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Recent developments in the theory of the fractional Fourier and linear canonical transforms

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Abstract

In recent years, there has been an enormous effort put in the definition and analysis of fractional or fractal operators. Fractional calculus is for example a flourishing field of active research. In this paper we restrict ourselves to the fractional Fourier operator and friends that are traditionally used in optics, mechanical engineering and signal processing. The book by H.M. Ozaktas, Z. Zalevsky, and M.A. Kutay, *The fractional Fourier transform*, John Wiley, 2001 gives a state of the art of 2001. Because this field is still in full expansion, we want to summarize in this survey paper some of the recent developments that appeared in the literature since then, revealing some unexplored aspects.

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1 Introduction

The idea of fractional powers of the Fourier operator appears in the mathematical literature as early as 1929 [120, 33, 58]. It has been rediscovered in quantum mechanics [76, 70], optics [73, 84, 13] and signal processing [14]. The boom in publications started in the early years of the 1990's and it is still going on. A recent state of the art can be found in [85] which contains an extensive bibliography. See also [57].

However, it is not only the Fourier transform that has been fractionalized. The term fractal or fractional is now available in almost everywhere: geometry, optics, mechanics, signal processing, numerical analysis, calculus. We shall restrict ourselves in this paper to a summary of the recent developments in the literature since 2000 concerning fractional operators used in signal processing. chemistry, dynamical systems, stochastic processes, ...

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2 The fractionalization of a linear operator

If $M \in \mathbb{R}^{n\times n}$ is a square diagonalizable matrix M then we may write its eigenvalue decomposition $M = E\Lambda E^{-1}$. Clearly for any integer a it holds that

$$M^a = E\Lambda^a E^{-1}. \tag{1}$$

So it is a natural generalization to use the same formula as a definition if a is not integer. We say that we have a fractional (in fact any real) power of M.

Exactly the same idea can be used for a linear operator $\mathcal{M}$ on a linear space if it has a sequence of eigenvectors that is complete in the whole space [129]. Let $\{\lambda_k, e_k\}_{k=0}^{\infty}$ be the sequence of eigenvalues and corresponding eigenvectors. Since the set of eigenvectors is complete, we can associate with each element f in the Hilbert space a unique set of coordinates and conversely, if we are given a sequence of coordinates, we know to which element f they belong. These mappings are called the *analysis* and the *synthesis* operators respectively. They are adjoint operators. If $\mathcal{E}$ is the synthesis operator and $\mathcal{E}^*$ the analysis operator, which for a given set of basis vectors $\{e_k\}$ are defined by

$$\mathcal{E} : \{c_k\}_{k=0}^{\infty} \mapsto f = \sum_{k=0}^{\infty} c_k e_k \quad \text{and} \quad \mathcal{E}^* : f \mapsto \{c_k\}_{k=0}^{\infty}, \tag{2}$$

then we can write

$$\mathcal{M} = \mathcal{E}\Lambda\mathcal{E}^* \tag{3}$$

where Λ is the simple diagonal scaling operator

$$\Lambda : \{c_k\}_{k=0}^{\infty} \mapsto \{\lambda_k c_k\}_{k=0}^{\infty}. \tag{4}$$

Its fractional power is then clearly $\mathcal{M}^a = \mathcal{E}\Lambda^a\mathcal{E}^*$ since $\mathcal{E}^*\mathcal{E}$ is the *expansion operator*, or if we identify the function with its series expansion, it can be considered as the identity operator.

If the space is not spanned by a countable set, but by a set of eigenfunctions depending on a continuous variable $\xi \in \mathbb{R}$, again the same scheme is applicable:

$$\mathcal{E} : \{c(\xi)\}_{\xi\in\mathbb{R}} \mapsto f = \int c(\xi)e(\xi)\mathrm{d}\xi \quad \text{and} \quad \mathcal{E}^* : f \mapsto \{c(\xi)\}_{\xi\in\mathbb{R}}, \tag{5}$$

then we can write

$$\mathcal{M} = \mathcal{E}\Lambda\mathcal{E}^* \tag{6}$$

where Λ is the simple diagonal scaling operator

$$\Lambda : \{c(\xi)\}_{\xi\in\mathbb{R}} \mapsto \{\lambda(\xi)c(\xi)\}_{\xi\in\mathbb{R}}. \tag{7}$$

Of course all these relations will only hold under suitable conditions for the sums and integrals to converge.

An example is the fractional derivative. The derivative operator has eigenvectors $\{e^{i\xi x} : \xi \in \mathbb{R}\}$ and eigenvalues $\{j\xi : \xi \in \mathbb{R}\}$. So the analysis operator is the Fourier transform, Λ is the scaling operator with the eigenvalue $j\xi$ and the synthesis operator is the inverse Fourier transform. This is indeed a well known property of the Fourier transform: a derivative in the x-domain corresponds to a multiplication with $j\xi$ in the ξ-domain. The fractional form of the derivative is thus obvious: we multiply with $(j\xi)^a$ in the Fourier domain instead of

with $j\xi$. This is the basis of the whole theory of fractional calculus [102], a domain that goes back to a question raised by Leibniz in 1695. See [80, 74] for an introduction and some history.

The subject of fractional calculus is broad and still growing. In this survey, we shall not follow this path any further. Instead we shall concentrate on another set of fractional operators that evolved from the fractional Fourier transform and other integral transforms that are often used in signal processing applications. Their spectrum will be discrete as opposed to the continuous spectrum of the differentiation operator.

3 The fractional Fourier transform

The fractional form of the classical Fourier operator

$$\mathcal{F} : f \mapsto \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} f(x) \mathrm{d}x. \tag{8}$$

will be at the heart of this survey. The fractionalization fits into the second scheme mentioned in the previous section when we restrict ourselves to the Hilbert space $L^2 = L^2(\mathbb{R})$. A set of eigenfunctions that is conventionally agreed upon is given by the Gauss-Hermite functions $\psi_n(t)$

$$\psi_n(x) = \frac{2^{1/4}}{\sqrt{2^n n!}} e^{-x^2/2} H_n(x), \quad \text{where} \quad H_n(x) = (-1)^n e^{x^2} \frac{d^n e^{-x^2}}{dx^n}, \tag{9}$$

and the eigenvalues are $\lambda_n = e^{-in\pi/2}$, $n = 0, 1, \ldots$. Since $\{\psi_n\}$ forms an orthonormal set, $c_n = \langle f, \psi_n \rangle$. Thus $\mathcal{E}^* : L^2 \to \ell^2 : f \mapsto \{\langle \psi_n, f \rangle\}$. So for $a \in \mathbb{R}$, the FrFT of f is defined as

$$[\mathcal{F}^a f](\xi) = \sum_{n=-\infty}^{\infty} \langle f, \psi_n \rangle \, \lambda_n^a \psi_n(\xi) = \int_{-\infty}^{\infty} f(x) k_a(\xi, x) \mathrm{d}x, \tag{10}$$

where the kernel $k_a(\xi, x)$ of the integral representation

$$k_a(\xi, x) = \sum_{n=0}^{\infty} \lambda_n^a \psi_n^*(x) \psi_n(\xi). \tag{11}$$

Using Mehler's formula [62, p. 61] it turns out that the kernel equals

$$k_a(\xi, x) = C_a K_a(\xi, x) \exp\left\{-ix\xi \csc\alpha\right\}, \tag{12}$$

$$K_a(\xi, z) = \exp\left\{\frac{i}{2}(x^2 + \xi^2)\cot\alpha\right\}, \quad C_a = \frac{\exp(i\alpha/2)}{\sqrt{2\pi i \sin\alpha}} = \sqrt{\frac{1 - i\cot\alpha}{2\pi}}, \quad \alpha = a\frac{\pi}{2}. \tag{13}$$

If $\sin\alpha = 0$, then it is defined by a limiting process, which reduces the kernel to a Dirac delta: $\delta(x \pm \xi)$.

The fractional Fourier transform is very popular in optical system analysis. The reason is the following. Consider an object illuminated by some light source and place several optical components (lenses, and other optical media) on the axis formed by source and object. Then, moving a screen along this axis, it is possible that under certain ideal conditions, we see an image or an inverted image of the object on the screen. This is an illustration of the fact that the Fourier operator is just inverting the axis when it is applied twice and it is the

identity operator when applied four times. In between the place of the upright and the inverted image, there is a place where the Fourier transform or the inverse Fourier transform of the image can be seen. But when moving the screen continuously, we shall not only see projections of the Fourier transform, the inverse object, the inverse Fourier transform, and the upright object. The projection on the screen will visualize all the fractional powers of the Fourier transform.

Quantum mechanics is another domain where the FrFT appears naturally. The wave function of an harmonic oscillator satisfies the Schrödinger equation. The wave function at instant t can be obtained from the initial wave function at time $t = 0$ by an integral transform whose kernel is the Green function. It can be shown [68] that this Green kernel is essentially the kernel of the FrFT. See also [122].

When considering the class of all FrFT, which is parameterized by the parameter a, then it has a group structure, called the elliptic group [122]. It is like a rotation group since $\mathcal{F}^a\mathcal{F}^b = \mathcal{F}^{a+b}$ and $\mathcal{F}^0$ is the identity and the inverse is obviously $(\mathcal{F}^a)^{-1} = \mathcal{F}^{-a}$. It does indeed operate in the (x, ξ)-plane like a rotation (see below) which may be characterized by a rotation matrix

$$R_a = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}, \quad \text{where } \alpha = a\pi/2 \tag{14}$$

In the case of the Fourier transform, x is often a time variable and ξ is the frequency variable when $a = 1$. So when looked at in the time-frequency plane, then the Fourier transform will map the representation of the signal as a function of time into a representation of the same signal as a function of frequency, which corresponds to a a rotation over an angle $\pi/2$ (assuming time and frequency axes are orthogonal). The FrFT generalizes this to a rotation over any angle $a\pi/2$ in the time-frequency plane.

