General Polynomial Chaos in the Current-Voltage Formulation of the Optimal Power Flow Problem

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Abstract-Mathematical optimization techniques play a key role in enabling the power system transition to sustainable energy and are used for a variety of applications such as scenario analysis, optimal planning and operational decision making. Power flow optimization, a.k.a., optimal power flow, is a building block for many applications in network operations and planning. This paper discusses the treatment of general polynomial chaos expansion for the current-voltage formulation of the optimal power flow problem. The power flow equations of the current-voltage formulation are linear, making their Galerkin projection significantly more tractable compared to formulations in the power-voltage space, while still being exact. Furthermore, auxiliary variables and quadratic constraints enable chance constraints as second-order-cone constraints. An additional advantage of this approach is that the Galerkin projection of the quadratic constraints is significantly less complex compared to those of non-linear constraints with a polynomial degree higher than two, as would be needed for expressing the original variables' variance without the auxiliary variables. On average, the current-voltage formulation using auxiliary variables shows more than an order of magnitude speed-up with respect to the power-voltage formulation without auxiliary variables.

Index Terms-AC optimal power flow, uncertainty, general polynomial chaos expansion, chance-constrained, current-voltage formulation.

I. INTRODUCTION

A. Background and Motivation

Given the rise in renewable energy sources, the operation of power systems has become increasingly uncertain. Incorporating uncertainty in decision making problems benefits stakeholders and decreases operational costs. Stochastic optimization is a framework to make optimal decisions under uncertainty. Stochastic optimal power flow (OPF) models build the basis for a number of possible applications. One such application is the minimisation of the post-market clearing redispatch for congestion management in zonal markets which can be very costly. In Germany, the cost of congestion management in the transmission system has been approximately 1.4 billion \in in 2020, of which 443 million \in are attributed to re-dispatch alone [1]. By accurately considering uncertainty from renewable generation in the determination of re-dispatch

actions, significant operational cost savings can be achieved. Further, stochastic OPF builds the basis for stochastic securityconstrained OPF models, essential for the secure operation of the transmission system. For instance, Capitanescu et al. [2] propose an algorithmic approach for computing day-ahead operational decisions in order to guarantee feasibility of the nextday security management for a range of possible operating conditions representing the uncertainties.

Commonly, in stochastic OPF models, random variables are assumed to be Gaussian, exploiting their mathematical properties for the sake of computational efficiency. However, many phenomena in power systems do not cohere to Gaussian uncertainty, e.g., wind speed follows a Weibull distribution and solar irradiation follows a Beta distribution. The handling of the power flow equations under stochastic behavior is famously challenging, because it requires, (i) propagating uncertainties through a set of implicit non-linear equations; and (ii) algorithms that provide probabilistic or robust guarantees for constraint satisfaction [3]. Furthermore, most methods rely on convexity of the original deterministic problem, however, this does not hold for the full AC OPF. In this light, general polynomial chaos expansion (gPC) has been applied in the literature to obtain stochastic solutions to the non-linear OPF problem.

Polynomial choas expansion (PCE) enables development of structured representation of random variables in terms of orthogonal base polynomials weighted by coefficients. Essentially, PCE is to random variables what a Fourier series is to a periodic signal: a representation of an infinite-dimensional object in terms of finitely many scalar coefficients [3]. The polynomials are orthogonal with respect to the joint distribution of the input random variables. This is also called Wiener chaos expansion [4]. The original PCE introduced by Wiener in 1938 was limited to the Gaussian distribution and the corresponding Hermite polynomials. Xiu and Karniadakis generalized the PCE introduced by Wiener to all Askey scheme members [5], expanding its use to general non-Gaussian distributions [6] and henceforth gained popularity as gPC. Furthermore, Wan and Karniadakis proposed an expansion beyond Wiener-Askey scheme to handle the arbitrary probability density functions based on Stielties procedure [7] and called it arbitrary polynomial chaos expansion (aPC). Oladyshkin and Nowak proposed the moment based orthogonal polynomial construction and



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named it as data-driven aPC [8]. In this work, we deal with non-Gaussian distributions defined by Askey schemes and henceforth refer it as gPC as correct abbreviation even-though different authors use PCE for the same. Depending on the chosen polynomial basis, well-known univariate distributions, e.g., the normal or Beta distribution, are exactly represented by a first-order polynomial, hence by just two polynomial chaos expansion coefficients. It has been shown that polynomial chaos methods are computationally superior to Monte-Carlo based methods [9]. An in-depth treatment of polynomial chaos is out of scope for this paper, however, the interested reader is referred to excellent text books, for instance: "Numerical methods for stochastic computations" by Xiu [10] or "Introduction to uncertainty quantification" by Sullivan [11].

Compared to Monte-Carlo-based, i.e., scenario-based, stochastic optimization, the primary advantage of gPC is its ability to accurately and efficiently handle equality constraints that involve random continuous variables such as the power flow equations under uncertain load or generation values. At the same time, gPC also helps enforcing inequality constraints using moment-based reformulations of chance constraints.

B. Significance of Variable Spaces for OPF

The power flow physics can be equivalently represented in a variety of variable spaces. In the OPF literature, the power-voltage variable space is commonly used, which allows for elimination of the power variables for (fixed) loads. However, other variable spaces are also used, e.g. currentvoltage is the natural choice in Kirchhoff's circuit laws [12], and power-lifted-voltage ('W') is a choice that enables tight convex relaxations [13]. Computational experiments have been developed by Sadat and Kim for deterministic OPF [14]. They observe, for transmission data sets, that the rectangular powervoltage (ACR) form computationally outperforms the rectangular current-voltage (IVR) formulation for the deterministic OPF.

