The Mellin Transform Method: Electromagnetics, Complex Analysis, and Educational Potential

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***Abstract*—The Mellin Transform Method (MTM) is a method of analytical evaluation of definite integrals that utilizes key concepts of Complex Analysis (CA). Though sometimes laborious, the algorithmic, step-wise character of the method as well as its vast field of applications render it powerful and versatile. In particular, certain integrals that are presented in the literature without proof (or with proofs that are ingenious and non-intuitive) can be directly attacked in a standard manner. This is true for the three examples presented herein, all taken from standard and advanced electromagnetics textbooks. The particular goal of this paper is two-fold. For one, the effectiveness of the MTM is demonstrated for the three aforesaid examples. It becomes obvious that the treatment of the most elementary example is no way different than that of the other two. For another, the educational virtues of the MTM in demonstrating key concepts of CA are highlighted. CA is a difficult mathematical area, usually abstract, and often sidelined in worldwide engineering curricula. In this work, it is proposed to incorporate the MTM evaluations—as a mathematical digression—into typical microwaves or advanced electromagnetics courses by breaking down the method into 7 steps, with each one exhibiting rich—elementary or advanced—CA concepts such as analytic functions, the gamma function, generalized hypergeometric functions, contour integrals involving meromorphic function, as well as the important idea of analytic continuation. The MTM is more likely to benefit students who will end up working in research environments (rather than engineers who will work in industrial settings).**

***Index Terms*—Mellin Transform, Complex Analysis, Analytical Integration, Mathematical Methods of Electromagnetics**

I. Introduction

INTEGRAL-transform methods are pervasive in engineering practice, especially because many integral transforms can reduce complexity vis-à-vis the solution of canonical boundary value problems. However, it is fair is fair to say that many members of the electrical engineering community know very little about the *Mellin Transform Method* (MTM): the method is underutilized in standard textbooks and often sidelined, a problem diagnosed by many of those who have encountered the MTM; see for example [1], where Ablowitz and Fokas state "This method, although often quick and easy to apply, is not widely known."

But what is this method? The MTM is a highly effective and (after some initial study) *simple* method for the *analytical* calculation of one-dimensional, definite integrals. Actually, a quite wide class of integrals can be directly confronted by the method; these integrals are called *Mellin convolutions* (soon to be formally defined) and they appear in many interesting and popular electrical engineering problems. In this paper, three examples from microwave and electromagnetic theory will be presented. Aside from its *versatility*, another salient characteristic of the method is that it is not likely to surprise the user. In other words, the process is completely *algorithmic* and not contingent upon the integral's complexity. In this paper, the procedure is segmented into (at maximum) 7 discrete steps to facilitate the introduction of the method to interested engineers and students. It is interesting to note that the aforesaid algorithmic nature of the method has enabled it to be utilized in symbolic integration routines in popular software packages (e.g., Mathematica).

While many mathematicians have written about transform methods [2, 3, 4, 5] and transform methods are popular among engineers, it is also true that what we call the MTM is not particularly well-known in the engineering community. Still, it is dealt with by a vast mathematical literature driven by the pioneering works of A. P. Prudnikov, O. I. Marichev and J. A. Brychkov, see indicatively the seminal [6], the more recent work [7] and the references therein. In tandem, the core element and first step of the MTM, the *Mellin Transform* (MT), that coined the name of the method after all, has gained a lot of attention recently in rather applied scientific regimes. For example, the MT is a popular tool in economics [8] and image

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processing [9] (especially under the guise of the related Fourier-Mellin Transform [10]); it has also found application in acoustics [11], wireless-channel fading studies [12], as well as in a problem concerning equilibrium of an object supported by surface tension [13,14]. All these disciplines are of immediate interest to the modern engineer.

Its efficiency and versatility render the MTM a useful tool in the hands of research engineers with more theoretical endeavors or for scientists that like to cross-check results that can be acquired via specialized software. Yet (as has been hinted, but not fully supported in previous works) also valuable are the educational virtues of MTM. In particular, the method's use of important Complex Analysis (CA) concepts makes it an ideal vehicle for reviewing and understanding fundamental CA ideas in practical—rather than abstract—settings. Such ideas include: analytic functions; meromorphic functions (as opposed to functions with branch points and/or essential singularities, which are more difficult to understand); power series; contour integrals; the gamma function; and classes of special functions including the generalized hypergeometric function. As will be revealed, the method treats special functions (such as Bessel functions) in ways no more difficult than elementary functions (such as trigonometric functions). Finally, the all-important idea of analytic continuation, which often perplexes students, permeates the MTM. This is particularly important in an era when CA is often sidelined in favor of topics that are considered more catchy, more modern, and easier to understand. This applies to both advanced-undergraduate and beginning-graduate engineering curricula, and remains true even in particular instances where CA could greatly help one obtain results of immediate engineering interest.

The present paper can be considered as a continuation of the tutorial/review work [15] and its expanded version [16], which will be referred to frequently. It is the aspiration of these works to illuminate the various MTMs. The main differences with [15] and [16] are as follows: (i) The three examples herein are original in the sense that (to the authors' best knowledge) they have not been previously treated by the MTM. (ii) The first and rather elementary example has the educational value of teaching all the aforesaid CA concepts; it is suggested to be used as a useful digression from a typical microwaves course, perhaps as a self-contained part of a lecture (in Section III the context within which the integrals showcased here arise, is discussed in details). As such, Section IV A pertaining to this example is filled with numerous comments of educational flavor. (iii) The second and third are examples of the efficiency and, in a sense, simplicity of the method, in complex integrals encountered in realistic and frontier research settings, that can be used in tandem for demonstration purposes to more advanced, graduate students.