The definition of the FrFT that we gave here is the standard one. However, there are other eigenfunctions possible, which lead to alternative definitions. A systematic investigation of the eigenfunctions of the Fourier operator is undertaken in [26] which gives rise to nonstandard definitions of the FrFT.

The best handbook available for the moment on the fractional Fourier transform is [85]. However, there do appear many papers in this domain, and the purpose of this paper is to sketch the additional achievements of the last couple of years.

4 The linear canonical transform

Besides the fractional Fourier transform (FrFT), a much more general class of linear canonical transforms (LCT) has been studied [122, 1, 22, 95]. The linear canonical transforms are characterized by a general matrix A which need not be restricted to a rotation matrix, but it will represent any affine linear transformation in the (x, ξ)-plane that is characterized by a general 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \det(A) = ad - bc = 1. \tag{15}$$

However, for typographical convenience, we shall often denote this matrix as $A = (a, b, c, d)$ in the text, but all operations have to be understood in the usual matrix sense. Obviously, there are 3 free parameters. It is an integral transform denoted as $[\mathcal{F}^A f](\xi) = \int_{-\infty}^{\infty} k_A(\xi, x) f(x) \mathrm{d}x$ whose kernel is

$$k_A(\xi, x) = C_A K_A(\xi, x) \exp\{-i\xi x/b\} \tag{16}$$

$$K_A(\xi,x)=\exp\left\{\frac{i}{2}\left(\frac{d}{b}\xi^2+\frac{a}{b}x^2\right)\right\},\quad C_A=\frac{1}{\sqrt{2\pi ib}}. \tag{17}$$

Clearly for $A=R_a$, we get the FrFT of order a, except for a constant unimodular factor $e^{i\alpha/2}$. I.e., $\mathcal{F}^a=e^{i\alpha/2}\mathcal{F}^A$ if $A=R_a$.

If $b=0$, the transform is defined by a limiting process, which reduces the kernel to

$$k_a(\xi,x)=\pm a^{-1/2}\exp\{ic\xi^2/2a\}\delta(x-\xi/a). \tag{18}$$

Unless otherwise stated, we shall restrict ourselves to the case of real matrices A. In that case the LCT $\mathcal{F}^A$ is a unitary operator in $L^2(\mathbb{R})$.

The FrFT is not the only particular case for real A. Four other special cases are

- *The Fresnel transform:*
  It is defined as
  $$[\mathcal{F}resnel^z f](\xi):=\frac{e^{i\pi z/l}}{\sqrt{ilz}}\int_{-\infty}^{\infty}e^{i(\pi/lz)(\xi-x)^2}f(x)\mathrm{d}x \tag{19}$$
  This is obtained for $A=(a,b,c,d)=(1,b,0,1)$ in the sense that with $b=\frac{zl}{2\pi}$ we have $\mathcal{F}resnel^z=e^{i\pi z/l}\mathcal{F}^A$.

- *Dilation:* Defining the dilation operation as $f(x)\mapsto\sqrt{s}f(s\xi)$, then it can be obtained as with $A=(a,b,c,d)=(1/s,0,0,s)$ writing the operation as $\sqrt{\mathrm{sgn}(s)}\mathcal{F}^A$.

- *Gauss-Weierstrass transform or chirp convolution:*
  This is the transform obtained for $A=(a,b,c,d)=(1,b,0,1)$:
  $$[\mathcal{F}^A f](\xi)=C_A\int_{-\infty}^{\infty}\exp\{i(x-\xi)^2/2b\}f(x)\mathrm{d}x. \tag{20}$$

- *Multiplication by a Gaussian or chirp multiplication:*
  A transformation obtained for $A=(a,b,c,d)=(1,0,c,1)$:
  $$[\mathcal{F}^A f](\xi)=\exp\{ic\xi^2/2\}f(\xi). \tag{21}$$

The eigenfunctions and eigenvalues of the LCT can be found in [122, 54, 95].

Since the kernel is Hermitian, the kernel for the inverse transform is given by

$$k_A^{-1}(\xi,x)=k_A^*(\xi,x)=k_A(x,\xi). \tag{22}$$

Moreover, the group structure implies that

$$\mathcal{F}^A\mathcal{F}^B=\mathcal{F}^C\quad\text{with}\quad C=AB. \tag{23}$$

We shall call this the additivity property for LCTs.

Note that this definition is given for a signal that is nonperiodic and depending on a continuous variable x. Essentially the same type of definition can be used for periodic signals and/or signals that depend on a discrete variable. See also [87] for fractional Fourier series and discrete time fractional Fourier transforms. See also [25] for a uniform treatment in the case of the FrFT. It is clear by that by the general procedure explained above, we can also fractionalize the LCT itself.

5 Geometric interpretation

The meaning of the transformation made by the LCT can be illustrated by its effect on the Wigner distribution.

Let f be a signal, then its *Wigner distribution* or *Wigner transform* $\mathcal{W}f$ is defined as

$$[\mathcal{W}f](x,\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x+u/2)\overline{f(x-u/2)}e^{-i\xi u}\mathrm{d}u. \tag{24}$$

Its meaning is roughly speaking one of energy distribution of the signal in the (x,ξ)-plane. Indeed, setting $f_1 = \mathcal{F}f$, we have

$$\int_{-\infty}^{\infty} [\mathcal{W}f](x,\xi)\mathrm{d}\xi = |f(x)|^2 \quad \text{and} \quad \int_{-\infty}^{\infty} [\mathcal{W}f](x,\xi)\mathrm{d}x = |f_1(\xi)|^2, \tag{25}$$

so that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [\mathcal{W}f](x,\xi)\mathrm{d}\xi\mathrm{d}x = \|f\|^2 = \|f_1\|^2, \tag{26}$$

which is the energy of the signal f.

The effect of the LCT on the Wigner distribution is that if $f_A(\xi) = [\mathcal{F}^A f](\xi)$ is the LCT of f and if the Wigner distributions of f and f_A are given by W_f and W_{f_A} respectively, then

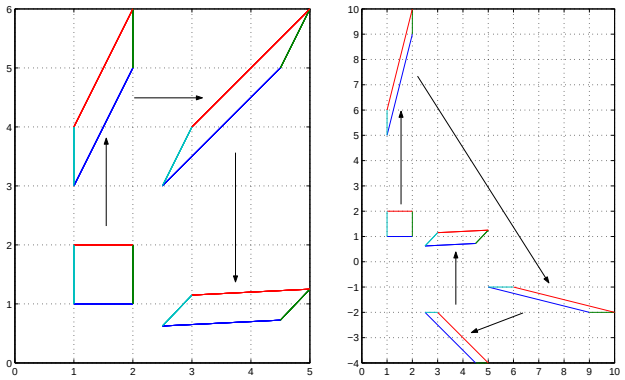
$$W_{f_A}(x',\xi') = W_f(x,\xi) \tag{27}$$

where

$$\mathbf{x}' := \begin{bmatrix} x' \\ \xi' \end{bmatrix} = A \begin{bmatrix} x \\ \xi \end{bmatrix} =: A\mathbf{x}. \tag{28}$$

In other words if $\mathcal{M}_A : w(\mathbf{x}) \mapsto w(A\mathbf{x})$, then $\mathcal{M}_A\mathcal{W}\mathcal{F}^A = \mathcal{W}$. Thus in the case of the FrFT, the Wigner distribution of f is *rotated* over and angle α. In the case of the LCT, the Wigner distribution has been transformed by a more general affine transform as illustrated in Figure 1.

Figure 1: The effect of a LCT on a square. Left when the matrix A is decomposed as in (46) and right when it is decomposed as in (47). The eventual result is of course the same.



The *ambiguity function*, defined as

$$[\mathcal{A}f](x,\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u+x/2)\overline{f(u-x/2)}e^{-i\xi u}\mathrm{d}u \tag{29}$$

is essentially the two-dimensional Fourier transform of the Wigner distribution: if $\mathcal{F}$ is the two-dimensional Fourier transform, then $[\mathcal{A}f](x\xi) = 2\pi[\mathcal{F}\mathcal{W}f](-x,\xi)$. Whereas the Wigner distribution gives an idea about how the energy of the signal is distributed in the (x,ξ)-plane, the ambiguity function will have a correlative interpretation. Indeed $[\mathcal{A}f](x,0)$ is the autocorrelation function of f and $[\mathcal{A}f](0,\xi)$ is the autocorrelation function of $f_1 = \mathcal{F}f$. It can be shown that also for the ambiguity transform we have $\mathcal{M}_A\mathcal{A}\mathcal{F}^A = \mathcal{A}$.

In fact such a relation holds for any (x,ξ) representation of the Cohen class [32].

For a more detailed analysis of the effect of the parameters of the LCT on the Wigner distribution and ambiguity function see [93]. Some properties about the signal can be obtained by considering the Wigner or the ambiguity transform of the Wigner or the ambiguity transform [50], which are then called quartic transforms.