The choice of variable space changes the opportunities for elimination of certain categories of variables, which is a way to influence the problem size, i.e., number of constraints and variables, without impacting accuracy. Those opportunities depend on the relative abundance of system elements such as loads and branches. Table I provides a comparison between IVR and ACR forms. The ACR form has an advantage when there is a higher amount of fixed demand, as no variables are required for constant power loads. Conversely, the power flow in a branch requires four real scalar variables in ACR but only two in IVR¹. In IVR, the non-linearity is confined to the transformation of generator/load power set points to current variables, while the power flow equations themselves are linear. In ACR, the non-linearity is concentrated in the expressions for power flow through a branch. A final key difference is the formulation of Kirchhoff's current law (KCL) at the buses. IVR develops the KCL expressions in the

¹Eliminating expressions with total current in favor of those with series current only.

TABLE I

	ACR	IVR
KCL at bus <i>i</i> in	power	current
power flow equations	quadratic	linear
voltage variable at bus i	$U^{\mathrm{re}}_i + j U^{\mathrm{im}}_i$	$U^{\mathrm{re}}_i + jU^{\mathrm{im}}_i$
flow variable in branch <i>lij</i>	power	current
-sending	$P_{lij} + j Q_{lij}$	$I^{\mathrm{re}}_{lij} + j I^{\mathrm{im}}_{lij}$
-receiving	$P_{lji} + j Q_{lji}$	$I_{lji}^{\mathrm{re}} + j I_{lji}^{\mathrm{im}}$
reduced flow variable in branch lij		0 0
-sending	$P_{lij} + j Q_{lij}$	$I_{lij}^{\mathrm{s,re}} + j I_{lij}^{\mathrm{im}}$
-receiving	$P_{lji} + j Q_{lji}$	-
variable per fixed load d	-	$I_d^{\mathrm{re}} + j I_d^{\mathrm{im}}$
variable per generator g	$P_g + jQ_g$	$I_{g}^{\mathrm{re}} + j I_{g}^{\mathrm{im}}$

COMPARISON OF ACR AND IVR FORMULATIONS

namesake current variables, whereas ACR lifts them to the power variables,

$$IVR: \sum I = 0 \stackrel{U \neq 0}{\longleftrightarrow} \sum S = 0: ACR.$$
(1)

C. Literature Review on Stochastic (O)PF

The literature review focuses on stochastic power flow simulation and optimization, and categorizes it according to the presence of:

- chance constraints (CC),
- general polynomial chaos expansion (gPC), and
- power flow (PF) simulation versus optimization (OPF).

1) CC-OPF: Bienstock et al. [15] consider the availability of a reliable wind forecast and the distribution function of the uncertain generation, in their CC-OPF to satisfy all the network limits with a high probability while simultaneously minimizing the cost of economic re-dispatch. They demonstrate the scalability on a 2746-bus network, run on a personal laptop. Venzke et al. [16] propose a convex reformulation of chance-constrained OPF, considering two types of uncertainty sets, i.e., a rectangular one and a multivariate Gaussian distribution of forecast errors. The authors propose a novel analytical reformulation of the linear chance constraints and a tractable approximation to the semi-definite chance constraints and provide numerical illustration on a 9-bus system with two wind farms. Roald et al. [17] propose an accurate but tractable analytical reformulation of the chance constraints in the context of OPF in polar power-voltage variables. The reformulation maintains the non-linear power flow equations for a forecasted operating point, and models the impact of uncertainty through a linearization in that point.

2) gPC-PF: Mühlpfordt et al. [18] demonstrate the application of gPC to the stochastic power flow problem. They develop numerical results for the polynomial chaos for the rectangular power-voltage formulation of the power flow equations, applied to the IEEE 14 bus system. Métivier et al. [19] propose a modified algorithm based on gPC that delivers significantly improved computational efficiency, while retaining the high level of accuracy of the standard polynomial



chaos expansion. The key contribution is that they exploit sparsity and algebraic properties of the power flow equations. Sheng and Wang [20] propose a computationally efficient and accurate algorithm to evaluate stochastic gPC-based power flow, exploiting sparsity and a continuation method.

3) gPC-OPF: Engelmann et al. [21] combine uncertainty propagation via gPC with the augmented lagrangian alternating direction inexact Newton method to solve stochastic OPF problems with non-Gaussian uncertainties, in a context of distributed computation.

4) gPC-CC-OPF: Arguably, the most advanced and conceptually complete framework for stochastic optimization is that of gPC with chance constraints. Mühlpfordt et al. [22] present a constructive approach to chance-constrained linearized 'dc' OPF that does not assume a specific probability distribution and demonstrate its use on a 300-bus case study. Lastly, Mühlpfordt et al. [23] propose a framework to formulate chance-constrained ACR-OPF using gPC and constraint generation, accounting for voltage magnitude and current magnitude limits, but without relying on samples, relaxations or linearizations.

D. Contributions and Organization of the Paper

While a lot of work has been done on stochastic power flow simulation using the gPC technique, optimization-focused works that include chance constraints are comparatively rare. Previous work in the context of gPC-CC-OPF universally use the ACR formulation. However, the impact of variable spaces for the power flow physics has not yet been explored in the context of stochastic OPF through gPC. Therefore, this paper develops the gPC extension of the IVR formulation of the deterministic OPF problem. Our work therefore complements the ACR results of Mühlpfordt et al. [23]. Additionally, using auxiliary variables and constraints, chance constraints are formulated as convex second-order cones, and the Galerkin projection the constraints for the necessary probabilistic moments is simplified from order four to order two.

