $\mathcal{O}(n^{-5})$.

The corresponding expansion for the weights is

$$
w_k = \frac{4 x_k^\alpha \, e^{-x_k}}{\rho J_{\alpha-1}^2 (j_{\alpha,k})} \left(1 + \frac{2(\alpha^2 + j_{\alpha,k}^2 - 1)}{3\rho^2} + \frac{1}{45\rho^4} \left[46\alpha^4 + 33 j_{\alpha,k}^4 + 6 j_{\alpha,k}^2 (11\alpha^2 - 19) - 140\alpha^2 + 94 \right] \right.
$$

$$
+ \frac{4}{3^4 35 \rho^6} \Big[657 j_{\alpha,k}^6 + 27 j_{\alpha,k}^4 (73\alpha^2 - 181) + j_{\alpha,k}^2 (2459\alpha^4 - 10750\alpha^2 + 14051) \tag{A.1}
$$

$$
+ 1493\alpha^6 - 9303\alpha^4 + 19887\alpha^2 - 12077 \Big]
$$

$$
+ \frac{1}{3^5 5^2 7 \rho^8} \Big[215918\alpha^8 - 2292440\alpha^6 + 10190964\alpha^4 - 20174360\alpha^2
$$

$$
+ 12059918 + 53220 j_{\alpha,k}^8 + 240 j_{\alpha,k}^6 (887\alpha^2 - 2879) + 3 j_{\alpha,k}^4 (125671\alpha^4 - 729422\alpha^2 + 1456807)
$$

$$
\left. + 6 j_{\alpha,k}^2 (63299\alpha^6 - 507801\alpha^4 + 1678761\alpha^2 - 2201939) \Big] + \mathcal{O}(n^{-10}) \right).
$$

Note that the $\mathcal{O}(n^{-10})$ term is inside the parenthesis, i.e., the asymptotic order is relative to the size of the leading order term.

Bulk. For the asymptotics in the bulk, one first has to determine the roots of the leading order term of the asymptotic expansion of the polynomials as expressed by (4.2). For standard associated Laguerre this results in having to solve (4.4) to obtain t_k. A computational procedure is described in § 4.5.

The asymptotic expansion of the nodes in the bulk (region I in Figure 1) we arrive at is:

$$
x_k = \rho t_k - \frac{1}{12\rho} \left(\frac{5}{(1 - t_k)^2} - \frac{4}{1 - t_k} + 12\alpha^2 - 4 \right) \tag{A.2}
$$

$$+\frac{1-t_k}{720t_k\rho^3}\left(\frac{1600}{(1-t_k)^6}-\frac{3815}{(1-t_k)^5}+\frac{2814}{(1-t_k)^4}-\frac{576}{(1-t_k)^3}\right.$$

$$\left.-\frac{16}{(1-t_k)^2}+16(15\alpha^4-30\alpha^2+7)\left[2-\frac{3}{1-t_k}\right]\right)$$

$$+\frac{-(1-t_k)^2}{2^6 3^4 35 t_k^2\rho^5}\left[\frac{-1727136}{(1-t_k)^5}+\frac{16131880}{(1-t_k)^6}+\frac{-48469876}{(1-t_k)^7}\right.$$

$$+175\left(\frac{379569}{(1-t_k)^8}+\frac{-246416}{(1-t_k)^9}+\frac{61700}{(1-t_k)^{10}}\right)+4608\frac{3+2t_k}{1-t_k}\left(31-147\alpha^2+105\alpha^4-21\alpha^6\right)$$

$$+\frac{384}{(1-t_k)^2}\left(-1346+6405\alpha^2-4620\alpha^4+945\alpha^6\right)$$

$$\left.+\frac{320}{(1-t_k)^3}\left(-43+126\alpha^2-63\alpha^4\right)+\frac{80}{(1-t_k)^4}\left(-221-630\alpha^2+315\alpha^4\right)\right]$$

$$+\frac{(1-t_k)^3}{2^8 3^5 5^2 7 t_k^3\rho^7}\left[\frac{43222750000}{(1-t_k)^{14}}-\frac{241928673000}{(1-t_k)^{13}}+\frac{566519158800}{(1-t_k)^{12}}\right.$$

$$-\frac{714465642135}{(1-t_k)^{11}}+\frac{518401904799}{(1-t_k)^{10}}-\frac{212307298152}{(1-t_k)^9}+\frac{672}{(1-t_k)^8}\left(12000\alpha^4-24000\alpha^2+64957561\right)$$