6 Fractional operations

Several operations like convolution, correlation, x-shift, ξ-shift etc can be defined in the fractional domain. See [5, 25, 38]. As the LCT is unitary we can use the same approach as in [5] to define shifts in the LCT domain. Define the unitary operator denoting a shift in the x-domain as follows

$$[\mathcal{T}_0(x')f](x) = f(x-x'). \tag{30}$$

Then we may define a unitary shift in the LCT domain as

$$\mathcal{T}_A(x_A) = (\mathcal{F}^A)^{-1}\mathcal{T}_0(x_A)\mathcal{F}^A. \tag{31}$$

A frequency shift $\mathcal{T}_1(\xi') : f(x) \mapsto e^{i\xi' x}f(x)$ can then be represented by $\mathcal{T}_1(\xi') = \mathcal{F}^{-1}T_0(\xi')\mathcal{F}$.

Other operations can be defined in a similar way. Defining the $L^2(\mathbb{R})$ inner product as

$$\langle f,g\rangle = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\overline{g(x)}\mathrm{d}x, \tag{32}$$

the convolution in the x-domain can be defined as

$$(f *_0 g)(x) = \left\langle f, \mathcal{F}^2\mathcal{T}_0(x)g^*\right\rangle = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x')g(x-x')\mathrm{d}x', \tag{33}$$

where $[\mathcal{F}^2\mathcal{T}_0(x)g^*](x') = g^*(x-x')$.

With this notation, it is easy to denote the ambiguity function as follows. Set

$$F(x,u,\xi) = \mathcal{T}_1(\xi/2)\mathcal{T}_0(u/2)f(x) = e^{ix\xi/2}f(x-u/2), \tag{34}$$

then

$$[\mathcal{A}f](u,\xi) = \langle F(x,-u,-\xi), F(x,u,\xi)\rangle \tag{35}$$

where the inner product is with respect to the variable x.

The convolution in the LCT domain can then be defined as

$$(f *_A g)(x) = \left\langle f, \mathcal{F}^2\mathcal{T}_A(x)g^*\right\rangle. \tag{36}$$

Note that for $A = I$, $\mathcal{F}^A$ is the identity and therefore the classical convolution is recovered as $f *_I g = f *_0 g$. Note that this definition is inspired by [5] and differs from the more common definition (37) given below. Similar definitions can be given for correlation operations etc.

It is well known that $h = f *_0 g$ if and only if $\mathcal{F}h = \mathcal{F}f \times \mathcal{F}g$. Therefore the convolution in the x-domain and the multiplication in the domain of its Fourier transform are called dual operations. Similarly, one can define a dual operation "$\circ_A$" for a general fractional operation "$\bullet_A$". For a general framework to obtain such dualities in the case of the FrFT, we refer to [59].

In fact such duality is the basis for giving alternative definitions of fractional operations. Again we restrict ourselves to the convolution. A proposal by Mustard [75] (see also [85, p. 420]) is essentially to define the fractional convolution as

$$h = f *_A g \Leftrightarrow \mathcal{F}^A h = \mathcal{F}^A f \times \mathcal{F}^A g. \tag{37}$$

Note that we may consider $G = \mathcal{F}^A g$ as the transfer function of a linear system. This is useful in analyzing the role of the parameters (a, b, c, d) of the LCT.

Also other more general operations can be defined in such a way. For example a canonical correlation of f and g can be defined as (see [91])

$$\mathcal{F}^{A_3}[(\mathcal{F}^{A_1} f) \times (\mathcal{F}^{A_2} g)^*], \tag{38}$$

where it is required that

$$\frac{d_1}{b_1} + \frac{a_3}{b_3} = \frac{d_2}{b_2}. \tag{39}$$

The latter type of fractional operations can be seen as general filtering operations in the fractional domain. Indeed, writing $g_A = \mathcal{F}^A g$ then $h = f *_A g$ means that

$$h = (\mathcal{F}^A)^{-1}\big([\mathcal{F}^A f] \times g_A\big) \tag{40}$$

i.e., we multiply with a filter function g_A in the x_A domain. In this sense, the *Hilbert transform* is also of this type since with $g(x) = 1/x$,

$$[\mathcal{H}ilbert\, f](x) = \sqrt{\frac{2}{\pi}}(f *_0 g)(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(x')}{x - x'}\mathrm{d}x' \tag{41}$$

(integral in the sense of principal value). So in the frequency domain, we have to multiply with the signum function. Thus some fractional generalization along these lines is obvious. See section 11.

A somewhat related problem is the so called windowed fractional Fourier transform. Here the windowed Fourier transform is applied to the FrFT of the signal. Thus it is defined as $\mathcal{F}^{-a}[h \times \mathcal{F}^a f]$, where h is the window function. Applying a window in the fractional Fourier domain may have advantages if in that domain the signal is more concentrated, and thus better caught in a narrow window. See [112]. An optimal fractional domain with minimal spread can be computed from the moments [9]. When the optimal rotation angle is found, there will be a better reduction of cross terms from the different components in the signal [11]. What has been done for the FrFT could also be done for the LCT, but we are not aware of publications on this aspect.

7 Simplified linear canonical transform

Often in optical applications, of the 3 free parameters in the LCT, it is only the ratio a/b that is important. Consider an optical systems whose image is given by the transformation

$f_{A_1} = \mathcal{F}^{A_1} f$ and suppose we want to model it by a (simpler) system described by $f_{A_2} = \mathcal{F}^{A_2} f$, where $A_1 = (a_1, b_1, c_1, d_1)$ and $A_2 = (a_2, b_2, c_2, d_2)$. In defining the model, it will be important to fix the characteristics of the image that we want to match. The following property holds [6]. If $s = a_1/a_2 = b_1/b_2$, then

$$|\mathcal{F}^{A_2} f| = |\mathcal{F}^S \mathcal{F}^{A_1} f| \tag{42}$$

where $S = (1/s, 0, 0, s)$ represents a scaling of the variable. This implies that the results of the two transforms $\mathcal{F}^{A_1}$ and $\mathcal{F}^{A_2}$ will have the same intensity if $(a_1, b_1) = (a_2, b_2)$. If the scaling is ignored, and only the intensity is important, then it is sufficient to choose $a_1/b_1 = a_2/b_2$. In the latter case, the remaining variables in the model are chosen freely (subject to the constraint $a_2 d_2 - b_2 d_2 = 1$).

Another approach which leads to the same conclusion can be found in [91]. There it is shown that if we define two filtering operations in the x_A-domain (see the previous section)

$$h_i = (\mathcal{F}^{A_i})^{-1}[(\mathcal{F}^{A_i} f) \times G_i], \quad i = 1, 2 \tag{43}$$

then $h_1 = h_2$ if $a_1/b_1 = a_2/b_2$ and $G_2(x_A) = G_1(b_1 x_A/b_2)$. Thus if $a_1/b_1 = a_2/b_2$, then whatever operation that can be obtained with A_1 can also be obtained by A_2. I.e., A_1 and A_2 are equivalent.

Also the effect of a/b on the Wigner distribution of $\mathcal{F}^A f$ discussed in [93] leads to a similar conclusion.

The freedom to choose the parameters of the LCT for a given ratio a/b can lead to a simplification in the computation.

For example, if in the FrFT we want to keep the ratio $a/b = \cot\alpha$, then this can be realized as a LCT with parameters $A = (\cot\alpha, 1, -1, 0)$ (type 1) or $A = (1, \tan\alpha, -2\cot\alpha, -1)$ (type 2) or still other types which are inspired by applications or by its optical realization. See [91].

8 Computational aspects

Note that the kernel of the LCT contains a factor of the form

$$\exp\left\{\frac{i}{2b}(d\xi^2 + ax^2 - 2x\xi)\right\} = \exp\left\{\frac{i}{2b}\left[(d-1)\xi^2 + (a-1)x^2 + (x-\xi)^2\right]\right\} \tag{44}$$

which means that the LCT can be decomposed into several steps. Let us define chirp functions

$$c_r(x) = \exp\left\{i\frac{r}{2}x^2\right\} \tag{45}$$

then the successive steps are

1. chirp multiplication: $\mathcal{C}_{\frac{a-1}{b}} : f(x) \mapsto g(x) = f(x) c_{\frac{a-1}{b}}(x)$
2. chirp convolution: $g(x) \mapsto h(\xi) = C_A[\mathcal{F}g](\xi)[\mathcal{F}c_{1/b}](\xi)$
3. chirp multiplication: $\mathcal{C}_{\frac{d-1}{b}} : h(\xi) \mapsto f_A(\xi) = h(\xi) c_{\frac{d-1}{b}}(\xi)$.

This corresponds to the following decomposition of the LCT matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ (d-1)/b & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ (a-1)/b & 1 \end{bmatrix}. \tag{46}$$

As we have mentioned above, the rightmost and leftmost factors define chirp multiplications, while the middle factor corresponds to a chirp convolution.

Another interesting decomposition is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ db^{-1} & 1 \end{bmatrix} \begin{bmatrix} b & 0 \\ 0 & b^{-1} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ b^{-1}a & 1 \end{bmatrix}. \tag{47}$$

In this case the computation can be read off from right to left:

1. chirp multiplication: $\mathcal{C}_{b^{-1}a} : f(x) \mapsto g(x) = f(x)c_{b^{-1}a}(x)$
2. Fourier transform: $\mathcal{F} : g(x) \mapsto h(\xi) = [\mathcal{F}g](\xi)$
3. dilation: $\mathcal{D}_b : h(\xi) \mapsto k(\xi) = g(\xi/b)/\sqrt{b}$
4. chirp multiplication: $\mathcal{C}_{db^{-1}} : k(\xi) \mapsto f_A(\xi) = k(\xi)c_{db^{-1}}(\xi)$

so that $\mathcal{F}^A = \mathcal{C}_{db^{-1}}\mathcal{D}_b\mathcal{F}\mathcal{C}_{b^{-1}a}$. The latter is the decomposition that was exploited in [59].