The remainder of the paper is organized as follows: Section II introduces the deterministic IVR-OPF. Section III extends the IVR-OPF to allow for random variables through gPC. Furthermore, its fundamental theorems, integration with the Galerkin projection and application towards chance constraints are discussed. Section IV provides numerical results for a number of test cases. Finally, Section V concludes the paper.

II. DETERMINISTIC OPF

The feasible set of the deterministic IVR-OPF problem is introduced. Special attention is paid to its features, focusing on the differences compared to the ACR formulation.

Buses $i \in \mathcal{I}$ are the vertices of the graph representing the power system of interest. The bus voltage in rectangular coordinates is,

$$\mathsf{U}_i = \mathsf{U}_i^{\mathrm{re}} + j \,\mathsf{U}_i^{\mathrm{im}}.\tag{2}$$

Voltage magnitude $|U_i|$ is bounded between U_i^{\min} and U_i^{\max} . The shunt admittance at a bus *i* is given by y_i^{sh} , where g_i^{sh}



Fig. 1. π -model of a branch lij, with series impedance z_l^s and from- and to-side shunt admittances y_{lij}^{sh} and y_{lji}^{sh} .

and b_i^{sh} denote the corresponding conductance and susceptance, respectively.

Branches $l \in \mathcal{L}$ are the edges of the graph, represented by a π -model (Fig. 1). A tuple $lij \in \mathcal{T}^{1}$ links a branch l to its from- i and to-bus j; $\mathcal{T}^{1 \leftarrow}$ contains the equivalent tuples with *i* and *j* reversed, and the union set is $\mathcal{T}^{1} = \mathcal{T}^{1} \cup \mathcal{T}^{1}$. The current flowing through a branch l from bus i to j is,

$$_{lij} = \mathsf{I}_{lij}^{\mathrm{re}} + j \,\mathsf{I}_{lij}^{\mathrm{m}} \tag{3}$$

$$=\mathsf{I}_{lij}^{\mathrm{sh}}+\mathsf{I}_{lij}^{\mathrm{s}} \tag{4}$$

$$= (\mathsf{I}_{lij}^{\mathrm{sh,re}} + \mathsf{I}_{lij}^{\mathrm{s,re}}) + j (\mathsf{I}_{lij}^{\mathrm{sh,im}} + \mathsf{I}_{lij}^{\mathrm{s,im}}).$$
(5)

Note that this notation implies $I_{lij}^{s} + I_{lji}^{s} = 0$. The current magnitude $|I_{lij}|$ is bounded by I_{l}^{rated} . The series impedance of a branch l is given by z_l^s , where r_l^s and x_l^s denote the corresponding series resistance and reactance, respectively. The shunt admittance of a branch l at bus i is given by y_{lij}^{sh} , where $g_{lij}^{\rm sh}$ and $b_{lij}^{\rm sh}$ denote the corresponding shunt conductance and susceptance, respectively.

Units $u \in \mathcal{U}$ generalize loads and generators. A tuple $ui \in$ \mathcal{T}^{u} links a unit u to a specific bus i. The current flowing from the bus i to the unit u is,

$$\mathbf{I}_u = \mathbf{I}_u^{\rm re} + j \, \mathbf{I}_u^{\rm im}.\tag{6}$$

The complex power flowing into a unit is therefore,

$$\mathsf{S}_u = \mathsf{U}_i \mathsf{I}_u^* \tag{7}$$

$$= \left(\mathsf{U}_{i}^{\mathrm{re}}\mathsf{I}_{u}^{\mathrm{re}} + \mathsf{U}_{i}^{\mathrm{im}}\mathsf{I}_{u}^{\mathrm{im}}\right) + j\left(\mathsf{U}_{i}^{\mathrm{im}}\mathsf{I}_{u}^{\mathrm{re}} - \mathsf{U}_{i}^{\mathrm{re}}\mathsf{I}_{u}^{\mathrm{im}}\right)$$
(8)

$$= \mathsf{P}_u + j \,\mathsf{Q}_u. \tag{9}$$

A unit u is either a load d or a generator g belonging to subsets \mathcal{D} or \mathcal{G} , respectively, and with set-points P_d^{ref} and Q_d^{ref} or limits P_g^{\min} , P_g^{\max} , Q_g^{\min} and Q_g^{\max} , respectively. Consequently, the feasible set of the IVR formulation of the

deterministic OPF problem is,

Reference Bus Constraint:

$$\mathsf{U}_i^{\mathrm{im}} = 0, \qquad \qquad \forall i \in \mathcal{I}^{\mathrm{ref}}, \quad (10a)$$

Bus Constraints:

I

$$\sum_{lij\in\mathcal{T}^1} \mathsf{I}_{lij}^{\mathrm{re}} + \sum_{ui\in\mathcal{T}^{\mathrm{u}}} \mathsf{I}_u^{\mathrm{re}} + g_i^{\mathrm{sh}} \mathsf{U}_i^{\mathrm{re}} - b_i^{\mathrm{sh}} \mathsf{U}_i^{\mathrm{im}} = 0, \ \forall i \in \mathcal{I},$$
(10b)

$$\sum_{lij\in\mathcal{T}^{l}} \lim_{ij} + \sum_{ui\in\mathcal{T}^{u}} \lim_{u} + g_{i}^{\text{sh}} \mathsf{U}_{i}^{\text{im}} + b_{i}^{\text{sh}} \mathsf{U}_{i}^{\text{re}} = 0, \, \forall i \in \mathcal{I}, \quad (10c)$$

$$(U_i^{\min})^2 \le (\mathsf{U}_i^{\mathrm{re}})^2 + (\mathsf{U}_i^{\mathrm{im}})^2 \le (U_i^{\max})^2, \quad \forall i \in \mathcal{I}, \quad (10d)$$
Branch Constraints:

