

Distribution-free P-box processes based on translation theory: definition and simulation

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Abstract

Typically, non-deterministic models of spatial or time dependent uncertainty are modelled using the well-established random field framework. However, while tailored for exactly these types of time and spatial variations, stochastic processes and random fields currently have only limited success in industrial engineering practice. This is mainly caused by its computational burden, which renders the analysis of industrially sized problems very challenging, even when resorting to highly efficient random field analysis methods such as EOLE. Apart from that, also the methodological complexity, high information demand and rather indirect control of the spatial (or time) variation has limited its cost-benefit potential for potential end-users. This data requirement was recently relaxed by some of the authors with the introduction of imprecise random fields, but so far the method is only applicable to parametric p-box valued stochastic processes and random fields. This paper extends these concepts by expanding the framework towards distribution-free p-boxes. The main challenges addressed in this contribution are related to both the non-Gaussianity of realisations of the imprecise random field in between the p-box bounds, as well as maintaining the imposed auto-correlation structure while sampling from the p-box. Two case studies involving a dynamical model of a car suspension and the settlement of an embankment are included to illustrate the presented concepts.

Keywords: Stochastic process; imprecise probability; probability box; random field; scarce data

1. Introduction

1.1. General rationale

Stochastic processes are applied and studied in various domains, reaching from engineering [1] to financial economics, to represent stochastic quantities that vary over time- and/or space [2]. However,

20 due to the theoretical and computational difficulties, usually these processes are assumed to be Gaussian,
21 which might not always be a truthful representation of reality [10]. Furthermore, the definition of a
22 stochastic process requires the rigorous description of the governing distribution function. This includes
23 selecting the appropriate distribution family as well as the governing hyper-parameters. In practice, an
24 accurate definition of these quantities might not be possible due to limitations on the available data
25 (quantity of the data, corrupted or missing data, etc.), but also conflicting sources of information (e.g.,
26 expert opinions). Recently introduced approaches based on Bayesian compressed sensing alleviate this
27 problem (see e.g., [3, 4] by making the estimated stochastic process robust to missing data, even when
28 these processes are non-Gaussian and have an unknown non-stationary auto-correlation function [5].
29 Furthermore, also de-trending is not required in these cases [5]. In [6], these methods were extended
30 to also account for the general case of multivariate, uncertain, unique, sparse, incomplete spatial data
31 (denoted MUSIC-X by the authors).

32 As a possible alternative pathway to account for low data availability is to resort to the more general
33 framework of imprecise probabilities [7]. According to this framework, epistemic uncertainty that results
34 from data deficiencies are taken explicitly into account in the analysis. In the context of stochastic
35 processes, parametric p-box valued stochastic processes have been introduced in e.g., [13], [14] and [15].
36 However, such approaches still require the definition of a (Gaussian) distribution family, which is not
37 always possible. This paper aims to go further than the available methods for simulating from imprecise
38 stochastic processes by introducing a type of distribution-free p-box stochastic process. Such process
39 is obtained by applying an interval-valued translation map to a standard normal Gaussian stochastic
40 process. As such, a distribution-free p-box stochastic process is obtained. Two examples are included
41 to illustrate the definition and propagation of these structures. The paper is constructed as follows: the
42 remainder of this section retakes some important concepts concerning the definition and simulation of
43 non-Gaussian stochastic processes; Section 3 introduces the approach for defining distribution-free p-box
44 processes; Section 4 briefly discusses a double-loop approach to propagate these structures; Section 5 and
45 Section 6 provide two case studies as illustration of the approach; Section 7 lists the conclusions of the
46 work.

47 **2. Background theory**

48 This section retakes some background theory on non-Gaussian stochastic processes to facilitate the
49 understanding of the developments presented further in the paper. Specifically, the general theory behind
50 stochastic processes is reproduced for the sake of unifying some definitions and highlighting some key
51 concepts. Further, translation theory is explained in detail since the ensuing developments are largely

52 based on this concept.

53 2.1. Stochastic processes

54 A finite-dimensional stochastic process $x(t, \omega)$ describes a set of correlated random variables $x(\omega)$,
 55 which are assigned to a countable number n_t of locations $t \in \Omega_d$ in the model domain $\Omega_d \subset \mathbb{R}^d$ with
 56 dimension $d \in \mathbb{N}$. Note that Ω_d may comprise both space and/or time dimensions. Each random
 57 variable $x(\omega) : (\Omega, \varsigma, P) \mapsto \mathbb{R}$ as such maps from a complete probability space to the real domain,
 58 with $\omega \in \Omega$ a coordinate in sample space Ω and ς the sigma-algebra. This condition holds as long
 59 as $x(t, \omega) \in \mathcal{L}^2(\omega, P)$, with $\mathcal{L}^2(\omega, P)$ the Hilbert space of second-order random variables (i.e., finite
 60 variance). For a given event $\omega_i \in \Omega$, the corresponding $x(t, \omega_i)$ is a realization of the stochastic process.
 61 A stochastic process is considered Gaussian if the distribution of $(x(t_1, \omega), x(t_2, \omega), \dots, x(t_{n_t}, \omega))$, with
 62 $n_t \in \mathbb{N}$, is jointly Gaussian $\forall t \in \Omega_d$. In this case, $x(t, \omega)$ is completely described by its mean function
 63 $\mu_x(t) : \Omega_d \mapsto \mathbb{R} = E_\omega[x(t, \omega)]$ and its auto-covariance function $\mathbf{C}_{xx}(\tau) : \Omega_d \times \Omega_d \mapsto \mathbb{R}$, given by $\mathbf{C}_{xx}(\tau) =$
 64 $E_\omega[(x(t, \omega) - \mu_x(t)), (x(t', \omega) - \mu_x(t'))]$ [2]. In the remainder of the paper, uni-variate scalar stochastic
 65 processes (i.e., $\Omega \in \mathbb{R}^1$) are considered for the sake of conciseness of notation. Note that the concepts
 66 explained in this paper scale straightforwardly to **multidimensional random fields as well (i.e., random**
 67 **fields defined over $\mathbb{R}^d, d > 1$, with the only difference that t and τ become vector-valued. Also the**
 68 **extension towards multi-variate and multi-index random fields should not pose too many challenges,**
 69 **since the concepts upon which the methods are built (i.e., translation theory) are well-understood in this**
 70 **context (see e.g., [34]). An in-depth discussion of such extension is left for further work.**

71 Generally, when applying stochastic processes in an engineering context, for instance to represent
 72 a spatially uncertain input quantity of a finite element model, the process has to be discretized over
 73 Ω_d . In this context, the Karhunen-Loève expansion is a very powerful tool to represent stochastic pro-
 74 cesses [8]. Specifically, following the Karhunen-Loève (KL) series expansion, a stochastic process $x(t, \omega)$
 75 is represented as:

$$x(t, \omega) = \mu_x(t) + \sigma_x \sum_{i=1}^{\infty} \sqrt{\lambda_i} \psi_i(t) \xi_i(\omega), \quad (1)$$

76 with σ_x the standard deviation of the random process and where the quantities $\lambda_i \in (0, \infty)$ and $\psi_i(t) :$
 77 $\Omega_d \mapsto \mathbb{R}$ are respectively the eigenvalues and eigenfunctions of the continuous, bounded, symmetric and
 78 positive (semi-)definite auto-correlation function $\rho_{xx}(\tau) : \Omega_d \times \Omega_d \mapsto [0, 1]$, in accordance with Mercer's
 79 theorem:

$$\rho_{xx}(\tau) = \sum_{i=1}^{\infty} \lambda_i \psi_i(t) \psi_i(t'), \quad (2)$$