II. Mellin-Transform Basics and Nomenclature

This section presents the fundamentals of the MT (see [16] for more details). The MT of a complex-valued function $f(x)$ of the positive variable x is denoted by $\tilde{f}(z)$ and is given by

$$\tilde{f}(z) = \int_0^\infty f(x) x^{z-1} dx. \tag{1}$$

If the MT in (1) exists at all, it is defined in a vertical strip in the complex-z plane, most often referred to as the *strip of analyticity* (SOA). The analytic continuation of $\tilde{f}(z)$ to other complex values of z is usually straightforward and often useful (even necessary) within the context of practical applications. Several elementary properties of the MT can be deduced directly from the definition (1) and from properties of the related Laplace and Fourier Transforms (LTs and FTs) [16]. In particular, the inversion formula for the MT is

$$f(x) = \frac{1}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} \tilde{f}(z) x^{-z} dz, \tag{2}$$

where δ is such that the integration path, which is a vertical line, lies within the aforementioned SOA.

In the context of the MT, the *Mellin convolution* is defined as follows

$$f(x) = \int_0^\infty g(xy) h(y) dy. \tag{3}$$

Many integrals have the form of the right-hand side (RHS) of (3) (e.g., all LTs and Fourier cosine transforms) or can be expressed in terms of integrals having this form (e.g., all FTs). Eqn. (3) is the starting point for the MTM.

As long as the MTs $\tilde{g}(z)$ and $\tilde{h}(z)$ exist and the SOAs of $\tilde{g}(z)$ and $\tilde{h}(1-z)$ overlap, the convolution in (3) can also be written as

$$f(x) = \frac{1}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} \tilde{g}(z) \tilde{h}(1-z) x^{-z} dz, \tag{4}$$

in which δ belongs to the SOAs of both $\tilde{g}(z)$ and $\tilde{h}(1-z)$. For one to be able to proceed, it is necessary that the integrand in (4) be written as a *standard product*, meaning that the RHS can be represented by a *Mellin-Barnes integral* as follows,

$$f(x) = \frac{1}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} \frac{\prod_{l=1}^{m} \Gamma(a_l + A_l z) \prod_{l=1}^{r} \Gamma(b_l - B_l z)}{\prod_{l=1}^{p} \Gamma(c_l + C_l z) \prod_{l=1}^{q} \Gamma(d_l - D_l z)} x^{-z} dz \tag{5}$$

in which A_l, B_l, C_l, and D_l are strictly positive constants and $\Gamma(z)$ is the gamma function. The above MB-integral representation is the heart of the MTM. Once derived, one can proceed in many ways. For example, a representation in terms of the *Meijer G-function* $G_{p,q}^{m,r}$ is often obtainable from (5) with little effort [16]. Alternatively, (5) can lead to useful series expansions or representations via the *generalized hypergeometric function* ${}_pF_q$. To this end, one first examines if it is possible to *close the integration contour at left or right* by calculating the following quantities Δ and Π

$$\Delta = \sum_{l=1}^{m} A_l - \sum_{l=1}^{r} B_l - \sum_{l=1}^{p} C_l + \sum_{l=1}^{q} D_l, \tag{6}$$

$$\Pi = -\ln|x| + \sum_{l=1}^{m} A_l \ln A_l - \sum_{l=1}^{r} B_l \ln B_l - \sum_{l=1}^{p} C_l \ln C_l + \sum_{l=1}^{q} D_l \ln D_l. \tag{7}$$

If $\Delta > 0$, the contour can be closed at left for all real x. If $\Delta <$

0, the contour can be closed at right for all real x. If $\Delta = 0$, the contour can be closed at left if $\Pi > 0$ and at right if $\Pi < 0$. (Chapt. 8 of [16] cites works discussing the more complicated case $\Pi = \Delta = 0$.) After determining the singularities to the left or right of the contour in (5), which will hopefully be poles (not necessarily simple poles) located at $z = z_n$, the residue theorem gives the series representation

$$f(x) = \sum_{n=0}^{\infty} \operatorname{Res}\{\tilde{f}(z)x^{-z}, z = z_n\}, \tag{8}$$

which can often be identified with the ${}_pF_q$, defined as

$$\begin{aligned} {}_pF_q\big(\alpha_1, \alpha_2, \dots, \alpha_p\,; \beta_1, \beta_2, \dots, \beta_q; z\big) \\ = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n \cdots (\alpha_p)_n}{(\beta_1)_n(\beta_2)_n \cdots (\beta_q)_n} \frac{z^n}{n!}. \end{aligned} \tag{9}$$

where $(z)_n = \Gamma(z+n)/\Gamma(z)$ is the *Pochhammer's symbol*. Eqn. (9), of course, makes sense provided that the series converges (more on this below).

Expressions for $f(x)$ involving ${}_pF_q$ or $G_{p,q}^{m,r}$ are often intermediate results, as long as they can often be rewritten in terms of the more usual special functions, such as Bessel functions (${}_pF_q$ and $G_{p,q}^{m,r}$ are actually wide classes of special functions [16]). Even when no simple closed form expression can be arrived at, it is usually more efficient to use the aforesaid expressions instead of applying quadrature rules to the original integral (3) or to the transformed integral (5). In this regard, however, let us note that there exists a number of recent works on the numerical calculation of Mellin-Barnes integrals, such as [17], where proper modifications of the integration contours are proposed. Today, some numerical packages automatically evaluate the ${}_pF_q$, and even $G_{p,q}^{m,r}$.