Branch Constraints

$$\begin{array}{ll} \mathsf{U}_{i}^{\mathrm{re}}\mathsf{I}_{d}^{\mathrm{re}} + \mathsf{U}_{i}^{\mathrm{im}}\mathsf{I}_{d}^{\mathrm{im}} = P_{d}^{\mathrm{ref}}, & \forall di \in \mathcal{T}^{\mathrm{d}}, \quad (10j) \\ \mathsf{U}_{i}^{\mathrm{im}}\mathsf{I}_{d}^{\mathrm{re}} - \mathsf{U}_{i}^{\mathrm{re}}\mathsf{I}_{d}^{\mathrm{im}} = Q_{d}^{\mathrm{ref}}, & \forall di \in \mathcal{T}^{\mathrm{d}}, \quad (10k) \\ \text{Generator Constraints:} \\ P_{g}^{\mathrm{min}} \leq \mathsf{U}_{i}^{\mathrm{re}}\mathsf{I}_{g}^{\mathrm{re}} + \mathsf{U}_{i}^{\mathrm{im}}\mathsf{I}_{g}^{\mathrm{im}} \leq P_{g}^{\mathrm{max}}, & \forall gi \in \mathcal{T}^{\mathrm{g}}, \quad (10l) \\ Q_{g}^{\mathrm{min}} \leq \mathsf{U}_{i}^{\mathrm{im}}\mathsf{I}_{g}^{\mathrm{re}} - \mathsf{U}_{i}^{\mathrm{re}}\mathsf{I}_{g}^{\mathrm{im}} \leq Q_{g}^{\mathrm{max}}, & \forall gi \in \mathcal{T}^{\mathrm{g}}, \quad (10l) \\ \end{array}$$
The following aspects of the IVR formulation with respect to its ACR counterpart are highlighted for the subsequent application of the gPC method,
 1) the neuver flow equations, i.e., Obm's law and Kirghhoff's

$$\begin{split} \mathsf{U}_{j}^{\mathrm{re}} &= \mathsf{U}_{i}^{\mathrm{re}} - r_{l}^{\mathrm{s}} \mathsf{l}_{ij}^{\mathrm{s,re}} + x_{l}^{\mathrm{s}} \mathsf{l}_{ij}^{\mathrm{s,im}}, \\ \mathsf{U}_{j}^{\mathrm{im}} &= \mathsf{U}_{i}^{\mathrm{im}} - r_{l}^{\mathrm{s}} \mathsf{l}_{lij}^{\mathrm{s,im}} - x_{l}^{\mathrm{s}} \mathsf{l}_{lij}^{\mathrm{s,re}} \end{split}$$

 $\mathsf{I}_{lij}^{\mathrm{re}} = g_{lij}^{\mathrm{sh}} \mathsf{U}_i^{\mathrm{re}} - b_{lij}^{\mathrm{sh}} \mathsf{U}_i^{\mathrm{im}} + \mathsf{I}_{lij}^{\mathrm{s,re}},$

 $\mathsf{I}_{lij}^{\mathrm{im}} = g_{lij}^{\mathrm{sh}} \mathsf{U}_{i}^{\mathrm{im}} + b_{lij}^{\mathrm{sh}} \mathsf{U}_{i}^{\mathrm{re}} + \mathsf{I}_{lij}^{\mathrm{s,im}}$

 $(\mathsf{I}_{lij}^{\mathrm{re}})^2 + (\mathsf{I}_{lij}^{\mathrm{im}})^2 \le (I_l^{\mathrm{rated}})^2,$

Demand Constraints:

1) the power flow equations, i.e., Ohm's law and Kirchhoff's current law, in (10e)-(10h) are linear, whereas these are quadratic in the ACR formulation;

 $\forall lij \in \mathcal{T}^{l}$,(10e)

 $\forall lij \in \mathcal{T}^{l}$, (10f)

 $\forall lij \in \mathcal{T}^{l}, (10g)$

 $\forall lij \in \mathcal{T}^{l}, (10h)$

 $\forall lij \in \mathcal{T}^{l}, (10i)$

 $\forall di \in \mathcal{T}^{d}, (10j)$

 \mathcal{T}^{g} , (101)

 \mathcal{T}^{g} . (10m)

- 2) unit power is enforced through quadratic expressions (10j)-(10m), whereas they are linear in the ACR formulation; and
- 3) bounds on bus voltage magnitude (10d) and line current magnitude (10i) are quadratic, similar to the ACR formulation.

III. STOCHASTIC OPF THROUGH GPC

The formulation presented in the previous Section II assumes deterministic load and generator set-points. However, in reality, these set-points are influenced by random variables, and consequently are random variables themselves. In turn, all variables in (10) become random variables. In general, all stochastic drivers are captured by a real-valued finitevariance stochastic germ $\omega = [\omega_1, ..., \omega_{N_{\omega}}]^{\mathrm{T}}$ with $N_{\omega} \in \mathbb{N}$, and corresponding set of possible realizations $\Omega \subset \mathbb{R}^{N_{\omega}}$. One method of modeling non-linear behavior of under uncertainty is through gPC expansion. The crucial features of gPC are summarized in the next sections, including: § III-A) its fundamental theorems, § III-B) integration with the Galerkin projection, and §III-C) application to chance constraints. Finally, Section III-D presents the IVR formulation of the gPC-CC-OPF.