80 These quantities are in practice obtained by solving the homogeneous Fredholm integral equation of

81 the second kind:

$$\int_{\Omega_d} \rho_{xx}(\tau) \psi_i(t') dt' = \lambda_i \psi_i(t), \quad (3)$$

82 where $t' = t + \tau$ for which many efficient discretization schemes exist [9]. Since $\rho_{xx}(\tau)$ is bounded,
 83 symmetric and positive semi-definite, and furthermore in most practical cases can be assumed positive
 84 definite, these eigenvalues λ_i are non-negative and the eigenfunctions $\psi_i(t)$ satisfy the following orthog-
 85 onality condition:

$$\langle \psi_i(t), \psi_j(t) \rangle = \int_{\Omega_d} \psi_i(t) \psi_j(t) dt = \delta_{ij} \quad (4)$$

86 with δ_{ij} the Kronecker delta and $\langle \cdot, \cdot \rangle : \Omega_d \times \Omega_d \mapsto \mathbb{R}$ an inner product space. Hence, the eigenfunctions
 87 form a complete orthogonal basis on an \mathcal{L}_2 Hilbert space. In this case, the series expansion in Eq. 2 can
 88 be shown to be optimally convergent [8].

89 The variables $\xi_i(\omega), i = 1, \dots, \infty$, introduced in Eq. 1, are uncorrelated random variables, which are
 90 determined according to:

$$\xi_i(\omega) = \frac{1}{\sqrt{\lambda_i}} \int_{\Omega_d} [x(t, \omega) - \mu_x(t)] \psi_i(t) dt, \quad (5)$$

91 which can be shown to be independent standard normally distributed in the case of a Gaussian random
 92 process. For practical reasons, the infinite series expansion in Eq. 1 is usually truncated after a finite
 93 number of terms $n_{KL} \in \mathbb{N}$:

$$x(t, \omega) = \mu_x(t) + \sigma_x \sum_{i=1}^{n_{KL}} \sqrt{\lambda_i} \psi_i(t) \xi_i(\omega), \quad (6)$$

94 where n_{KL} should be selected such that a well-chosen variance error metric is minimized [11].

95 In the context of applying stochastic processes in an imprecise probabilistic context, the Gaussian-
 96 assumption is no longer generally valid. This has several implications for their discretization. For
 97 non-Gaussian processes, the $\xi_i(\omega), i = 1, \dots, n_{KL}$ represented in Eq. 5 are non-Gaussian too and their
 98 distribution needs to be solved for explicitly. Furthermore, in this case, the corresponding random
 99 variables $\xi_i(\omega)$ may exhibit higher order dependencies that are difficult to quantify [10]. Finally, Eq. 5
 100 reveals that the distribution of $\xi_i(\omega)$ depends on sample path realisations $x(t, \omega)$ of the stochastic process,
 101 and hence, iterative methods such as presented by [11] need to be applied. Alternatively, also translation
 102 theory as introduced by Grigoriu [21] provides a viable approach towards simulating from (strongly)
 103 non-Gaussian stochastic processes.

104 2.2. Translation stochastic processes

105 Translation process theory, as introduced by Grigoriu [21], provides a different pathway for the sim-
 106 ulation of non-Gaussian stochastic processes. Specifically, a Gaussian stochastic process $\eta(t, \omega)$ with
 107 autocorrelation function $\rho_{xx}(\tau)$ is transformed using a nonlinear transformation into a non-Gaussian

108 process $x(t, \omega)$. This is formally expressed as:

$$x(t, \omega) = F_X^{-1} \circ \Phi(\eta(t, \omega)) = g(\eta(t, \omega)), \quad (7)$$

109 with $g := F_X^{-1} \circ \Phi$ the so-called translation mapping, F_X^{-1} the inverse of the target non-Gaussian cumula-
 110 tive distribution function (CDF) that represents the distribution of the non-Gaussian stochastic process
 111 and Φ the marginal standard normal CDF, i.e., $\Phi(\eta) = P(\eta(t, \omega) < \eta)$.

112 It can be shown (see [21]) that the mean μ_x , variance σ_x^2 and correlation function $r_{xx}(\tau)$ of $x(t, \omega)$
 113 have closed-form expressions, that are given respectively by:

$$\mu_x = E_\eta[x(t, \omega)] = E_\eta[g(\eta(t, \omega))] = \int_{-\infty}^{\infty} g(\eta)\phi(\eta)d\eta \quad (8)$$

$$\sigma_x^2 = E_\eta[x(t, \omega) - \mu_x]^2 = \int_{-\infty}^{\infty} (g(\eta) - \mu_x)^2\phi(\eta)d\eta \quad (9)$$

$$r_{xx}(\tau) = E[x(t, \omega) - \mu_x]E[x(t + \tau, \omega) - \mu_x] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (g(\eta_1) - \mu_x)(g(\eta_2) - \mu_x)\phi(\eta_1, \eta_2, \rho_{xx}(\tau))dydz \quad (10)$$

116 with $\phi(\eta_1, \eta_2, \rho_{xx}(\tau))$ the density of a bivariate standard Gaussian vector with correlation coefficient
 117 $\rho_{xx}(\tau)$, given by:

$$\phi(\eta_1, \eta_2, \rho_{xx}(\tau)) = \frac{1}{\sqrt{2\pi(1 - \rho_{xx}(\tau)^2)}} \exp -\frac{\eta_1^2 + \eta_2^2 - 2\eta_1\eta_2\rho_{xx}(\tau)}{2(1 - \rho_{xx}(\tau))^2} \quad (11)$$

118 These equations can be readily solved by standard numerical quadrature schemes. Furthermore, it can
 119 be shown that the forward transformation of a Gaussian process to a non-Gaussian process is always
 120 possible ([21]).

121 On the other hand, given a non-Gaussian $r_{xx}(\tau)$, it is not always possible to determine the corre-
 122 sponding Gaussian autocorrelation $\rho_{xx}(\tau)$ that, when transformed, yields $r_{xx}(\tau)$ [21]. This happens when
 123 either the inverse of Eq. 10 yields an autocorrelation function that is not positive semi-definite, or when
 124 the normalized autocorrelation $\xi(\tau)$ has values that lie outside of the admissible range $[\xi^{min}(\tau), \xi^{max}(\tau)]$
 125 which can be found by setting $\rho_{xx}(\tau)$ to respectively 1 and -1 in Eq. 10 [22]. In literature, iterative meth-
 126 ods have been introduced to find the closest approximation of $r_{xx}(\tau)$ that is an admissible autocorrelation
 127 function (see e.g., [12]).

128 3. P-box stochastic processes

129 In practical engineering cases, it is not always possible to define a **precise** distribution function $F_X(x)$
 130 to construct the non-Gaussian stochastic process. In this context, p-boxes can provide a valuable tool to

131 represent the uncertainty an analyst has on the specification of the appropriate distribution function
 132 (see [16] for a recent review on computational methods). This section introduces p-box-valued stochastic
 133 processes.