For the digital computation, we have to discretize the variables under the restriction of the Nyquist sampling theorem. Using a fast Fourier transform, we can derive a fast algorithm for the computation [35, 83, 45, 60].

Suppose that the Wigner distribution of the original signal f is supported in the (x, ξ)-plane within a circle with $\|\mathbf{x}\| < \Delta/2$ where $\mathbf{x} = [x, \xi]^T$. Then the Wigner distribution of $f_A = \mathcal{F}^A f$ will be supported in a circle with radius $m\Delta/2$ where $m = \|A\|$. For example in the case of the FrFT where $A = R_\alpha$, it is not difficult to show that $\|A\| \leq 2$. Thus if we want to recover the result from discrete samples, we need to have at least mN samples. Therefore some sinc interpolation is used to insert samples in between two given samples before the chirp convolution is computed. For more details in the case of the FrFT we refer to the cited references. The case of the LCT is very similar. See for example [90, 96] for the use of the DCT technique to evaluate all types of fractional and linear canonical transforms efficiently. For a filter bank implementation see [53].

9 Multidimensional linear canonical transform

So far we considered functions f in only one variable x. However the same ideas can be used to obtain definitions for the multivariate case.

The most simple solution is a tensor product form. This means that we define the n-dimensional (n-D) LCT as the result of applying subsequently n LCTs to each of the variables separately. Thus, if we denote in bold face the n-tuples of the corresponding one-dimensional variables, then the n-dimensional LCT can be defined as

$$f_{\mathbf{A}}(\boldsymbol{\xi}) = [\mathcal{F}^{\mathbf{A}} f](\boldsymbol{\xi}) = \int_{-\infty}^{\infty} k_{\mathbf{A}}(\boldsymbol{\xi}, \mathbf{x}) f(\mathbf{x}) \mathrm{d}\mathbf{x}, \tag{48}$$

where $k_{\mathbf{A}}(\boldsymbol{\xi}, \mathbf{x}) = \prod_{j=1}^{n} k_{A_j}(\xi_j, x_j)$. It is convenient to define the matrix $\mathbf{A}$ as the square matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{bmatrix}, \quad \text{where} \quad A_j = \begin{bmatrix} a_j & b_j \\ c_j & d_j \end{bmatrix}, \quad j = 1, \ldots, n \quad \text{and}$$
$$\mathbf{a} = \mathrm{diag}(a_1, \ldots, a_n), \quad \mathbf{b} = \mathrm{diag}(b_1, \ldots, b_n),$$
$$\mathbf{c} = \mathrm{diag}(c_1, \ldots, c_n), \quad \mathbf{d} = \mathrm{diag}(d_1, \ldots, d_n).$$

Such a definition implies that almost all the properties of the 1-D case are maintained. For example we have the composition rule $\mathcal{F}^{\mathbf{A}}\mathcal{F}^{\mathbf{B}} = \mathcal{F}^{\mathbf{C}}$ with $\mathbf{C} = \mathbf{AB}$, which means that $C_j = A_jB_j$, $j = 1, \ldots, n$. Also, the affine transformation of the Wigner distribution is as before. Indeed the n-D Wigner distribution is defined as

$$[\mathcal{W}f](\mathbf{x}, \boldsymbol{\xi}) = (2\pi)^{-n/2} \int_{-\infty}^{\infty} f(\mathbf{x} + \mathbf{u}/2)\overline{f(\mathbf{x} - \mathbf{u}/2)}e^{-i\boldsymbol{\xi}\cdot\mathbf{u}}\mathrm{d}\mathbf{u} \tag{49}$$

where we have used the dot to denote the inner product of the two vectors: $\boldsymbol{\xi}\cdot\mathbf{u} = \sum_{j=1}^{n} \xi_j u_j$. It can then be shown along the same lines as in the scalar case that

$$[\mathcal{W}f_{\mathbf{A}}](\mathbf{x}', \boldsymbol{\xi}') = [\mathcal{W}f](\mathbf{x}, \boldsymbol{\xi}), \quad \begin{bmatrix} \mathbf{x}' \\ \boldsymbol{\xi}' \end{bmatrix} = \mathbf{A} \begin{bmatrix} \mathbf{x} \\ \boldsymbol{\xi} \end{bmatrix}. \tag{50}$$

The nonseparable (affine) transform corresponds to taking general instead of diagonal matrices $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$. The kernel does not factor anymore. The problem has been considered in [114], where the authors are looking for unitary transforms $\mathcal{F}^{\mathbf{A}}$ that have the linear transformation effect (50) on the Wigner distribution. These are called metaplectic transforms. (See also [34, 43].) It turns out that $\mathbf{A}$ needs to be symplectic (hence $|\det \mathbf{A}| = 1$) and so they came to the following definition based on the decomposition (47):

$$\mathcal{F}^{\mathbf{A}} = \mathcal{C}_{\mathbf{db}^{-1}}\mathcal{D}_{\mathbf{b}}\mathcal{F}\mathcal{C}_{\mathbf{b}^{-1}\mathbf{a}}. \tag{51}$$

where the n-D Fourier transform is the classical one:

$$[\mathcal{F}f](\boldsymbol{\xi}) = (2\pi)^{-n/2} \int_{-\infty}^{\infty} f(\mathbf{x})e^{-i\mathbf{x}\cdot\boldsymbol{\xi}}\mathrm{d}\boldsymbol{\xi}, \tag{52}$$

the dilation operator stands for

$$[\mathcal{D}_{\mathbf{b}}f](\mathbf{x}) = \frac{1}{\sqrt{\det\ \mathbf{b}}} f(\mathbf{b}^{-1}\mathbf{x}). \tag{53}$$

and an n-D chirp is $c_{\mathbf{r}}(\mathbf{x}) = \exp\{\frac{i}{2}\mathbf{x}^T\mathbf{r}\mathbf{x}\}$. However, besides the standard condition $\det\ \mathbf{A} = \pm 1$, it is required that $\mathbf{b}$ is symmetric. The reason why $\mathbf{b}$ has to be symmetric is that in the block case, the dilation is described by a matrix of the form

$$\begin{bmatrix} \mathbf{b}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{b}^T \end{bmatrix} \tag{54}$$

which does not fit into the decomposition (47) of $\mathbf{A}$. The definition of [114] also requires $\mathbf{db}^{-1}$ and $\mathbf{b}^{-1}\mathbf{a}$ to be symmetric. But that is no real restriction because these matrices appear in chip expressions and if $\mathbf{r}$ is not symmetric, then the the quadratic form $\mathbf{x}^T\mathbf{r}\mathbf{x}$ can always be rewritten as $\frac{1}{2}\mathbf{x}^T(\mathbf{r} + \mathbf{r}^T)\mathbf{x}$ (if $\mathbf{r}$ is real). The previous definition requires $\mathbf{b}$ to be invertible, but that condition can be removed [115].

For non separable 2-D FrFT definitions and properties see also [109, 108, 115, 94].

10 Radial canonical transforms

It is well known that the Hankel transform appears naturally as the radial part of the Laplace operator expressed in cylindrical coordinates [122, sec. 8.4]. This makes a link with the n-D

FrFT. Indeed, when dealing with a circular symmetry in the problem, it is advantageous to write the FrFT in appropriate coordinates.

In general the canonical transforms for such a situation are known as *radial canonical transforms* [121, 116].

Such a radial canonical transform is obtained as the result of applying the LCT in n-D and assume spherical symmetry, so that we may use the variables $x = \|\mathbf{x}\|$ and $\xi = \|\boldsymbol{\xi}\|$ instead of the vectors $\mathbf{x}$ and $\boldsymbol{\xi}$ because the function will only depend on x and its transform will only depend on ξ. In this way a LCT leads to a *canonical Hankel transform* characterized by the matrix A of the LCT. Its definition is an integral transform

$$[\mathcal{H}^A f](\xi) = \int_0^\infty k_A(\xi, x) f(x) \mathrm{d}x. \tag{55}$$

with kernel

$$k_A(\xi, x) = x^{n-1} \frac{e^{-i\frac{\pi}{2}(\frac{n}{2}+\nu)}}{b} (x\xi)^{1-n/2} \exp\left\{\frac{i}{2b}(ax^2 + d\xi^2)\right\} J_{n/2+\nu-1}\left(\frac{x\xi}{b}\right), \tag{56}$$

with J_ν the Bessel function of the first kind of order ν. The dimension of the original problem is n. For $n = 2$ and for A the rotation matrix, we obtain the fractional Hankel transform

$$[\mathcal{H}^a f](\xi) = \frac{e^{i(1+\nu)(\pi/2-\alpha)}}{\sin\alpha} \int_0^\infty f(x) \exp\left[-\frac{i(\xi^2 + x^2)}{2} \cot\alpha\right] J_\nu\left(\frac{x\xi}{\sin\alpha}\right) x\mathrm{d}x. \tag{57}$$

The eigenfunctions are *Gauss-Laguerre functions*

$$\psi_n^\nu(x) = \sqrt{\frac{n!}{\Gamma(n+\nu+1)}} e^{-x^2/2} L_n^\nu(x), \quad \nu > -1 \tag{58}$$

where $L_n^\nu(x)$ is the nth Laguerre polynomial of order ν.

