Computing $\mathscr{H}_2$ norms and their derivatives for time-delay systems, using Krylov based model order reduction

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1 Introduction

The $\mathscr{H}_2$ norm plays an important role in the field of systems and control as a robustness measure with respect to noise or external disturbances and has numerous applications in many fields, such as robust optimization and $\mathscr{H}_2$ optimal control. For time-delay systems with transfer function

$$\gamma(s) := C(sI - \Sigma_{i=0}^{m} A_i e^{-s\tau_i})^{-1} B,$$

it was shown in [1], that the $\mathscr{H}_2$ norm $||\gamma(s)||_2$ can be calculated by rewriting the time-delay system in a linear infinite-dimensional form and applying a spectral discretization to become a system without delay. The $\mathscr{H}_2$ norm of this (larger) standard LTI system, $||\gamma_N(s)||_2$, can be easily calculated, resulting in an approximation for $||\gamma(s)||_2$. Downside of this method is the fact that the discretized model is of order nN, with n the size of the original time-delay system and N the number of discretization points. In [2], a Krylov based model order reduction technique for time-delay systems is suggested which allows to reduce the order of the discretized model further down to a chosen value $k << N$, matching several moments of $\gamma(s)$ and $\gamma_N(s)$ around zero and infinity. Our approach builds on this technique and extends it into a two-sided Krylov method. This not only allows to choose smaller values of k as we match double the amount of moments, but more importantly also to efficiently calculate derivatives of the $\mathscr{H}_2$ norm w.r.t. matrices of the original time-delay system.

2 Contributions

The idea behind the approach consists of discretizing a delay system of order n into a standard linear system of the form

$$G_N \dot{z}(t) = z(t) + H_N u(t), \quad y(t) = F_N z(t), \quad z(t) \in \mathbb{R}^{nN \times nN},$$

in such a way that several moments at zero and infinity are preserved. Subsequently, the system matrices F_N, G_N, H_N are projected on the Krylov spaces $\mathscr{K}_k(G_N, H_N)$ and $\mathscr{K}_k(G_N^T, (G_N^T)^{-1} F_N^T)$ as

$$\hat{F} = F_N V, \qquad \hat{G} = W^T G_N V, \qquad \hat{H} = W^T H_N,$$

where V, W contain the vectors respectively spanning the right and left Krylov spaces and $\hat{F}, \hat{G}, \hat{H}$ the system matrices of the reduced model of dimension k, such that these moments are again preserved, as shown in Fig 1. Moreover, our construction is such that the matrices of the reduced system are *independent of* N as long as $N > 2k$, hence the implementation does not involve a choice of N. As the reduced order system closely approximates the original time-delay system, we can approximate $||\gamma(s)||_2$ in function of the small matrices $\hat{F}, \hat{G}, \hat{H}$ of the reduced model.

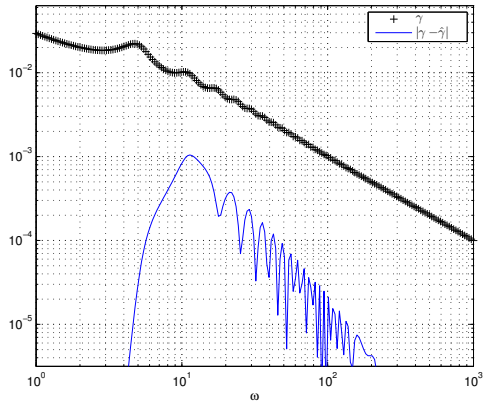


Figure 1: Frequency response and error after reducing a time-delay system or order $n = 100$ to a reduced model of order $k = 10$. Notice a smaller error around zero and towards infinity.

By constructing both a right and left Krylov space of G_N, we are able to not only evaluate the $\mathscr{H}_2$ norm, but also its *derivatives* w.r.t. the system matrices A_i, B, C in function of $\hat{F}, \hat{G}, \hat{H}$, thus again independently of N. For low-order feedback control, control parameters typically appear in the system matrices A_i, B, C, therefore being able to express and efficiently evaluate elements of $\frac{\partial(\gamma^2)}{\partial A_i}, \frac{\partial(\gamma^2)}{\partial B}, \frac{\partial(\gamma^2)}{\partial C}$ is most interesting from optimization and control point of view.

References

[1] J. Vanbiervliet, W. Michiels and E. Jarlebring, *Using spectral discretization for the optimal $\mathscr{H}_2$ design of time-delay systems*, Int. J. Control, 84(2):228-241, 2011.

[2] W. Michiels, E. Jarlebring and K. Meerbergen, *Krylov based model order reduction of time-delay systems*, SIAM J. Matrix Anal. Appl. 32(4):1399-1421, 2011.