A. General Polynomial Chaos Expansion

Consider a basis of N_{ω} -variate polynomial basis functions $[\psi_k(\omega)]_{k\in\mathcal{K}}$ that is orthogonal with probability function $\mathbb{P}(\omega)$ such that,

$$\langle \psi_l, \psi_k \rangle = \mathbb{E}[\psi_l, \psi_k] = \int \psi_l \psi_k d\mathbb{P}(\omega) = \gamma_l \delta_{lk}, \, \forall l, k \in \mathcal{K} \subseteq \mathbb{N},$$
(11)

where γ_l and $\delta_{l,k}$ denote a positive scalar and the Kronecker delta, respectively. Polynomial chaos expansion approximates any real-valued random variable x of finite variance that is a

function of the stochastic germ ω as a linear combination \hat{x} of the orthogonal polynomial basis $[\psi_k(\omega)]_{k\in\mathcal{K}}$,

$$\hat{\mathbf{x}} = \sum_{k \in \mathcal{K}} x_k \psi_k = \sum_{k \in \mathcal{K}} \frac{\langle \mathbf{x}, \psi_k \rangle}{\langle \psi_k, \psi_k \rangle} \psi_k, \qquad (12)$$

where $x_k \in \mathbb{R}$ denote the gPC coefficients. For a given random variable x, a gPC coefficient x_k is given by the inner product of the random variable x and the corresponding polynomial basis function ψ_k divided by the inner product of the polynomial basis function ψ_k with itself, see (12). The expectation \mathbb{E} and variance \mathbb{V} of an approximated random variable \hat{x} are,

$$\mathbb{E}[\hat{\mathbf{x}}] = x_0,\tag{13}$$

$$\mathbb{V}[\hat{\mathsf{x}}] = \sum_{k \in \mathcal{K} \setminus \{0\}} \langle \psi_k, \psi_k \rangle x_k^2.$$
(14)

The truncation error $\|x - \hat{x}\|$ decays to zero for $|\mathcal{K}| \to \infty$. The cardinality of \mathcal{K} is,

$$|\mathcal{K}| = \frac{(N_{\omega} + N_d)!}{N_{\omega}! N_d!},\tag{15}$$

where N_d denotes the maximum degree of the polynomial basis functions. It has been shown that polynomial basis functions with degree two capture the non-linear nature of the power flow equations without significant loss of accuracy while solving gPC-CC-OPF [23].

B. Stochastic Galerkin Method

An intrusive approach is required to exploit gPC in an optimization context. This implies that pre-existing numerical solution schemes for the deterministic problem cannot be used as they are, and must be coupled or otherwise modified to solve the stochastic problem [11]. The stochastic Galerkin method uses the formalism of weak solutions, expressed in terms of inner products, to form systems of equations for the stochastic modes, which are generally coupled together.

Concretely, a vector of approximated prescribed random variables, e.g., load set-points, is propagated through a set of implicit non-linear equations (10) to obtain a vector of unknown random variables, e.g., bus voltages. Taking a closer look at (10), two distinct operations stand out: summation and multiplication. Consider three approximated random variables \hat{x} , \hat{y} and \hat{z} defined using the same gPC basis, the Galerkin projection of both operations is,

$$\hat{\mathbf{z}} = \hat{\mathbf{x}} + \hat{\mathbf{y}} \xrightarrow{\mathrm{gp}} z_k = x_k + y_k, \qquad \forall k \in \mathcal{K}, (16)$$

$$\hat{\mathbf{z}} = \hat{\mathbf{x}} \cdot \hat{\mathbf{y}} \xrightarrow{\text{gp}} z_k = \sum_{k_1, k_2 \in \mathcal{K}} M(x_{k_1} \cdot y_{k_2}), \quad \forall k \in \mathcal{K}, \ (17)$$

where M denotes the multiplication tensor,

$$M = \frac{\langle \psi_{k_1}, \psi_{k_2}, \psi_k \rangle}{\langle \psi_k, \psi_k \rangle},\tag{18}$$

and can be computed ahead-of-time. Furthermore, note that if one of the terms in the multiplication is a deterministic parameter, i.e., $x_k = 0, \ \forall k \in \mathcal{K} \setminus \{0\} = \mathcal{K}_0$, the Galerkin projection of the multiplication simply reduces to,

$$z_k = x_0 \cdot y_k, \quad \forall k \in \mathcal{K}.$$
⁽¹⁹⁾

Concretely, the Galerkin projection permits solution of the stochastic problem by means of $|\mathcal{K}|$ deterministic and 'tractable' relations.

However, besides the Galerkin projection adding $|\mathcal{K}| - 1$ additional constraints for each constraint in (10), each Galerkin multiplication significantly reduces the sparsity of the corresponding constraints following the summation over all secondorder permutations of \mathcal{K} . Consequently, using the OPF formulation containing the smallest number of quadratic terms improves the sparsity of the resulting stochastic problem. The quadratic terms of the ACR formulation occur in the branch power flow equations, whereas for the IVR formulation, they occur in the unit power equations. Analysis of the benchmarking library PGLIB [24] shows that, on average, the branches outnumber the units by a factor 1.6. Therefore, from the perspective of sparsity, the IVR formulation seems a promising choice for improving the gPC-CC-OPF scalability.