See also [58, 77, 7, 61, 85, 39, 40, 126].

See also [125] for the example of the 2-D FrFT. Similar investigations could be undertaken for problems with other types of symmetry.

11 Other fractional transforms

The idea that was explained in section 2, has been the inspiration to define several other fractional operators. As opposed to the LCT, we shall call them *fractional angular transforms* because, like the FrFT, they will depend on only one parameter, which can be given an angular interpretation.

Alieva and coworkers [8, 12, 10] have derived general definitions and properties for what they call *cyclic transforms*. These are operators $\mathcal{M}$ such that for some integer N the power $\mathcal{M}^N$ becomes the identity. For example, $N = 4$ for Fourier and Hilbert transforms and $N = 2$ for Hankel and Hartley transforms. If in the LCT $a + b = 2\cos(2\pi m/N)$ with m and N integers, then the period is N.

The operator $\mathcal{M}$ being cyclic implies that its eigenvalues are unimodular of the form $\lambda_n = \exp(2\pi m/N)$. Requiring that $\mathcal{M}^a\mathcal{M}^b = \mathcal{M}^{a+b}$ and $\mathcal{M}^0 = \mathcal{M}^N = \mathcal{I}$ implies a specific form for the kernel yet leaving a lot of freedom. By this general strategy it is possible to generate several fractional forms for a single operator $\mathcal{M}$.

In this way they are able to define fractional forms of the Hilbert, Hankel, Sine, Cosine and Hartley transform.

These were fractional angular operators that were inspired by a theoretical motivation. It is easy to derive certain properties of the eigenfunctions for example, but they are in most cases identical to results obtained by different approaches.

Zayed [129] gave an extra twist to this idea. Instead of using the definition (11) for the kernel of the fractional transform, he proposes to use as a kernel the radial limit along a ray (defined by $\alpha = a\pi/2$), of some "artificially" constructed kernel, namely

$$k_a(\xi, x) = \lim_{|\lambda|\to 1^-} \sum_{n=0}^{\infty} |\lambda|^n e^{in\alpha} \psi_n^*(x)\psi_n(\xi). \tag{59}$$

Note that the eigenvalues λ_n^a are replaced by $|\lambda|^n e^{in\alpha}$. In this way, we can just choose some complete set of orthogonal functions ψ_n and a value a to define a fractional operator. The form of the kernel immediately implies that $\mathcal{M}^a\mathcal{M}^b = \mathcal{M}^{a+b}$ and that it has period 4 (or less) in a. Since the FrFT has eigenvalues $e^{in\alpha}$, it will be no surprise that this is a trivially recovered as a special case of Zayed's definition.

In a similar way he can also obtain a fractional form of the *Mellin transform*, the *Hankel transform*, the *Riemann-Liouville* derivative and integral. For the space of functions square integrable on the interval $[-1, 1]$, Jacobi-functions can be defined to play the role of the ψ_n, leading to a new fractional transform [129]. It is not clear though what effect these transforms will have on the Wigner distribution of the function or for what particular applications they will be useful..

Again these definitions are inspired by theoretical considerations. It is however one of the very few exceptions where the fractional transforms that we consider in this survey and the fractional derivative and fractional integral feature in the same paper.

Let us look somewhat closer at some of the more important fractional transforms in the recent literature.

• *The Hilbert transform*

It is defined as the principal value of the integral

$$\frac{1}{\pi}\int_{-\infty}^{\infty} (\xi - x)^{-1} f(x)\mathrm{d}x. \tag{60}$$

Its Fourier transform is $-i\,\mathrm{sgn}(\xi)[\mathcal{F}f](\xi)$. So that we could define it as $\mathcal{F}^{-1}[-i\,\mathrm{sgn}(\xi)[\mathcal{F}f](\xi)]$. Observe that $-i\,\mathrm{sgn}(\xi) = e^{-i\pi/2}h(\xi) + e^{i\pi/2}h(-\xi)$ where $h(\xi)$ is the Heaviside step function: $h(\xi) = 1$ for $\xi \geq 0$ and $h(\xi) = 0$ for $\xi < 0$. This formulation allows for a fractional canonical generalization [93], namely

$$[\mathcal{H}ilbert_{\phi}^{A} f](\xi) = (\mathcal{F}^A)^{-1}\Big[\Big(e^{-i\phi}h(\xi) + e^{i\phi}h(-\xi)\Big)[\mathcal{F}^A f](\xi)\Big]. \tag{61}$$

For $\phi = \pi/2$ and $A = I$ we get the classical formula. For $A = R_a$ we obtain an angular fractional Hilbert transform. This transform has not a tractable effect on the Wigner distribution [93].

The multidimensional case is not so well developed, unless when it is defined relative to some direction $\mathbf{e} \in \mathbb{R}^n$, $\|\mathbf{e}\| = 1$ as the principal value of

$$[\mathcal{H}ilbert_{\mathbf{e}} f](\boldsymbol{\xi}) = \frac{1}{\pi}\int_{-\infty}^{\infty} t^{-1} f(\boldsymbol{\xi} - t\mathbf{e})\mathrm{d}t. \tag{62}$$

In [31], a "fractional form" is defined where the t^{-1} in the integrand is replaced by t^{-a} with $0 < a \leq 1$.

See also [67, 128, 86] for some other aspects of the fractional Hilbert transform. In section 13 we consider an application.

• *The Cosine, Sine and Hartley transforms*

In the Fourier transform the original function is multiplied by $e^{ix\xi} = \cos x\xi + i \sin x\xi$ and the product is integrated. In the Cosine, Sine and Hartley transform, the exponential factor is replaced by $\cos x\xi = \mathrm{Re}(e^{ix\xi})$, by $\sin x\xi = \mathrm{Im}(e^{ix\xi})$, and by $\mathrm{cas}\, x\xi = \cos x\xi + \sin x\xi$ respectively. These classical transforms are therefore used when the signal is real, and this also holds for the derived transforms that we shall discuss in this section. The only advantage of the generalizations is that they contain more parameters and are therefore more flexible for whatever application they will be used sooner or later.

So, let us see how we can fractionalize these transforms. One possibility to obtain angular fractionalizations of these (see [49]) is to replace the kernel k_a of the FrFT by its real part, its imaginary part or the sum of its real and imaginary part. However this will not respect the angular additivity property and hence the inverse of the transform with power a is not just the transformation with power $-a$.

With the arsenal of fractionalization techniques, it is not difficult though to give several fractionalizations that do respect the additivity property [96]. If ψ_n is the nth Gauss-Hermite function, then it is an eigenfunction of the Fourier operator $\mathcal{F}$ with eigenvalue $(-i)^n$. Moreover, ψ_n is even for n even and it is odd for n odd. This allows us to conclude that they are also eigenfunctions of the cosine, sine and Hartley operators but with sightly adapted eigenvalues. Using the general technique of section 2, we obtain fractional forms of these operators. The kernel for the fractional cosine transform (FrCT) is exactly like the kernel (12) of the FrFT, except that in the last factor we have to replace exp by cos. For the fractional sine transform (FrST) we have to replace exp by sin, but we have to add an extra phase shift by adding an extra factor to the kernel of the form $-ie^{i\alpha}$, with $\alpha = a\pi/2$. The fractional Hartley tranform (FrHT) turns out to be the sum of the fractional cosine and fractional sine transform. With formulas:

$$\mathcal{C}os = \frac{1}{2}[\mathcal{F} + \mathcal{F}^2\mathcal{F}], \quad \mathcal{S}in = \frac{i}{2}[\mathcal{F} - \mathcal{F}^2\mathcal{F}] \tag{63}$$

and

$$\mathcal{H}artley = \mathcal{C}os + \mathcal{S}in = \frac{1+i}{2}\mathcal{F} + \frac{1-i}{2}\mathcal{F}^2\mathcal{F} \tag{64}$$

generalize to

$$\mathcal{C}os^a = \frac{1}{2}[\mathcal{F}^a + \mathcal{F}^2\mathcal{F}^a], \quad \mathcal{S}in^a = \frac{i^a}{2}[\mathcal{F}^a - \mathcal{F}^2\mathcal{F}^a] \tag{65}$$

and

$$\mathcal{H}artley^{\,a} = \mathcal{C}os^a + \mathcal{S}in^a = \frac{1+i^a}{2}\mathcal{F}^a + \frac{1-i^a}{2}\mathcal{F}^2\mathcal{F}^a. \tag{66}$$

All of them are cyclic of order $N = 2$. To some extend, they satisfy the additivity property. That they are not invertible in the whole space follows from the fact that for example the cosine transform has eigenvalues $\lambda_{2n+1} = 0$ for the odd eigenfunctions ψ_{2n+1}, and the sine transform has eigenvalues $\lambda_{2n} = 0$ for the even eigenfunctions ψ_{2n}. Thus the cosine will kill all the components of the function in the space of the odd eigenvectors and thus it will not be possible to recover these by inverting the operation, except if we know that there are no components along these odd eigenfunctions, i.e., if the given signal is

even. Similarly for the sine transform. There is no problem for the Hartley transforms since there are no zero eigenvalues in that case. Thus $\mathcal{C}os^{-a}\mathcal{C}os^a f = f$ only if f is even and $\mathcal{S}in^{-a}\mathcal{S}in^a f = f$ only if f is odd.