Eqns. (5) and (9) already exhibit the strong link between the MT and $\Gamma(z)$. The latter is itself perhaps the simplest instance of a MT, as it is the MT of the elementary function e^{-x}. The gamma function has simple poles at $z_n = -n$, for $n = 0,1,2, \dots$, with corresponding residues $(-1)^n/n!$. The gamma function has many important and useful properties; the ones most useful for the MTM are the *recurrence formula* $\Gamma(z+1) = z\Gamma(z)$, the *reflection formula* $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$, and the *duplication formula* $\Gamma(2z) = \big(1/\sqrt{\pi}\big)2^{2z-1}\Gamma(z)\Gamma(z+1/2)$. This last formula is a specialized case of the so-called *multiplication formula*, which can be found in [16].

It is evident from (5) that the poles and the corresponding residues of the function $\Gamma(uz+v)x^{-z}$ are important for the MTM. The said poles are located at $z = p_n$ with $p_n = -(v+n)/u$, $n = 0,1,2, \dots$, while the corresponding residues are given by

$$\operatorname{Res}\{\Gamma(uz+v)x^{-z}, z = p_n\} = \frac{1}{u}\frac{(-1)^n}{n!}x^{-p_n}. \tag{10}$$

III. Examples

The first integral to be showcased here originates from the theory of *cylindrical cavities* and is given by

$$I_1 = \int_0^a rJ_0^2(k_c r)dr, \quad k_c = x_{0,1}/a, \tag{11}$$

where a is the radius of the cavity, $x_{0,1}$ is the first root of $J_0(x)$, J_0 is the Bessel function of the first kind and of zeroth order, and $k_c^2 = k^2 - k_z^2$, with k the wavenumber and k_z the propagation constant along the cylindrical axis. The integral I_1 is involved in the calculation of the stored electromagnetic energy inside the cavity, but also of the power loss in the conducting walls. It can thus be used for to calculate the quality factor of the TM_{010} mode, which is dominant when $d/a < 2$, where d is the cavity length. As such, every undergraduate electrical engineer will confront (11) in a typical microwaves course (and possibly in a more exotic one, see [18], which comes from a nuclear engineering course in MIT). The simplicity of the calculation of the electromagnetic fields inside the cavity combined with the vast applications of the device make it a popular introductory example, as can be seen from well-established textbooks in the field [19, 20].

The integral in (11), which is also dealt with in more mathematical works, can be evaluated via the orthonormality relation associated with Fourier-Bessel series (e.g., see [21] and [22], which prove the aforesaid relation using Bessel-function properties). Most students, however, are not likely to be familiar with such series. The evaluated form of I_1 is important for the understanding of cylindrical cavities, which can in turn be succeeded naturally by discussions on more general topics such as arbitrarily shaped cavities and the orthonormality and/or orthonormalization of solutions to the Helmholtz equation. (Some textbooks prefer to first discuss the aforementioned more general topics, but it is not always trivial to deduce the more specialized results).

How do the engineering textbooks treat (11)? In [18], the TM_{010} mode is particularly emphasized, due to its importance for the book's scope, which pertains to particle acceleration. Eqn. (11) corresponds to [18, eqn. 12.49], but the final result for I_1 is given without proof. In their classic books, Collin and Pozar treat cylindrical cavities in similar ways. Both focus on the TE modes; Pozar [19] does not include a discussion on TM modes, while Collin [20] gives Q for TM modes. However, (11) appears rather indirectly: [19, eqn. C.14], which is an extension of (11) to Bessel functions of higher order, is presented as a "useful integral relation involving Bessel functions" without proof; in [20], only general integral expressions for the energy and the power loss are presented, and the author states that "these integrals may be evaluated by substituting the expressions given earlier for the fields" (similar integrals were treated during the study of the circular disk resonator, see [20, eqn. 7.36], which is given without proof). In the context of Hertz potential formalism, Collin's [23] treatment resembles that of Pozar. The relevant equation is [23, chapt. 5, eqn. 71b], which is identical to [19, eqn. C.14]. The result is again given without proof. Balanis [24], finally, treats both TE and TM modes, but focuses on the latter, presenting a complete analysis of TM_{010}. The integral I_1 is in [24, eqn. 9.52], where the final result is given without proof; instead, [24] cites the mathematical handbook [25].

The evaluation of I_1 serves as a straightforward introduction to the proposed method. Two more integrals are also considered below. Both originate from [26]. The integral I_2, which appears as [26, eqn. (7.100)], pertains to the *quadrupole lens system* of [26, fig. 7.4], and measures the coupling strength between the two dipoles that make up the quadrupole:

$$I_2 = \int_0^\infty \lambda^{-1} e^{-\lambda t} J_{2n+1}(\lambda) J_{2k+1}(\lambda) d\lambda\,, \qquad n,k \in \mathbb{N}, \quad (12)$$

where $2t$ is the distance between the upper and lower electrodes. Multipole lenses play a dominant role in particle beam focusing, deflection and particle acceleration. Quadrupoles, in particular, are used in high-energy applications due to their strong focusing abilities, and are of great importance, as multipoles of higher order can be studied by considering a quadrupole series expansion [27].

The integral I_3 is drawn from the theory of *open spherical shells* and is actually the capacitance of the two shells [26, eqns. 3.95 and 3.100]

$$I_3 = 4a^2 \times \int_0^\infty \left(\frac{2}{\pi b \nu}\right)^{\frac{1}{2}} \sum_{n=0}^{\infty} (4n+1)^{\frac{1}{2}} x_n J_{2n+1/2}(b\nu) e^{-2\nu a} d\nu, \quad (13)$$

where a is the radius of both spheres, $b = 2a\tan(\theta_0/2)$ with $0 < \theta_0 < \pi/2$ the two shells' identical subtending angles (see [26, fig. 3.13]), and x_n are Fourier coefficients, irrelevant to this analysis.