C. Moment-Based Reformulation of Chance Constraints

In a stochastic context, it is infeasible to enforce deterministic bounds on continuous random variables. Alternatively, chance constraints ensure that the probability of a random variable × violating a deterministic bound x^{\min} or x^{\max} is below a certain level ε :

$$\mathbb{P}(\mathsf{x} \ge x^{\min}) \ge (1 - \varepsilon) \text{ or } \mathbb{P}(\mathsf{x} \le x^{\max}) \ge (1 - \varepsilon).$$
 (20)

The moment-based reformulation equivalently states this as,

$$x^{\min} \le \mathbb{E}(\mathsf{x}) \pm \lambda(\varepsilon) \sqrt{\mathbb{V}(\mathsf{x})} \le x^{\max},$$
 (21)

where $\lambda(\varepsilon) > 0$ is chosen based on knowledge of the random variable. For example, for a Gaussian random variable, the reformulation is exact with $\lambda(\varepsilon) = \lambda_{\Phi}(\varepsilon) := \Phi^{-1}(1-\varepsilon)$, where $\Phi(\cdot)$ is the quantile function of the standard Gaussian distribution $\mathcal{N}(0, 1)$.

Using (13) and (14), (21) is exactly reformulated as,

$$x^{\min} \le x_0 \pm \lambda(\varepsilon) \sqrt{\sum_{k \in \mathcal{K}_0} \langle \psi_k \psi_k \rangle x_k^2} \le x^{\max}, \qquad (22)$$

which are two second-order cone constraints.

However, note that deterministic bounds (10d), (10i), (10l) and (10m) are not expressed using a single variable, rather by using the sum of product of variables, e.g., $P_g^{\min} \leq U_i^{\operatorname{re}}|_g^{\operatorname{re}} + U_i^{\operatorname{in}}|_g^{\operatorname{in}} \leq P_g^{\operatorname{max}}$, preventing direct application of (22). Mühlpfordt et al. [23] address this by expressing the expectation and the variance of the voltage and current magnitude² through a Galerkin projection (Table II). Note that this introduces a sum over all fourth-order permutations of \mathcal{K} and corresponding fourth-order multiplication tensor.

As an alternative, this paper proposes auxiliary variables and constraints to avoid the need for a fourth-order multiplication tensor, consequently keeping the feasible set purely quadratic. The chance constraints are enforced on the auxiliary variables.

²Note that there is no need to extend this for the active and reactive generator bounds as these variables are naturally part of the ACR formulation.

Concretely, the following auxiliary variables and constraints are introduced:

- 1) lifted bus voltage magnitude: $W_i = (U_i^{re})^2 + (U_i^{im})^2$,
- 2) lifted branch current magnitude: $J_{lij} = (I_{lij}^{re})^2 + (I_{lij}^{im})^2$,
- 3) active generator power: $P_g = U_i^{re} I_q^{re} + U_i^{m} I_q^{m}$, and,
- 4) reactive generator power: $Q_g = U_i^{im} I_g^{re} U_i^{re} I_g^{im}$.

These equations are subsequently pushed through a Galerkin projection as described in Section III-B, resulting in $|\mathcal{K}|$ constraints for each auxiliary variable. Although the auxiliary approach introduces a factor $|\mathcal{K}|$ more constraints, three features make it an appropriate choice for the gPC-CC-OPF:

- 1) individual constraints are significantly sparser given the summation over all second-order permutations of \mathcal{K} , rather than all fourth-order permutations;
- individual constraints are quadratic rather than fourthorder polynomial, avoiding the need for automatic differentiation; and
- it avoids compounding truncation errors from repeated orthogonal projection associated with the non-associative property of the Galerkin projection [11].

D. GPC-CC-OPF Using IVR Formulation

Finally, the feasible set of the IVR formulation of the gPC-CC-OPF is,

Reference Bus Constraints – $i \in \mathcal{I}^{\text{ref}}$:

$$\mathsf{U}_{i,k}^{\mathrm{re}} = 0, \qquad \qquad \forall k \in \mathcal{K}_0, \, (23a)$$

$$\bigcup_{i,k}^{\text{im}} = 0, \qquad \qquad \forall k \in \mathcal{K}, \quad (23b)$$

Bus Constraints – $\forall i \in \mathcal{I}$:

$$\sum_{lij\in\mathcal{T}^{l}} \mathsf{I}^{\mathrm{re}}_{lij,k} + \sum_{ui\in\mathcal{T}^{\mathrm{u}}} \mathsf{I}^{\mathrm{re}}_{u,k} + g^{\mathrm{sh}}_{i} \mathsf{U}^{\mathrm{re}}_{i,k} - b^{\mathrm{sh}}_{i} \mathsf{U}^{\mathrm{im}}_{i,k} = 0, \quad \forall k \in \mathcal{K}, \quad (23c)$$

$$\sum_{lij\in\mathcal{T}^{1}} \mathsf{I}_{lij,k}^{\text{im}} + \sum_{ui\in\mathcal{T}^{u}} \mathsf{I}_{u,k}^{\text{im}} + g_{i}^{\text{sh}}\mathsf{U}_{i,k}^{\text{im}} + b_{i}^{\text{sh}}\mathsf{U}_{i,k}^{\text{re}} = 0, \quad \forall k \in \mathcal{K}, \quad (23d)$$

$$\sum_{k_1,k_2 \in \mathcal{K}} M(\mathsf{U}_{i,k_1}^{\mathrm{re}} \mathsf{U}_{i,k_2}^{\mathrm{re}} + \mathsf{U}_{i,k_1}^{\mathrm{im}} \mathsf{U}_{i,k_2}^{\mathrm{im}}) = \mathsf{W}_{i,k}, \qquad \forall k \in \mathcal{K}, \quad (23e)$$

$$(U_i^{\min})^2 \le \mathbb{E}(\overline{W}_i) \pm \lambda(\varepsilon^{\mathsf{v}}) \sqrt{\mathbb{V}(\overline{W}_i)} \le (U_i^{\max})^2, \tag{23f}$$