For this reason the cosine and sine transforms are usually defined as one-sided transforms. This just means that it is assumed that the given function is even respectively odd and the integral over $\mathbb{R}$ in the integral representation is replaced by 2 times the integral over $\mathbb{R}^+$, thus $\int_{-\infty}^{\infty} = 2\int_0^{\infty}$.

Exactly the same procedure can be used to define canonical cosine, sine or Hartley transforms. We only need to recall from [95] that for any LCT there is always a complete set of orthogonal eigenfunctions ψ_n that can be chosen such that the ψ_{2n} are even and the ψ_{2n+1} are odd. Thus, we can in the above derivation replace the exponent a by A except for the factor i^a. Therefore, this factor is skipped in [96] and replaced by 1, which leaves us with the definitions

$$\mathcal{C}os^A = \frac{1}{2}[\mathcal{F}^A + \mathcal{F}^2\mathcal{F}^A], \quad \mathcal{S}in^A = \frac{1}{2}[\mathcal{F}^A - \mathcal{F}^2\mathcal{F}^A], \tag{67}$$

Just like $\mathcal{F}^A = e^{i\alpha/2}\mathcal{F}^a$, we also have $\mathcal{C}os^A = e^{i\alpha/2}\mathcal{C}os^a$ and $\mathcal{S}in^A = e^{i3\alpha/2}\mathcal{S}in^a$ when A is the rotation matrix R_a. The additivity properties in the sense of LCTs will hold for the $\mathcal{C}os^A$ and $\mathcal{S}in^A$ transform if we restrict ourselves to the even, respectively odd functions. However, defining $\mathcal{H}artley^A = \mathcal{C}os^A + \mathcal{S}in^A$ does not give the additivity property like in the LCT. Using instead $\mathcal{H}artley^A = \frac{1}{2}[\mathcal{F}^A \pm i\mathcal{F}^2\mathcal{F}^A]$ would give the additivity property, but now we do not get $\mathcal{H}artley^a$ when $A = R_a$.

We also note that $\mathcal{C}os^A f = \mathcal{F}^A f$ and a fortiori $\mathcal{C}os^a f = \mathcal{F}^a f$ if f is even.

Furthermore simplified canonical cosine, sine and Hartley transforms can be defined just like the simplified LCTs were defined. There are however several possibilities depending on how the matrix A is chosen. One of the advantages of the simplified versions is that in general, when f is real, then we can arrange the simplified fractional or canonical cosine transform such that the transform will also be real. The additivity property is lost though. We refer to [95, 92] for further details.

• *LCT with complex matrix A*

Although it is not the main theme in this survey, we briefly mention some elements of the theory where in the LCT, the matrix A is allowed to be complex. Such transformations appear in quantum physics, differential equations or more advanced optics. See [122].

When A is complex, we loose the unitarity of the LCT as an operator on $L^2(\mathbb{R})$. It is however possible to define a measure $\mu_A(z)$ in the complex plane such that with respect to the inner product

$$\langle f, g\rangle_A = \int_{\mathbb{C}} \overline{f(z)}g(z)\mathrm{d}\mu_A(z) \tag{68}$$

the LCT $\mathcal{F}^A$ becomes unitary when A is complex. Some examples.

- *The Bragman transform*

  This transform corresponds to the matrix

$$A_B = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}. \tag{69}$$

  The kernel is

$$\frac{1}{\sqrt{\sqrt{2}\pi}}\exp(-\xi^2/2 + \sqrt{2}x\xi - x^2/2) \tag{70}$$

and

$$\mathrm{d}\mu_A(x) = \sqrt{\frac{2}{\pi}}\exp(-|x|^2)\mathrm{d}x. \tag{71}$$

- *The bilateral Laplace transform*

  In this case we have a particularly simple matrix

$$A_L = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}. \tag{72}$$

  The transform kernel is

$$-\frac{i}{\sqrt{2\pi}}\exp(-x\xi). \tag{73}$$

  The corresponding inner product collapses to an integral along the imaginary axis with weight 1.

- In the case of circular symmetry, one may switch to spherical coordinates and the *Barut-Girardello transform* will appear in the Bragman case and the *Hankel transform* in the bilateral Laplace case.

Note also the relation between the Bragman and bilateral Laplace transform and the Fourier transform. Using the dilation matrix $A_D = (e^{i\pi/4}, 0, 0, e^{-i\pi/4})$ (to be read as a 2×2 matrix), we have $A_L = A_D A_F A_D^*$ and $A_B = A_F^{-1/2}$ where $A_F = R_1 = (0, 1, -1, 0)$ is the matrix corresponding to the Fourier transform. This relation allows for the definition of fractional Laplace transforms by defining their LCT matrix as

$$A_{L^a} = A_D A_{F^a} A_D^* = \begin{bmatrix} \cos\alpha & i\sin\alpha \\ i\sin\alpha & \cos\alpha \end{bmatrix}, \tag{74}$$

where $\alpha = a\pi/2$. The Bragman and bilateral Laplace transforms are special cases obtained for $a = -1/2$ and $a = 1$ respectively.

Just like the Hankel transform was obtained from the FrFT in the case of circular symmetry, it is possible to exploit this type of symmetry in the case of the bilateral Laplace transform. This results in the *Barut-Girardello transform*. It is essentially like the Hankel transform but replacing the Bessel functions of the first kind J_ν appearing in the kernel by I_ν, which are Bessel functions of the second kind. Its fractional variant is then defined as the radial counterpart of the fractional bilateral Laplace transform. For more details eigenfunctions and integral kernel, see [116, 119].

• *Offset transforms*

The offset versions of the previous transforms add a shift in both the x and the ξ domain to the usual definition. For example the offset LCT is the integral transform with kernel

$$C_A e^{i\xi\eta}\exp\left\{\frac{i}{2}\left(\frac{d}{b}(\xi-\tau)^2 - 2(\xi-\tau)x + \frac{a}{b}x^2\right)\right\} \tag{75}$$

where η and τ are the ξ and x shifts respectively. It is shown in [97] that if ψ is an eigenfunction of the LCT, then for example if $a + d \neq 2$

$$\psi_{\tau,\eta}(x) = \exp\left\{i\frac{c\tau + (1-a)\eta}{2-a-d}x\right\}\psi\left(x - \frac{(1-d)\tau + b\eta}{2-a-d}\right) \tag{76}$$

is an eigenfunction of the offset LCT. It is further shown that there is always a complete set of orthonormal eigenfunctions of the offset LCT (and hence also of the offset FrFT).

The offset transforms and their generalizations are useful when in optical systems one wants to investigate self-imaging phenomena, that is when the output is a (possibly scaled) duplicate of the original. If in such a system, a lens is shifted up or down, or when a prism is inserted, then the self-imaging phenomenon will remain. Both of these modifications can be modelled by an offset transform and more generally, all inputs that will result in a self-imaging phenomenon can be computed. In this way resonance phenomena can also be analysed. The more general forms of the offset transforms have more parameters and will thus be better equipped to model more general situations.

• *Other transforms*

Fractionalization of all types of transforms have been undertaken. The most useful ones are based on eigenvalue decomposition. For example the *Hadamard transform* [99]. *Walsh* and *Haar transform* [71, 72], *Mellin transform* [129, 4], *Gabor transform* [2, 132, 21, 30, 3], *Radon transform* [127] etc. To keep this survey within a reasonable number of pages we shall not elaborate further on these.

12 Discrete fractional transforms

During recent years a lot of effort went to the design of discrete analogs of the fractional transforms. The definitions are however not unique and many approaches exist. We give some elements of recent developments.

12.1 Discrete fractional Fourier transform

The discrete Fourier transform (DFT) multiplies a vector $\mathbf{f}$ with the DFT matrix F. This matrix has eigenvalues $\lambda_k \in \{1, -1, i, -i\}$ with certain multiplicities. It is possible to choose the eigenvectors orthogonal such that the eigenvalue decomposition of M is $F = E\Lambda E^*$. As we have explained before, the discrete fractional Fourier transform (DFrFT) will then be defined as $F^a\mathbf{f} = E\Lambda^a E^*\mathbf{f}$. So the problem is reduced to an appropriate computation of the eigenvectors of the DFT matrix. To approximate the Gauss-Hermite functions we also want them to be even or odd vectors, just like the even and odd Gauss-Hermite functions. The method proposed in [27, 28, 29] is to construct a symmetric matrix S with simple eigenvalues that commutes with F, so that it has the same eigenvectors as F. This matrix is obtained by discretizing the continuous differential equation whose eigenfunction solutions are the Gauss-Hermite functions:

$$(\mathcal{D}^2 + \mathcal{F}\mathcal{D}^2\mathcal{F}^{-1})f(x) = \lambda f(x), \quad \mathcal{D} = \frac{d}{dx}. \tag{77}$$

It is proved that S is given by $S = A + B$ with A a circulant matrix whose first row is $(2, 1, 0, \ldots, 0, 1)$ and $B = FAF^{-1}$. Using the symmetry properties, we can even more simplify the problem with the matrix

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & & \\ & I_r & J_r \\ & J_r & -I_r \end{bmatrix} \quad \text{or} \quad P = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & & \\ & I_r & J_r \\ & J_r & -I_r \end{bmatrix} \tag{78}$$

where $r = \lfloor (N-1)/2 \rfloor$ depending on N being odd or even I_r is the $r \times r$ unit matrix and J_r is I_r with its columns in reversed order. Indeed,

$$PSP^{-1} = \begin{bmatrix} Ev & 0 \\ 0 & Od \end{bmatrix} \tag{79}$$

and the problem is thus reduced to finding eigenvectors for Ev and Od. It is proved that the eigenvectors constructed in this way can be uniquely defined to be even and odd. These eigenvectors are the so called discrete Gauss-Hermite functions (vectors) and they represent approximate samples of the continuous counterparts. It is however only true if the size N of the matrix tends to infinity. For finite N, the approximation will become worse when the order of the discrete Gauss-Hermite functions approaches N. Therefore, a more accurate discretization of the continuous equation (77) is proposed. The matrix A is somewhat more complicated, but the same arguments can be used to come to the same result with better approximating properties. An investigation of these approximating properties is undertaken in [19, 18].