The integrals I_2 and I_3 are (or seem to be) more advanced than I_1 and appear in [26], a textbook containing delicate mathematical manipulations and thus better suited for doctoral students. (R. Scharstein states that ref. [26] should be "required reading of every new PhD graduate who professes expertise in linear wave scattering." [28].)

Even if I_2 and I_3 arise from more advanced applications, they can be discussed briefly, as a complement to a detailed evaluation of an integral such as I_1. Digressing from engineering topics to give mathematical prerequisites is not unusual. When discussing cylindrical cavities, for example, a self-contained tutorial in Bessel functions is more generally useful, as such functions arise in many applications. Similarly, a digression into the MTM (using I_1 as a first example, while stressing that the evaluation of I_2 and I_3 is actually no more difficult) can be highly beneficial, especially for learning the CA concepts required for the MTM evaluations.

IV. Method Demonstration

The evaluations themselves, as well as the associated CA concepts that arise naturally are now dealt with. The procedure followed, which is provided in the form of simple algorithmic steps, will be seen to be not contingent upon the problem's complexity. All the required theory has been presented in Section II.

A. Cylindrical Cavities

Step 1: Recognize/Cast Integral as Mellin Convolution

It is obvious that (11) is a Mellin convolution, but with a finite upper integration limit. To this end, one can define

$$g(x) = J_0^2(x), \quad x > 0, \quad (14)$$

$$h(x) = \begin{cases} x, & 0 < x < a \\ 0, & x > a, \end{cases} \quad (15)$$

so that I_1 can be written as in (3) with $x \to k_c$ and $y \to r$.

Step 2: Recast as MB Integral

One can use (4) as long as the MTs of both $g(x)$ and $h(x)$ exist and their respective SOAs overlap. Existence boils down to the convergence of the involved integral, with the z- variable as a parameter. Simple rules (see [16, Appendix A] or [29, Appendix B]) can be used for this purpose, even for elaborate integrals. The ability to eyeball the convergence, gained from practice, is very important; a convergence test is a common first step when studying a new integral. The convergence test will also yield the SOA. In simple cases, the MT can be calculated manually. This, however, is not true in general, as it strongly depends on the function to be transformed. Most frequently, one can rely on software packages (see, e.g., the `MellinTransform` command of Mathematica) or on extensive tables. This fact allows students and interested researchers to be introduced to the monumental work [6], and to the relevant Soviet mathematical literature, which is fundamental to the MTM. For (14) and (15), the MTs involve gamma functions and are

$$\tilde{g}(z) = 2^{z-1} \frac{\Gamma(1-z)\Gamma\left(\frac{z}{2}\right)}{\left[\Gamma\left(1-\frac{z}{2}\right)\right]^3}, \qquad 0 < \Re\mathrm{e}\{z\} < 1, \quad (16)$$

$$\tilde{h}(z) = \frac{a^{z+1}}{z+1} = \frac{a^{z+1}\Gamma(z+1)}{\Gamma(z+2)}, \qquad \Re\mathrm{e}\{z\} > -1. \quad (17)$$

In the MT context, the second form in (17) (which follows from the first via the recurrence formula for the gamma function), is preferable because of the aforementioned importance of MB integrals.

Since the two MTs exist and the SOAs of $\tilde{g}(z)$ and $\tilde{h}(1-z)$ overlap, one can calculate the MT of the convolution as

$$I_1 = \frac{1}{2\pi i} \frac{a^2}{2} \int_{\delta-i\infty}^{\delta+i\infty} \frac{\Gamma(1-z)\Gamma(2-z)\Gamma\left(\frac{z}{2}\right)}{\left[\Gamma\left(1-\frac{z}{2}\right)\right]^3 \Gamma(3-z)} \left(\frac{ax}{2}\right)^{-z} dz, \quad (18)$$

with $0 < \delta < 1$, so that the vertical contour lies within the region of overlap. While one might think that the initial integral has been transformed into a much more complicated one, impressive simplification will be achieved via the residue theorem.

Step 3: Find Integrand Singularities

This step is usual when performing contour integrations. However, as long as the integrand has been expressed in terms of gamma functions, the procedure is trivial. This is a good opportunity to discuss the well-known series expansion of the gamma function [16, eqn. (2.15)], [30, eqn. (5.9.4)] (the proof is itself a simple exercise on integration by parts) because the series representation is very illuminating when it comes to the determination of the singularities of the several involved gamma functions. The said representation makes obvious that the only singularities of a gamma function are simple poles, as already mentioned in Section II. It is convenient to tabulate the singularities as follows.

TABLE I: SINGULARITIES OF (18). D/N STANDS FOR DENOMINATOR/NUMERATOR, P STANDS FOR A SIMPLE POLE, Z FOR A SIMPLE ZERO, AND Z^3 FOR A TRIPLE ZERO; n IS A NATURAL NUMBER. THE DOTTED LINE GIVES THE POSITION OF THE VERTICAL INTEGRATION CONTOUR.

