Stabilisation of linear time-delay systems using nonsmooth optimisation techniques

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1 Introduction

We address the stabilisation of the general linear time-delay system

$$\dot{x}(t) = \sum_{i=0}^{m} A_i x(t-\tau_i) + \sum_{j=0}^{n} B_j u(t-\tau_j), \qquad (1)$$

controlled by a linear static state feedback

$$u(t) = Kx(t). \qquad (2)$$

Here, $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^p$ represent the respective state and input vector, A_j, B_j and K are matrices of appropriate dimensions and $\tau_i, \tau_j \geq 0$ are the fixed delays.

The stability of the closed-loop system (1)-(2) is determined by the real part of the rightmost eigenvalue, or spectral abscissa $\alpha(K)$. Indeed, a system is stable only if $\alpha(K) < 0$.

2 Stabilisation method

The proposed stabilisation method consists of solving directly the optimisation problem

$$\min_K \alpha(K) = \min_K \sup_\lambda \{\Re(\lambda) : \lambda \text{ is an eigenvalue}\} \qquad (3)$$

in function of the controller parameters contained in K. This is a hard problem for two reasons. Firstly, time-delay systems exhibit infinitely many eigenvalues, whereas only a finite number of degrees of freedom is available. Secondly, the objective function $\alpha(K)$ is a nonsmooth and even non-Lipschitz function of the parameters in K, and its minimisers are likely to be found in points of nondifferentiability. This prohibits the use of standard optimisation algorithms.

If the minimum of (3) is negative, the resulting system is stabilised and its trajectories follow a maximal exponential decay rate to the steady state.

The main advantage of this method over other approaches, as the one based on Lyapunov's second method, is that it does not introduce any conservatism, implying that stabilising controller parameters can in principle be found as long as they exist.

3 Optimisation algorithm

We use the gradient sampling algorithm presented in [1], a recently developed bundle gradient optimisation method that is able to optimise nonsmooth functions. It has already been successfully applied to finite-dimensional eigenvalue optimisation problems, where the dimension of the system is much larger than the number of controller parameters. Considering the infinite nature of time-delay systems, our results can be regarded as an extension to an infinite-dimensional setting.

The gradient sampling algorithm is essentially a generalisation of the steepest descent method that can deal with nonsmoothness, as it constructs at every iteration point x_k an approximation to the *nonsmooth* steepest descent direction. It does so by sampling several points in the near neighbourhood of x_k, and combining the gradients of these sampled points into a direction of common descent, along which a standard line search is then performed.

To evaluate the objective function $\alpha(K)$ itself, we must determine the supremum of all solutions to the nonlinear characteristic equation of the time-delay system

$$\det\left(\lambda I - \sum_{i=0}^{m} A_i e^{-\lambda\tau_i} - \sum_{j=0}^{n} B_j K e^{-\lambda\tau_j}\right) = 0.$$

For this, we have DDE-BIFTOOL at our disposal, a MATLAB package for the bifurcation analysis of nonlinear delay equations. It includes a routine that, using a linear multi-step approach, is able to compute the stability determining (i.e. rightmost) eigenvalues of a linear or linearised time-delay system with fixed delays, from which the value of $\alpha(K)$ is of course easily obtained.

We illustrate the effectiveness of our method by applying it to several numerical examples.

References

[1] J.V. Burke, A.S. Lewis, and M.L. Overton, "A robust gradient sampling algorithm for nonsmooth, nonconvex optimization", SIAM Journal on Optimization, 2005