$$-\frac{192}{(1-t_k)^7}\left(103425\alpha^4-206850\alpha^2+15948182\right)+\frac{3360}{(1-t_k)^6}\left(4521\alpha^4-9042\alpha^2-7823\right)$$

$$-\frac{1792}{(1-t_k)^5}\left(3375\alpha^6-13905\alpha^4+17685\alpha^2-1598\right)+\frac{16128}{(1-t_k)^4}\left(450\alpha^6-2155\alpha^4+2960\alpha^2-641\right)$$

$$-\frac{768}{(1-t_k)^3}\left(70875\alpha^8-631260\alpha^6+2163630\alpha^4-2716980\alpha^2+555239\right)$$

$$+\frac{768}{(1-t_k)^2}\left(143325\alpha^8-1324260\alpha^6+4613070\alpha^4-5826660\alpha^2+1193053\right)$$

$$-\frac{5806080}{1-t_k}\left(15\alpha^8-140\alpha^6+490\alpha^4-620\alpha^2+127\right)+24883200\alpha^8-232243200\alpha^6$$

$$\left.+812851200\alpha^4-1028505600\alpha^2+210677760\right]+\mathcal{O}(n^{-9}).$$

The first two terms in (A.2) are exactly those found by Tricomi [35, (56)], where the error bound was reported to be $\mathcal{O}(n^{-2})$, while [46] reported a stricter $\mathcal{O}(n^{-3})$.

The corresponding expansion for the weights is

$$w_k = x_k^\alpha\, \mathrm{e}^{-x_k} 2\pi\sqrt{\frac{t_k}{1-t_k}}\left(1-\frac{1}{6\rho^2}\left[5(1-t_k)^{-3}-2(1-t_k)^{-2}\right]\right. \tag{A.3}$$

$$+\frac{(1-t_k)^2}{720\rho^4 t_k^2}\left[\frac{8000}{(1-t_k)^8}-\frac{24860}{(1-t_k)^7}+\frac{27517}{(1-t_k)^6}-\frac{12408}{(1-t_k)^5}+\frac{1712}{(1-t_k)^4}\right.$$

$$\left.+\frac{32}{(1-t_k)^3}+\frac{16}{(1-t_k)^2}(15\alpha^4-30\alpha^2+7)\right]$$

$$-\frac{(1-t_k)^3}{90720\rho^6 t_k^3}\left[\frac{43190000}{(1-t_k)^{12}}-\frac{204917300}{(1-t_k)^{11}}+\frac{393326325}{(1-t_k)^{10}}-\frac{386872990}{(1-t_k)^9}\right.$$

$$+\frac{201908326}{(1-t_k)^8}-\frac{50986344}{(1-t_k)^7}+\frac{80}{(1-t_k)^6}\left(315\alpha^4-630\alpha^2+53752\right)$$

$$-\frac{320}{(1-t_k)^5}\left(189\alpha^4-378\alpha^2-89\right)+\frac{480}{(1-t_k)^4}\left(63\alpha^4-126\alpha^2+43\right)$$

$$-\frac{384}{(1-t_k)^3}\left(315\alpha^6 - 1470\alpha^4 + 1995\alpha^2 - 416\right)$$

$$+\frac{2304}{(1-t_k)^2}\left(21\alpha^6 - 105\alpha^4 + 147\alpha^2 - 31\right)\Bigg] + \mathcal{O}(n^{-8})\Bigg).$$

Soft edge. The soft edge for the standard associated Gauss–Laguerre rule is around $x = 4n$. The weights rapidly decay in this regime and even underflow for moderate n. For that reason we restrict the number of terms given here, but we note that more terms could be computed [1].

The expansion of the large nodes in region III is, where a_k are the zeros of the Airy function,

$$x_k = (4n + 2\alpha + 2) + 2^{2/3}a_{n-k+1}(4n + 2\alpha + 2)^{1/3} + \frac{2^{4/3}}{5}a_{n-k+1}^2(4n + 2\alpha + 2)^{-1/3}$$

$$+\left(\frac{11}{35} - \alpha^2 - \frac{12}{175}a_{n-k+1}^3\right)(4n + 2\alpha + 2)^{-1}$$

$$+\left(\frac{16}{1575}a_{n-k+1} + \frac{92}{7875}a_{n-k+1}^4\right)2^{2/3}(4n + 2\alpha + 2)^{-5/3}$$

$$-\left(\frac{15152}{3031875}a_{n-k+1}^5 + \frac{1088}{121275}a_{n-k+1}^2\right)2^{1/3}(4n + 2\alpha + 2)^{-7/3} + \mathcal{O}(n^{-3}).$$

This expression was adapted from [46, (4.9)], while we derived the following leading order term of the weight:

$$w_k = 4^{1/3}x_k^{\alpha+1/3}\exp(-x_k)\,\mathrm{Ai}'(a_{n-k+1})^{-2}\left[1 + \mathcal{O}(n^{-2/3})\right]. \tag{A.4}$$

Here, Ai' represents the derivative of the Airy function. Like above, the $\mathcal{O}$ term is in the parentheses, hence relative accuracy is obtained.