Gamma Functions	P	D/N	-4	-3	-2	-1	0	1	2	3
$\Gamma(1-z)$	$n+1$	N						P	P	P
$\Gamma(z/2)$	$-2n$	N	P		P		P			
$\Gamma(2-z)$	$n+2$	N							P	P
$[\Gamma(1-z/2)]^3$	$2n+2$	D							Z^3	
$\Gamma(3-z)$	$n+3$	D								Z

Step 4: Close the Contour

After the singularities have been obtained, an appropriate closed contour must be selected. This actually spans a very long and often difficult chapter in CA, as there is no general recipe when it comes to finding proper contour closures; within the context of the MTM, however, the discussion is much simplified, due to the nature of the MB integrals arising in Step 2. As stressed earlier, the individual MTs of the convolution are cast as products and quotients of gamma functions, as in (18). Under the conditions below (6) and (7), the contour for a MB integral can be an infinite semi-circle, lying in either half-plane ($\Re\mathrm{e}\{z\} < 0$ or $\Re\mathrm{e}\{z\} > 0$.) In cases where both choices are permissible, a particular choice (e.g., the contour surrounding fewer singularities) may be preferable to the other. Here $\Delta > 0$, so the contour must be closed at left, which happens to be the most convenient choice.

Step 5: Calculate Residues

Having defined the contour, one can proceed with application of the residue theorem, another hallmark of CA. Again, there are no omnipotent recipes and the procedure may prove itself a challenge. This task here is simplified because the residues due to the gamma functions are given by the simple expressions mentioned in Section II. Thus, the application of the residue theorem here leads to

$$I_1 = \frac{a^2}{2}\sum_{n=0}^{\infty}\frac{\Gamma(2n+1)\Gamma(2n+2)}{[\Gamma(n+1)]^3\Gamma(2n+3)} \times \mathrm{Res}\left\{\Gamma\left(\frac{z}{2}\right)\left(\frac{ax}{2}\right)^{-z}, z = -2n\right\}, \tag{19}$$

since only $\Gamma(z/2)$ contributes to the singularities. Using (10), (19) yields

$$I_1 = a^2\sum_{n=0}^{\infty}\frac{\Gamma(2n+1)\Gamma(2n+2)}{[\Gamma(n+1)]^3\Gamma(2n+3)}\frac{(-1)^n}{n!}\left(\frac{ax}{2}\right)^{2n}. \tag{20}$$

Step 6: Recast as Appropriate Special Function

While the series in (20) appears to be complicated (Step 5 often produces such series), elementary manipulations can lead to a representation by a generalized hypergeometric function. To achieve this, the gamma functions must depend only on n (not $2n,\ 3n,$ etc.); the required transformations are achieved via the multiplication formula (in our case, the duplication formula suffices) and use of Pochhammer's symbol. At this point, students are naturally led to discover some specific values of the gamma function, e.g., $\Gamma(1/2) = \sqrt{\pi}$. In this manner, one is led to

$$I_1 = \frac{a^2}{2\sqrt{\pi}}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)\Gamma(2)}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n}{(1)_n(2)_n}\frac{(-a^2x^2)^n}{n!} \tag{21}$$

and then, via (9), to

$$I_1 = \frac{a^2}{2}\,{}_1F_2\left(\frac{1}{2}; 1,2; -(ax)^2\right). \tag{22}$$

In addition to being useful for numerics, the generalized hypergeometric functions constitute a very versatile class of functions because, in many cases, they reduce to more common special functions; therefore, they are an excellent tool in the hands of every theoretician [6]. The MTM is a very natural and straightforward way to introduce them to students.

Step 7: Simplify Result

This step is optional in the sense that results like (22) readily lend themselves to numerical evaluation; however, they do not always give a physical interpretation, and may not be amenable to further asymptotic/analytic treatments. For the particular example at hand, one can actually simplify (22). This can be achieved either via software packages (see Mathematica's `FullSimplify` command) or tables (again, [6] has an extensive one). Either way, (22) reduces to

$$I_1 = \frac{a^2}{2}\left[J_0^2(ax) + J_1^2(ax)\right] \tag{23}$$

or (given that $x = k_c$ and $k_c a = x_{0,1}$)

$$I_1 = \frac{a^2}{2}J_1^2(x_{0,1}), \tag{24}$$

which is the same result reported in the literature (apply [19, eqn. C.14] and [23, chapt. 5, eqn. 71b] for $n = 0$ or simply compare with [24, eqn. 9.52]). As will be seen, the seemingly more complicated integrals I_2 and I_3 can be evaluated by the same method. Therefore, the MTM, while somewhat laborious, is systematic. For completeness, the reader can compare to the nonstandard, and perhaps more innovative, derivations provided in the Appendix.

The above evaluation of I_1 forms an excellent introduction to the MTM and illustrates many important CA concepts. At this point, the instructor can discuss I_2 and I_3, or terminate the digression and allow the course to resume its normal path.

The energy stored inside the cavity is simply $W = |A|^2\pi d\epsilon k_c^4 I_1$, which yields

$$W = |A|^2\pi d\epsilon k_c^4\frac{a^2}{2}J_1^2(x_{0,1}), \tag{25}$$

where ϵ is the electric permittivity and A is a constant that depends on the particular excitation (but A is irrelevant to this analysis). The power loss in the conducting walls is

$$P_d = |A|^2R_m\pi\omega^2\epsilon^2\left(\frac{x_{0,1}}{a}\right)^2\left[2\int_0^a rJ_1^2\left(\frac{x_{01}}{a}r\right)dr + adJ_1^2(x_{0,1})\right]. \tag{26}$$

Here, ω is the angular frequency and R_m is the wall resistance. The first term accounts for the losses in the end-caps, while the second for the losses in the body of the cavity.

