Received: Added at production | Revised: Added at production | Accepted: Added at production

DOI: xxx/xxxx

RESEARCH ARTICLE

Prescribed-Time Sensor Fault Estimation for Linear Systems with Unknown Inputs by Periodic Delayed Observers

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Summary

This paper studies the problem of sensor fault estimation for linear systems in the presence of unknown inputs in both input and output channels. Relying on matrix equation theory, the unknown inputs are first removed from the output channel. Second, an augmented descriptor system is constructed by treating both system state and sensor fault as (pseudo)-state variables. Third, the augmented descriptor system is transformed into a regular one, followed by the elimination of the unknown input from the state equation. Finally, a reduced-order prescribed-time sensor fault estimator is obtained by using periodic delayed observers. This matrix equation based approach is then complemented by an alternative one, which relies on successive transformations of state, input and outputs. This approach sheds light on the adopted solvability conditions throughout the paper and on the role of different directions in the output space in handling unknown inputs, sensor faults and the core state estimation problem. Finally, two numerical examples are presented to illustrate the effectiveness of the proposed methodologies.

KEYWORDS:
Linear system, Fault estimation, Unknown input, Periodic delayed output, Prescribed-time observer.

1 | INTRODUCTION

Due to the rapid development of industrial technology, the complexity of control system structures has increased a lot[4], which may result in devices and components to be more susceptible to sudden changes in the external environment or imperfect behavior. In such scenarios, the components of a system are more likely prone to unforeseen faults, which may lead to performance degradation and accidents[2,16]. Thus, it is highly relevant to develop effective fault diagnosis techniques to improve safety and reliability of industrial control systems[13]. In the last decades, fault diagnosis has been actively studied by many scholars and some excellent results have been obtained, see for instance references[3,8,18,23,24,26,28]. Generally speaking, fault diagnosis concerns three tasks, namely, fault detection, fault isolation, and fault identification (estimation). A standard approach for the first two tasks consists of generating a residual signal and comparing it to a prescribed threshold to determine whether or not a fault has occurred[27]. Within the scope of the third task, the shape and magnitude of the fault can be obtained[31]. Well-known fault estimation approaches rely on sliding mode observers[5,7], adaptive observers[21] and unknown input observers[1].

It is worth mentioning that most of the available results are devoted to improving the estimation accuracy. In reality, the estimation time is also a leading performance index in fault estimation. The longer it takes the estimator states to converge, the longer is the estimation time. Unfortunately, the above mentioned fault estimators only guarantee asymptotic convergence. This observation has been a driving force of recent research on finite-time fault estimation[25,14,22]. However, the convergence time in

these results is dependent on the initial conditions for the state variables. Fixed-time convergence can be viewed as a stronger notion of finite-time convergence, which guarantees that the convergence time is not only independent of the initial conditions but can be determined a priori. To the best of our knowledge, methods to achieve fixed-time convergence include time-varying high gain methods, sliding mode techniques and other approaches relying on Lyapunov inequalities. In these methods, the proposed controllers are nonlinear, non-smooth, or they become singular in the sense that the feedback gain will grow unbounded when the convergence time is approached [32]. However, in [35] it has been shown that non-smooth feedback, unbounded gains, and even nonlinear feedback, can be avoided when using periodic delayed output feedback. The proposed methodology has been applied to address the problem of prescribed-time fault estimation for linear system with unknown input in [15]. It should be noted that, in this reference, the second-order derivative of the sensor fault is assumed to be identically zero, which may limit the types of failures occurring in industrial processes. In addition, the case where the output channel may also contain unknown inputs was not considered. It is obvious that the sensor estimator design becomes more challenging if unknown inputs are present in the output equation.

In this paper, we propose a novel approach to design prescribed-time sensor fault estimators for linear systems with unknown inputs appearing in both input and output channels. In the proposed fault estimators, the gains are continuous, continuously differentiable, and even smooth in the sense of infinitely many times differentiable. The estimation time is independent of the initial conditions and can be prescribed. We adopt periodic delayed observers to realize these properties. A reduced-order sensor fault estimator is designed. We consider two complementary methodologies. The first one starts from a state-augmentation with the sensor fault and heavily builds on the theory of matrix equations. The second one, inspired by [10,6], relies on successive transformations of states, inputs and outputs, revealing the underlying system structure and dependencies. Compared to the setting taken in article [30], sensor faults and the presence of unknown inputs in the output channel are considered. The effectiveness of the proposed approach is illustrated by two numerical examples.

This paper is organized as follows. The problem formulation and some preliminaries are spelled out in Section 2. The prescribed-time estimators are then designed in Sections 3 and 4, relying on two distinct methodologies. Numerical examples are presented in Section 5. In Section 6 some concluding remarks are formulated.

Notation: In this paper, $\mathbf{R}^n$ denotes the n-dimensional Euclidean space and $\mathbf{R}_p^{n\times m}$ stands the set of all $n\times m$ real matrices with rank being p. For an integer $k \geq 0$, a function f is said to be of class $\mathbf{C}^k$ if the derivatives $f^{(\iota)}$, $\iota = 0, 1, \ldots, k$ exist and are continuous. The set $\mathbf{C}^k_{n\times m}$ denotes the set of $n\times m$ matrices whose elements are of class $\mathbf{C}^k$.

2 | PROBLEM FORMULATION AND PRELIMINARIES

As in articles [20,29] we model the sensor fault by means of an additive perturbation w to the output equation as

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Ed(t), \\ y(t) = Cx(t) + Du(t) + Fw(t) + Jd(t), \end{cases} \tag{1}$$

where $x(t) \in \mathbf{R}^n, u(t) \in \mathbf{R}^m, d(t) \in \mathbf{R}^l, y(t) \in \mathbf{R}^p$, and $w(t) \in \mathbf{R}^q$ are, respectively, the state, the known input, the unknown input, the output, and the sensor fault at time $t \geq 0$. Matrices $A \in \mathbf{R}^{n\times n}, B \in \mathbf{R}^{n\times m}, C \in \mathbf{R}^{p\times n}, D \in \mathbf{R}^{p\times m}, E \in \mathbf{R}^{n\times l}, J \in \mathbf{R}^{p\times l}$, and $F \in \mathbf{R}^{p\times q}$ are known constant matrices. We assume that d and u are Lebesgue measurable and locally essentially bounded functions. Within the approach of Section 3, an additional condition on function w will be imposed.

In this paper, we are devoted to design sensor fault estimators such that the estimation errors converge to zero in a prescribed time T ($T > 0$), as described below.

Problem 1. Design reduced-order sensor fault estimator in the form

$$\begin{cases} \dot{\chi}(t) = f(\chi_t, u_t, y_t), \\ \hat{w}(t) = g(\chi_t, u_t, y_t), \end{cases}$$

such that $w(t) = \hat{w}(t), \forall t \geq T$.

Before solving Problem 1 using two complementary approaches, we firstly present the two required assumptions.

Assumption 1. [12] The matrices C, J, E and F satisfy

$$\operatorname{rank}(\Delta) \triangleq \operatorname{rank}\begin{bmatrix} I_n & 0 & 0 & E \\ C & F & J & 0 \\ 0 & 0 & 0 & J \end{bmatrix} = n + q + \operatorname{rank}(J) + \operatorname{rank}\begin{bmatrix} E \\ J \end{bmatrix}.$$

Assumption 2. [12] The matrices A, C, E, F, and J satisfy

$$\operatorname{rank}\begin{bmatrix} A - sI_n & 0 & E \\ C & F & J \end{bmatrix} = n + q + \operatorname{rank}\begin{bmatrix} E \\ J \end{bmatrix}, \quad \forall s \in \mathbf{C}.$$

Remark 1. When there is no sensor fault and unknown input in the output channel, i.e. $F = J = 0$, Assumptions 1 and 2 reduce to

$$\operatorname{rank}(CE) = \operatorname{rank}(E)$$

and

$$\operatorname{rank}\begin{bmatrix} A - sI_n & E \\ C & 0 \end{bmatrix} = n + \operatorname{rank}(E),$$

respectively, where we have used

$$\operatorname{rank}\begin{bmatrix} I_n & 0 \\ -C & I_p \end{bmatrix}\begin{bmatrix} I_n & E \\ C & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} I_n & E \\ 0 & -CE \end{bmatrix} = n + \operatorname{rank}(CE).$$

These two conditions are the same as those in [15]. Thus, Assumptions 1 and 2 in this paper are a natural generalization. Moreover, the approach of Section 4 will shed light on the necessity of these assumptions and the solvability conditions of Problem 1. We refer to the discussion at the end of that section.

Regarding fixed-time estimation, realized by periodic feedback, we give a definition borrowed from [35].

Definition 1. [35] Let $h > 0$ be a given constant and $k > 0$ be a given integer or infinity. A symmetric matrix function $H_r : \mathbf{R} \to \mathbf{R}^{p\times p}$ is called an $\mathbf{S}_p^{(k)}(r)$ function if

(1) H_r is $2r$-periodic and $H_r(t) = 0, \forall t \in [0, r]$.

(2) $H_r(t) \geq 0, \forall t \in [r, 2r]$ and there exists an $r^* \in (r, 2r)$ such that $H_r(r^*) > 0$.

(3) $H_r \in \mathbf{C}^k_{p\times p}$, which implies that $H_r^{(i)}(r) = H_r^{(i)}(2r) = 0, i = 0, 1, \ldots, k$.

It follows that $\mathbf{S}_p^{(k)}(r) \subset \mathbf{C}^k_{p\times p}$. Note that it is easy to construct an $\mathbf{S}_p^{(k)}(r)$ function. For example, for any integer $k \in [1, \infty)$, an $\mathbf{S}_p^{(k)}(r)$ function $H_r(t)$ can be constructed as [35]

$$H_r(t) = \begin{cases} 0, & t \in [0, r], \\ \left(\left(\mathrm{e}^{a_1(t-r)} - 1\right)\left(1 - \mathrm{e}^{a_2(t-2r)}\right)\right)^{k+1} I_p, & t \in (r, 2r), \end{cases} \tag{2}$$

where $a_1 > 0$ and $a_2 > 0$.

Finally, instrumental in the approach of the next section, we need the following technical lemma.

Lemma 1. [34] Assume that $\Psi \in \mathbf{R}^{m\times n}, \xi \in \mathbf{R}^n$ and $\eta \in \mathbf{R}^m$. If equation $\Psi\xi = \eta$ is solvable for ξ, then all solutions can be parameterized by

$$\xi = \Psi^-\eta + (I_n - \Psi^-\Psi)\delta,$$

where $\delta \in \mathbf{R}^n$ is a vector of free parameters, and Ψ^- is the generalized inverse of Ψ, satisfying $\Psi\Psi^-\Psi = \Psi$.

3 | APPROACH BASED ON MATRIX EQUATIONS AND STATE AUGMENTATION

3.1 | Elimination and augmentation

In [34], where the case $J = 0$ and $F = 0$ is considered, the output derivative $\dot{y}$ is utilized to achieve the goal. However, in this paper, the unknown input d also appears in the output channel. Inspired by [9], we use well-known matrix equation theory to first eliminate d from the output equation. More precisely, the solution of d to the output equation, interpreted as an under-determined algebraic equation in variable $d(t)$, is given by

$$d(t) = J^-(y(t) - Cx(t) - Du(t) - Fw(t)) + D_1\delta(t), \tag{3}$$

where $\delta(t)$ is some vector parameterizing all solutions and $D_1 = I_l - J^- J$ with J^- satisfying $JJ^-J = J$. In what follows, δ can can be considered as a new unknown input. Substituting (3) into (1) and pre-multiplying the second equation of (1) with projector $I_p - JJ^-$ give

$$\begin{cases} \dot{x}(t) = (A - EJ^-C)\,x(t) + (B - EJ^-D)\,u(t) + EJ^-y(t) - EJ^-Fw(t) + ED_1\delta(t), \\ Y(t) = (I_p - JJ^-)Cx(t) + (I_p - JJ^-)Fw(t) + (I_p - JJ^-)Du(t), \end{cases} \tag{4}$$

where $Y(t) = (I_p - JJ^-)y(t)$. The second equation of (4) is always effectively present since $I_p - JJ^- \neq 0$. Assumptions 1-2 namely imply that matrix J is not of full row rank. Note that if J were of full row rank, then the full measured output could be directly driven by disturbance d, removing all information from the measured output. To achieve our goal of estimating eventually the sensor fault w, we define

$$\tilde{x}(t) = \begin{bmatrix} x^{\mathrm{T}}(t) & w^{\mathrm{T}}(t) \end{bmatrix}^{\mathrm{T}}. \tag{5}$$

Then it follows from (4) that

$$\begin{cases} G\dot{\tilde{x}}(t) = A_{\mathrm{a}}\tilde{x}(t) + B_{\mathrm{a}}u(t) + J_{\mathrm{a}}y(t) + E_{\mathrm{a}}\delta(t), \\ Y(t) = C_{\mathrm{a}}\tilde{x}(t) + D_{\mathrm{a}}Du(t), \end{cases} \tag{6}$$

where

$$G = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{\mathrm{a}} = \left[\begin{array}{c|c} A - EJ^-C & -EJ^-F \\ \hline 0 & 0 \end{array}\right], \quad B_{\mathrm{a}} = \begin{bmatrix} B - EJ^-D \\ 0 \end{bmatrix}, \quad E_{\mathrm{a}} = \begin{bmatrix} ED_1 \\ 0 \end{bmatrix},$$
$$J_{\mathrm{a}} = \begin{bmatrix} EJ^- \\ 0 \end{bmatrix}, \quad D_{\mathrm{a}} = I_p - JJ^-, \quad C_{\mathrm{a}} = \begin{bmatrix} D_{\mathrm{a}}C & D_{\mathrm{a}}F \end{bmatrix}.$$

We denote $\mathrm{rank}(C_{\mathrm{a}}) = p_0$ and $\mathrm{rank}(E_{\mathrm{a}}) = l_0$. Then, one can select p_0 linearly independent row vectors and l_0 linearly independent column vectors in respectively matrix C_{a} and E_{a} to form two matrices as

$$\check{C}_{\mathrm{a}} = \begin{bmatrix} C_{\mathrm{a}1}^{\mathrm{T}} & \cdots & C_{\mathrm{a},p_0}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} = P_0C_{\mathrm{a}}, \quad \check{E}_{\mathrm{a}} = \begin{bmatrix} E_{\mathrm{a}1}^{\mathrm{T}} & \cdots & E_{\mathrm{a},p_0}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} = E_{\mathrm{a}}Q_0,$$

in which $P_0 \in \mathbf{R}_{p_0}^{p_0\times p}$ and $Q_0 \in \mathbf{R}_{l_0}^{l\times l_0}$ are two permutation matrices. We let $\check{Y}(t) = P_0Y(t)$ and $\delta(t) = Q_0\check{\delta}(t)$. Then (6) becomes

$$\begin{cases} G\dot{\tilde{x}}(t) = A_{\mathrm{a}}\tilde{x}(t) + B_{\mathrm{a}}u(t) + J_{\mathrm{a}}y(t) + \check{E}_{\mathrm{a}}\check{\delta}(t), \\ \check{Y}(t) = \check{C}_{\mathrm{a}}\tilde{x}(t) + P_0D_{\mathrm{a}}Du(t). \end{cases} \tag{7}$$

Our objective can now be translated into the problem of designing a prescribed-time observer for the augmented system (7).