134 As a practical example on how such a P-box can be established in geotechnical analysis, consider
 135 Figure 1. This figure shows a normal probability plot for the Rock Mass Rating (RMR) values at the
 136 Izmir subway site (taken from [19]). As may be seen, the 95% probability bounds corresponding to a
 137 performed normality fit indicate that the estimation of the Normal distribution with these data is the
 138 most accurate around the mean value, and that the uncertainty of the fit increases towards the tails of the
 139 distribution. This particular point was also made by Ching et al., who showed that statistical uncertainty
 140 for random field parameters can be very big due to strong data constraints in geotechnical engineering [20].
 141 In this case, one might consider to rather make calculations with the bounds provided by the confidence
 142 interval, and all therein included distributions, rather than with the precise Normal estimate, as it relaxes
 143 the quite restricting Normality assumption. Indeed, by considering a set of distributions, a more robust
 144 and natural approach to deal with the existing uncertainties is followed, yielding more objective estimates.
 145 Similar scarce data examples can be found in multiple engineering applications, ranging from modelling
 146 wind loads on buildings, predicting earthquakes, or dealing with highly advanced engineered materials
 147 such as encountered in Additive Manufacturing.

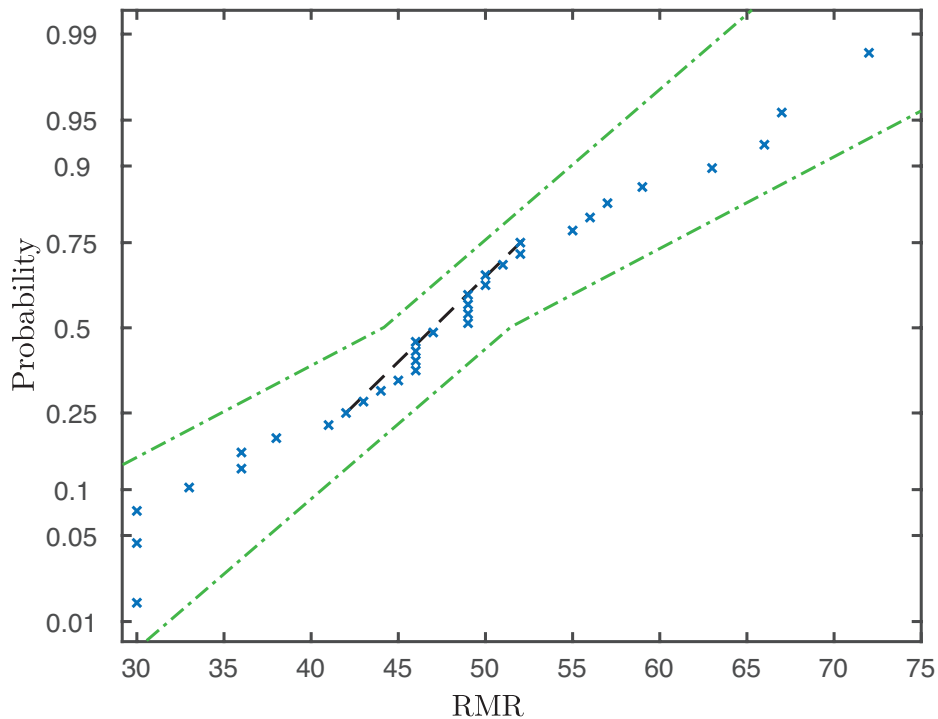


Figure 1: Normal probability plot, including the confidence interval bounds for the Rock Mass Rating data, taken from the Izmir subway site (taken from [19]). The black dashed line indicates the fitted normal distribution, the green dash-dotted lines indicate the confidence interval on this fit.

148 A scalar distribution-free p-box is usually described by a lower CDF $\underline{F}_X \in \mathbb{F}$ and an upper CDF
149 $\overline{F}_X \in \mathbb{F}$, where \mathbb{F} expresses the the set of all CDFs on \mathbb{R} . They are collected as a pair $[\underline{F}_X, \overline{F}_X]$ which
150 yields a set of possible CDFs via $\underline{F}_X(x) \leq F_X(x) \leq \overline{F}_X(x)$, $x \in \mathbb{R}$. A distribution-free p-box as such
151 corresponds to defining a lower probability \underline{P} and upper probability \overline{P} on events $\{X \leq x\} = (-\infty, x]$,
152 i.e., $\underline{P}(X \leq x) = \underline{F}_X(x)$ and $\overline{P}(X \leq x) = \overline{F}_X(x)$ for $x \in \mathbb{R}$, which define a credal set of probability
153 measures. In case additional information on the uncertainty is available, constraints on the p-box can
154 be enforced. For instance, if the (class of) distribution functions \mathcal{F} is known, the set of possible CDFs
155 $\{F_X(\cdot, \boldsymbol{\theta}) \in \mathcal{F} \mid \boldsymbol{\theta} \in D_{\boldsymbol{\theta}}\}$ can be defined conditional on a vector of hyper-parameters $\boldsymbol{\theta}$. Since this is a
156 special case of the distribution-free p-box, the following discussion on distribution-free p-box stochastic
157 is equally applicable. An important side-note should be made with respect to distribution-free p-boxes.
158 Due to the relaxed constraints on the set of admissible CDFs, generally, also non-physical distribution
159 functions are explicitly modelled in the p-box. It should be noted that developments in this context
160 have been made, focusing on imposing some constraints on the realisations within the distribution-free
161 p-box [18, 17].

162 As explained in Section 2, a (precise) stochastic process $x(t, \omega)$ can be considered as a collection
163 of n_t correlated random variables distributed throughout the model domain Ω_d . Imprecise stochastic
164 processes $\hat{x}(t, \omega)$ can be regarded as a natural extension of this idea, where for each discrete location
165 $t_i \in \Omega_d$, a scalar p-box is defined. However, since all $x(t_i, \omega)$, $i = 1, \dots, n_t$ are correlated according to
166 $\rho_{xx}(\tau)$, these scalar p-boxes also are correlated to each other. This observation complicates the analysis
167 of distribution-free imprecise stochastic processes significantly, since the direct simulation from a set of
168 correlated distribution-free p-boxes is far from trivial from both a theoretical as well as from a numerical
169 point of view. Furthermore, since distribution-free p-boxes are considered, also non-Gaussian processes
170 are inherently included in the imprecise stochastic description.

171 A potential solution to this issue is to start from a precise standard normal Gaussian stochastic process
172 $\eta(t, \omega)$ with predefined correlation function $\rho_{xx}(\tau)$ and pass this representation through an imprecisely
173 defined translation map which is defined as:

$$\hat{x}(t, \omega) = \left[\underline{F}_X^{-1}, \overline{F}_X^{-1} \right] \circ \Phi(\eta(t, \omega)), \quad (12)$$

174 which can be further expanded as:

$$\hat{x}(t, \omega) = \left[\underline{F}_X^{-1}, \overline{F}_X^{-1} \right] \circ \Phi \left(\sum_{i=1}^{n_{KL}} \sqrt{\lambda_i} \boldsymbol{\psi}_i(t) \xi_i(\omega) \right), \quad (13)$$

175 where $\Phi \left(\sum_{i=1}^{n_{KL}} \sqrt{\lambda_i} \boldsymbol{\psi}_i(t) \xi_i(\omega) \right)$ represents a re-scaling of $\eta(t, \omega)$ to the interval $[0, 1]$. Since a CDF is

176 by definition a monotonic function, i.e., $F_X(x_1) < F_X(x_2) \iff x_1 < x_2$, the bounds of $\hat{x}(t, \omega)$ are
 177 determined by $[\underline{F}_X^{-1}, \overline{F}_X^{-1}]$. It can be noted that the outer part of the interval-valued translation map,
 178 i.e., $[\underline{F}_X^{-1}, \overline{F}_X^{-1}]$ in effect represents the inverse of a distribution-free p-box. As such, the interval-valued
 179 translation map at first scales $\eta(t, \omega)$ to $[0, 1]$, after which a set-valued transformation is applied to
 180 transform this stochastic process into an imprecise stochastic process according to the p-box that is
 181 bounded by $[\underline{F}_X^{-1}, \overline{F}_X^{-1}]$.