Appendix A.2. Gauss–Jacobi

We present the results using, for brevity of notation, the definition

$$\rho = n + \frac{1}{2}(\alpha + \beta + 1).$$

Following (6.2) we suggest to use the expressions for the hard edge at the left end up to $k = \lceil\sqrt{n}\rceil$ and at the right end starting from $k = n - \lceil\sqrt{n}\rceil$.

Hard edge at $x = -1$. Let $j_{\beta,k}$ be the k-th zero of the Bessel function J_β of order β. Then we have the following terms, up to $\mathcal{O}(n^{-12})$:

$$x_k^{JIV} \sim -1 + \frac{2j_{\beta,k}^2}{4\rho^2} + \frac{-2j_{\beta,k}^2}{2^4 3\rho^4}\left\{j_{\beta,k}^2 - 3\alpha^2 - \beta^2 + 1\right\} \tag{A.5}$$

$$+\frac{2j_{\beta,k}^2}{45\rho^6}\left\{2j_{\beta,k}^4 - 3j_{\beta,k}^2(5\alpha^2 + 3\beta^2 - 2) + 45\alpha^4 + 7\beta^4 + 20(3\alpha^2 - 1)\beta^2 - 60\alpha^2 + 13\right\}$$

$$+\frac{-2j_{\beta,k}^2}{2835\rho^8}\left\{9j_{\beta,k}^6 - 18(7\alpha^2 + 5\beta^2 - 3)j_{\beta,k}^4 - 2835\alpha^6 - 247\beta^6\right.$$

$$+(328\beta^4 + [1512\alpha^2 - 575]\beta^2 + 567\alpha^2 - 113)j_{\beta,k}^2 - 1407(3\alpha^2 - 1)\beta^4 + 8505\alpha^4$$

$$\left.-21(405\alpha^4 - 600\alpha^2 + 133)\beta^2 - 8379\alpha^2 + 1633\right\}$$

$$+\frac{4j_{\beta,k}^2}{2^{10}3^5 5^2 7\rho^{10}}\left\{6j_{\beta,k}^8 - 15(9\alpha^2 + 7\beta^2 - 4)j_{\beta,k}^6 + \left[6615\alpha^4 + 769\beta^4 + 2(1620\alpha^2 - 589)\beta^2\right.\right.$$

$$-12150\alpha^2 + 2668\big]\,j_{\beta,k}^4 + 3\big[9450\alpha^6 - 999\beta^6 - 23(400\alpha^2 - 147)\beta^4 - 40635\alpha^4$$

$$- (2835\alpha^4 + 1850\alpha^2 - 294)\beta^2 + 50650\alpha^2 - 10236\big]j_{\beta,k}^2$$

$$+ 42525\alpha^8 + 2327\beta^8 + 22340(3\alpha^2 - 1)\beta^6 - 226800\alpha^6 + 168(1530\alpha^4 - 2415\alpha^2 + 542)\beta^4 + 517860\alpha^4$$

$$+ 20(11340\alpha^6 - 38745\alpha^4 + 42399\alpha^2 - 8488)\beta^2 - 509280\alpha^2 + 98717\bigg\} + \mathcal{O}(n^{-12}),$$

The corresponding weights are, up to $\mathcal{O}(n^{-8})$:

$$\frac{w_k^{J^{\mathrm{IV}}}}{w(x_k^{J^{\mathrm{IV}}})} \sim \frac{2}{J_{\beta-1}^2(j_{\beta,k})\rho^2}\bigg[1 + \frac{3\alpha^2 + \beta^2 - 1 - 2j_{\beta,k}^2}{12\rho^2} \tag{A.6}$$

$$+ \frac{45\alpha^4 + 7\beta^4 + 6j_{\beta,k}^4 + 20(3\alpha^2 - 1)\beta^2 - 6(5\alpha^2 + 3\beta^2 - 2)j_{\beta,k}^2 - 60\alpha^2 + 13}{2^4 3^2 5\rho^4}$$

$$+ \frac{1}{2^6 3^4 35\rho^6}\bigg\{2835\alpha^6 + 247\beta^6 - 36j_{\beta,k}^6 + 1407(3\alpha^2 - 1)\beta^4$$

$$+ 54(7\alpha^2 + 5\beta^2 - 3)j_{\beta,k}^4 - 8505\alpha^4 + 21(405\alpha^4 - 600\alpha^2 + 133)\beta^2$$

$$- 2\big[328\beta^4 + (1512\alpha^2 - 575)\beta^2 + 567\alpha^2 - 113\big]j_{\beta,k}^2 + 8379\alpha^2 - 1633\bigg\} + \mathcal{O}(n^{-8})\bigg].$$