The integral in the first term can be evaluated using the MTM. The procedure, which differs only in its details from the one just described (viz, for the evaluation of I_1), can be assigned to students as a completely straightforward first exercise in the

MTM. Alternatively, proceeding from first principles, the Appendix shows that the first integral in (26) equals I_1 (the reader may wish to compare the two procedures). Either way, one obtains

$$P_d = |A|^2 R_m \omega^2 \epsilon^2 a\pi \left(\frac{x_{0,1}}{a}\right)^2 J_1^2(x_{0,1})(a+d). \tag{27}$$

Combining (25) with (27), the quality factor $Q = \omega W / P_d$ is

$$Q = \frac{k_c a \zeta}{2R_m(1 + a/d)}, \tag{28}$$

where $\zeta = k_c/(\omega\epsilon)$ is the wave impedance of the cavity.

Next, the same 7-step procedure will be used for the second and third examples.

B. Quadrupole Lens

Step 1

This step is more straightforward than in the previous example, because the upper integration limit in (12) is infinite. One thus defines

$$g(x) = e^{-x}, \tag{29}$$

$$h(x) = \frac{J_{2n+1}(x) J_{2k+1}(x)}{x}, \tag{30}$$

so that (12) can be written in the form (3), where $x \to t > 0$ and $y \to \lambda > 0$.

Step 2

The MT of $g(x)$ can be easily recognized as the definition of $\Gamma(x)$. Thus,

$$\tilde{g}(z) = \Gamma(z), \qquad \Re\mathrm{e}\{z\} > 0. \tag{31}$$

The question of existence of the MT of $h(x)$ can be addressed via the aforementioned tests of integral convergence. The MT can be calculated by software packages or tables, leading to

$$\tilde{h}(z) = \left(\frac{1}{2}\right)^{-z+2} \frac{\Gamma(2-z)}{\Gamma\left(n - k + \frac{3}{2} - \frac{z}{2}\right)} \times \frac{\Gamma\left(n + k + \frac{1}{2} + \frac{z}{2}\right)}{\Gamma\left(n + k + \frac{5}{2} - \frac{z}{2}\right)\Gamma\left(-n + k + \frac{3}{2} - \frac{z}{2}\right)}, \quad -(2n+2k+1) < Re\{z\} < 2. \tag{32}$$

Since the two MTs exist and the SOAs of $\tilde{g}(z)$ and $\tilde{h}(1-z)$ overlap, one can calculate the MT of the convolution as

$$I_2 = \frac{1}{2\pi i}\int_{\delta - i\infty}^{\delta + i\infty} \frac{1}{2} \frac{\Gamma(z)\Gamma(1+z)}{\Gamma\left(n - k + 1 + \frac{z}{2}\right)} \times \frac{\Gamma\left(n + k + 1 - \frac{z}{2}\right)}{\Gamma\left(n + k + 2 + \frac{z}{2}\right)\Gamma\left(-n + k + 1 + \frac{z}{2}\right)} (2x)^{-z} dz, \tag{33}$$

with $0 < \Re\mathrm{e}\{z\} < 2(n + k + 1)$.

Step 3

Table II shows the singularities of the various gamma functions in (33). Note that the SOA, as well as the ensuing δ, do not depend on the relation between n and k.

TABLE II: SINGULARITIES OF THE GAMMA FUNCTIONS OF (33). ALL SINGULARITIES ARE EITHER SIMPLE POLES OR SIMPLE ZEROS; m IS A NATURAL NUMBER.

Gamma Functions	z
$\Gamma(z)$	$-m$
$\Gamma(1+z)$	$-(m+1)$
$\Gamma(n+k+1-z/2)$	$2(m+n+k+1)$
$\Gamma(n-k+1+z/2)$	$-2(m+n-k+1)$
$\Gamma(n+k+2+z/2)$	$-2(m+n+k+2)$
$\Gamma(-n+k+1+z/2)$	$-2(m-n+k+1)$

To the right of the contour, there is only one semi-infinite lattice, at $z = 2(m + n + k + 1)$, and this lattice consists entirely of poles. To the left, there are more than one lattice. Furthermore, zeros sometimes cancel out poles, producing removable singularities. Thus, it is more convenient, if possible, to close the contour at right.

Step 4

The integral is already in the MB form. A simple calculation leads to $\Delta = 0$. Therefore, one must examine $\Pi = \ln(4/x)$, giving the result that the contour should be closed at right for $x > 4$, but at left for $0 < x < 4$. It is natural to first proceed with the simpler choice, which is $x > 4$, as already mentioned. The obtained results will thus be *initially* limited to $x > 4$, a point to be further discussed shortly.

Step 5

Having selected the contour, the residue theorem can be applied to (33). The selected contour encloses only the simple poles of $\Gamma(n + k + 1 - z/2)$. Therefore, (33) yields

$$I_2 = -\frac{1}{2}\sum_{m=0}^{\infty} \frac{\Gamma(2m+2n+2k+2)}{\Gamma(m+2n+2)} \times \frac{\Gamma(2m+2n+2k+3)}{\Gamma(m+2n+2k+3)\Gamma(m+2k+2)} \times \mathrm{Res}\left\{\Gamma\left(n+k+1-\frac{z}{2}\right)(2x)^{-z}, z = 2(m+n+k+1)\right\}, \tag{34}$$

in which the overall minus sign results from the *clockwise* orientation of the contour. Next, one can substitute the residues of the gamma functions to get

$$I_2 = \sum_{m=0}^{\infty} \frac{\Gamma(2m+2n+2k+2)}{\Gamma(m+2n+2)} \times \frac{\Gamma(2m+2n+2k+3)}{\Gamma(m+2n+2k+3)\Gamma(m+2k+2)} \times \frac{1}{m!}[-(2x)^{-2}]^m (2x)^{-2(n+k+1)}. \tag{35}$$