182 In essence, for each x , the p-box $[\underline{F}_X, \overline{F}_X]$ provides an interval $[\underline{F}_X(x), \overline{F}_X(x)]$. Therefore, $[\underline{F}_X^{-1}, \overline{F}_X^{-1}] \circ$
 183 Φ analogously represents an interval-valued mapping of each realisation of $\eta(t, \omega)$. As such, each reali-
 184 sation of the stochastic process $\eta(t, \omega_j)$, corresponding to the event ω_j is translated towards an interval
 185 process that is consistent with the bounds on the CDF (due to the monotonicity of the CDF), and which
 186 is given as:

$$x^I(t, \omega_j) = [\underline{F}_X^{-1}, \overline{F}_X^{-1}] \circ \Phi \left(\sum_{i=1}^{n_{KL}} \sqrt{\lambda_i} \psi_i(t) \xi_i(\omega_j) \right), \quad (14)$$

187 with the lower bound given as:

$$\underline{x}(t, \omega_j) = \overline{F}_X^{-1} \circ \Phi \left(\sum_{i=1}^{n_{KL}} \sqrt{\lambda_i} \psi_i(t) \xi_i(\omega_j) \right), \quad (15)$$

188 and the upper bound defined as:

$$\overline{x}(t, \omega_j) = \underline{F}_X^{-1} \circ \Phi \left(\sum_{i=1}^{n_{KL}} \sqrt{\lambda_i} \psi_i(t) \xi_i(\omega_j) \right). \quad (16)$$

189 It should be noted that this is not an explicit interval process, i.e., an interval process that is repre-
 190 sented as a series expansion with interval-valued coefficients as described in [23], since the interval-valued
 191 nature in this process stems from the mapping that is performed on a single realisation of the **precise**
 192 Gaussian process, rather than from a series expansion with interval-valued weights. As such, typically
 193 applied interval propagation methods, as described in [24], cannot be applied to propagate this interval
 194 process in a straightforward way. Furthermore, the auto-dependence function of realisations within this
 195 interval process becomes inter-valued too. Intermediate realisations of $x^k(t, \omega_j) \in x^I(t, \omega_j)$ can be gener-
 196 ated by drawing admissible CDFs $(F_X^k)^{-1} \in [\underline{F}_X^{-1}, \overline{F}_X^{-1}]$. For reasons of clarity, the explanation of a
 197 possible procedure to do so is deferred to a later section. **For more information about interval processes**
 198 **(and -fields), the reader is referred to the work of [25], [26] or [27], who all defined interval processes**
 199 **according to different formalisms.**

200 Conversely, when collecting all $x^k(t, \omega)$, that corresponds to a certain realisation of the p-box $(F_X^k)^{-1} \in$
 201 $[\underline{F}_X^{-1}, \overline{F}_X^{-1}]$, this becomes again a **precise** random process, the properties of which can be computed by

202 virtue of translation process theory as:

$$\mu_{x^k(t,\omega)} = \int_{-\infty}^{\infty} (F_X^k)^{-1} \circ \Phi(\eta) \phi(\eta) d\eta \quad (17)$$

$$\sigma_{x^k(t,\omega)}^2 = \int_{-\infty}^{\infty} ((F_X^k)^{-1} \circ \Phi(\eta) - \mu_{x^k(t,\omega)})^2 \phi(\eta) d\eta \quad (18)$$

$$r_{x^k}(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} ((F_X^k)^{-1} \circ \Phi(\eta_1) - \mu_{x^k(t,\omega)}) ((F_X^k)^{-1} \circ \Phi(\eta_2) - \mu_{x^k(t,\omega)}) \phi(\eta_1, \eta_2, \rho_{xx}(\tau)) d\eta_1 d\eta_2 \quad (19)$$

205 with $\phi(\eta_1, \eta_2, \rho_{xx}(\tau))$ as defined in Eq. (11). Note that this random process is stationary in case the
 206 underlying Gaussian process is stationary since it is mapped through a **precise** $(F_X^k)^{-1}$. Evidently every
 207 $(F_X^k)^{-1}$ will yield a random process with generally different central moments for every $(F_X^k)^{-1} \in [\underline{F}_X^{-1},$
 208 $\overline{F}_X^{-1}]$.

209 Based on the preceding discussion, it can be seen that an imprecise stochastic process can jointly be
 210 regarded as a stochastic collection of interval processes as well as as a credal set of stochastic processes.
 211 Since each stochastic realisation of the imprecise stochastic process is an interval process, and vice versa,
 212 each realisation within the p-box corresponds to a stochastic process, also the mean of the corresponding
 213 imprecise process is interval-valued. Due to the monotonicity of the translation map, this mean is formally
 214 given as:

$$\mu_x^I = [\underline{\mu}_x, \overline{\mu}_x] = [E_\omega [x(t, \omega)], E_\omega [\overline{x}(t, \omega)]] \quad (20)$$

215 These bounds can be computed in a straightforward manner by invoking Eq. 17 twice: once on the
 216 ensemble of lower bounds and once on the ensemble of upper bounds. As an illustration of these concepts,
 217 consider the example in Figure 2. This figure shows two stochastic realisations of the **precise** zero-mean
 218 Gaussian stochastic process, as well as their transformation into two realisations of the distribution-free
 219 p-box random process, which manifest themselves in the shape of two interval processes. From this figure,
 220 it can also be seen that the collection of all upper bounds of these interval processes, denoted $\overline{x}(t, \omega)$,
 221 represent a stochastic process that has $\overline{F}_X(x)$ as a distribution. The same obviously holds for the lower
 222 bound, as well as any intermediate realisation $(F_X^k)^{-1} \in [\underline{F}_X^{-1}, \overline{F}_X^{-1}]$.

223 **It should further** be noted that the auto-correlation function of $x^I(t, \omega)$, being $r_{xx}(\tau)$ also has become
 224 an **interval-valued function that may be modelled as an interval process** due to the interval-valued trans-
 225 lation map. This might have important implications for structural dynamical problems, where the match
 226 of a dominant frequency of the loading process might interfere with a natural frequency of the structure.
 227 The detailed treatment of this issue and a potential solution hereto however fall outside the scope of this
 228 paper.

229 **As a final comment, Figure 1 hints at a possible, very straightforward, approach to infer the bounds**

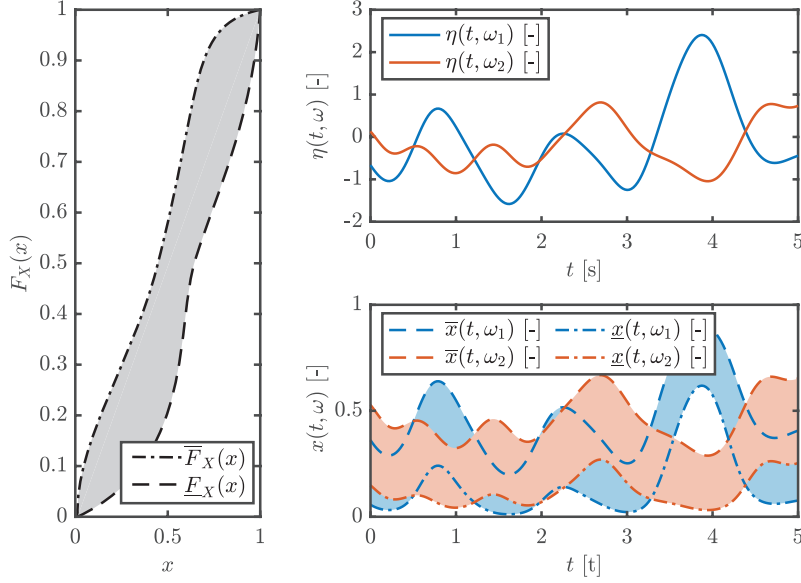


Figure 2: Illustration of the transformation of two stochastic realisations of the **precise** zero-mean Gaussian stochastic process into two realisations of the distribution-free p-box random process, which manifest themselves in the shape of two interval processes.