Before formulating a solution with a reduced-order observer, we present two technical lemmas, whose proofs can be found in the Appendix.

Lemma 2. Under Assumption 1, there exist nonsingular matrix $P_{\mathrm{a}} \in \mathbf{R}^{(n+q)\times(n+q)}$ and matrix $Q_{\mathrm{a}} \in \mathbf{R}^{(n+q)\times p_0}$ such that

$$P_{\mathrm{a}}G + Q_{\mathrm{a}}\check{C}_{\mathrm{a}} = I_{n+q}.$$

Lemma 3. If Assumption 1 holds, then $\tilde{W}_{\mathrm{a}}P_{\mathrm{a}}\check{E}_{\mathrm{a}} = 0$, where

$$\tilde{W}_{\mathrm{a}} = I_{n+q} - W_{\mathrm{a}}\check{C}_{\mathrm{a}}, \quad W_{\mathrm{a}} = P_{\mathrm{a}}\check{E}_{\mathrm{a}}(\check{C}_{\mathrm{a}}P_{\mathrm{a}}\check{E}_{\mathrm{a}})^-. \tag{8}$$

in which P_{a} is defined in (55) in Appendix.

3.2 | Reduced-order prescribed-time fault estimation

We will develop a reduced-order prescribed-time sensor fault estimator, where we will rely on notation introduced in the proof of Lemma 2 in the appendix. Let

$$z(t) = K_{\mathrm{a}}\tilde{x}(t), \tag{9}$$

where K_{a} is defined in (51). Then it follows from (7) and (54) that

$$\begin{cases} \dot{z}(t) = V_{\mathrm{a}}^{-1}A_{\mathrm{a}}K_{\mathrm{a}}^{-1}z(t) + V_{\mathrm{a}}^{-1}B_{\mathrm{a}}u(t) + V_{\mathrm{a}}^{-1}J_{\mathrm{a}}y(t) - \bar{G}P_0D_{\mathrm{a}}D\dot{u}(t) + V_{\mathrm{a}}^{-1}\check{E}_{\mathrm{a}}\hat{\check{\delta}}(t) + \bar{G}\dot{\check{Y}}(t), \\ \check{Y}(t) = \check{C}_{\mathrm{a}}K_{\mathrm{a}}^{-1}z(t) + P_0D_{\mathrm{a}}Du(t), \end{cases} \tag{10}$$

where V_{a}^{-1} and $\bar{G}$ are respectively defined in (52) and (53). Taking the derivative of the second equation of (10) yields

$$\dot{\check{Y}}(t) = \check{C}_{\mathrm{a}}P_{\mathrm{a}}A_{\mathrm{a}}K_{\mathrm{a}}^{-1}z(t) + \check{C}_{\mathrm{a}}P_{\mathrm{a}}B_{\mathrm{a}}u(t) + \check{C}_{\mathrm{a}}P_{\mathrm{a}}J_{\mathrm{a}}y(t) + \check{C}_{\mathrm{a}}P_{\mathrm{a}}\check{E}_{\mathrm{a}}\hat{\check{\delta}}(t) + \check{C}_{\mathrm{a}}Q_{\mathrm{a}}\dot{\check{Y}}(t) + (I_{p_0} - \check{C}_{\mathrm{a}}Q_{\mathrm{a}})P_0D_{\mathrm{a}}D\dot{u}(t).$$

Then, applying Lemma 1 gives

$$\begin{aligned}\hat{\delta}(t) =&(\check{C}_{\rm a}P_{\rm a}\check{E}_{\rm a})^{-}(I_{p_0}-\check{C}_{\rm a}Q_{\rm a})\dot{\check{Y}}(t)-(\check{C}_{\rm a}P_{\rm a}\check{E}_{\rm a})^{-}\check{C}_{\rm a}P_{\rm a}A_{\rm a}K_{\rm a}^{-1}z(t)-(\check{C}_{\rm a}P_{\rm a}\check{E}_{\rm a})^{-}\check{C}_{\rm a}P_{\rm a}J_{\rm a}y(t)\\ &-(\check{C}_{\rm a}P_{\rm a}\check{E}_{\rm a})^{-}\check{C}_{\rm a}P_{\rm a}B_{\rm a}u(t)-(\check{C}_{\rm a}P_{\rm a}\check{E}_{\rm a})^{-}(I_{p_0}-\check{C}_{\rm a}Q_{\rm a})P_0D_{\rm a}D\dot{u}(t)+\left(I_l-(\check{C}_{\rm a}P_{\rm a}\check{E}_{\rm a})^{-}\check{C}_{\rm a}P_{\rm a}\check{E}_{\rm a}\right)\tilde{\delta}(t).\end{aligned}\tag{11}$$

Substituting (11) into the first equation of (10) and combining with the second equation of (10) produces

$$\begin{cases}\dot{z}(t)=\mathcal{A}_{\rm a}z(t)+\mathcal{B}_{\rm a}u(t)+\mathcal{J}_{\rm a}y(t)+\mathcal{C}_{\rm a}\dot{\check{Y}}(t)+\mathcal{D}_{\rm a}\dot{u}(t),\\ \check{Y}(t)=\check{C}_{\rm a}K_{\rm a}^{-1}z(t)+P_0D_{\rm a}Du(t),\end{cases}\tag{12}$$

where we have used Lemma 3 and

$$\mathcal{A}_{\rm a}=\tilde{W}_{\rm a1}V_{\rm a}^{-1}A_{\rm a}K_{\rm a}^{-1},\mathcal{B}_{\rm a}=\tilde{W}_{\rm a1}V_{\rm a}^{-1}B_{\rm a},\mathcal{J}_{\rm a}=\tilde{W}_{\rm a1}V_{\rm a}^{-1}J_{\rm a},\mathcal{C}_{\rm a}=W_{\rm a1}+\tilde{W}_{\rm a1}\bar{G},\mathcal{D}_{\rm a}=-(W_{\rm a1}+\tilde{W}_{\rm a1}\bar{G})P_0D_{\rm a},$$

in which $\tilde{W}_{\rm a1}=I_{n+q}-W_{\rm a1}\check{C}_{\rm a}K_{\rm a}^{-1},W_{\rm a1}=V_{\rm a}^{-1}E_{\rm a}(\check{C}_{\rm a}P_{\rm a}\check{E}_{\rm a})^{-}$. Let's partition the coefficient matrices of system (12) as

$$V_{\rm a}^{-1}A_{\rm a}K_{\rm a}^{-1}=\begin{bmatrix}\bar{A}_{\rm a11}&\bar{A}_{\rm a12}\\ \bar{A}_{\rm a21}&\bar{A}_{\rm a22}\end{bmatrix},\quad V_{\rm a}^{-1}B_{\rm a}=\begin{bmatrix}\bar{B}_{\rm a1}\\ \bar{B}_{\rm a2}\end{bmatrix},\quad V_{\rm a}^{-1}J_{\rm a}=\begin{bmatrix}\bar{J}_{\rm a1}\\ \bar{J}_{\rm a2}\end{bmatrix},\tag{13}$$

where $\bar{A}_{\rm a11}\in\mathbf{R}^{p_0\times p_0}$, $\bar{B}_{\rm a1}\in\mathbf{R}^{p_0\times m}$ and $\bar{J}_{\rm a1}\in\mathbf{R}^{p_0\times l_0}$. Note that

$$\tilde{W}_{\rm a1}=\begin{bmatrix}I_p-Y_{\rm a1}&0\\ -Y_{\rm a2}&I_{n+q-p_0}\end{bmatrix},\tag{14}$$

where $Y_{ai}=\bar{E}_{ai}(\check{C}_{\rm a}P_{\rm a}E_{\rm a})^{-}$, $i=1,2$. By (13) and (14), we can calculate the coefficient matrices of system (12) as

$$\begin{aligned}\mathcal{A}_{\rm a}&=\begin{bmatrix}(I_{p_0}-Y_{\rm a1})\bar{A}_{\rm a11}&(I_{p_0}-Y_{\rm a1})\bar{A}_{\rm a12}\\ \bar{A}_{\rm a21}-Y_{\rm a2}\bar{A}_{\rm a11}&\bar{A}_{\rm a22}-Y_{\rm a2}\bar{A}_{\rm a12}\end{bmatrix}\triangleq\begin{bmatrix}\tilde{A}_{\rm a11}&\tilde{A}_{\rm a12}\\ \tilde{A}_{\rm a21}&\tilde{A}_{\rm a22}\end{bmatrix},\\ \mathcal{B}_{\rm a}&=\begin{bmatrix}(I_{p_0}-Y_{\rm a1})\bar{B}_{\rm a1}\\ \bar{B}_{\rm a2}-Y_{\rm a2}\bar{B}_{\rm a1}\end{bmatrix}\triangleq\begin{bmatrix}\tilde{B}_{\rm a1}^{\rm T}&\tilde{B}_{\rm a2}^{\rm T}\end{bmatrix}^{\rm T},\\ \mathcal{J}_{\rm a}&=\begin{bmatrix}(I_{p_0}-Y_{\rm a1})\bar{J}_{\rm a1}\\ \bar{J}_{\rm a2}-Y_{\rm a2}\bar{J}_{\rm a1}\end{bmatrix}\triangleq\begin{bmatrix}\tilde{J}_{\rm a1}^{\rm T}&\tilde{J}_{\rm a2}^{\rm T}\end{bmatrix}^{\rm T},\\ \mathcal{C}_{\rm a}&=\begin{bmatrix}I_{p_0}+(Y_{\rm a1}-I_{p_0})V_{\rm a1}G_1\\ (Y_{\rm a2}V_{\rm a1}-V_{\rm a2})G_1\end{bmatrix}\triangleq\begin{bmatrix}\tilde{C}_{\rm a1}^{\rm T}&\tilde{C}_{\rm a2}^{\rm T}\end{bmatrix}^{\rm T},\\ \mathcal{D}_{\rm a}&=-\begin{bmatrix}(I_{p_0}+(Y_{\rm a1}-I_{p_0})V_{\rm a1}G_1)P_0D_{\rm a}D\\ (Y_{\rm a2}V_{\rm a1}-V_{\rm a2})G_1P_0D_{\rm a}D\end{bmatrix}\triangleq\begin{bmatrix}\tilde{D}_{\rm a1}^{\rm T}&\tilde{D}_{\rm a2}^{\rm T}\end{bmatrix}^{\rm T}.\end{aligned}$$

In this way, system (12) can be written as

$$\begin{cases}\begin{bmatrix}\dot{z}_{\rm a1}(t)\\ \dot{z}_{\rm a2}(t)\end{bmatrix}=\begin{bmatrix}\tilde{A}_{\rm a11}&\tilde{A}_{\rm a12}\\ \tilde{A}_{\rm a21}&\tilde{A}_{\rm a22}\end{bmatrix}\begin{bmatrix}z_{\rm a1}(t)\\ z_{\rm a2}(t)\end{bmatrix}+\begin{bmatrix}\tilde{B}_{\rm a1}\\ \tilde{B}_{\rm a2}\end{bmatrix}u(t)+\begin{bmatrix}\tilde{C}_{\rm a1}\\ \tilde{C}_{\rm a2}\end{bmatrix}\dot{\check{Y}}(t)+\begin{bmatrix}\tilde{D}_{\rm a1}\\ \tilde{D}_{\rm a2}\end{bmatrix}\dot{u}(t)+\begin{bmatrix}\tilde{J}_{\rm a1}\\ \tilde{J}_{\rm a2}\end{bmatrix}y(t),\\ \check{Y}(t)=z_{\rm a1}(t)+P_0D_{\rm a}Du(t).\end{cases}\tag{15}$$