From Branch Constraints – $\forall lij \in \mathcal{T}^{l}$:

$$\begin{aligned} \mathsf{U}_{j,k}^{\rm re} &= \mathsf{U}_{i,k}^{\rm re} - r_l^{\rm s} {}_{lij,k}^{\rm s,re} + x_l^{\rm s} {}_{lij,k}^{\rm s,im}, & \forall k \in \mathcal{K}, \ \text{(23g)} \\ \mathsf{U}_{j,k}^{\rm im} &= \mathsf{U}_{i,k}^{\rm im} - r_l^{\rm s} {}_{lij,k}^{\rm s,im} - x_l^{\rm s} {}_{lij,k}^{\rm s,re}, & \forall k \in \mathcal{K}, \ \text{(23h)} \end{aligned}$$

Branch Constraints – $\forall lij \in \mathcal{T}^1$:

$$\mathsf{I}_{lij,k}^{\mathsf{re}} = g_{lij}^{\mathsf{sh}} \mathsf{U}_{i,k}^{\mathsf{re}} - b_{lij}^{\mathsf{sh}} \mathsf{U}_{i,k}^{\mathsf{in}} + \mathsf{I}_{lij,k}^{\mathsf{s,re}}, \qquad \forall k \in \mathcal{K}, \quad (23i)$$

$$\mathsf{I}_{lij,k}^{\mathrm{im}} = g_{lij}^{\mathrm{sh}} \mathsf{U}_{i,k}^{\mathrm{im}} + b_{lij}^{\mathrm{sh}} \mathsf{U}_{i,k}^{\mathrm{re}} + \mathsf{I}_{lij,k}^{\mathrm{s,im}}, \qquad \forall k \in \mathcal{K}, \quad (23j)$$

$$\sum_{k_1,k_2 \in \mathcal{K}} M(\mathsf{I}^{\mathsf{re}}_{lij,k_1}\mathsf{I}^{\mathsf{e}}_{lij,k_2} + \mathsf{I}^{\mathsf{im}}_{lij,k_1}\mathsf{I}^{\mathsf{im}}_{lij,k_2}) = \mathsf{J}_{lij,k}, \quad \forall k \in \mathcal{K}, \quad (23k)$$

$$\mathbb{E}(\overline{J}_{lij}) \pm \lambda(\varepsilon^{i}) \sqrt{\mathbb{V}(\overline{J}_{lij})} \le (I_{l}^{\text{rated}})^{2},$$
(231)

Demand Constraints – $\forall di \in \mathcal{T}^{d}$:

$$\sum_{k_1,k_2 \in \mathcal{K}} M(\mathsf{U}_{i,k_1}^{\mathrm{re}} \mathsf{I}_{d,k_2}^{\mathrm{re}} + \mathsf{U}_{i,k_1}^{\mathrm{im}} \mathsf{I}_{d,k_2}^{\mathrm{im}}) = P_{d,k}^{\mathrm{ref}}, \qquad \forall k \in \mathcal{K}, \ (23\mathrm{m})$$

$$\sum_{k_1,k_2 \in \mathcal{K}} M(\mathsf{U}_{i,k_1}^{\mathsf{im}}|_{d,k_2}^{\mathsf{re}} - \mathsf{U}_{i,k_1}^{\mathsf{re}}|_{d,k_2}^{\mathsf{im}}) = Q_{d,k}^{\mathsf{ref}}, \qquad \forall k \in \mathcal{K}, \quad (23\mathsf{n})$$

TABLE II REFORMULATION OF THE EXPECTATION AND VARIANCE OF THE BUS VOLTAGE AND LINE CURRENT MAGNITUDE FOR ACR [23].

 $\mathbb{E}[|\mathsf{U}_i|^2] = \sum_{k \in \mathcal{K}} \langle \psi_k, \psi_k \rangle \left((\mathsf{U}_{i,k}^{\mathrm{re}})^2 + (\mathsf{U}_{i,k}^{\mathrm{im}})^2 \right)$ $\mathbb{V}[|\mathsf{U}_{i}|^{2}] = \sum_{k_{1},k_{2},k_{3},k_{4}\in\mathcal{K}}^{i,k_{1},i_{1},i_{2},i_{3},k_{4}\in\mathcal{K}}^{i,k_{1},i_{1},i_{2},i_{3},i_{4},i$ $\mathbb{V}[|\mathbf{I}_{lij}|^2] = |y_l^{\mathrm{s}}|^2 \sum_{k_1,k_2,k_3,k_4 \in \mathcal{K}} \langle \psi_{k_1}, \psi_{k_2}, \psi_{k_3}, \psi_{k_4} \rangle \left(\mathsf{U}_{ij,k_1}^{\mathrm{re}} \mathsf{U}_{ij,k_2}^{\mathrm{re}} \mathsf{U}_{ij,k_4}^{\mathrm{re}} + 2\mathsf{U}_{ij,k_1}^{\mathrm{re}} \mathsf{U}_{ij,k_2}^{\mathrm{re}} \mathsf{U}_{ij,k_3}^{\mathrm{in}} \mathsf{U}_{ij,k_4}^{\mathrm{in}} + \mathsf{U}_{ij,k_1}^{\mathrm{in}} \mathsf{U}_{ij,k_2}^{\mathrm{in}} \mathsf{U}_{ij,k_3}^{\mathrm{in}} \mathsf{U}_{ij,k_4}^{\mathrm{in}} \right)$