230 of the p-box valued translation map based on a small data-set. As is clear in the Figure, the confidence
 231 bounds in the normality fit of rock data are clearly expanding towards the tails of the distribution. One
 232 possible approach could be to use those confidence bounds as the bounds of the p-box to ensure robustness
 233 of the results with respect to the (lack of) confidence in the normal fit. Of course, also more elaborate
 234 schemes can be imagined, for instance based on Kolmogorov-Smirnov bounds or robust Bayesian analysis.
 235 This is however left for future work.

236 4. Propagation of distribution-free imprecise stochastic processes

237 Usually, an analyst who is confronted with imprecise probabilistic model quantities is concerned with
 238 finding the bounds on some probabilistic measure \mathcal{P} of the model's responses of interest. In case a **precise**
 239 density function $f_{\mathbf{X}}$ is known, the n^{th} central moment of the model's response $E_{\mathbf{y}}[Y^n]$ or the probability
 240 of failure p_F is determined by solving an integral equation of the following form:

$$\mathcal{P}(F_X^k) = \int_{\mathbb{R}^{n_x}} \mathcal{H}(\mathbf{x} | F_X^k) f_{\mathbf{X}}^k(\mathbf{x}) d\mathbf{x}, \quad (21)$$

241 where \mathcal{P} denotes, depending on the context, the n^{th} central moment of the model's response $E_{\mathbf{y}}[Y^n]$
 242 or the probability of failure p_F . In case $\mathcal{P} \equiv E_{\mathbf{y}}[Y^n]$ is considered, $\mathcal{H} \equiv g^n(\mathbf{x})$, where $g(\mathbf{x})$ represents
 243 the so-called performance function of the model. On the other hand, in case the calculation is aimed at
 244 computing p_F , $\mathcal{H} \equiv I_g(\mathbf{x})$, where I_g is the indicator function which is 1 in case $g(\mathbf{x}) \leq 0$, $\mathbf{x} \in \mathbb{R}^{n_x}$, and
 245 0 otherwise.

246 To infer the bounds on \mathcal{P} , two optimization problems need to be solved to actively search the param-
 247 eter space spanned by $[\underline{F}_X(x), \overline{F}_X(x)]$. Specifically, the lower bound is obtained as:

$$\underline{\mathcal{P}} = \min_{F_X^k \in [\underline{F}_X(x), \overline{F}_X(x)]} \mathcal{P}(F_X^k), \quad (22)$$

248 whereas the upper bound is determined as:

$$\overline{\mathcal{P}} = \max_{F_X^k \in [\underline{F}_X(x), \overline{F}_X(x)]} \mathcal{P}(F_X^k) \quad (23)$$

249 Note that each realisation F_X^k drawn from this interval represents a non-Gaussian random process with
 250 auto-correlation structure as described in Eq. (10). As such, this is effectively a double-loop approach,
 251 which might entail a non-negligible computational cost to solve. The main difficulty associated with
 252 solving these optimization problems lies in the fact that the optimisation has to be performed over the
 253 infinite-dimensional space of bounded, strictly monotonically increasing functions in the interval $[0, 1]$
 254 (i.e., càdlàg functions). Such calculation is intractable, even for the most simple cases. In this paper, it
 255 is therefore proposed to approximate these optimization problems as discrete problems. Specifically, it is
 256 aimed at solving following problems:

$$\begin{aligned} \underline{\mathcal{P}} &= \min_{\mathbf{F}_X \in \mathbf{F}_X^I} \mathcal{P}(F_X^k) \\ \overline{\mathcal{P}} &= \max_{\mathbf{F}_X \in \mathbf{F}_X^I} \mathcal{P}(F_X^k) \end{aligned} \quad (24)$$

257 subject to:

$$\mathbf{A}\mathbf{F}_X \leq 0 \quad (25)$$

258 where $\mathbf{F}_X = F_X(\mathbf{x}_s)$, with $\mathbf{x}_s \in \mathbb{R}^{n_s}$ representing n_s equally spaced sample points throughout the
 259 support of $[\underline{F}_X(x), \overline{F}_X(x)]$. Similarly, $\mathbf{F}_X^I = [\underline{F}_X(\mathbf{x}_s), \overline{F}_X(\mathbf{x}_s)] \in \mathbb{I}\mathbb{R}^{n_s}$ represents an n_s dimensional
 260 interval vector collecting the bounds of the p-box for each sample point in the support. The inequality
 261 shown in Eq. (23) enforces the realisations drawn from the interval vector \mathbf{F}_X^I to be strict monotonic to
 262 ensure that they represent admissible CDFs, where $\mathbf{A} \in \mathbb{R}^{n_s-1 \times n_s}$ represents an upper-triangular band
 263 matrix with $A_{1,:} = [1 \ -1 \ 0 \ \dots \ 0]$. As such, the infinite-dimensional optimisation problem is converted
 264 to a linear-inequality-constrained optimization problem over n_s variables. Finally, it can be noted that
 265 the translation mapping explained in section 3 requires the calculation of the inverse of the CDF (see
 266 e.g., Eq. 17). This is for instance required to generate the required sample paths of $x^k(t, \omega)$ to estimate

267 the performance function $g(\mathbf{x}^k(t, \omega))$. Hereto, a piece-wise cubic Hermite polynomial interpolation is
 268 performed using the Fritsch-Carlson algorithm to estimate a functional relationship between \mathbf{x}_s and \mathbf{F}_X .
 269 This approach is selected as this allows for generating a strictly monotonic, C^1 continuous interpolation
 270 of the inverse of $\mathbf{F}_X(\mathbf{x}_s)$. For a detailed treatment of this algorithm, the reader is referred to [28].

271 5. Case study 1: a quarter car model

272 The case study represents a quarter-car model, which is a 2-DOF idealisation of the realistic dynamics
 273 of the suspension of a car. Specifically, this case study is concerned with assessing the bounds on several
 274 comfort metrics of a vehicle suspension, given a p-box process-valued base excitation. The quarter-car
 275 dynamics can be represented as a set of two ordinary differential equations:

$$m_s \ddot{x}_s + c_s (\dot{x}_s - \dot{x}_{us}) + k_s (x_s - x_{us}) = 0 \quad (26)$$

$$m_{us} \ddot{x}_{us} - c_s (\dot{x}_s - \dot{x}_{us}) - k_s (x_s - x_{us}) + c_t (\dot{x}_{us} - \dot{x}_0) + k_t (x_{us} - x_0) = 0 \quad (27)$$

277 with $\dot{\bullet}$ the time derivative of \bullet , x_{us} the displacement of the unsprung mass (i.e., the suspension com-
 278 ponents, wheel and other components directly connected to them), x_s the displacement of the sprung
 279 mass (i.e., all components resting on the suspension), m_{us} and m_s the unsprung and sprung mass of a
 280 quarter of the car, c_s and c_t respectively the damping coefficients of the suspension and tire, k_s and k_t
 281 respectively the stiffness coefficients of the suspension and tire. Finally, x_0 and \dot{x}_0 are the displacement
 282 and velocity in vertical direction that excite the bottom of the wheel (i.e., the road profile). The com-
 283 plete road profile is denoted $x_0(t)$. The dynamics of the car are simulated over a distance of 50 (m),
 284 when the car is travelling at a speed of 10 (m/s). The one dimensional spatial domain is discretized into
 285 1000 equidistant points and the time domain is discretized into time intervals of 0.005 (s). A schematic
 286 representation of the model is given in figure 3.