Inspired by[35], we design $2r$-periodic gain $L_{(A_{\rm ao},r)}:\mathbf{R}\to\mathbf{R}^{(n+q-p_0)\times p_0}$ as

$$L_{(A_{\rm ao},r)}(t)=\mathrm{e}^{A_{\rm ao}t}M_{\rm ao}\mathrm{e}^{A_{\rm ao}^{\rm T}(t-r)}\tilde{A}_{\rm a12}^{\rm T}H_r(t),\quad t\in[0,2r],\tag{16}$$

where $A_{\rm ao}=\tilde{A}_{\rm a22}-L_{\rm ao}\tilde{A}_{\rm a12}$ with $L_{\rm ao}\in\mathbf{R}^{(n+q-p_0)\times p_0}$, $M_{\rm ao}\in\mathbf{R}^{(n+q-p_0)\times(n+q-p_0)}$ is a matrix to be determined, and $H_r(t)$ is an $\mathbf{S}_{n+q-p_0}^{(k)}(r)$ function for some integer $k\geq 1$. In what follows we use $L_{A_{\rm ao}}(t)$ instead of $L_{(A_{\rm ao},r)}(t)$ to simplify the notations, and denote

$$\begin{aligned}G_{\rm a1}&=A_{\rm ao}\Omega_{\rm a0}+\Omega_{\rm a4},H_{\rm a1}=\Omega_{\rm a1}-L_{\rm ao}\Omega_{\rm a2}+A_{\rm ao}(\tilde{D}_{\rm a2}-L_{\rm ao}\Omega_{\rm a3}),\\ G_{\rm a2}(t)&=A_{\rm ao}\Omega_{\rm a}(t)-\dot{\Omega}_{\rm a}(t)-L_{A_{\rm ao}}(t)\Omega_{\rm a5},\\ H_{\rm a2}(t)&=(\dot{L}_{A_{\rm ao}}(t)-A_{\rm ao}L_{A_{\rm ao}}(t))\Omega_{\rm a3}-L_{A_{\rm ao}}(t)(\Omega_{\rm a2}+\tilde{A}_{\rm a12}(\tilde{D}_{\rm a2}-L_{\rm ao}\Omega_{\rm a3})),\\ M_{\rm a1}&=K_{\rm a}^{-1}\begin{bmatrix}0^{\rm T}&I_{n+q-p_0}\end{bmatrix}^{\rm T},N_{\rm a1}=K_{\rm a}^{-1}\begin{bmatrix}(P_0D_{\rm a})^{\rm T}&\Omega_{\rm a0}^{\rm T}\end{bmatrix}^{\rm T},N_{\rm a2}(t)=K_{\rm a}^{-1}\begin{bmatrix}0^{\rm T}&\Omega_{\rm a}^{\rm T}(t)\end{bmatrix}^{\rm T},\\ P_{\rm a1}&=K_{\rm a}^{-1}\begin{bmatrix}-(P_0D_{\rm a}D)^{\rm T}&(\tilde{D}_{\rm a2}-L_{\rm ao}\Omega_{\rm a3})^{\rm T}\end{bmatrix}^{\rm T},\ P_{\rm a2}(t)=-K_{\rm a}^{-1}\begin{bmatrix}0^{\rm T}&(L_{A_{\rm ao}}(t)\Omega_{\rm a3})^{\rm T}\end{bmatrix}^{\rm T},\end{aligned}\tag{17}$$

where

$$\begin{aligned}&\Omega_{\rm a0}=(L_{\rm ao}+\tilde{C}_{\rm a2}-L_{\rm ao}\tilde{C}_{\rm a1})P_0D_{\rm a},\quad\Omega_{\rm a1}=\tilde{B}_{\rm a2}-\tilde{A}_{\rm a21}P_0D_{\rm a},\\ &\Omega_{\rm a2}=\tilde{B}_{\rm a1}-\tilde{A}_{\rm a11}P_0D_{\rm a}D,\quad\Omega_{\rm a3}=\tilde{D}_{\rm a1}+P_0D_{\rm a}D,\Omega_{\rm a}(t)=L_{A_{\rm ao}}(t)(I_{p_0}-\tilde{C}_{\rm a1})P_0D_{\rm a},\end{aligned}$$

$$\Omega_{\mathrm{a}4}=\tilde{A}_{\mathrm{a}21}P_0D_{\mathrm{a}}+\tilde{J}_{\mathrm{a}2}-L_{\mathrm{ao}}(\tilde{A}_{\mathrm{a}11}P_0D_{\mathrm{a}}+\tilde{J}_{\mathrm{a}1}),\quad \Omega_{\mathrm{a}5}=\tilde{A}_{\mathrm{a}11}P_0D_{\mathrm{a}}+\tilde{J}_{\mathrm{a}1}+\tilde{A}_{\mathrm{a}12}\Omega_{\mathrm{a}0}.$$

Then we can state the following result of the paper.

Theorem 1. Let Assumptions 1-2 hold and assume that the fault w is a locally absolutely continuous function. Let T be a constant satisfying $T>0$, and let the matrices $\{G_{\mathrm{a}1},G_{\mathrm{a}2}(t),H_{\mathrm{a}1},H_{\mathrm{a}2}(t),M_{\mathrm{a}1},N_{\mathrm{a}1},N_{\mathrm{a}2}(t),P_{\mathrm{a}1},P_{\mathrm{a}2}(t)\}$ be described in (17) in which the gain $L_{A_{\mathrm{ao}}}(t)$ is given by (16). Then, for any $r>0$, matrix

$$W_{\mathrm{o}}\left(A_{\mathrm{ao}},r\right)=\int_0^r \mathrm{e}^{A_{\mathrm{ao}}^{\mathrm{T}}s}\tilde{A}_{\mathrm{a}12}^{\mathrm{T}}H_r\left(s+r\right)\tilde{A}_{\mathrm{a}12}\mathrm{e}^{A_{\mathrm{ao}}s}\mathrm{d}s \tag{18}$$

is invertible. In addition, if we choose M_{ao} as

$$M_{\mathrm{ao}}=W_{\mathrm{o}}^{-1}\left(A_{\mathrm{ao}},r\right), \tag{19}$$

where $r=\frac{1}{2}T$, then a prescribed-time sensor fault estimator of reduced-order p_0 can be constructed as

$$\begin{cases}\dot{\chi}_{\mathrm{a}}(t)=A_{\mathrm{ao}}\chi_{\mathrm{a}}(t)-L_{A_{\mathrm{ao}}}(t)\tilde{A}_{\mathrm{a}12}\chi_{\mathrm{a}}(t-r)+G_{\mathrm{a}2}(t)y(t-r)+H_{\mathrm{a}1}u(t)+G_{\mathrm{a}1}y(t)+H_{\mathrm{a}2}(t)u(t-r),\\ \hat{\tilde{x}}(t)=M_{\mathrm{a}1}\chi_{\mathrm{a}}(t)+N_{\mathrm{a}1}y(t)+N_{\mathrm{a}2}\left(t\right)y(t-r)+P_{\mathrm{a}1}u(t)+P_{\mathrm{a}2}(t)u(t-r),\\ \hat{w}(t)=\left[\begin{array}{cc}0 & I_q\end{array}\right]\hat{\tilde{x}}(t),\end{cases} \tag{20}$$

where the initial conditions can be chosen arbitrarily.

Proof. In light of (15), we know that

$$\begin{aligned}\dot{z}_{\mathrm{a}2}(t)=&\tilde{A}_{\mathrm{a}22}z_{\mathrm{a}2}(t)+\tilde{A}_{\mathrm{a}21}z_{\mathrm{a}1}(t)+\tilde{B}_{\mathrm{a}2}u(t)+\tilde{C}_{\mathrm{a}2}\dot{\check{Y}}(t)+\tilde{D}_{\mathrm{a}2}\dot{u}(t)+\tilde{J}_{\mathrm{a}2}y(t)\\ =&\tilde{A}_{\mathrm{a}22}z_{\mathrm{a}2}(t)+\tilde{A}_{\mathrm{a}21}(\check{Y}(t)-P_0D_{\mathrm{a}}Du(t))+\tilde{B}_{\mathrm{a}2}u(t)+\tilde{C}_{\mathrm{a}2}\dot{\check{Y}}(t)+\tilde{D}_{\mathrm{a}2}\dot{u}(t)+\tilde{J}_{\mathrm{a}2}y(t)\\ =&\tilde{A}_{\mathrm{a}22}z_{\mathrm{a}2}(t)+(\tilde{A}_{\mathrm{a}21}P_0D_{\mathrm{a}}+\tilde{J}_{\mathrm{a}2})y(t)+(\tilde{B}_{\mathrm{a}2}-\tilde{A}_{\mathrm{a}21}P_0D_{\mathrm{a}}D)u(t)+\tilde{C}_{\mathrm{a}2}P_0D_{\mathrm{a}}\dot{y}(t)+\tilde{D}_{\mathrm{a}2}\dot{u}(t),\end{aligned} \tag{21}$$

and

$$\begin{aligned}\dot{\check{Y}}(t)=&\tilde{A}_{\mathrm{a}11}(\check{Y}(t)-P_0D_{\mathrm{a}}Du(t))+\tilde{A}_{\mathrm{a}12}z_{\mathrm{a}2}(t)+\tilde{J}_{\mathrm{a}1}y(t)+\tilde{B}_{\mathrm{a}1}u(t)+\tilde{C}_{\mathrm{a}1}\dot{\check{Y}}(t)+(\tilde{D}_{\mathrm{a}1}+P_0D_{\mathrm{a}}D)\dot{u}(t)\\ =&\tilde{A}_{\mathrm{a}12}z_{\mathrm{a}2}(t)+(\tilde{A}_{\mathrm{a}11}P_0D_{\mathrm{a}}+\tilde{J}_{\mathrm{a}1})y(t)+(\tilde{B}_{\mathrm{a}1}-\tilde{A}_{\mathrm{a}11}P_0D_{\mathrm{a}}D)u(t)+\tilde{C}_{\mathrm{a}1}P_0D_{\mathrm{a}}\dot{y}(t)+(\tilde{D}_{\mathrm{a}1}+P_0D_{\mathrm{a}}D)\dot{u}(t).\end{aligned} \tag{22}$$

Define

$$y_{\mathrm{a}2}(t)=\dot{\check{Y}}(t),\quad u_{\mathrm{a}2}(t)=\left[\begin{array}{cccc}y^{\mathrm{T}}(t) & \dot{y}^{\mathrm{T}}(t) & u^{\mathrm{T}}\left(t\right) & \dot{u}^{\mathrm{T}}\left(t\right)\end{array}\right]^{\mathrm{T}}.$$

Then, combining (21) and (22) leads to

$$\begin{cases}\dot{z}_{\mathrm{a}2}(t)=\tilde{A}_{\mathrm{a}22}z_{\mathrm{a}2}(t)+\tilde{\Phi}_{\mathrm{a}}u_{\mathrm{a}2}(t),\\ y_{\mathrm{a}2}(t)=\tilde{A}_{\mathrm{a}12}z_{\mathrm{a}2}(t)+\tilde{\Psi}_{\mathrm{a}}u_{\mathrm{a}2}(t),\end{cases} \tag{23}$$

where

$$\tilde{\Phi}_{\mathrm{a}}=\left[\begin{array}{cccc}\tilde{A}_{\mathrm{a}21}P_0D_{\mathrm{a}}+\tilde{J}_{\mathrm{a}2} & \tilde{C}_{\mathrm{a}2}P_0D_{\mathrm{a}} & \Omega_{\mathrm{a}1} & \tilde{D}_{\mathrm{a}2}\end{array}\right],\quad \tilde{\Psi}_{\mathrm{a}}=\left[\begin{array}{cccc}\tilde{A}_{\mathrm{a}11}P_0D_{\mathrm{a}}+\tilde{J}_{\mathrm{a}1} & \tilde{C}_{\mathrm{a}1}P_0D_{\mathrm{a}} & \Omega_{\mathrm{a}2} & \Omega_{\mathrm{a}3}\end{array}\right].$$

Now, we show that $(\tilde{A}_{\mathrm{a}22},\tilde{A}_{\mathrm{a}12})$ is observable. For this purpose, we only need to prove that $(\tilde{A}_{\mathrm{a}},\check{C}_{\mathrm{a}})$ is observable since

$$\operatorname{rank}\left[\begin{array}{c}sI_{n+q}-\mathcal{A}_{\mathrm{a}}\\ \check{C}_{\mathrm{a}}K_{\mathrm{a}}^{-1}\end{array}\right]=\operatorname{rank}\left[\begin{array}{cc}K_{\mathrm{a}}^{-1} & 0\\ 0 & I_p\end{array}\right]\left[\begin{array}{c}\left(sI_{n+q}-\mathcal{A}_{\mathrm{a}}\right)K_{\mathrm{a}}\\ \check{C}_{\mathrm{a}}\end{array}\right]=\operatorname{rank}\left[\begin{array}{c}sI_{n+q}-\tilde{A}_{\mathrm{a}}\\ \check{C}_{\mathrm{a}}\end{array}\right].$$

We first prove that Assumption 2 is equivalent to

$$\left[\begin{array}{cc}A_{\mathrm{a}}-sG & \check{E}_{\mathrm{a}}\\ \check{C}_{\mathrm{a}} & 0\end{array}\right]=n+q+\operatorname{rank}(\check{E}_{\mathrm{a}}),\forall s\in\mathbf{C}, \tag{24}$$

if Assumption 1 holds. We rely on results and notations introduced in the proof of Lemma 2 in the Appendix. According to (47), we have

$$\operatorname{rank}\left[\begin{array}{ccc}A-sI_n & 0 & E\\ C & F & J\end{array}\right]=n+q+\operatorname{rank}(J)+\operatorname{rank}(\check{E}_{\mathrm{a}}),\forall s\in\mathbf{C}. \tag{25}$$

Denote

$$\Delta_1^i=\operatorname{diag}\{I_i,[J^-J\ D_1]\},\quad \Delta_{\mathrm{a}}^i=\operatorname{diag}\left\{I_i,\left[\begin{array}{c}JJ^-\\ D_{\mathrm{a}}\end{array}\right]\right\},\quad \Delta_1=\left[\begin{array}{cccc}I_n & 0 & 0 & 0\\ 0 & I_q & 0 & 0\\ -J^-C & -J^-F & I_l & 0\\ 0 & 0 & 0 & I_l\end{array}\right]. \tag{26}$$