Step 6

In what is admittedly the most laborious and error-prone part of the MTM, one uses the duplication formula for all the gamma functions in (35) that depend on $2m$ and then introduces the Pochhammer's symbol to obtain

$$I_2 = \frac{1}{2\pi} \frac{\Gamma(n+k+1)\left[\Gamma\left(n+k+\frac{3}{2}\right)\right]^2}{\Gamma(2n+2)} \times \frac{\Gamma(n+k+2)}{\Gamma(2n+2k+3)\Gamma(2k+2)} \left(\frac{2}{x}\right)^{2n+2k+2} \tag{36}$$

$$\times\ {}_4F_3\left(n+k+1, n+k+\frac{3}{2}, n+k+\frac{3}{2}, n+k+2; 2n+2, 2n+2k+3, 2k+2; -\frac{4}{x^2}\right).$$

Though (36) can be viewed a final result, use of the duplication formula anew leads to the more polished formula

$$I_2 = (2t)^{-2(n+k+1)} \frac{\Gamma(2n+2k+2)}{\Gamma(2n+2)\Gamma(2k+2)} \times\ {}_4F_3\left(n+k+1, n+k+\frac{3}{2}, n+k+\frac{3}{2}, n+k+2; 2n+2, 2n+2k+3, 2k+2; -\frac{4}{t^2}\right). \quad (37)$$

in which the initial variable $t = x$ was reintroduced. To the authors' knowledge, this is a new expression; [26] stops at (35): see [26, eqn. 7.103], which is given without proof[1].

Eqn. (37) was derived subject to $t > 4$. However, the original integral (12), which is a LT, converges for all $\Re\{t\} > 0$ and is a complex-analytic function of t in the right-half plane $\Re\{t\} > 0$. The result in (37) is itself analytic[2] when $\Re\{t\} > 0$. *Thus, by analytic continuation,* (37) *holds for all* $\Re\{t\} > 0$. Note that, for the quadrupole lens, only real and positive values of t make physical sense. Yet a mention of complex values re-emphasizes the ubiquitous idea of analytic continuation.

There seems to be no further simplification of (37) and the procedure terminates at this point.

C. Open Spherical Shells

Step 1

First, cast (13) as a Mellin convolution by introducing

$$g(x) = e^{-x}, \quad (38)$$

$$h(x) = \left(\frac{2}{\pi b x}\right)^{\frac{1}{2}} \sum_{n=0}^{\infty} (4n+1)^{\frac{1}{2}} x_n J_{2n+1/2}(bx), \quad (39)$$

where $x \to 2a > 0$ and $y \to v > 0$.

Step 2

The MT of $g(x)$ is given in (31). In order to investigate the existence of the MT of $h(x)$, the aforementioned integral convergence tests can be employed, leading to $-2n < \Re\{z\} < 2$, which defines the corresponding SOA. The MT itself can be calculated symbolically or by combining appropriate tables with basic properties of the MT, leading to

$$\tilde{h}(z) = \left(\frac{2}{\pi b}\right)^{\frac{1}{2}} \times \sum_{n=0}^{\infty} (4n+1)^{\frac{1}{2}} \frac{x_n}{2} \left(\frac{b}{2}\right)^{-z+\frac{1}{2}} \frac{\Gamma\left(n+\frac{z}{2}\right)}{\Gamma\left(n+\frac{3}{2}-\frac{z}{2}\right)}, \quad (40)$$
$$-2n < \Re\{z\} < 2.$$

The MT of the convolution exists, since the individual MTs exist and the SOAs of $\tilde{g}(z)$ and $\tilde{h}(1-z)$ overlap. This is expressed as

$$I_3 = \frac{4a^2}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} \left(\frac{2}{\pi b}\right)^{\frac{1}{2}} \times \sum_{n=0}^{\infty} (4n+1)^{\frac{1}{2}} \frac{x_n}{2} \left(\frac{b}{2}\right)^{z-\frac{1}{2}} \frac{\Gamma(z)\Gamma\left(n+\frac{1}{2}-\frac{z}{2}\right)}{\Gamma\left(n+1+\frac{z}{2}\right)} x^{-z} dz, \quad (41)$$

with $0 < \Re\{z\} < 2n+1$.

Step 3

Due to the existence of n in (41), it is preferable to construct a table similar to Table III.

TABLE III: AS IN TABLE II, BUT FOR (41).

Gamma Functions	z
$\Gamma(z)$	$-m$
$\Gamma(n+(1-z)/2)$	$2(m+n)+1$
$\Gamma(n+1+z/2)$	$-2(m+n+1)$

The situation is less complicated to the right of the vertical contour, where only the simple poles of $\Gamma(n+(1-z)/2)$ exist.

Step 4

Simple calculations lead to $\Delta = 0$ and $\Pi = \ln(2/x)$, independent of n. Thus, one can close the contour at left for $0 < x < 2$ and at right for $x > 2$. For reasons already discussed, the contour is closed at right.