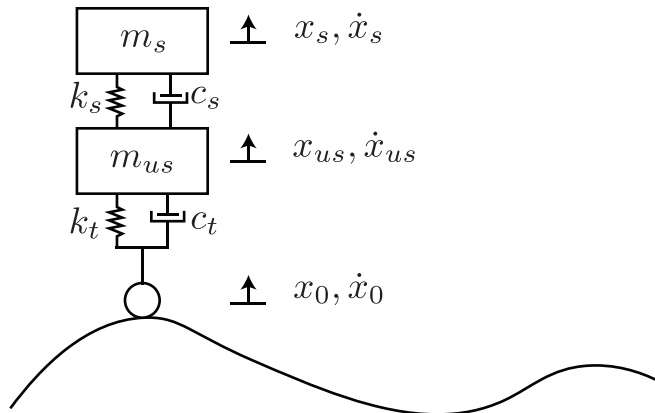


Figure 3: Schematic illustration of the quarter-car model

For the solution of this coupled system of ODEs, a state-space model is employed:

$$\frac{d}{dt} \begin{bmatrix} x_{us} - x_0 \\ \dot{x}_{us} \\ x_s - x_{us} \\ \dot{x}_s \end{bmatrix} = A \begin{bmatrix} x_{us} - x_0 \\ \dot{x}_{us} \\ x_s - x_{us} \\ \dot{x}_s \end{bmatrix} + \begin{bmatrix} -1 \\ \frac{4c_t}{m_{us}} \\ 0 \\ 0 \end{bmatrix} \dot{x}_0 \quad (28)$$

with the matrix A equal to:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-4k_t}{m_{us}} & \frac{-4(c_s+c_t)}{m_{us}} & \frac{4k_s}{m_{us}} & \frac{4c_s}{m_{us}} \\ 0 & -1 & 0 & 1 \\ 0 & \frac{4c_s}{m_s} & \frac{-4k_s}{m_s} & \frac{-4c_s}{m_s} \end{bmatrix} \quad (29)$$

Four state variables are considered, being respectively the tire deflection ($x_{us} - x_0$); the unsprung mass velocity \dot{x}_{us} ; the suspension stroke $x_s - x_{us}$, and sprung mass velocity \dot{x}_s . Typically, in the context of assessing the dynamical comfort of a car, two parameters are of interest: the suspension stroke (i.e., the relative displacement of the car body with respect to the unsprung mass) and the acceleration of the sprung mass. In the proceeding study, the damping effect of the tire, c_t is considered negligible.

Table 1: Considered case studies for the distribution free p-box process. \mathcal{B} and \mathcal{N} represent respectively the Beta and Normal distribution.

Case	$\underline{F}_X(x)$	$\overline{F}_X(x)$	b_t
1	$\min[\mathcal{B}(1, 3), \mathcal{B}(5, 5)]$	$\max[\mathcal{B}(1, 3), \mathcal{B}(5, 5)]$	0.5
2	$\min[\mathcal{B}(3, 1), \mathcal{B}(5, 5)]$	$\max[\mathcal{B}(3, 1), \mathcal{B}(5, 5)]$	0.5
3	$\min[\mathcal{B}(1, 1), \mathcal{B}(2, 5)]$	$\max[\mathcal{B}(1, 1), \mathcal{B}(2, 5)]$	0.5
4	$\min[\mathcal{B}(1, 0.2), \mathcal{B}(5, 5)]$	$\max[\mathcal{B}(1, .02), \mathcal{B}(5, 5)]$	0.5
5	$\min[\mathcal{N}(0, 0.75), \mathcal{B}(1, 0.2)]$	$\max[\mathcal{N}(0, 0.75), \mathcal{B}(1, 0.2)]$	2

The complete road profile $x_0(t)$ is modelled as a p-box valued stochastic process. The auto-correlation of the underlying Gaussian process $\eta(t, \omega)$ is governed by a squared exponential auto-correlation function with a correlation length of 0.5 m. Sample path realisations of $\eta(t, \omega)$ are generated using the Karhunen–Loève series expansion, while retaining 32 terms. The stochastic content of the imprecise stochastic process is represented using via a distribution-free p-box. For illustrative reasons, the bounds of the p-box, $[\underline{F}_X(x), \overline{F}_X(x)]$ are generated by taking the extremes of a set of distributions. Hereto, 5 different case studies are considered, which are summarized in Table 1. The corresponding p-boxes area also visualized in figure 4. Each random process corresponding to a realisations of these p-boxes is given as:

$$x_0^k(t, \omega) = (F_X^k)^{-1} \circ \Phi \left(\sum_{i=1}^{n_{KL}} \sqrt{\lambda_i} \psi_i(t) \xi_i(\omega) \right) \quad (30)$$

303 which are generated by the optimization algorithms introduced in Section 4. Applying the double-loop
 304 optimization algorithm introduced in Section 4, the bounds on the probability of failure of the structure
 305 are computed. In this context, the performance function $g(x)$ of the car model is given as:

$$g(\mathbf{x}, \mathbf{y}) = 1 - \max_{i=1, \dots, m} \left(\frac{|x_s(\mathbf{x}, t_i) - x_{us}(\mathbf{x}, t_i)|}{d} \right) \quad (31)$$

306 where the threshold value b_t is also given in Table 1. This corresponds to a first passage probability.
 307 Since the process is non-Gaussian, highly efficient and dedicated sampling methods such as Directional
 308 Importance Sampling [29], as also applied in the context of imprecise probabilities in [30] or [31], are
 309 not applicable. Therefore, the integral equation in the inner loop of the optimization is solved using
 310 Subset- ∞ , as presented in [32], with an initial sample size of 5000 and a proposal standard deviation of
 311 0.1. The discretisation of $F_X(s)$, as described in Section 4, is performed using $n_s = 40$ slices, yielding a
 312 40-dimensional optimization problem, which is solved using a gradient-free pattern search optimization
 313 algorithm. Pattern search is specifically selected to avoid the need to calculate gradients of p_F .

314 The results of performing the double-loop optimization problem are shown in Table 2. In this table,
 315 \underline{p}_F^* and \bar{p}_F^* indicate the bounds on p_F obtained by means of optimization, whereas \underline{p}_F and \bar{p}_F are the
 316 failure probabilities corresponding to the bounds of the p-box. As is clear, the bounds obtained by just
 317 propagating $\underline{F}_X(x)$ and $\bar{F}_X(x)$ are not conservative. This is a direct result from the fact that the car
 318 model acts as a filter on the excitation towards the responses of interest.

Table 2: Bounds on the probability of failure based on propagating the bounds, as well as performing optimization. Herein, \underline{p}_F^* and \bar{p}_F^* indicate the bounds on p_F obtained by means of optimization, whereas \underline{p}_F and \bar{p}_F are the failure probabilities corresponding to the bounds of the p-box.