For any $J \in \mathbf{R}^{p\times l}$ of rank r, its singular value decomposition and generalized inverse can be expressed as

$$J = U\begin{bmatrix}\Xi & 0\\ 0 & 0\end{bmatrix}V^{\mathrm{T}},\ J^{-} = V\begin{bmatrix}\Xi^{-1} & 0\\ 0 & 0\end{bmatrix}U^{\mathrm{T}},$$

where U and V are orthogonal matrices, $\Xi = \mathrm{diag}\left(\sigma_1, \sigma_1, \ldots, \sigma_r\right), \sigma_i > 0, i = 1, 2, \ldots, r$. Thus we get

$$\mathrm{rank}\left[\begin{array}{cc} J^{-}J & D_{\mathrm{l}}\end{array}\right] = \mathrm{rank}\, V\begin{bmatrix} I_r & 0\,0 & 0\\ 0\,0\,0 & I_{l-r}\end{bmatrix}V^{\mathrm{T}} = l,$$

which shows that the matrix $[J^{-}J\ D_{\mathrm{l}}]$ is of full row rank. Since the matrix Δ_{a}^{n} is of full column rank, we can get

$$\begin{aligned}
\mathrm{rank}\begin{bmatrix} A - sI_n & 0 & E\\ C & F & J\end{bmatrix} =& \mathrm{rank}\begin{bmatrix} A - sI_n & 0 & E\\ C & F & J\end{bmatrix}\Delta_{\mathrm{l}}^{n+q}\\
=& \mathrm{rank}\begin{bmatrix} A - sI_n - EJ^{-}C & -EJ^{-}F & 0 & ED_{\mathrm{l}}\\ C & F & J & 0\end{bmatrix}\\
=& \mathrm{rank}\Delta_{\mathrm{a}}^{n}\begin{bmatrix} A - sI_n - EJ^{-}C & EJ^{-}F & 0 & ED_{\mathrm{l}}\\ C & F & J & 0\end{bmatrix}\\
=& \mathrm{rank}\begin{bmatrix} A - sI_n - EJ^{-}C & -EJ^{-}F & 0 & ED_{\mathrm{l}}\\ JJ^{-}C & JJ^{-}F & J & 0\\ D_{\mathrm{a}}C & D_{\mathrm{a}}F & 0 & 0\end{bmatrix}\Delta_{\mathrm{l}}\\
=& \mathrm{rank}(J) + \mathrm{rank}\begin{bmatrix} A_{\mathrm{a}} - sG & \check{E}_{\mathrm{a}}\\ \check{C}_{\mathrm{a}} & 0\end{bmatrix},
\end{aligned} \tag{27}$$

where we have used Assumption 1. In view of (25) and (27), it is obvious that Assumption 2 is equivalent to (24). Next, we show that (24) is equivalent to the observability of $(\tilde{A}_{\mathrm{a}}, \check{C}_{\mathrm{a}})$. Note that

$$\begin{aligned}
\mathrm{rank}\begin{bmatrix} A_{\mathrm{a}} - sG & \check{E}_{\mathrm{a}}\\ \check{C}_{\mathrm{a}} & 0\end{bmatrix} =& \mathrm{rank}\begin{bmatrix} P_{\mathrm{a}}A_{\mathrm{a}} - sP_{\mathrm{a}}G & P_{\mathrm{a}}\check{E}_{\mathrm{a}}\\ \check{C}_{\mathrm{a}} & 0\end{bmatrix}\\
=& \mathrm{rank}\begin{bmatrix} I_{n+q} & -sQ_{\mathrm{a}}\\ 0 & I_p\end{bmatrix}\begin{bmatrix} P_{\mathrm{a}}A_{\mathrm{a}} - sP_{\mathrm{a}}G & P_{\mathrm{a}}\check{E}_{\mathrm{a}}\\ \check{C}_{\mathrm{a}} & 0\end{bmatrix}\\
=& \mathrm{rank}\begin{bmatrix} \tilde{W}_{\mathrm{a}} & -sW_{\mathrm{a}}\\ 0 & I_p\\ W_{\mathrm{a}}\check{C}_{\mathrm{a}} & sW_{\mathrm{a}}\end{bmatrix}\begin{bmatrix} P_{\mathrm{a}}A_{\mathrm{a}} - sI_{n+q} & P_{\mathrm{a}}\check{E}_{\mathrm{a}}\\ \check{C}_{\mathrm{a}} & 0\end{bmatrix}\\
=& \mathrm{rank}\begin{bmatrix} \tilde{A}_{\mathrm{a}} - sI_{n+q} & 0\\ \check{C}_{\mathrm{a}} & 0\\ W_{\mathrm{a}}\check{C}_{\mathrm{a}}P_{\mathrm{a}}A_{\mathrm{a}} & P_{\mathrm{a}}\check{E}_{\mathrm{a}}\end{bmatrix}\begin{bmatrix} I_{n+q} & 0\\ -(\check{C}_{\mathrm{a}}P_{\mathrm{a}}E_{\mathrm{a}})^{-}\check{C}_{\mathrm{a}}P_{\mathrm{a}}A_{\mathrm{a}} & I_l\end{bmatrix}\\
=& \mathrm{rank}\begin{bmatrix} \tilde{A}_{\mathrm{a}} - sI_{n+q}\\ \check{C}_{\mathrm{a}}\end{bmatrix} + \mathrm{rank}(\check{E}_{\mathrm{a}}).
\end{aligned}$$

From the final expression we conclude that Assumption 2 is satisfied if and only if $(\tilde{A}_{\mathrm{a}}, \check{C}_{\mathrm{a}})$ is observable.

Besides, the observability of $(\tilde{A}_{\mathrm{a22}}, \tilde{A}_{\mathrm{a12}})$ implies that matrix $W_{\mathrm{o}}\left(A_{\mathrm{ao}}, r\right)$ is invertible. Then an observer for system (23) can be designed as

$$\begin{cases}\dot{\hat{z}}_{\mathrm{a2}}(t) = \tilde{A}_{\mathrm{a22}}\hat{z}_{\mathrm{a2}}(t) + \tilde{\boldsymbol{\Phi}}_{\mathrm{a}}u_{\mathrm{a2}}(t) + L_{\mathrm{ao}}\left(y_{\mathrm{a2}}(t) - \hat{y}_{\mathrm{a2}}(t)\right) + L_{A_{\mathrm{ao}}}(t)(y_{\mathrm{a2}}(t-r) - \hat{y}_{\mathrm{a2}}(t-r)),\\ \hat{y}_{\mathrm{a2}}(t) = \tilde{A}_{\mathrm{a12}}\hat{z}_{\mathrm{a2}}(t) + \tilde{\Psi}_{\mathrm{a}}u_{\mathrm{a2}}(t),\end{cases} \tag{28}$$

where

$$\hat{y}_{\mathrm{a2}}(t) = \begin{cases} y_{\mathrm{a2}}(t), & t \geq 0,\\ \tilde{v}_{\mathrm{a1}}(t), & t \in [-r, 0),\end{cases}$$

in which $\tilde{v}_{\mathrm{a1}}(t)$ is an arbitrarily piecewise continuous function. Denote $e_{\mathrm{a2}}(t) = z_{\mathrm{a2}}(t) - \hat{z}_{\mathrm{a2}}(t)$. Then from (23) and (28) we can obtain a similar expression as (23) in [15]. Thus $e_{\mathrm{a2}}(t) = 0$ for $t \geq 2r$ if and only if the choice (19) is taken. Furthermore, we have from (28) that

$$\begin{aligned}
\dot{\hat{z}}_{\mathrm{a2}}(t) =& A_{\mathrm{ao}}\hat{z}_{\mathrm{a2}}(t) - L_{A_{\mathrm{ao}}}(t)\tilde{A}_{\mathrm{a12}}\hat{z}_{\mathrm{a2}}(t-r) + (\tilde{A}_{\mathrm{a21}}P_0D_{\mathrm{a}} + \tilde{J}_{\mathrm{a2}} - L_{\mathrm{ao}}(\tilde{A}_{\mathrm{a11}}P_0D_{\mathrm{a}} + \tilde{J}_{\mathrm{a1}}))y(t) + (\Omega_{\mathrm{a1}} - L_{\mathrm{ao}}\Omega_{\mathrm{a2}})u(t)\\
&- L_{A_{\mathrm{ao}}}(t)(\tilde{A}_{\mathrm{a11}}P_0D_{\mathrm{a}} + \tilde{J}_{\mathrm{a1}})y(t-r) + (\tilde{C}_{\mathrm{a2}} - L_{\mathrm{ao}}\tilde{C}_{\mathrm{a1}} + L_{\mathrm{ao}})P_0D_{\mathrm{a}}\dot{y}(t) + (L_{A_{\mathrm{ao}}}(t) - L_{A_{\mathrm{ao}}}(t)\tilde{C}_{\mathrm{a1}})P_0D_{\mathrm{a}}\dot{y}(t-r)
\end{aligned}$$

$$
\begin{aligned}
&- L_{A_{\mathrm{ao}}}(t)\Omega_{\mathrm{a2}}u(t-r) + (\tilde{D}_{\mathrm{a2}} - L_{\mathrm{ao}}\Omega_{\mathrm{a3}})\dot{u}(t) - L_{A_{\mathrm{ao}}}(t)\Omega_{\mathrm{a3}}\dot{u}(t-r)\\
=&A_{\mathrm{ao}}\hat{z}_{\mathrm{a2}}(t) - L_{A_{\mathrm{ao}}}(t)\tilde{A}_{\mathrm{a12}}\hat{z}_{\mathrm{a2}}(t-r) - L_{A_{\mathrm{ao}}}(t)(\tilde{A}_{\mathrm{a11}}P_0D_{\mathrm{a}} + \tilde{J}_{\mathrm{a1}})y(t-r) + (\Omega_{\mathrm{a1}} - L_{\mathrm{ao}}\Omega_{\mathrm{a2}})u(t) + \Omega_{\mathrm{a4}}y(t) + \Omega_{\mathrm{a0}}\dot{y}(t)\\
&- L_{A_{\mathrm{ao}}}(t)\Omega_{\mathrm{a2}}u(t-r) + \Omega_{\mathrm{a}}(t)\dot{y}(t-r) + (\tilde{D}_{\mathrm{a2}} - L_{\mathrm{ao}}\Omega_{\mathrm{a3}})\dot{u}(t) - L_{A_{\mathrm{ao}}}(t)\Omega_{\mathrm{a3}}\dot{u}(t-r).
\end{aligned}
$$

Let

$$
\chi_{\mathrm{a}}(t) = \hat{z}_{\mathrm{a2}}(t) - \Omega_{\mathrm{a0}}y(t) - (\tilde{D}_{\mathrm{a2}} - L_{\mathrm{ao}}\Omega_{\mathrm{a3}})u(t) - \Omega_{\mathrm{a}}(t)y(t-r) + L_{A_{\mathrm{ao}}}(t)\Omega_{\mathrm{a3}}u(t-r). \tag{29}
$$

Then we can get

$$
\begin{aligned}
\dot{\chi}_{\mathrm{a}}(t) =&A_{\mathrm{ao}}\chi_{\mathrm{a}}(t) - L_{A_{\mathrm{ao}}}(t)\tilde{A}_{\mathrm{a12}}\chi_{\mathrm{a}}(t-r) + (A_{\mathrm{ao}}\Omega_{\mathrm{a0}} + \Omega_{\mathrm{a4}})y(t) + (A_{\mathrm{ao}}(\tilde{D}_{\mathrm{a2}} - L_{\mathrm{ao}}\Omega_{\mathrm{a3}}) + (\Omega_{\mathrm{a1}} - L_{\mathrm{ao}}\Omega_{\mathrm{a2}}))u(t)\\
&+ (A_{\mathrm{ao}}\Omega_{\mathrm{a}}(t) - L_{A_{\mathrm{ao}}}(t)\tilde{A}_{\mathrm{a12}}\Omega_{\mathrm{a0}} - (\dot{\Omega}_{\mathrm{a}}(t) + L_{A_{\mathrm{ao}}}(t)(\tilde{A}_{\mathrm{a11}}P_0D_{\mathrm{a}} + \tilde{J}_{\mathrm{a1}})))y(t-r)\\
&+ ((\dot{L}_{A_{\mathrm{ao}}}(t) - A_{\mathrm{ao}}L_{A_{\mathrm{ao}}}(t))\Omega_{\mathrm{a3}} - L_{A_{\mathrm{ao}}}(t)(\Omega_{\mathrm{a2}} + \tilde{A}_{\mathrm{a12}}(\tilde{D}_{\mathrm{a2}} - L_{\mathrm{ao}}\Omega_{\mathrm{a3}})))u(t-r),
\end{aligned}
$$

which is the first equation of (20). Moreover, it follows from (29) and (15) that

$$
\begin{aligned}
\hat{\tilde{x}}(t) =&K_{\mathrm{a}}^{-1}\begin{bmatrix} z_{\mathrm{a1}}(t) \\ \hat{z}_{\mathrm{a2}}(t)\end{bmatrix} = K_{\mathrm{a}}^{-1}\begin{bmatrix} 0 \\ I_{n+q-p_0}\end{bmatrix}\chi_{\mathrm{a}}(t) + K_{\mathrm{a}}^{-1}\begin{bmatrix} P_0D_{\mathrm{a}} \\ \Omega_{\mathrm{a0}}\end{bmatrix}y(t) + K_{\mathrm{a}}^{-1}\begin{bmatrix} 0 \\ \Omega_{\mathrm{a}}(t)\end{bmatrix}y(t-r)\\
&+ K_{\mathrm{a}}^{-1}\begin{bmatrix} -P_0D_{\mathrm{a}}D \\ \tilde{D}_{\mathrm{a2}} - L_{\mathrm{ao}}\Omega_{\mathrm{a3}}\end{bmatrix}u(t) - K_{\mathrm{a}}^{-1}\begin{bmatrix} 0 \\ L_{A_{\mathrm{ao}}}(t)\Omega_{\mathrm{a3}}\end{bmatrix}u(t-r),
\end{aligned}
$$

which is the second equation of (20), while the last equation is immediately obtained from (5). The proof is finished. □

Remark 2. The condition of continuous differentiability of the fault stems from the methodology, more precisely from its embedding in the (pseudo) state of (7).

4 | REDUCED-ORDER ESTIMATION BASED ON STATE, INPUT AND OUTPUT TRANSFORMATIONS

In this section, we outline an alternative approach, relying on successive transformations. As we shall see, the construction allows to highlight the role of different directions in the output space in order to handle sensor faults, handle unknown inputs and solve the core estimation task.

Using a preliminary transformation of output y and unknown input d, system (1) can always be brought into the form

$$
\begin{cases}
\dot{x}(t) &= Ax(t) + Bu(t) + E_1d_1(t) + E_2d_2(t),\\
y_1(t) &= C_1x(t) + D_1u(t) + F_1w(t) + d_1(t),\\
y_2(t) &= C_2x(t) + D_2u(t) + F_2w(t),\\
y_3(t) &= C_3x(t) + D_3u(t),
\end{cases} \tag{30}
$$

where $d_1(t) \in \mathbf{R}^{l_1}$, with $l_1 = \mathrm{rank}(J)$, $d_2(t) \in \mathbf{R}^{l-l_1}$, and matrix F_2 is of full row rank. More precisely, using the singular value decomposition of J in (1), a maximal number of output directions that do not depend on unknown input d can be separated (reducing y to y_1) and, at the same time, the effective number of unknown inputs in the output equation can be reduced to a minimum. In a similar way, within the subspace spanned by these output directions, a maximum number of directions can be formed that do not depend on the sensor faults w, inducing the distinction between y_2 and y_3. Note that the conditions on the system matrices in Assumptions 1-2 are invariant with respect to these transformations.