Since the dimension of the 4 eigenspaces of F is approximately $N/4$, there are several possibilities to choose the orthogonal eigenvectors in each eigenspace. So there are many different possible choices for the decomposition $F = E\Lambda E^*$, hence also many possible definitions of $F^a = E\Lambda^a E^*$. For example, depending on the order of approximation in the discretisation of (77), we get a different matrix S, and hence different eigenvectors. Another element that makes the definition ambiguous is the choice of the fractional power of the eigenvalues. In general, the complex value $(-i)^a$ is not uniquely defined when a is real.

Modifications are proposed in [89] where approximations of the discrete Gauss-Hermite vectors are obtained by sampling the continuous Gauss-Hermite functions and then they are projected and orthogonalized in the appropriate eigenspaces. A somewhat similar technique was used in [88]. Other techniques based on eigenvalue decompositions are [70, 98]. For an implementation suited for implementation on a parallel computer see [123].

Thus variants to define the DFrFT were proposed. One may consult [90] for a survey with advantages and disadvantages of the different approaches. We mention briefly

- There are methods based on linear combinations of F^k, $k = 0, 1, 2, 3$ [110, 26]. This has some disadvantages similar to the disadvantages of the early definition of the FrFT given in [76] which was also based on a combination of $\mathcal{F}^k$, $k = 0, 1, 2, 3$.

- Some methods are based on sampling the continuous FrFT [83].

- In [15] a definition was given based on group theory.

- The authors of [107] define the DFrFT as the result of the continuous FrFT whose input is a pulse train.

- Closed form solutions [90] are obtained by considering the input and output as samples of the continuous transform. The discrete matrix F^a is then constructed by sampling the continuous transform kernel and then force it to satisfy $(F^a)^*(F^a) = I$. The number of samples needed in the output will depend on a and will be large when a is close to 0. Using $F^a = F^{a-1}F$, this disadvantage can be overcome. If N is the number of given samples, then we will have $M \geq N$ samples for the output and the sampling distance in the x and ξ domain should satisfy $\Delta x\, \Delta\xi = s2\pi \sin\alpha/(2M+1)$ with s

an integer relative prime to $2M+1$. A completely similar technique can be used to define a discrete form of the LCT. This definition will not have the additivity property. It is however efficient in computing samples of the continuous transform. For other applications some simplifications are possible. A comparison with all the other types of DFrFT can be found in [90].

- In [37] orthogonal eigenvectors are constructed by imposing certain symmetries. The idea is the following. If the eigenvector has to satisfy a certain "symmetry" property like being periodic or like having certain zero entries, then the subspace of eigenvectors has a smaller dimension. Therefore, the eigenvectors can be found by solving a small eigenvalue problem, which results in a small eigenvector, which is then "expanded" to a full size eigenvector of the DFT matrix. Choosing the symmetries such that orthogonality is obtained, these authors obtain explicit forms of orthogonal eigenvectors that can be computed efficiently if N is of the form $N = rM^2$ with r small (typically 1 or 2). The discrete cosine transform of [24] is an example of such an algorithm. Of course, when the eigenvectors are fixed, the DFrFT is immediately defined.

- A somewhat different approach is taken in [17, 16]. The DFrFT (also called *Fourier-Kravchuk transform*) is still defined by the decomposition $F^a = E\Lambda^a E^T$ with $\Lambda = \mathrm{diag}(1, e^{-ia}, \ldots, e^{-iNa})$. The matrix E is an orthogonal matrix which contains N Kravchuk functions. The nth Kravchuk function is a vector of dimension N whose kth entry is given by

$$2^{n-N/2}\sqrt{\binom{N}{N/2+k}/\binom{N}{n}}\,\phi_n(k+N/2) \tag{80}$$

where ϕ_n is the nth symmetric Kravchuk polynomial:

$$\phi_n(k) = \frac{(-1)^n}{2^n}\binom{N}{n}\,{}_2F_1(-n,-k;N;2). \tag{81}$$

The Kravchuk polynomials are orthogonal with respect to summation over the points $\{0,1,\ldots,N\}$ with respect to the binomial weight $\left\{2^{-N}\binom{N}{k}\right\}_{k=0}^{N}$ so that the Kravchuk functions are orthogonal over the same points with weight 1. In other words, the matrix E is orthogonal. The Kravchuk functions appear as solutions of an eigenvector of the harmonic oscillator equation in which the Hamiltonian is replaced by a finite difference operator on equispaced points. Moreover, as $N \to \infty$, these Kravchuk functions converge to the Gauss-Hermite functions that solve this eigenequation with continuous Hamiltonian. The additivity property is obviously satisfied by construction. The computation of this Fourier-Kravchuk transform can be easily computed by first multiplying by E^T which is possibly implemented by the Feinsilver-Schott algorithm [41] (not a fast algorithm!), then multiplying by the diagonal matrix Λ^a and finally multiplying by E.

12.2 Other discrete transforms and related problems

It is clear from the previous section that there is not a unique approach to the discrete fractional Fourier transform. Perhaps this is the reason why the discrete versions of other

related transforms is still so much underdeveloped. Because of page limitations and because there is only little to be said we just quickly summarize what is known so far. There is still room for many new research results here.

- *Discrete cosine, sine and Hartley transform and general LCT*
  These transforms are most directly related to the Fourier transform, and their fractional counterparts were fairly easily obtained in the continuous case. However, there are some complications for the discrete case.

  As is well known there are eight types of discrete cosine (DCT) and sine (DST) transforms [113]. The DFT matrix has eigenvectors that are even if they are in the eigenspaces of ± 1 and they are odd when they are in the eigenspaces of $\pm i$. The DCT-I and DST-I matrices have only nonzero eigenvalues ± 1 and the DCT-I and DST-I eigenvectors of size N can be easily obtained from the first half of the entries of the Gauss-Hermite eigenvectors of a DFT matrix of size that is approximately twice the size. See [100] for details. A similar technique is used in [117] for the discrete Hartley, DCT-IV and DST-IV transforms. See also [88]. Thus, for those cases, we can basically rely on the computations that were proposed for the discrete fractional Fourier transform.

  For the DCT and DST of types II and III, the eigenstructure is more complicated. Besides the eigenvalues ± 1, which will be simple (if they appear at all), there are a number of complex conjugate pairs of eigenvalues. The advantage is that the eigenvectors are uniquely defined, so it only remains to choose the real powers of the eigenvalues. This has been explored in [24]. The DCT's of type I, IV, V, VIII all behave similar (only multiple eigenvalues ± 1 and 0), and those of type II, III, VI, VII are similar in that they have distinct eigenvalues. An in depth analysis of all the eigenvalues and eigenfunctions is given in [26].

  It should also be possible to fractionalize the multidimensional Hartley transform [131].

  We mentioned before the closed form technique for the discrete LCT given in [90]. Apart from this, the definition and computation of a discrete LCT seems to be largely unexplored.

- *Discrete Wigner distribution*
  One of the basic facts about the fractional Fourier transform is its rotational effect on the Wigner distribution of the signal. However, the effect of discrete transforms on the Wigner distribution is not so clear anymore. First of all one needs to define a discrete Wigner distribution. Again, there are several approaches possible. See [106, 81, 82, 101, 105]. We also note that the Wigner distribution is redundant so that it can be recovered from a relatively small number of sample values [104]. The rotation interpretation of the FrFT, and more generally, the transformation resulting from a LCT, is not trivial either [107].

- *Sampling theorems and uncertainty principle*
  Let Δ_a^2 be the spread of the function $f_a = \mathcal{F}^a f$, then the well known uncertainty principle says $\Delta_0^2 \Delta_1^2 \geq 1/4$. Thus a more precise localization in the x-domain will imply a less precise localization in the Fourier domain. It should then be clear that we may expect an intermediate result for a fractional domain that is in between. This

is also what has been obtained. The previous uncertainty principle is generalized to $\Delta_0^2\Delta_a^2 \geq (\sin^2\alpha)/4$ [111] for fractional Fourier transforms over an angle α. Of course such results could be placed in a much broader perspective of uncertainty principles for operators. See [47, 48] and the references therein.

Sampling theorems say how many samples of a signal are needed to recover the signal exactly. Obviously, it is only possible to formulate such theorems if we know something about the signal. More precisely, the bandwidth, i.e., the maximal frequencies that are allowed in the signal will essentially fix the number of samples that are needed, as in the classical Shannon sampling theorem. But what if we have only information about the FrFT instead of the Fourier transform? Also in that case several sampling theorems were obtained like e.g., in [38, 44, 130].