The bulk. The leading order term of the roots t_k is given by (6.1). The nodes in the interior of the interval expand asymptotically up to $\mathcal{O}(n^{-10})$ as

$$x_k \sim t_k + \frac{2\alpha^2 - 2\beta^2 + (2\alpha^2 + 2\beta^2 - 1)t_k}{8\rho^2} + \frac{1}{2^7 3\rho^4(1 - t_k^2)} \tag{A.7}$$

$$\bigg(32\alpha^4 - 32\beta^4 - \big[16\alpha^4 + 16\beta^4 + 4(12\alpha^2 - 5)\beta^2 - 20\alpha^2 + 5\big]t_k^3 - 24(\alpha^2 - \beta^2)t_k^2 - 40\alpha^2 + 40\beta^2$$

$$+ 3\big[16\alpha^4 + 16\beta^4 + 4(4\alpha^2 - 7)\beta^2 - 28\alpha^2 + 11\big]t_k\bigg)$$

$$+ \frac{1}{2^{10}15\rho^6(1 - t_k^2)^2}\bigg(576\alpha^6 - 576\beta^6$$

$$+ \big[96\alpha^6 + 96\beta^6 + 80(8\alpha^2 - 3)\beta^4 - 240\alpha^4 + 2(320\alpha^4 - 440\alpha^2 + 101)\beta^2 + 202\alpha^2 - 39\big]t_k^5$$

$$- 320(\alpha^2 - 6)\beta^4 + 240(\alpha^2 - \beta^2)t_k^4 - 1920\alpha^4 + 16(20\alpha^4 - 127)\beta^2$$

$$- 10\big[32(5\alpha^2 + 3)\beta^4 + 96\alpha^4 + 2(80\alpha^4 - 152\alpha^2 - 97)\beta^2 - 194\alpha^2 + 99\big]t_k^3$$

$$+ 160\big[6\alpha^6 - 6\beta^6 + 2(\alpha^2 + 15)\beta^4 - 30\alpha^4 - (2\alpha^4 + 41)\beta^2 + 41\alpha^2\big]t_k^2 + 2032\alpha^2$$

$$+ 15\big[96\alpha^6 + 96\beta^6 + 16(4\alpha^2 - 23)\beta^4 - 368\alpha^4 + 2(32\alpha^4 - 72\alpha^2 + 223)\beta^2 + 446\alpha^2 - 173\big]t_k\bigg)$$

$$- \frac{1}{2^{15}3^2 35\rho^8(1 - t_k^2)^3}\bigg(219648\alpha^8 - 219648\beta^8 - \big[9728\alpha^8 + 9728\beta^8$$

$$+ 896(138\alpha^2 - 49)\beta^6 - 43904\alpha^6 + 224(1160\alpha^4 - 1720\alpha^2 + 389)\beta^4$$

$$+ 87136\alpha^4 + 8(15456\alpha^6 - 48160\alpha^4 + 49364\alpha^2 - 9785)\beta^2 - 78280\alpha^2 + 14921\big]t_k^7$$

$$- 10752(14\alpha^2 - 127)\beta^6 - 40320(\alpha^2 - \beta^2)t_k^6 - 1365504\alpha^6$$

$$+ 21\big[2048\alpha^8 + 2048\beta^8 + 128(146\alpha^2 - 123)\beta^6 - 15744\alpha^6 + 32(1320\alpha^4 - 1760\alpha^2 + 2023)\beta^4 + 64736\alpha^4$$

$$+ 8(2336\alpha^6 - 7040\alpha^4 + 3644\alpha^2 - 11275)\beta^2 - 90200\alpha^2 + 37111\big]t_k^5 + 75264(5\alpha^2 - 49)\beta^4$$

$$+ 4480\big[44\alpha^8 - 44\beta^8 + 8(3\alpha^2 + 55)\beta^6 - 440\alpha^6 - 24(7\alpha^2 + 72)\beta^4 + 1728\alpha^4$$