In what follows, we assume that Assumptions 1-2 hold. The former immediately implies that matrix F_2 in (30) is also of full column rank, and thus an invertible $q \times q$ matrix. Now, the first two output equations can be given the explicit role of elimination of unknown input d_1 and all sensor faults w, via

$$
\begin{bmatrix} d_1(t) \\ w(t)\end{bmatrix} = \begin{bmatrix} I & F_1 \\ 0 & F_2\end{bmatrix}^{-1}\begin{bmatrix} y_1(t) - C_1x(t) - D_1u(t) \\ y_2(t) - C_2x(t) - D_2u(t)\end{bmatrix}. \tag{31}
$$

Clearly, w can be determined once x has been estimated. Substituting the expression for d_1 in (30) reduces the estimation problem to the one for

$$
\begin{cases}
\dot{x}(t) &= \bar{A}x(t) + \bar{B}u(t) + \bar{G}y_1(t) + \bar{H}y_2(t) + E_2d_2(t),\\
y_3(t) &= C_3x(t) + D_3u(t),
\end{cases} \tag{32}
$$

with

$$\bar{A} = A - E_1C_1 + E_1F_1F_2^{-1}C_2, \quad \bar{B} = B - E_1D_1 + E_1F_1F_2^{-1}D_2, \ \ \bar{G} = E_1, \ \ \bar{H} = -E_1F_1F_2^{-1}.$$

Assumptions 1-2, when applied to (30), can be rephrased in terms of the following requirements for "reduced" system (32):

$$\operatorname{rank}(C_3E_2) = \operatorname{rank}(E_2), \quad \operatorname{rank}\begin{bmatrix} \bar{A} - sI_n & E_2 \\ C_3 & 0 \end{bmatrix} = n + \operatorname{rank}(E_2), \quad \forall s \in \mathbf{C}. \tag{33}$$

With (32) and (33), we are in the setting of article[15], addressing the fixed-time estimation problem in the presence of an unknown input, in this case d_2. As explained in Section III-C of that reference, there always exist transformations of state ($x \mapsto (\zeta_1, \zeta_2)$) and output $y_3 \mapsto (\tilde{y}_1, \tilde{y}_2)$ that bring (32) into the following form, which is reminiscent of a normal form,

$$\begin{cases} \dot{\zeta}_1(t) = \bar{A}_{11}\zeta_1(t) + \bar{A}_{12}\zeta_2(t) + \bar{B}_1u(t) + \bar{G}_1y_1(t) + \bar{H}_1y_2(t), \\ \dot{\zeta}_2(t) = \bar{A}_{21}\zeta_1(t) + \bar{A}_{22}\zeta_2(t) + \bar{B}_2u(t) + \bar{G}_2y_1(t) + \bar{H}_2y_2(t) + \mathcal{E}d_2(t), \\ \tilde{y}_1(t) = \bar{C}\zeta_1(t) + \bar{D}_1u(t), \\ \tilde{y}_2(t) = \zeta_2(t) + \bar{D}_2u(t), \end{cases} \tag{34}$$

having the properties that the pair $(\bar{C}, \bar{A}_{11})$ is observable and that matrix $\mathcal{E}$ is of full row rank. The advantage of the form (34) is that the only state variables that are directly affected by unknown input d_2, namely these represented by ζ_2, are available from measured output $\tilde{y}_2$ and measured input u, by the second output equation. As a consequence, the (reduced-order fixed-time) estimation problem can again be reduced, namely to the one for ζ_1:

$$\begin{cases} \dot{\zeta}_1(t) = \bar{A}_{11}\zeta_1(t) + \bar{A}_{12}\zeta_2(t) + \bar{B}_1u(t) + \bar{G}_1y_1(t) + \bar{H}_1y_2(t), \\ \tilde{y}_1(t) = \bar{C}\zeta_1(t) + \bar{D}_1u(t), \end{cases} \tag{35}$$

for which the methodology of[35] can be used as ζ_2, u, y_1, y_2 are measured or directly based on the measurements. Recall that the estimate for (ζ_1, ζ_2), and thus for x, implies an estimate of w by the second equation of (31). The minimum dimension of the observer is equal to the dimension of ζ_1, reduced by the number of independent observations, which amounts to

$$(n - \operatorname{rank}(E_2)) - \operatorname{rank}(\bar{C}) = n - \operatorname{rank}(C_3),$$

which corresponds exactly to the dimension $n + q - p_0$ of observer (20).

We highlight some properties of the transformation based approach. The above construction sheds light on the role in the estimation of specific directions in the p-dimensional output space, represented by $(y_1, y_2, \tilde{y}_1, \tilde{y}_2)$:

- outputs y_1 and y_2 serve to eliminate unknown input d_1 and to obtain eventually the estimate for the sensor fault w;
- output $\tilde{y}_1$ serves to reconstruct all state-variables whose dynamic equation in (34) is directly affected by unknown input d_2;
- output $\tilde{y}_2$ finally serves for the *core* dynamic estimation task (of state variables ζ_1 in (34)).

Clearly, for every considered sensor fault and (independent) unknown input, the number of required outputs for solving Problem 1 increases by one. At the same time, the construction translates the solvability conditions expressed by Assumptions 1-2 into (i) the invertibility of matrix F_2, necessary for elimination of d_1 and w, (ii) the fact that $\tilde{y}_1$ in (34) does not depend on ζ_2 and the full rank of $\mathcal{E}$, allowing an isolation of d_2 to a measurable part of the state of minimal dimension, and (iii) the observability of $(\bar{C}, \bar{A}_{11})$ such that the remaining information in the measurements can be used to estimate the non-measured directions in the state-space in a prescribed-time (note that for asymptotic estimation, detectability is sufficient).

Let us know compare this methodology with the one of Section 3, illustrating the complementarity.

- The approach proposed in the present section is fundamentally different from the one in Section 3 in the way in which the estimate of the sensor fault is obtained. While the estimation problem in Section 3 is dynamic in nature, in the sense that w is part of an extended state and along x modeled by differential equation (10), relation (31) is static in nature. As a consequence, the approach of this section does not require the sensor fault to be continuously differentiable. This will be illustrated by the third example in the next section.
- The approach of Section 4, on the one hand, complements the one of Section 3 by giving a concrete meaning to the satisfaction of Assumptions 1-2 and making the role of different output directions explicit. On the other hand, the methodology of Section 3, which naturally gives rise to conditions formulated as in Assumptions 1-2, has the merit of expressing solvability conditions in terms of "clean" conditions the original system matrices, in the spirit of Luenberger's foundational work[17].

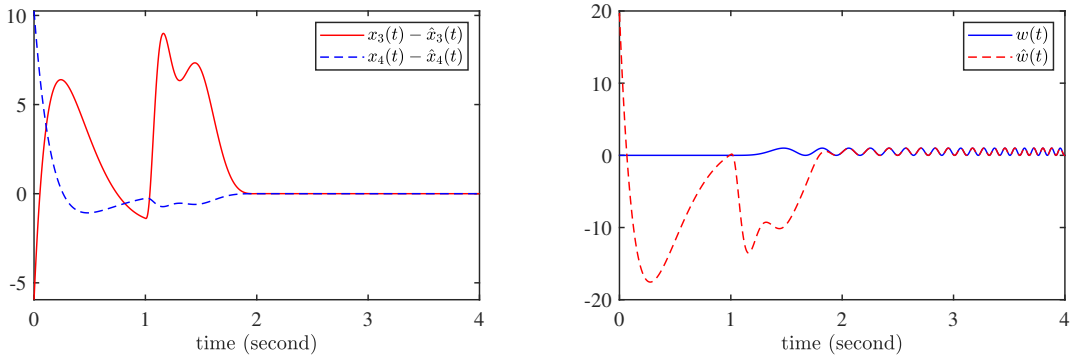


FIGURE 1 The errors $x_3(t) - \hat{x}_3(t)$ and $x_4(t) - \hat{x}_4(t)$, the sensor fault $w(t)$ and its estimate $\hat{w}(t)$ associated with the estimator (20) for Example 1

Remark 3. Consider now the case where output delay is added, such that system (1) becomes

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Ed(t), \\ y(t) = Cx(t-\gamma) + Du(t-\gamma) + Fw(t) + Jd(t). \end{cases} \tag{36}$$

If $d \equiv 0$, then Problem 1 is solvable for any value of $\gamma > 0$, under the condition that the prescribed convergence time satisfies $T \geq 3\gamma$. To see this, part of the output can be used to eliminate w (second equation of (31)). The other directions in the output can then be used to solve the prescribed-time observation problem of the plant in the presence of output delay γ. According to article[35], *any* convergence time for the estimate of x larger than 2γ can be realized using smooth periodic feedback. From the second equation of (31), the correct estimate of w is obtained γ secs later.

If $d \neq 0$ the adopted approach cannot be used, since the elimination of the unknown input from the output equation, using both the methodology of Section 3 and Section 4, creates a term in the state equation with state-delay, while the fixed-time stabilization mechanisms using periodic feedback is specific for systems with input/output delay.

5 | SIMULATION RESULTS

Example 1: The dynamics of a helicopter in the vertical plane are modeled in article[19] by

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{37}$$

where $x(t) = [x_1(t), x_2(t), x_3(t), x_4(t)]^{\mathrm{T}}$, with $x_1(t)$, $x_2(t)$, $x_3(t)$ and $x_4(t)$ denoting the horizontal velocity, the vertical velocity, the pitch rate and the pitch angle, respectively, and $u(t) = [u_1(t), u_2(t)]^{\mathrm{T}}$ with $u_1(t)$ and $u_2(t)$ denoting the collective and the longitudinal cyclic, respectively. The matrices A and B are

$$A = \begin{bmatrix} -0.0336 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.0100 & 0.0024 & -4.0208 \\ 0.1002 & 0.3681 & -0.7070 & 1.4200 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.4422 & 0.1761 \\ 3.5446 & -7.5922 \\ -5.5446 & 4.4900 \\ 0 & 0 \end{bmatrix}.$$

The helicopter flight will be inevitably affected by unknown factors and may also suffer from sensor failure. Taking these factors into account, we rewrite the above model as system (1). Then, other parameters for system (1) are chosen as

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{\mathrm{T}}, \quad F = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^{\mathrm{T}}, \quad J = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}}.$$

It is assumed that the known input, unknown input and sensor fault take the following form

$$u(t) = [0.1, 0]^{\mathrm{T}}, \quad d(t) = \sin(2t), \quad w(t) = \begin{cases} 0, & t \leq 1, \\ \sin^2(5(t-1)^2), & t > 1. \end{cases}$$

We can verify that Assumption 1 holds and that $(\tilde{A}_{\mathrm{a}}, \check{C}_{\mathrm{a}})$ is observable, which means that Assumption 2 is satisfied. Based on the construction in Lemma 2, we obtain

$$P_{\mathrm{a}} = \begin{bmatrix} 1 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 5 & 0 & 0 & 0 \end{bmatrix}, \quad Q_{\mathrm{a}} = \begin{bmatrix} 0 & -5 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & -5 \end{bmatrix}.$$

To construct estimator (20), we firstly calculate the gain L_{ao}. For this, we follow the procedure spelled out in[35]. In particular, we select

$$\gamma = -2 \min_{i=1,2,\ldots,n} \{\mathrm{Re}\{\lambda_i(\tilde{A}_{\mathrm{a}22})\}\} + 0.001 = \frac{1359}{352}.$$

Then by solving the parametric Lyapunov equation

$$\tilde{A}_{\mathrm{a}22} Q + Q \tilde{A}_{\mathrm{a}22}^{\mathrm{T}} - Q \tilde{A}_{\mathrm{a}12}^{\mathrm{T}} \tilde{A}_{\mathrm{a}12} Q = -\gamma Q,$$

we determine gain L_{ao} as

$$L_{\mathrm{ao}} = \begin{bmatrix} -\frac{496}{265} & \frac{353}{1187} & -\frac{577}{2715} \\ 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}.$$

For calculating the time-varying gain $L_{(A_{\mathrm{ao}},r)}(t)$, we take an $\mathbf{S}^{(1)}_{n+q-p_0}(r)$ function $H_r(t)$ as (2) with $a_1 = a_2 = 0.0001$. Note that $n+q-p_0 = 3$. We choose $T = 2$. Then $r = \frac{1}{2}T = 1$. Let the initial conditions be $x(0) = [1,\ 2,\ 1,\ 5]^{\mathrm{T}}$ and $\chi_{\mathrm{a}}(0) = [0,\ 1,\ -1]^{\mathrm{T}}$. From the simulation of system (1) and estimator (20), the obtained error trajectories, the fault signal and its estimation are displayed in Fig. 1. By this simulation result we know that the proposed reduced-order estimator (20) can accurately estimate the original system states and fault signal from $t = 2$s on, which is consistent with the conclusion of Theorem 1.

Example 2: Consider system (1) with

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 & -2 & -3 & 1 \\ 1 & 2 & -2 & 3 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 3 & 0 & -1 & 0 \\ 1 & 2 & 0 & 0 & 3 & 0 & 0 \\ 1 & -1 & -2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.4 & 0.1 & 0.3 \\ 1 & 0 & 0 \\ 0 & 2 & -1 \\ -1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 0 & 1 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}.$$

We take the known input, the unknown input and the sensor fault as

$$u(t) = [0.1, 0, 0]^{\mathrm{T}}, \quad d(t) = [\sin(2t), \sin(t), \cos(2t)]^{\mathrm{T}}, \ w(t) = [w_1(t), w_2(t)]^{\mathrm{T}},$$

$$w_1(t) = \begin{cases} 0, & t \le 1, \\ \sin^2(5(t-1)^2), & t > 1, \end{cases} \qquad w_2(t) = \begin{cases} 0, & t \le 1, \\ -\sin^2(3(t-1)^2), & t > 1. \end{cases}$$

By calculation, one can verify that Assumption 1 holds and $(\tilde{A}_{\mathrm{a}}, \check{C}_{\mathrm{a}})$ is observable, which illustrates that Assumption 2 is satisfied. In the light of Lemma 2, we can obtain P_{a} and Q_{a} (we omit their values here).