Case	\underline{p}_F	\bar{p}_F	\underline{p}_F^*	\bar{p}_F^*
1	$3.25 \cdot 10^{-4}$	0.065	$2.77 \cdot 10^{-4}$	0.385
2	$3.25 \cdot 10^{-4}$	0.061	$9.95 \cdot 10^{-6}$	0.378
3	0.0045	0.400	0.0028	0.624
4	0.0038	0.469	$3.01 \cdot 10^{-4}$	0.571
5	0.0062	0.015	$9.43 \cdot 10^{-6}$	0.423

319 A further explanation of these results can be given based on Figure 4. This figure shows clearly
 320 that the CDF corresponding to the highest probability of failure pushes the probability mass as much
 321 as possible towards the bounds of the p-box. This makes sense from a physical standpoint since the
 322 performance function contains an absolute value operation, and hence, positive and negative responses
 323 contribute both equally to the failure. Furthermore, the considered system is a 2-degree-of-freedom
 324 oscillator where the quantity of interest is the relative displacement between the two masses. This as
 325 such constitutes a perfect symmetric system. The CDF that minimizes p_F on the other hand aims at
 326 getting as much of the probability mass as possible towards the centre of the support.

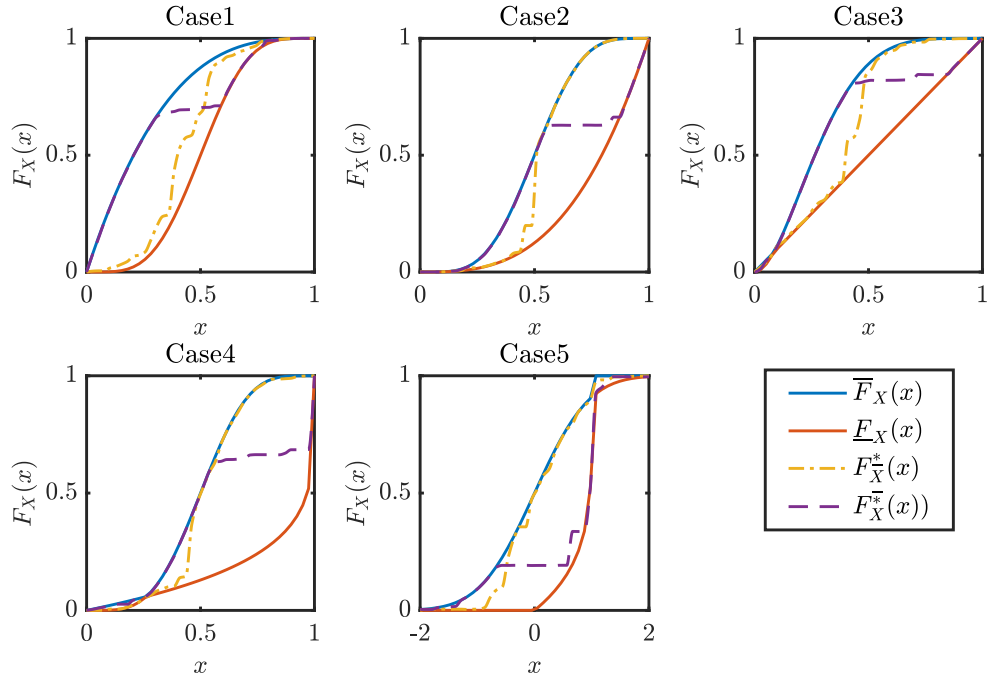


Figure 4: P-boxes corresponding to the 5 cases, as well as the realisations $F_{\bar{X}}^*$ and $F_{\tilde{X}}^*$ that yield respectively p_F^* and \bar{p}_F^* .

327 **6. Case study 2: an embankment settlement problem**

328 *6.1. Case introduction*

329 As a second case study, a 2-dimensional imprecise random field is considered as the source of uncer-
 330 tainty in an embankment settlement problem. Hereto, a 2D Finite Element model with random Young's
 331 modulus is constructed based on the geometrical description illustrated in Figure 5 under a plain strain
 332 assumption.

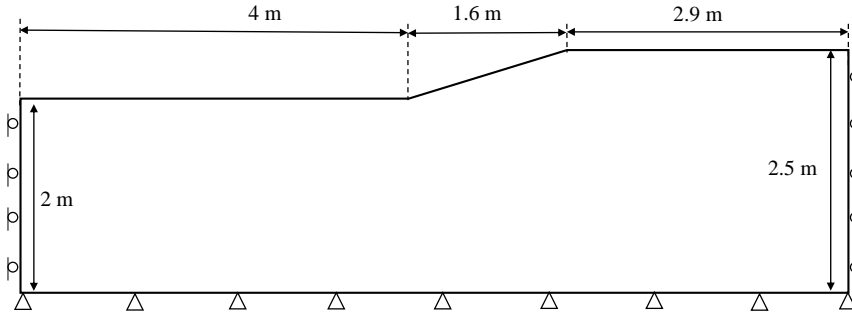


Figure 5: Schematic illustration of the embankment model

333 The boundary conditions of the model are that: the bottom-side is considered fully fixed, the left and
 334 right side of the geometry are free in vertical direction, and the top is free. The embankment settlement

335 is evaluated under self-gravity using an elastic constitutive model, which is represented as:

$$\epsilon_x = \frac{1 - \nu^2}{E} \left(\sigma_x - \frac{\nu}{1 - \nu} \sigma_y \right), \quad (32)$$

336

$$\epsilon_y = \frac{1 - \nu^2}{E} \left(\sigma_y - \frac{\nu}{1 - \nu} \sigma_x \right), \quad (33)$$

337 where ϵ and σ indicate respectively the strain and stresses in the model, E is Young's modulus and ν is
 338 the Poisson's ratio, which is set to 0.43. The earth gravity is calculated as:

$$\sigma_{eg} = \gamma - y, \quad (34)$$

339 where γ is the volumetric weight, which is 19.62 kg/m³ in this model. y is the depth of the soil. This
 340 geometry is discretized into 144 quadratic rectangular finite elements with a regular size of 22.22 ×
 341 53.28 [cm]. The mesh, together with the solution of a deterministic simulation is illustrated in Figure 6.
 342 This figure in addition shows the magnitude of the displacement field u_{max} . In the ensuing analysis,
 343 the maximum vertical displacement is utilised to evaluate the embankment settlement and failure is
 344 considered when this quantity exceeds 24.4 cm.

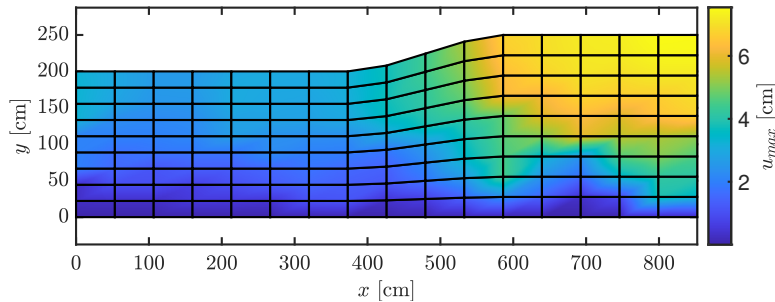


Figure 6: Finite element discretization of the geometry, together with the magnitude of the displacement fields

345 6.2. Imprecise random field modelling

346 As mentioned in the previous section, Young's modulus of the soil is modelled as an imprecise random
 347 field. The Gaussian random field $\eta(t, \omega)$ that is used as a basis for the analysis is modelled with a sym-
 348 metric squared exponential kernel with correlation length of 25 [cm]. The resulting correlation function
 349 is discretized over the midpoints of the elements and the eigenvalues and eigenvectors are calculated with
 350 an iterative Krylov-Shur procedure. The truncation of the random field is selected such that 99 % of the
 351 variance is retained.