$$
\begin{aligned}
&- (24\alpha^6 - 168\alpha^4 - 2405)\beta^2 - 2405\alpha^2\big]t_k^4 \\
&+ 3687936\alpha^4 + 105\big[6656\alpha^8 + 6656\beta^8 - 128(50\alpha^2 + 443)\beta^6 - 56704\alpha^6 - 32(296\alpha^4 - 696\alpha^2 - 6027)\beta^4 \\
&+ 192864\alpha^4 - 8(800\alpha^6 - 2784\alpha^4 + 3580\alpha^2 + 30285)\beta^2 - 242280\alpha^2 + 99933\big]t_k^3 \\
&+ 384(392\alpha^6 - 980\alpha^4 + 10527)\beta^2 + 2688\big[424\alpha^8 - 424\beta^8 + 4(4\alpha^2 + 783)\beta^6 \\
&- 3132\alpha^6 + 4(35\alpha^2 - 2407)\beta^4 + 9628\alpha^4 - (16\alpha^6 + 140\alpha^4 - 11429)\beta^2 - 11429\alpha^2\big]t_k^2 \\
&- 4042368\alpha^2 + 35\big[23552\alpha^8 + 23552\beta^8 + 128(90\alpha^2 - 1231)\beta^6 - 157568\alpha^6 + 32(328\alpha^4 - 1376\alpha^2 + 14095)\beta^4 \\
&+ 451040\alpha^4 + 8(1440\alpha^6 - 5504\alpha^4 + 9964\alpha^2 - 65439)\beta^2 - 523512\alpha^2 + 206379\big]t_k\Big) + \mathcal{O}(n^{-10}).
\end{aligned}
$$

Up to $\mathcal{O}(n^{-8})$ relative accuracy, relative to the value of the weight function $w(x_k)$ at the corresponding node, the weights expand as

$$
\begin{aligned}
\frac{w_k}{w(x_k)} \sim\ & \frac{\pi\sqrt{1-t_k^2}}{2\rho}\Bigg[2 - \frac{1 - 2\alpha^2 - 2\beta^2}{4\rho^2} \\
&+ \frac{1}{2^6 3(t_k^2 - 1)^2\rho^4}\bigg(\big[16\alpha^4 + 16\beta^4 + 4(12\alpha^2 - 5)\beta^2 - 20\alpha^2 + 5\big]t_k^4 + 48\alpha^4 + 48\beta^4 \\
&+ 12(4\alpha^2 - 7)\beta^2 - 6\big[4(4\alpha^2 + 1)\beta^2 + 4\alpha^2 - 3\big]t_k^2 - 84\alpha^2 + 64\big[\alpha^4 - \beta^4 - 2\alpha^2 + 2\beta^2\big]t_k + 33\bigg) \\
&+ \frac{1}{2^9 15(t_k^2 - 1)^3\rho^6}\bigg(\big[96\alpha^6 + 96\beta^6 + 80(8\alpha^2 - 3)\beta^4 - 240\alpha^4 \\
&+ 2(320\alpha^4 - 440\alpha^2 + 101)\beta^2 + 202\alpha^2 - 39\big]t_k^6 - 1440\alpha^6 - 1440\beta^6 - 240(4\alpha^2 - 23)\beta^4 \\
&- 5\big[96\alpha^6 + 96\beta^6 + 16(20\alpha^2 - 27)\beta^4 - 432\alpha^4 + 2(160\alpha^4 - 136\alpha^2 + 295)\beta^2 + 590\alpha^2 - 237\big]t_k^4 \\
&+ 5520\alpha^4 - 640\big[3\alpha^6 - 3\beta^6 + (\alpha^2 + 15)\beta^4 - 15\alpha^4 - (\alpha^4 + 22)\beta^2 + 22\alpha^2\big]t_k^3 \\
&- 30(32\alpha^4 - 72\alpha^2 + 223)\beta^2 - 15\big[288\alpha^6 + 288\beta^6 - 16(8\alpha^2 + 81)\beta^4 - 1296\alpha^4 \\
&- 2(64\alpha^4 - 88\alpha^2 - 863)\beta^2 + 1726\alpha^2 - 717\big]t_k^2 \\
&- 6690\alpha^2 - 128\big[33\alpha^6 - 33\beta^6 - 5(\alpha^2 - 27)\beta^4 - 135\alpha^4 + (5\alpha^4 - 166)\beta^2 + 166\alpha^2\big]t_k + 2595\bigg) + \mathcal{O}(n^{-8})\Bigg].
\end{aligned}
\tag{A.8}
$$

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