Following the same approach as for Example 1 to calculate L_{ao}, we have

$$L_{\mathrm{ao}} = \begin{bmatrix} -\frac{18983}{169} & \frac{51958}{271} & 0 & 0 \\ -\frac{6272}{157} & \frac{8319}{122} & 0 & 0 \\ -\frac{7123}{363} & \frac{7804}{233} & 0 & 0 \\ \frac{1511}{48} & -\frac{3600}{67} & 0 & 0 \\ \frac{1001}{34} & -\frac{8945}{178} & 0 & 0 \end{bmatrix}.$$

For calculating the time-varying gain $L_{(A_{ao},r)}(t)$, we take an $\mathbf{S}_5^{(2)}(r)$ function $H_r(t)$ as (2) with $a_1 = a_2 = 0.001$. We choose $T = 4$. Then $r = \frac{1}{2}T = 2$. Let the initial conditions be $x(0) = [1, 0, 1, 0, 1, 0, -1]^{\mathrm{T}}$ and $\chi_a(0) = [0, 0, 0, 0, 0]^{\mathrm{T}}$. As a result of the simulation of system (1) with the estimator (20), the states of system and their estimates, and the states of the sensor fault and their estimates are shown by Fig. 2. This figure demonstrates excellent estimation performance and confirm that the estimation time is equal to 4s, the value of T.

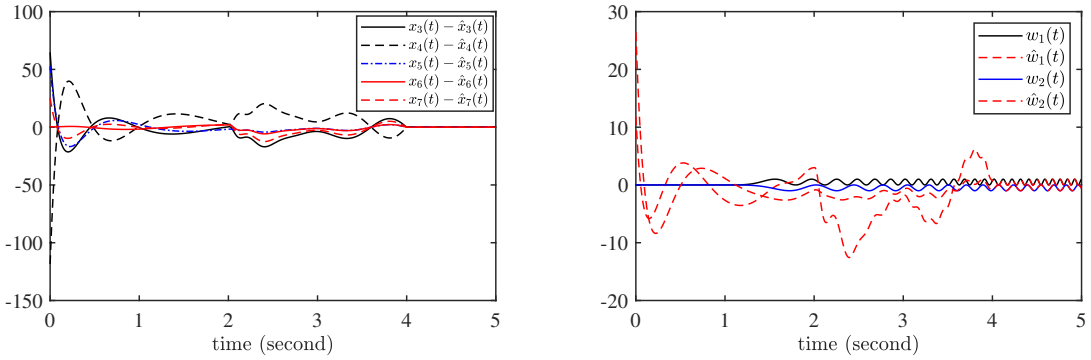


FIGURE 2 The errors $x_3(t) - \hat{x}_3(t)$, $x_4(t) - \hat{x}_4(t)$, $x_5(t) - \hat{x}_5(t)$, $x_6(t) - \hat{x}_6(t)$, and $x_7(t) - \hat{x}_7(t)$, the sensor fault $w(t)$ and its estimate $\hat{w}(t)$ associated with the estimator (20) for Example 2

Example 3: Finally, we illustrate the methodology of Section 4 by considering system (1) with

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 & -2 & -3 \\ 1 & 2 & -2 & 3 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 \\ 2 & 1 & 0 & 3 & 0 & -1 \\ 1 & 2 & 0 & 0 & 3 & 0 \\ 1 & -1 & -2 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.4 & 0.1 & 0.3 \\ 1 & 0 & 0 \\ 0 & 2 & -1 \\ -1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}, \quad E = \left[\begin{array}{c|c} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array}\right],$$

$$C = \left[\begin{array}{cccccc} 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{array}\right], \quad D = \left[\begin{array}{ccc} 0 & 0 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{array}\right], \quad F = \left[\begin{array}{c} 1 \\ \hline 1 \\ 0 \\ 0 \end{array}\right], \quad J = \left[\begin{array}{cc} 1 & 0 \\ \hline 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array}\right]. \tag{38}$$

We take the known input, unknown input and sensor fault as

$$u(t) = [1, 0.2, 0]^{\mathrm{T}}, \quad d(t) = \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}, \quad w(t) = \begin{cases} 0, & t \in [0.5 + k, k + 1), \quad k = 0, 1, \ldots, \\ 30, & t \in [k, 0.5 + k). \end{cases}$$

The system model is already in the form (30) with $E = \left[\, E_1 \,\middle|\, E_2 \,\right]$ and with the partitioning of the output matrices corresponding to y_1, y_2 and y_3. Assumptions 1-2 are satisfied.

From the first and the second output equation, we obtain

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} d_1(t) \\ w(t) \end{bmatrix} = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} x(t) - \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} u(t),$$

from which we get

$$\begin{cases} d_1(t) = y_1(t) - y_2(t) + x_1(t) - x_4(t) + u_1(t), \\ w(t) = y_2(t) - x_1(t) - u_1(t). \end{cases} \tag{39}$$

Substituting the expression for $d_1(t)$ into (30) and (38) yields system (32), with

$$\bar{A}=\begin{bmatrix} 1 & 0 & 0 & -1 & -2 & -3 \\ 2 & 2 & -2 & 2 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 \\ 2 & 1 & 0 & 3 & 0 & -1 \\ 1 & 2 & 0 & 0 & 3 & 0 \\ 1 & -1 & -2 & 0 & 0 & 0 \end{bmatrix},\ \bar{B}=\begin{bmatrix} 0.4 & 0.1 & 0.3 \\ 2 & 0 & 0 \\ 0 & 2 & -1 \\ -1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix},\ \bar{G}=\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ \bar{H}=\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ E_2=\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ C_3=\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}^{\mathrm{T}},\ D_3=\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}^{\mathrm{T}}.$$

Observe that only the first state equation explicitly depends on d_2, while the first output has relative degree two with respect to input d_2. When we define

$$\zeta_1=\begin{bmatrix} x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix},\ \zeta_2=x_1+x_5,\ \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} y_3,$$

we arrive at description (34), with state equation governed by

$$\bar{A}_{11}=\begin{bmatrix} 2 & -2 & 2 & -2 & 0 \\ 1 & -1 & -1 & -1 & 0 \\ 1 & 0 & 3 & -2 & -1 \\ 2 & 0 & 0 & 2 & 0 \\ -1 & -2 & 0 & -1 & 0 \end{bmatrix},\ \bar{A}_{12}=\begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix},\ \bar{B}_1=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & -1 \\ -1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix},\ \bar{G}_1=\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ \bar{H}_1=\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\bar{A}_{21}=\begin{bmatrix} 2 & 0 & -1 & -1 & -3 \end{bmatrix},\ \bar{A}_{22}=2,\ \ \bar{B}_2=\begin{bmatrix} -1.6 & 1.1 & 0.3 \end{bmatrix},\ \bar{G}_2=\bar{H}_2=0,\ \mathcal{E}=1,$$

and output equation governed by

$$\bar{C}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix},\ \bar{D}_1=\begin{bmatrix} 0 & 1 & 0 \end{bmatrix},\ \ \bar{D}_2=\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}.$$

From the last equation of (34) we obtain

$$\zeta_2(t)=\tilde{y}_2(t)-\bar{D}_2u(t)=[0\ 1]\ y_3(t)-u_2(t)-u_3(t). \tag{40}$$

Expressions (39) and (40) highlight that the joint estimation problem of states and unknown parameter is solved if we can estimate $\zeta_1=[x_2\ \cdots\ x_6]^{\mathrm{T}}$. So, in what follows we can focus on reduced system (35). Recall that the satisfaction of Assumptions 1-2 implies the observability of $\left(\bar{C},\bar{A}_{11}\right)$. Let's partition the sates and the coefficient matrices of system (35) as

$$\begin{bmatrix} x_2(t) \\ \hline x_3(t) \\ x_4(t) \\ x_5(t) \\ x_6(t) \end{bmatrix}=\begin{bmatrix} x_2(t) \\ \hline \zeta_{22}(t) \end{bmatrix},\ \bar{A}_{11}=\left[\begin{array}{c|cccc} 2 & -2 & 2 & -2 & 0 \\ \hline 1 & -1 & -1 & -1 & 0 \\ 1 & 0 & 3 & -2 & -1 \\ 2 & 0 & 0 & 2 & 0 \\ -1 & -2 & 0 & -1 & 0 \end{array}\right]\triangleq\begin{bmatrix} \breve{A}_{11} & \breve{A}_{12} \\ \breve{A}_{21} & \breve{A}_{22} \end{bmatrix},\ \bar{A}_{12}=\left[\begin{array}{c} 2 \\ \hline 1 \\ 2 \\ 1 \\ 1 \end{array}\right]\triangleq\begin{bmatrix} \breve{A}_1 \\ \breve{A}_2 \end{bmatrix},$$

$$\bar{B}_1=\left[\begin{array}{ccc} 2 & 0 & 0 \\ \hline 0 & 2 & -1 \\ -1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 2 & 0 \end{array}\right]\triangleq\begin{bmatrix} \breve{B}_1 \\ \breve{B}_2 \end{bmatrix},\ \bar{G}_1=\left[\begin{array}{c} 1 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right]=\begin{bmatrix} \breve{G}_1 \\ \breve{G}_2 \end{bmatrix},\ \bar{H}_1=\left[\begin{array}{c} -1 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right]\triangleq\begin{bmatrix} \breve{H}_1 \\ \breve{H}_2 \end{bmatrix}.$$

In this way, system (35) can be written as

$$\begin{cases} \begin{bmatrix} \dot{x}_2(t) \\ \hline \dot{\zeta}_{22}(t) \end{bmatrix}=\begin{bmatrix} \breve{A}_{11} & \breve{A}_{12} \\ \breve{A}_{21} & \breve{A}_{22} \end{bmatrix}\begin{bmatrix} x_2(t) \\ \hline \zeta_{22}(t) \end{bmatrix}+\begin{bmatrix} \breve{A}_1 \\ \breve{A}_2 \end{bmatrix}\zeta_2(t)+\begin{bmatrix} \breve{B}_1 \\ \breve{B}_2 \end{bmatrix}u(t)+\begin{bmatrix} \breve{G}_1 \\ \breve{G}_2 \end{bmatrix}y_1(t)+\begin{bmatrix} \breve{H}_1 \\ \breve{H}_2 \end{bmatrix}y_2(t), \\ \tilde{y}_1(t)=x_2(t)+\bar{D}_1u(t), \end{cases}$$

which implies

$$\dot{\zeta}_{22}(t)=\breve{A}_{21}\tilde{y}_1(t)+\breve{A}_{22}\zeta_{22}(t)+\breve{A}_2\zeta_2(t)+(\breve{B}_2-\breve{A}_{21}\bar{D}_1)u(t),$$

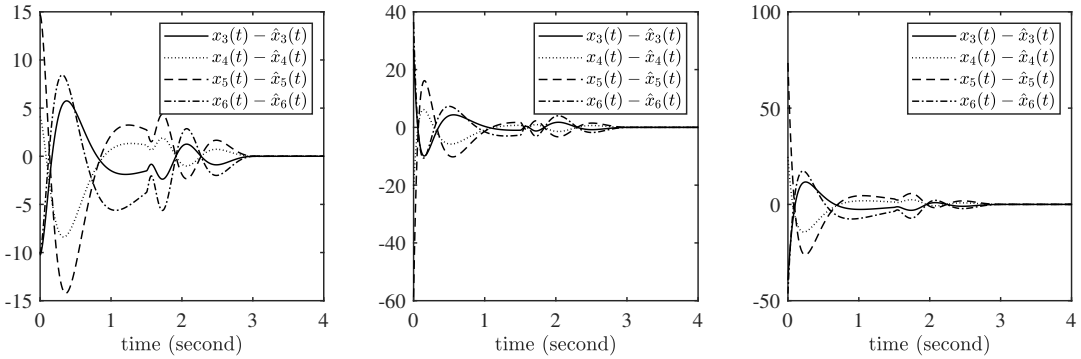


FIGURE 3 The errors associated with the estimator (45) for different initial conditions for Example 3

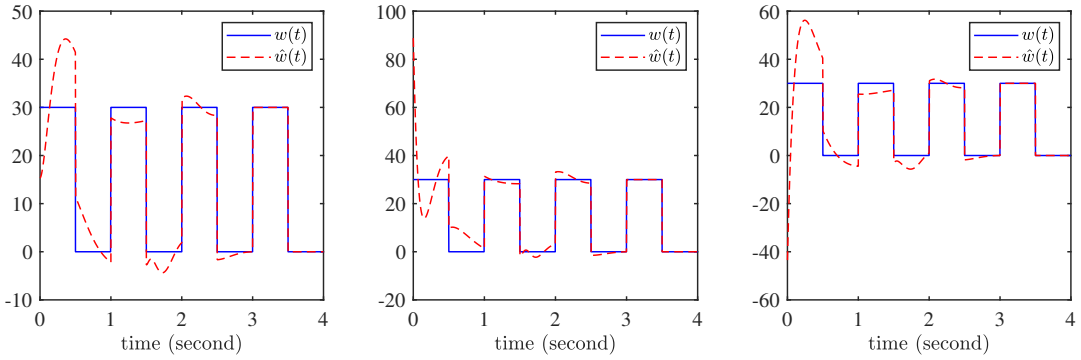


FIGURE 4 The sensor fault $w(t)$ and its estimate $\hat{w}(t)$ associated with the estimator (45) for different initial conditions Example 3

and

$$\dot{\tilde{y}}_1(t) = \dot{x}_2(t) + \bar{D}_1\dot{u}(t) = \breve{A}_{11}\tilde{y}_1(t) + \breve{A}_{12}\zeta_{22}(t) + \breve{A}_1\zeta_2(t) + (\breve{B}_1 - \breve{A}_{11}\bar{D}_1)u(t) + \breve{G}_1 y_1(t) + \breve{H}_1 y_2(t) + \bar{D}_1\dot{u}(t).$$