352 The imprecise random field is specifically defined as:

$$E(t, \omega) = 1 \cdot 10^{08} + 1 \cdot 10^{07} \cdot \left[\underline{F}_X^{-1}, \overline{F}_X^{-1} \right] \circ \Phi(\eta(t, \omega)), \quad (35)$$

353 where the lower and upper bound of the CDF are in this case defined as:

$$\underline{F}_X = \min [\mathcal{N}(0, 0.5), \mathcal{N}(0, 0.75)] \quad (36)$$

$$\overline{F}_X = \max [\mathcal{N}(0, 0.5), \mathcal{N}(0, 0.75)], \quad (37)$$

354 with $\mathcal{N}(\mu, \sigma)$ indicating a Normal distribution with mean μ and standard deviation σ . One sample of
 355 the underlying Gaussian random field $\eta(t, \omega)$, together with one realisation of the p-box random field,
 356 are shown in Figure 7.

357 The inverse of samples that are drawn within this p-box is calculated by fitting a piece-wise cubic
 358 Hermite polynomial to the realisation via the Fritsch-Carlson algorithm, which are generated by the
 359 optimization algorithms introduced in Section 4. Applying the double-loop optimization algorithm in-
 360 troduced in Section 4, the bounds on the probability of failure of the structure are computed, taking
 361 the failure criterion discussed in the previous subsection into account. Hereto, SubSet- ∞ is used in the
 362 'inner loop' with an initial sample size of 1000 samples and a proposal standard deviation for each level
 363 of $[0.1, 0.1, 0.2, 0.4, 0.5, \dots]$. The discretization of the p-box is performed using $n_s = 50$ slices, yield-
 364 ing a 50-dimensional optimization problem. This optimization problem is solved using a gradient-free
 365 patternsearch optimization algorithm.

366 Figure 8 illustrates the realisations in the p-box that yield the best- and worst-case response in the
 367 bank settlement. The corresponding bounds on the probability of failure, consistent with the p-box are
 368 respectively $5.03 \cdot 10^{-05}$ and $2.38 \cdot 10^{-02}$. Unsurprisingly, the worst and best case behaviour in of the
 369 embankment are to be expected at the boundaries of the p-box. This is caused by the linear relation
 370 between Young's modulus and the maximum displacement in the embankment.

371 7. Conclusions

372 This paper discusses the concept of distribution-free p-box stochastic processes and processes. To
 373 generate realisations of such a process, it is proposed to pass realisations of a standard Gaussian process
 374 through an imprecisely defined translation map such that the auto-correlation of the original process is
 375 largely maintained. Furthermore, an optimization approach is introduced to actively look for those real-
 376 isations inside the p-box that yield a stochastic process that yields an extreme in a probabilistic measure
 377 of a response of interest. Two case studies are included to illustrate the theoretical and computational

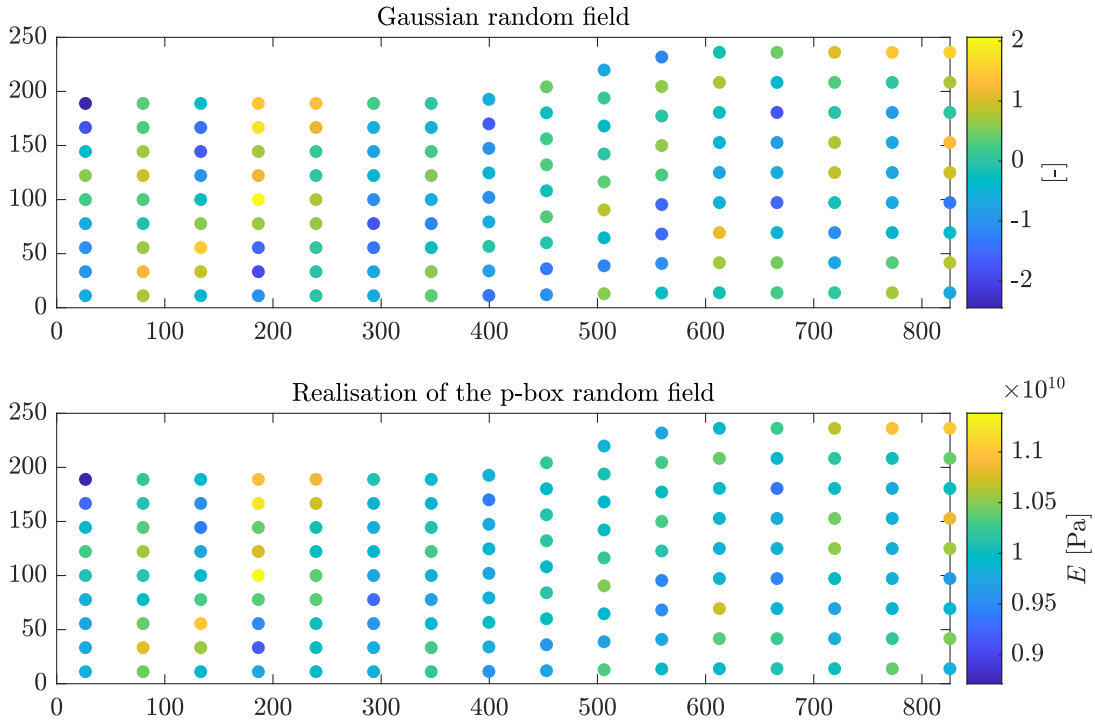


Figure 7: Top: Gaussian random field that is mapped through the interval-valued translation map. Bottom: one realisation of the resulting p-box random field.

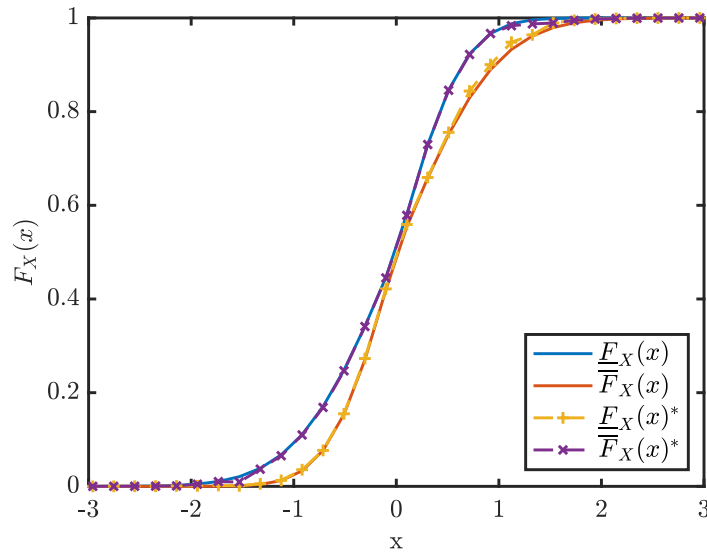


Figure 8: Imposed P-box together with the resulting realisations that yield worst and best case behaviour in the embankment.

378 aspects of the presented approach. The first case study on a quarter car model illustrated that the
 379 bounds of the P-box in fact do not necessarily coincide with the bounds on the probability of failure,
 380 which motivates the application of optimization algorithms. The second case study illustrates how the
 381 framework has to be applied in combination with 2-dimensional random fields in a Geotechnical case

382 study.

383 Future work will focus on propagating the p-box process such that a target auto-correlation of the
384 p-box process can be predefined by looking for an appropriate pre-mapped auto-correlation function for
385 each realisation F_x^k in the p-box. Furthermore, the application of more advanced approaches such as
386 e.g., based on sparse polynomial chaos expansions as discussed in [33] to propagate the p-box will be
387 investigated.

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