To the best of our knowledge, there has not been a systematic investigation of these topics in the general LCT case or the multidimensional case. Neither is there a theory for the fractional transforms for the case of unequally spaced data like for the classical FFT [103].

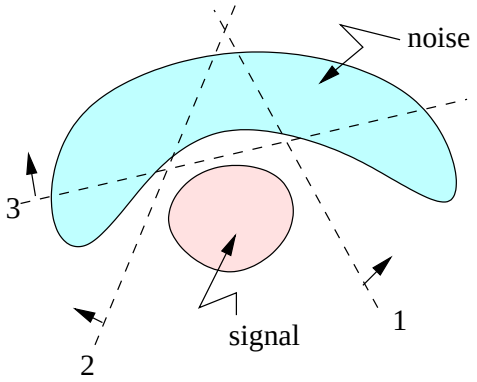
13 Applications

Traditionally, the FrFT is used as modelling tool in quantum mechanics and in optics systems. In fact, that is where the impulse for the development of the whole theory came from. So far we have written this survey with the signal and image processing applications in mind. The quantum mechanical applications usually require a more abstract and somewhat less intuitive approach. So we have chosen to neglect this part of the story. It is well documented in the literature [42, 122]. But even when we restrict ourselves to the optical, and the signal and image processing side of the story, there is still an overwhelming pile of literature where the previously surveyed transforms are used. So, we do not want or can elaborate on all of these because they often need a specific vocabulary, and because it would bring us far beyond the restrictive page limit. So we mention very briefly the scent some of the applications and invite the reader to look up the details in the literature. These are only intended as illustrations of what is possible with these transforms, and we are fully aware of the fact that we are far from complete and that we do not include many details. Moreover we only want to refer to the more recent papers, since the order ones have been covered in other surveys or books.

13.1 Filtering

One of the typical applications is filtering. If the components of a signal interact in the time and the frequency domain, then it may be difficult to separate them and filter out the noise. However using different rotations, it is possible to separate components from a signal as long as their Wigner distributions are disjunct. See for example [85]. Figure 2 shows schematically the Wigner distribution of a noisy signal. By successive rotations (3 in the example of the figure that bring the indicated axes in vertical position) and removing the energy that is on the indicated side of the axes using some low or high-pass filter, the noise can be completely removed. This is impossible with classical Fourier techniques.

Figure 2: Filtering operations to remove noise by rotations and bandpass filtering.



13.2 Compression

Another application is the compression of signals [124]. The idea is that one or more fractional Fourier transforms are computed and filtered (e.g. by thresholding) to obtain simpler representations of the signal. It may well be that the FrFT components are much simpler in one domain than in the other. Take for example a chirp function. An appropriate FrFT will transform it into a delta function, which allows for a very sparse representation. If more FrFTs are made, they can be done sequentially or in parallel, and in the case of an image, they can be different for the two spatial directions. As it is now, this technique does not seem to be competitive in quality or in computer time with classical techniques.

13.3 Image encryption

There are many papers on optical encryption using the FrFT. E.g., [51, 52, 64, 65, 79, 78, 118, 133] to mention just a few. A possible encryption technique for an image is to first multiply the image with a random phase (i.e., multiply it with a function of the form $e^{i\phi(x,y)}$), then apply a 2-D FrFT of some order that may be different in the two directions. Only the intensity of the result is stored. The same set of operations is applied a second time to the image but with a different phase and different orders. The original signal (intensity and phase) can be recovered from the two resulting images by a recursive scheme [51].

There are several variants to this. For example [65] proposes to do the phase multiplication and FrFT not in parallel but in cascade. Thus performing three subsequent FrFTs interlaced by a multiplication with a random phase.

Decryption is obtained by simply inverting all the operations in opposite order.

Moreover, because all these techniques can be implemented by optical systems, which are naturally two-dimensional, they become extremely performant when applied to images.

13.4 Digital watermarking

For copyright protection, a watermark can be embedded in a digital image. This watermark should be some signature embedded in the image without disturbing the original image visually. This embedding can be done by computing a (separable) 2-D FrFT of the image. Then the FrFT coefficients are ordered and some signature (e.g. a chirp-like signal $\cos(a_x x^2 +$

$a_y y^2)$) is added to a sequence of coefficients. These coefficients should not be the largest ones to not disturb the image, but also not the smallest ones in order not to be filtered out as noise. An inverse FrFT will restore an image that is practically not blurred by the watermark, and the watermark can only be removed by someone who knows the parameters of the signature and orders of the FrFT which provides an extra security and makes it much more robust for attacks such as translation, rotation, filtering, and cropping. See [36] for more details.

13.5 Neural networks and pattern recognition

Because of the extra parameter that is provided in the FrfFT by the rotation angle as compared with the ordinary Fourier transform gives an extra degree of freedom that can be used to optimize the performance of some system. In [20] for example, the FrFT is used as a preprocessor for a neural network that has to recognize certain situations. By testing the learning convergence of the neural network for different values of the FrFT order, one may find an optimal value of this order for which the recognition is best. See also [66] for pattern recognition application. A certain object will be "recognized" if there is a high correlation with a known object. Most often the correlation is computed in the original or in the classical Fourier domain. But in such applications again, it is possible that the correlation peak can be made more prominent or sharper by computing a correlation in an optimally chosen FrFT domain.

13.6 Edge detection

The Hilbert transform is known to be very effective for edge detection problems. The fractional Hilbert transform This is illustrated in [86]. The classical Hilbert transform will also detect the edges, but with the fractional order, it is possible to emphasize the rising or the falling edges. More specifically, if you take the amplitude of the fractional Hilbert transform of a rectangular function, then plotting the magnitude of the classical Hilbert transform will show some overshoot at the top corners of the rectangle. This overshoot will be quite symmetric. However, using a fractional Hilbert transform, this symmetry will be lost and depending on α being smaller or larger than 1, the overshoot will shrink on the left top corner (rising edge) and increase on the right top corner (falling edge) or vice versa. Thus the fractional parameter can be used to distinguish between the rising and falling edges.

13.7 Antennas, radar and sonar

In [56], the blind source separation problem is considered. Classically, the mixing coefficients (the received signal is a linear combination of unknown uncorrelated signals to be recovered) can be estimated using a weighted correlation matrix of the mixed signal. It is proposed to use different weights that correspond to windowing operations applied in different FrFT domains. That is, if D_k is a diagonal matrix of samples for the kth window or weight function and F^{a_k} is the DFrFT matrix, then the weight matrices chosen are $[F^{a_k}]^{-1} D_k F^{a_k}$. This is sometimes called a (discrete) short time FrFT. Joint diagonalization (approximately in least squares sense) of all these correlation matrices gives then an estimate of the mixing coefficients. For other applications in radar and sonar see [23, 63, 55].

13.8 Communication theory

In multicarrier communication systems, the limited time-frequency window within which a certain message has to be transferred, one may assign different symbols of the message to subcarriers. To avoid symbol interference, one should arrange for the carriers to have nonoverlapping support in this time-frequency window. By taking carriers that are orthogonal in the time or the frequency domain one may avoid that the symbols mutually influence each other in the time or the frequency domain. In wireless communication channels, the channel frequency response is rapidly time varying and then Doppler spread my cause interchannel interference. In [69], it is proposed to use chirp carriers. It is illustrated that with these carriers, the transmitter is essentially a block inverse FrFT and the receiver a block FrFT. The main idea is thus as follows. Suppose the symbols of the kth block are $\{a_{k,n} : n \in \mathbb{Z}\}$ and we assign each symbol to a carrier $f_{-\alpha,n}(t)$, which is a chirp $\mathcal{F}^{-a}[\delta(u - (n/T)\sin\alpha)]$, properly normalized such that $\int_{\mathbb{R}} f_{\alpha,n}(t) f_{\alpha,m}(t)dt = \delta_{n,m}$. Then for block k, we obtain a signal $s_k(t) = \sum_n a_{k,n} f_{-\alpha,n}(t)$, which is essentially an inverse FrFT. The $a_{k,n}$ can be recovered by sampling the FrFT of $s_k(t)$. For more details and a discrete time implementation see [69].

13.9 Tomography

Consider a plane wave that is scattered by an object. The tomography problem is to recover the object from the measurements of the scattered wave. In computed axial tomography (CAT), the wavelength is small compared to the size of the scattering objects (e.g., X-rays in medical applications) and then a geometric model for the scattering can be used. With ultrasound waves, this is not true anymore and the wavelength is of the same size as the scattering objects and quantum mechanical issues play a role. This is called diffraction tomography. The relation between the FrFT and quantum mechanics is known [68], and this is used in [46] to show that under certain conditions, the measured data function and the 3-D Fourier transform of the scattering potential are related by a 2-D FrFT. This is used to analyse how diffraction tomography transforms in CAT when the wavelength approaches zero, and hence it can be analysed in what range of the wavelength vs. the size of the scattering object CAT is still effective.

14 Conclusion

The FrFT and related fractional transforms play or can play an essential role in many applications. Much has been achieved, but the understanding of basics and generalizations is constantly growing. As an appendix to the book [85] we have tried to sketch some recent developments and new results. There are however numerous topics that we have not touched upon because we wanted to stay closely to the interest of readers working in signal processing and related topics. The survey leaves some space for open problems that become apparent when all the results are placed next to one another. We hope that it will trigger some researchers to try and fill the gaps.

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