If we set $\breve{y}_2(t) = \dot{\tilde{y}}_1(t) - \bar{D}_1\dot{u}(t)$ and $\breve{u}_2(t) = \left[\begin{array}{cccc} \tilde{y}_1^{\mathrm{T}} & y_1^{\mathrm{T}} & y_2^{\mathrm{T}} & \zeta_2^{\mathrm{T}} \end{array} u^{\mathrm{T}}(t)\right]^{\mathrm{T}}$, we have

$$\left\{\begin{array}{l} \dot{\zeta}_{22}(t) = \breve{A}_{22}\zeta_{22}(t) + \breve{B}\breve{u}_2(t), \\ \breve{y}_2(t) = \breve{A}_{12}\zeta_{22}(t) + \breve{D}\breve{u}_2(t), \end{array}\right. \tag{41}$$

where

$$\breve{B} = \left[\begin{array}{ccccc} \breve{A}_{21} & 0 & 0 & \breve{A}_2 & \breve{B}_2 - \breve{A}_{21}\bar{D}_1 \end{array}\right], \ \breve{D} = \left[\begin{array}{ccccc} \breve{A}_{11} & \breve{G}_1 & \breve{H}_1 & \breve{A}_1 & \breve{B}_1 - \breve{A}_{11}\bar{D}_1 \end{array}\right].$$

We can design the prescribed-time observer for system (41) as

$$\left\{\begin{array}{l} \dot{\hat{\zeta}}_{22}(t) = \breve{A}_{22}\hat{\zeta}_{22}(t) + \breve{B}\breve{u}_2(t) + \bar{L}_{\mathrm{o}}(\breve{y}_2(t) - \hat{\breve{y}}_2(t)) + L_{(\bar{A}_{\mathrm{o}},r)}(t)(\breve{y}_2(t-r) - \hat{\breve{y}}_2(t-r)), \\ \hat{\breve{y}}_2(t) = \breve{A}_{12}\hat{\zeta}_{22}(t) + \breve{D}\breve{u}_2(t), \end{array}\right. \tag{42}$$

with

$$L_{(\bar{A}_{\mathrm{o}},r)}(t) = \mathrm{e}^{\bar{A}_{\mathrm{o}}t}W_{\mathrm{o}}^{-1}(\bar{A}_{\mathrm{o}},r)\mathrm{e}^{\bar{A}_{\mathrm{o}}^{\mathrm{T}}(t-r)}\bar{C}^{\mathrm{T}}H_r(t), \quad \bar{A}_{\mathrm{o}} = \breve{A}_{22} - \bar{L}_{\mathrm{o}}\breve{A}_{12},$$

$$W_{\mathrm{o}}(\bar{A}_{\mathrm{o}},r) = \int_0^r \mathrm{e}^{\bar{A}_{\mathrm{o}}^{\mathrm{T}}s}\breve{A}_{12}^{\mathrm{T}}H_r(s+r)\breve{A}_{12}\mathrm{e}^{\bar{A}_{\mathrm{o}}s}\mathrm{d}s.$$

Then, we have $\zeta_{22}(t) - \hat{\zeta}_{22}(t) = 0, \forall t \geq 2r$. Moreover, it follows from (42) that

$$\begin{aligned}
\dot{\hat{\zeta}}_{22}(t) =& \breve{A}_{22}\hat{\zeta}_{22}(t) + \breve{B}\breve{u}_2(t) + \bar{L}_o\breve{y}_2(t) - \bar{L}_o\hat{\breve{y}}_2(t) + L_{(\bar{A}_o,r)}(t)\breve{y}_2(t-r) - L_{(\bar{A}_o,r)}(t)\hat{\breve{y}}_2(t-r)\\
=& \bar{A}_o\hat{\zeta}_{22}(t) + (\breve{B} - \bar{L}_o\breve{D})\breve{u}_2(t) + \bar{L}_o\breve{y}_2(t) + L_{(\bar{A}_o,r)}(t)\breve{y}_2(t-r) - L_{(\bar{A}_o,r)}(t)\breve{A}_{12}\hat{\zeta}_{22}(t-r) - L_{(\bar{A}_o,r)}(t)\breve{D}\breve{u}_2(t-r)\\
=& \bar{A}_o\hat{\zeta}_{22}(t) - L_{(\bar{A}_o,r)}(t)\breve{A}_{12}\hat{\zeta}_{22}(t-r) - L_{(\bar{A}_o,r)}(t)\breve{A}_1\zeta_2(t-r) + (\breve{A}_{21} - \bar{L}_o\breve{A}_{11})\tilde{y}_1(t) + (\breve{A}_2 - \bar{L}_o\breve{A}_1)\zeta_2(t) - \bar{L}_o\breve{H}_1y_2(t)\\
&+ (\breve{B}_2 - \breve{A}_{21}\bar{D}_1 - \bar{L}_o(\breve{B}_1 - \breve{A}_{11}\bar{D}_1))u(t) - L_{(\bar{A}_o,r)}(t)\breve{A}_{11}\tilde{y}_1(t-r) - L_{(\bar{A}_o,r)}(t)\breve{G}_1y_1(t-r) - L_{(\bar{A}_o,r)}(t)\breve{H}_1y_2(t-r)\\
&- \bar{L}_o\breve{G}_1y_1(t) - L_{(\bar{A}_o,r)}(t)(\breve{B}_1 - \breve{A}_{11}\bar{D}_1)u(t-r) + \bar{L}_o\dot{\tilde{y}}_1(t) - \bar{L}_o\bar{D}_1\dot{u}(t) + L_{(\bar{A}_o,r)}(t)\dot{\tilde{y}}_1(t-r) - L_{(\bar{A}_o,r)}(t)\bar{D}_1\dot{u}(t-r).
\end{aligned}$$

Let

$$\zeta(t) = \hat{\zeta}_{22}(t) - \bar{L}_o\tilde{y}_1(t) + \bar{L}_o\bar{D}_1u(t) - L_{(\bar{A}_o,r)}(t)\tilde{y}_1(t-r) + L_{(\bar{A}_o,r)}(t)\bar{D}_1u(t-r). \tag{43}$$

Then

$$\begin{aligned}
\dot{\zeta}(t) =& \bar{A}_o\zeta(t) - L_{(\bar{A}_o,r)}(t)\breve{A}_{12}\zeta(t-r) - \bar{L}_o\breve{G}_1y_1(t) - \bar{L}_o\breve{H}_1y_2(t) + (\breve{A}_2 - \bar{L}_o\breve{A}_1)\zeta_2(t) + (\bar{A}_o\bar{L}_o + \breve{A}_{21} - \bar{L}_o\breve{A}_{11})\tilde{y}_1(t)\\
&+ (\breve{B}_2 - \breve{A}_{21}\bar{D}_1 - \bar{L}_o(\breve{B}_1 - \breve{A}_{11}\bar{D}_1) - \bar{A}_o\bar{L}_o\bar{D}_1)u(t) + (\bar{A}_oL_{(\bar{A}_o,r)}(t) - L_{(\bar{A}_o,r)}(t)\breve{A}_{11} - L_{(\bar{A}_o,r)}(t)\breve{A}_{12}\bar{L}_o - \dot{L}_{(\bar{A}_o,r)}(t))\tilde{y}_1(t-r)\\
&+ (L_{(\bar{A}_o,r)}(t)\breve{A}_{12}\bar{L}_o\bar{D}_1 - \bar{A}_oL_{(\bar{A}_o,r)}(t)\bar{D}_1 + \dot{L}_{(\bar{A}_o,r)}(t)\bar{D}_1 - L_{(\bar{A}_o,r)}(t)(\breve{B}_1 - \breve{A}_{11}\bar{D}_1))u(t-r).\\
&- L_{(\bar{A}_o,r)}(t)\breve{A}_1\zeta_2(t-r) - L_{(\bar{A}_o,r)}(t)\breve{G}_1y_1(t-r) - L_{(\bar{A}_o,r)}(t)\breve{H}_1y_2(t-r)
\end{aligned} \tag{44}$$

Finally, in view of (39) and (43), the sensor fault estimator can be expressed as

$$\begin{cases}
\dot{\zeta}(t) = \bar{A}_o\zeta(t) - L_{(\bar{A}_o,r)}(t)\breve{A}_{12}\zeta(t-r) - \bar{L}_o\breve{G}_1y_1(t) - \bar{L}_o\breve{H}_1y_2(t) + (\breve{A}_2 - \bar{L}_o\breve{A}_1)\zeta_2(t) + (\bar{A}_o\bar{L}_o + \breve{A}_{21} - \bar{L}_o\breve{A}_{11})\tilde{y}_1(t)\\
\qquad -L_{(\bar{A}_o,r)}(t)\breve{A}_1\zeta_2(t-r) + (\breve{B}_2 - \breve{A}_{21}\bar{D}_1 - \bar{L}_o(\breve{B}_1 - \breve{A}_{11}\bar{D}_1) - \bar{A}_o\bar{L}_o\bar{D}_1)u(t)\\
\qquad -L_{(\bar{A}_o,r)}(t)\breve{G}_1y_1(t-r) - L_{(\bar{A}_o,r)}(t)\breve{H}_1y_2(t-r) + (\bar{A}_oL_{(\bar{A}_o,r)}(t) - L_{(\bar{A}_o,r)}(t)\breve{A}_{11} - L_{(\bar{A}_o,r)}(t)\breve{A}_{12}\bar{L}_o - \dot{L}_{(\bar{A}_o,r)}(t))\tilde{y}_1(t-r)\\
\qquad +(L_{(\bar{A}_o,r)}(t)\breve{A}_{12}\bar{L}_o\bar{D}_1 - \bar{A}_oL_{(\bar{A}_o,r)}(t)\bar{D}_1 + \dot{L}_{(\bar{A}_o,r)}(t)\bar{D}_1 - L_{(\bar{A}_o,r)}(t)(\breve{B}_1 - \breve{A}_{11}\bar{D}_1))u(t-r)\\
\hat{\zeta}_{22}(t) = \zeta(t) + \bar{L}_o\tilde{y}_1(t) - \bar{L}_o\bar{D}_1u(t) + L_{(\bar{A}_o,r)}(t)\tilde{y}_1(t-r) - L_{(\bar{A}_o,r)}(t)\bar{D}_1u(t-r),\\
\hat{u}(t) = y_2(t) - \zeta_2(t) + \begin{bmatrix}0 & 0 & 1 & 0\end{bmatrix}\hat{\zeta}_{22}(t) - u_1(t),
\end{cases} \tag{45}$$

where $\zeta_2(t)$ is shown in (40). Following the same method as that for estimator (45), the gain $\bar{L}_o$ is computed as

$$\bar{L}_o = \begin{bmatrix} -\frac{5319}{146} & \frac{15836}{617} & \frac{3600}{49} & -\frac{11500}{251}\end{bmatrix}^{\mathrm{T}}.$$

In order to calculate the time-varying gain $L_{(\bar{A}_o,r)}(t)$, we take an $\mathbf{S}^{(1)}_{p_1}(r)$ function $H_r(t)$ as (2) with $a_1 = 0.1$, $a_2 = 0.8$. Note that $n - l + l_1 - 1 = 4$. We set the initial condition be $\hat{\zeta}_{22}(0) = [0, 0, 0, 0]^{\mathrm{T}}$. To illustrate that the convergence time does not depend on the initial condition $x(0)$, we choose $T = 3$ and different initial conditions, $x(0) = [0.1, 0.2, -3, -1, 0, -1]^{\mathrm{T}}$, $x(0) = [0.4, -0.8, -2, -0.8, 0, -0.2]^{\mathrm{T}}$, $x(0) = [0.6, 1, -3, -1, 0, -0.5]^{\mathrm{T}}$. Then $r = \frac{1}{2}T = 2$. From a simulation of system (1) and estimator (45), the error trajectories, the fault signal and its estimate are shown in Fig. 3 and Fig. 4, which illustrate that, for different initial conditions $x(0)$, the observer errors converge to zero at $t = 3$s and the fault signal can be estimated at $t = 3$s.

With the adopted approach, the effect of the unknown input d is fully compensated and the sensor fault is exactly recovered within the prescribed time. In what follows, we add noise to the output equation, in such a way that only an approximate compensation or recovery is possible. In addition, we select noise not satisfying the assumption taken on d. More precisely, we consider two different instances of white noise ϖ. We assume that the output equation is expressed as $y(t) = Cx(t) + Du(t) + Fw(t) + Jd(t) + J\varpi(t)$.

Case I: For $\varpi = [\varpi_1, \varpi_2]^T$, where $\varpi_1(t)$ and $\varpi_2(t)$ are uniformly distributed on $[0,\ 1]$, the simulations are show in Fig. 5

Case II: We assume $\varpi = [\varpi_1, \varpi_2]^T$, with $\varpi_1(t)$ and $\varpi_2(t)$ obeying Bernoulli distribution, with probabilities $\mathrm{Prob}\{\varpi_1(t) = 1\} = 0.4$, $\mathrm{Prob}\{\varpi_1(t) = 0\} = 0.6$, $\mathrm{Prob}\{\varpi_2(t) = 1\} = 0.7$ and $\mathrm{Prob}\{\varpi_2(t) = 0\} = 0.3$. For this case, simulations are presented in Figs. 6. As expected, the noise slightly degrades the performance of the estimation system. A quantitative analysis is however beyond the scope of the paper.