Step 5

Application of the residue theorem to (41) yields

$$I_3 = -\frac{4a^2}{b\sqrt{\pi}} \sum_{n=0}^{\infty} (4n+1)^{\frac{1}{2}} x_n \times \sum_{m=0}^{\infty} \frac{\Gamma(2m+2n+1)}{\Gamma\left(m+2n+\frac{3}{2}\right)} \times \operatorname{Res}\left\{\Gamma\left(n+\frac{1}{2}-\frac{z}{2}\right)\left(\frac{2x}{b}\right)^{-z}, z = 2(m+n)+1\right\}. \quad (42)$$

Using the residues of gamma functions, one immediately gets

$$I_3 = \frac{8a^2}{b\sqrt{\pi}} \sum_{n=0}^{\infty} (4n+1)^{\frac{1}{2}} x_n \left(\frac{2x}{b}\right)^{-(2n+1)} \times \sum_{m=0}^{\infty} \frac{\Gamma(2m+2n+1)}{\Gamma\left(m+2n+\frac{3}{2}\right)} \frac{1}{m!} \left[-\left(\frac{2x}{b}\right)^{-2}\right]^m. \quad (43)$$

Step 6

[1] And has an erroneous $\Gamma(2m+n+2)$ factor in the denominator, instead of the correct $\Gamma(m+2n+2)$.

[2] This is true even when the defining series (9) diverges, in which case the ${}_4F_3$ is defined via analytic continuation. In such a case, modern computer programs use means other than the series (9) to numerically calculate ${}_4F_3$. Usually, the specific means need not be of concern to the user, which can treat the programs as black boxes. See [16, chapt. 3] for further details; in particular, Case F2 of [16, chapt. 3], which discusses the possible branch points/branch cuts of ${}_4F_3$, can serve as a starting point for a digression into the important (but more advanced) topic of multivalued functions.

Using the duplication formula for the numerator, and with the aid of Pochhammer's symbol, one can obtain

$$I_3 = \frac{4a^2}{b\pi}\sum_{n=0}^{\infty}(4n+1)^{\frac{1}{2}}x_n\left(\frac{x}{b}\right)^{-(2n+1)} \times \frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma(n+1)}{\Gamma\left(2n+\frac{3}{2}\right)} \times {}_2F_1\left(n+\frac{1}{2}, n+1; 2n+\frac{3}{2}; -\frac{b^2}{x^2}\right), \tag{44}$$

The equation above can become slightly more attractive after a second use of the duplication formula, again for the numerator; this, in combination with the reintroduction of the initial quantities $x = 2a$ and $b = 2a\tan(\theta_0/2)$, gives

$$I_3 = \frac{2a}{\sqrt{\pi}}\sum_{n=0}^{\infty}(4n+1)^{\frac{1}{2}}x_n \times \frac{\tan^{2n}\frac{\theta_0}{2}}{2^{2n}}\frac{\Gamma(2n+1)}{\Gamma\left(2n+\frac{3}{2}\right)} \times {}_2F_1\left(n+\frac{1}{2}, n+1; 2n+\frac{3}{2}; -\tan^2\frac{\theta_0}{2}\right), \tag{45}$$

which is the final result and is identical to [26, eqn. 3.101]. This particular ${}_2F_1$ function, apparently, cannot be further simplified, thus concluding the evaluation.

V. Conclusion

The MTM is a powerful, versatile, and modern method for the calculation of definite integrals. Such integrals are encountered by electrical engineers in a great variety of contexts. This paper used the MTM to evaluate three integrals found in well-known electromagnetics textbooks (and also took the opportunity to correct a superficial error). Though alternative evaluations are possible, the MTM demonstrates two undeniable advantages: for one, it is straightforward and algorithmic, like a *recipe*, and as such its mastery is easily transferable and applicable to many more general integrals that have the form of Mellin convolutions, for another it acquaints students to a number of ubiquitous CA techniques, and using them in practice. It also introduces students, in a natural manner, to the generalized hypergeometric function, which is widely useful in both analytical and numerical settings.

Appendix

In this Appendix, alternative derivations of the main results are delineated. While sometimes less laborious, the proofs herein require knowledge of a number of Bessel-function identities. Even if these identities are interesting in their own right, such knowledge is more specialized than the general CA principles required to master the MTM, a method presented in this paper in a more-or-less algorithmic manner.

As already mentioned, [21] and [22] discuss orthonormality relations which can be used to evaluate integrals (involving Bessel functions of arbitrary order) of which I_1 is a special case.

To directly show that the first integral in (26) equals I_1, one can first derive the more general formula

$$\int_0^a r[J_1^2(yr) - J_0^2(yr)]dr = -\frac{a}{y}J_0(ay)J_1(ay), \tag{46}$$

where y is an arbitrary constant (note in passing that $y = 0$ is a *removable singularity* of the right-hand side, a point worth mentioning to students). For the special case $y = x_{0,1}/a$, (46) gives (26). To derive (46), differentiate both sides with respect to a and replace the resulting derivatives using the well-known identities $J_0'(z) = -J_1(z)$ and $J_1'(z) = J_0(z) - J_1(z)/z$. Finally, (46) holds for $a = 0$, completing the derivation.

The integral I_3 can be evaluated using the well-known Maclaurin-series expansion of $J_{2n+1/2}(bx)$. Term-by-term integration is then permissible because of the exponential (uniform) convergence of the integral, and one is eventually led to (45).

A similar procedure can evaluate I_2, because the (less well-known) Maclaurin-series expansion of the *product* $J_{2n+1}(\lambda)J_{2k+1}(\lambda)$ can be found in the literature [22]. The relevant series is

$$J_\nu(z)J_\mu(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!} \times \frac{\Gamma(\nu+\mu+2n+1)}{\Gamma(\nu+\mu+n+1)\Gamma(\nu+n+1)\Gamma(\mu+n+1)} \times \left(\frac{z}{2}\right)^{\nu+\mu+2n}. \tag{47}$$

Once again, term-by-term integration eventually yields the final result. (This method can be applied to I_1 as well, but the proofs in [21] and [22] are more elegant.)

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