6 | CONCLUSION

In this paper, a prescribed-time observer-based estimation technique for sensor faults and the state of linear systems has been proposed, in the presence of unknown inputs affecting both the state and output equation. By injecting a periodic delayed output in an augmented linear system, a reduced-order sensor fault estimator has been developed not only to obtain an accurate sensor fault

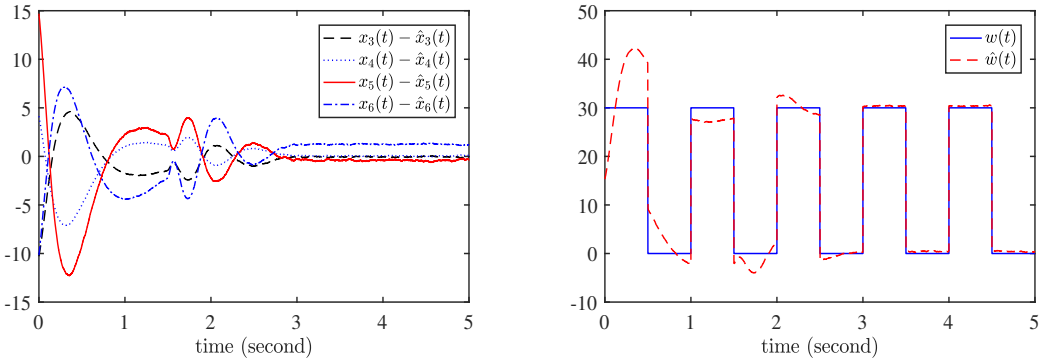


FIGURE 5 The response of $x(t)-\hat{x}(t)$, sensor fault $w(t)$ and its estimation $\hat{w}(t)$.

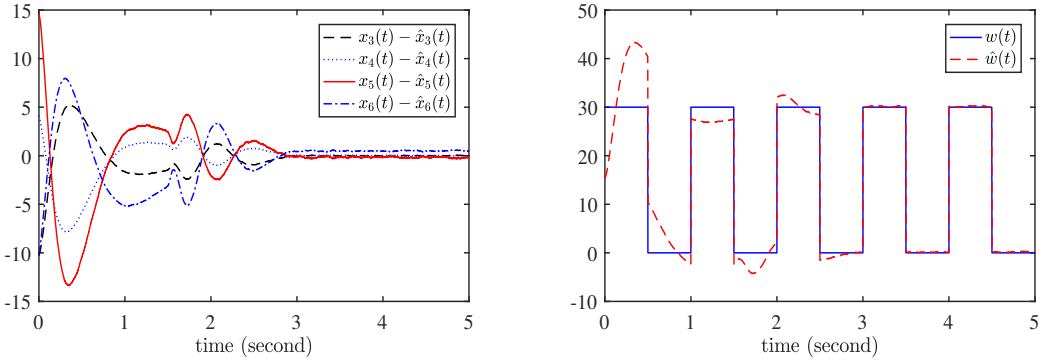


FIGURE 6 The response of $x(t)-\hat{x}(t)$, sensor fault $w(t)$ and its estimation $\hat{w}(t)$.

estimation, but also to make the estimation error converge to zero at a prescribed time. Different from exist results, the designed time-varying gains in the estimator are linear and can be chosen continuous, continuously differentiable, and even smooth. In addition, a complementary approach for the reduced-order estimation problem, based on structure revealing transformations of the system, has been outlined.

APPENDIX

Proof of Lemma 2

The proof is divided into two steps, which is inspired by[33].

Step 1: We prove that Assumption 1 is equivalent to

$$\operatorname{rank}\begin{bmatrix} G & \check{E}_{\mathrm{a}} \\ \check{C}_{\mathrm{a}} & 0 \end{bmatrix} = n+q+\operatorname{rank}(\check{E}_{\mathrm{a}}). \tag{46}$$

Duo to matrix $[J^{-}J \;\; D_{\mathrm{l}}]$ is of full row rank, we have

$$\operatorname{rank}\begin{bmatrix} E \\ J \end{bmatrix} = \operatorname{rank}\begin{bmatrix} E \\ J \end{bmatrix}\begin{bmatrix} J^{-}J & D_{\mathrm{l}} \end{bmatrix} = \operatorname{rank}\begin{bmatrix} EJ^{-}J & ED_{\mathrm{l}} \\ J & 0 \end{bmatrix} = \operatorname{rank}(J)+\operatorname{rank}(\check{E}_{\mathrm{a}}). \tag{47}$$

Thus, Assumption 1 becomes

$$\operatorname{rank}(\Delta) = n+q+2\operatorname{rank}(J)+\operatorname{rank}(\check{E}_{\mathrm{a}}). \tag{48}$$

Furthermore, in view of the fact that matrix Δ_a^n is of full column rank, we have

$$\operatorname{rank}(\Delta) = \operatorname{rank}(\Delta\Delta_{\mathrm{l}}^{n+q+l}) = \operatorname{rank}\begin{bmatrix} I_n & 0 & 0 & EJ^{-}J & ED_{\mathrm{l}} \\ C & F & J & 0 & 0 \\ 0 & 0 & 0 & J & 0 \end{bmatrix}$$

$$
\begin{aligned}
=&\operatorname{rank}\begin{bmatrix} I_n & 0 & 0 & 0 & ED_1 \\ C & F & J & 0 & 0 \\ 0 & 0 & 0 & J & 0 \end{bmatrix} \\
=&\operatorname{rank}(J)+\operatorname{rank}\begin{bmatrix} I_n & 0 & 0 & ED_1 \\ C & F & J & 0 \end{bmatrix} \\
=&\operatorname{rank}(J)+\operatorname{rank}\Delta_a^n\begin{bmatrix} I_n & 0 & 0 & ED_1 \\ C & F & J & 0 \end{bmatrix} \\
=&\operatorname{rank}(J)+\operatorname{rank}\begin{bmatrix} I_n & 0 & 0 & ED_1 \\ JJ^-C & JJ^-F & J & 0 \\ D_{\mathrm{a}}C & D_{\mathrm{a}}F & 0 & 0 \end{bmatrix}\Delta_1 \\
=&\operatorname{rank}(J)+\operatorname{rank}\begin{bmatrix} I_n & 0 & 0 & ED_1 \\ 0 & 0 & J & 0 \\ D_{\mathrm{a}}C & D_{\mathrm{a}}F & 0 & 0 \end{bmatrix} \\
=&2\operatorname{rank}(J)+\operatorname{rank}\begin{bmatrix} G & \check{E}_a \\ \check{C}_{\mathrm{a}} & 0 \end{bmatrix}. \qquad (49)
\end{aligned}
$$

It is clearly from (48) and (49) that Assumption 1 is equivalent to (46).

Step 2: We show that Lemma 2 is satisfied under condition (46). Since $\operatorname{rank}(\check{C}_{\mathrm{a}})=p_0$, there is a matrix $\check{C}_{\mathrm{a}0}\in\mathbf{R}^{(n+q)\times(n+q-p_0)}_{n+q-p_0}$ satisfying $\check{C}_{\mathrm{a}}\check{C}_{\mathrm{a}0}=0$. Let

$$K_{\mathrm{a}}=\begin{bmatrix}\check{C}_{\mathrm{a}}^{\mathrm{T}} & \check{C}_{\mathrm{a}0}\end{bmatrix}^{\mathrm{T}}. \qquad (50)$$

Notice that

$$K_{\mathrm{a}}\begin{bmatrix}\check{C}_{\mathrm{a}}^{\mathrm{T}}(\check{C}_{\mathrm{a}}\check{C}_{\mathrm{a}}^{\mathrm{T}})^{-1} & \check{C}_{\mathrm{a}0}(\check{C}_{\mathrm{a}0}^{\mathrm{T}}\check{C}_{\mathrm{a}0})^{-1}\end{bmatrix}=\begin{bmatrix} I_{p_0} & 0 \\ 0 & I_{n+q-p_0}\end{bmatrix},$$

which implies that

$$K_{\mathrm{a}}^{-1}=\begin{bmatrix}\check{C}_{\mathrm{a}}^{\mathrm{T}}(\check{C}_{\mathrm{a}}\check{C}_{\mathrm{a}}^{\mathrm{T}})^{-1} & \check{C}_{\mathrm{a}0}(\check{C}_{\mathrm{a}0}^{\mathrm{T}}\check{C}_{\mathrm{a}0})^{-1}\end{bmatrix}. \qquad (51)$$

Based on this we have

$$GK_{\mathrm{a}}^{-1}=\begin{bmatrix}G\check{C}_{\mathrm{a}}^{\mathrm{T}}(\check{C}_{\mathrm{a}}\check{C}_{\mathrm{a}}^{\mathrm{T}})^{-1} & G\check{C}_{\mathrm{a}0}(\check{C}_{\mathrm{a}0}^{\mathrm{T}}\check{C}_{\mathrm{a}0})^{-1}\end{bmatrix}\triangleq\begin{bmatrix}G_1 & G_2\end{bmatrix}.$$

From (46), we can get

$$n+q=\operatorname{rank}\begin{bmatrix}G\\ \check{C}_{\mathrm{a}}\end{bmatrix}K_{\mathrm{a}}^{-1}=\operatorname{rank}\begin{bmatrix}G_1 & G_2\\ I_{p_0} & 0\end{bmatrix},$$

which means that $\operatorname{rank}\left(G_2\right)=n+q-p_0$. Then there exists a matrix $G_0\in\mathbf{R}^{p_0\times(n+q)}_{p_0}$ satisfying $G_0G_2=0$. Denote

$$V_{\mathrm{a}}=\begin{bmatrix}G_0^{\mathrm{T}} & G_2\end{bmatrix}.$$

Then, it is obvious that

$$\begin{bmatrix}(G_0G_0^{\mathrm{T}})^{-1}G_0\\ (G_2^{\mathrm{T}}G_2)^{-1}G_2^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}G_0^{\mathrm{T}} & G_2\end{bmatrix}=\begin{bmatrix} I_{p_0} & 0\\ 0 & I_{(n+q)-p_0}\end{bmatrix},$$

which means

$$V_{\mathrm{a}}^{-1}=\begin{bmatrix}(G_0G_0^{\mathrm{T}})^{-1}G_0\\ (G_2^{\mathrm{T}}G_2)^{-1}G_2^{\mathrm{T}}\end{bmatrix}\triangleq\begin{bmatrix}V_{\mathrm{a}1}\\ V_{\mathrm{a}2}\end{bmatrix}. \qquad (52)$$

By defining

$$\bar{G}=\begin{bmatrix}\left(I_{p_0}-V_{\mathrm{a}1}G_1\right)^{\mathrm{T}} & -\left(V_{\mathrm{a}2}G_1\right)^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}, \qquad (53)$$

it is follows from (52) and (53) that

$$V_{\mathrm{a}}^{-1}GK_{\mathrm{a}}^{-1}+\bar{G}\check{C}_{\mathrm{a}}K_{\mathrm{a}}^{-1}=\begin{bmatrix}V_{\mathrm{a}1}G_1 & 0\\ V_{\mathrm{a}2}G_1 & I_{(n+q)-p_0}\end{bmatrix}+\begin{bmatrix} I_{p_0}-V_{\mathrm{a}1}G_1 & 0\\ -V_{\mathrm{a}2}G_1 & 0\end{bmatrix}=I_{n+q}. \qquad (54)$$

Finally, we can choose

$$P_{\mathrm{a}}=K_{\mathrm{a}}^{-1}V_{\mathrm{a}}^{-1}=\begin{bmatrix}\check{C}_{\mathrm{a}}^{\mathrm{T}}(\check{C}_{\mathrm{a}}\check{C}_{\mathrm{a}}^{\mathrm{T}})^{-1} & \check{C}_{\mathrm{a}0}(\check{C}_{\mathrm{a}0}^{\mathrm{T}}\check{C}_{\mathrm{a}0})^{-1}\end{bmatrix}\begin{bmatrix}V_1\\ V_2\end{bmatrix} \qquad (55)$$

and

$$Q_{\mathrm{a}}=K_{\mathrm{a}}^{-1}\bar{G}=\check{C}_{\mathrm{a}}^{\mathrm{T}}(\check{C}_{\mathrm{a}}\check{C}_{\mathrm{a}}^{\mathrm{T}})^{-1}\left(I_{p_0}-V_{\mathrm{a}1}G_1\right)-\check{C}_{\mathrm{a}0}(\check{C}_{\mathrm{a}0}^{\mathrm{T}}\check{C}_{\mathrm{a}0})^{-1}V_{\mathrm{a}2}G_1,$$

to obtain the conclusion.

Proof of Lemma 3

In the proof of Lemma 2, we have derived that Assumption 1 is equivalent to (46). Moreover, it follows from the proof of Lemma 1 in [11] that

$$\operatorname{rank}\begin{bmatrix} G & \check{E}_{\mathrm{a}} \\ \check{C}_{\mathrm{a}} & 0 \end{bmatrix} = n + q + \operatorname{rank}(\check{C}_{\mathrm{a}} P_{\mathrm{a}} \check{E}_{\mathrm{a}}).$$

Note that

$$\operatorname{rank}(\check{E}_{\mathrm{a}}) = \operatorname{rank}\begin{bmatrix} -I_{n+q} & C \\ 0 & I_{n+q} \end{bmatrix}\begin{bmatrix} \check{C}_{\mathrm{a}} P_{\mathrm{a}} \check{E}_{\mathrm{a}} \\ P_{\mathrm{a}} \check{E}_{\mathrm{a}} \end{bmatrix} = \operatorname{rank}\begin{bmatrix} \check{C}_{\mathrm{a}} P_{\mathrm{a}} \check{E}_{\mathrm{a}} \\ P_{\mathrm{a}} \check{E}_{\mathrm{a}} \end{bmatrix}.$$

Thus, if Assumption 1 holds, then

$$\operatorname{rank}\begin{bmatrix} \check{C}_{\mathrm{a}} P_{\mathrm{a}} \check{E}_{\mathrm{a}} \\ P_{\mathrm{a}} \check{E}_{\mathrm{a}} \end{bmatrix} = \operatorname{rank}(\check{C}_{\mathrm{a}} P_{\mathrm{a}} \check{E}_{\mathrm{a}}).$$

The remainder of the proof is similar to the proof of Lemma 1 in [15] and is thus omitted.

CONFLICT OF INTEREST

The authors declare no potential conflict of interest.

ACKNOWLEDGMENTS

This work was supported in part by the National Natural Science Foundation of China for Distinguished Young Scholars under Grant 62125303, the Science Center Program of National Natural Science Foundation of China under Grant 62188101, and the Fundamental Research Funds for the Central Universities under Grant HIT.BRET.2021008, by project C14/22/092 of the KU Leuven Research Council and by project G092721N of the Research Foundation-Flanders (FWO-Vlaanderen).

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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