Generator Constraints – $\forall gi \in \mathcal{T}^{g}$:

$$\sum_{k_1,k_2\in\mathcal{K}} M(\bigcup_{i,k_1}^{\mathrm{inn}}|_{g,k_2}^{\mathrm{re}} - \bigcup_{i,k_1}^{\mathrm{re}}|_{g,k_2}^{\mathrm{inn}}) = \mathsf{Q}_{g,k}, \qquad \forall k\in\mathcal{K}, \quad (23)$$

$$P_g^{\min} \le \mathbb{E}(\mathsf{P}_g) \pm \lambda(\varepsilon^{\mathsf{p}}) \sqrt{\mathbb{V}(\mathsf{P}_g)} \le P_g^{\max}, \tag{23}$$

$$Q_g^{\min} \le \mathbb{E}(\overline{\mathsf{Q}}_g) \pm \lambda(\varepsilon^{\mathsf{q}}) \sqrt{\mathbb{V}(\overline{\mathsf{Q}}_g)} \le Q_g^{\max},\tag{2}$$

where M denotes the multiplication tensor (18), and $\overline{X}_{.}$ = $[X_{k}]_{k \in \mathcal{K}}$ is employed for notational convenience. Observe the lower-order multiplication tensor in (23e), (23m)-(23p) compared to Table II. Finally, an objective is introduced to minimize the expected generation cost,

$$\min \sum_{g \in \mathcal{G}} \mathbb{E}\left[f_g(\overline{\mathsf{P}}_g)\right].$$
(24)

Typically, f_g is a quadratic function of the generator output.

IV. NUMERICAL ILLUSTRATION

The numerical illustration shows the impact of two aspects on the gPC-CC-OPF: the underlying OPF formulation, and auxiliary variable and constraints. To this end, the problem is solved for four networks: 14-bus, 30-bus, 57-bus and 118-bus systems³, where the 30-bus system is altered to match the case study in [23]⁴. In [23], the ACR formulation of the gPC-CC-OPF has been validated against Monte-Carlo simulations.

A stochastic germ is introduced with four distinct sources of uncertainty: two Beta distributions and two Normal distributions (Table III). The stochastic germ is used to represent load

TABLE III STOCHASTIC GERM AND AFFECTED BUSSES FOR EACH NETWORK.

		Affected buses									
ω	Distribution	14-bus	30-bus	57-bus	118-BUS						
1	$\mathcal{B}([0,1],2,2)$	3.	2,3.	3,15,18,29,41,	42,62,88,						
				51,52,53,55.	92,94,117.						
2	$\mathcal{B}([0,1],2,5)$	4.	4.	9,27,35,38,	56,59,70,95,104,105,						
				42,50,56.	110,112,115,118.						
3	$\mathcal{N}(0,1)$	2.	24.	12,14,25,31,	49,60,80,96,						
				44,49,54.	98,107.						
4	$\mathcal{N}(0,1)$	9.	10,21.	13,57.	82,99,103,106.						

³available at github.com/timmyfaraday/StochasticPowerModels.jl

⁴Assumptions: 1) shunt elements are neglected, 2) voltage magnitude at slack bus 1 is assumed constant at one, 3) line current limits of branches 15-23 and 25-27 are reduced to 11 and 12, respectively.

 $\sum_{k,k_2 \in \mathcal{K}} M(\bigcup_{i,k_1}^{\mathsf{re}} |_{g,k_2}^{\mathsf{re}} + \bigcup_{i,k_1}^{\mathsf{im}} |_{g,k_2}^{\mathsf{im}}) = \mathsf{P}_{g,k}, \qquad \forall k \in \mathcal{K}, \quad (230) \text{ and } \mathbb{V}[P_d^{\mathsf{ref}}] = (\sigma P_d^{\mathsf{nom}})^2, \text{ where } P_d^{\mathsf{nom}} \text{ denotes the nominal active power of the corresponding load } d^{\mathsf{T}}$ dard deviations σ are considered: 0.10 and 0.15. Three chance 3p) constraint levels ε are considered for all chance constraints: 0.05, 0.10 and 0.15. In similar work involving power flow [25] 3q) and three-phase unbalanced power flow [26], degree two

polynomials were deemed accurate enough for these problems. 3r) Henceforth, all numerical illustrations are limited till degree two polynomials.

The nonlinear solver used for the calculations is the opensource Ipopt v3.12.10 [27]. Simulations are run on a 64bit machine with Intel i7 CPU with 4 cores @2.80 GHz, 16 GB RAM, using JULIA 1.6.5. In contrast to the constraint generation method proposed in [23], all case studies in this paper are solved as a single-shot optimization. The maximum CPU time and iteration limit are set to 3600 s and 3000.

A. Results and Discussion

First, the effect of the power flow formulation and the auxiliary variables on the computation time is studied for a polynomial degree of one (Fig. 2). To this end, the following formulations are compared for each network:

- 1) reduced IVR formulation with auxiliary variables,
- 2) reduced IVR formulation without auxiliary variables,
- 3) ACR formulation with auxiliary variables,
- 4) ACR formulation without auxiliary variables.

Two things are highlighted: first, the computational time of the IVR formulation is consistently lower by approximately a factor five compared to the ACR. Second, the computational time of the formulation with auxiliary variables is consistently lower with approximately an order of magnitude compared to the one without. Note that at least some instances of the ACR and IVR formulations without auxiliary variables fail to converge within the set iteration limit. The ACR formulation with auxiliary variables is numerically more stable but fails to converge for one instance of the 118-bus network. The IVR formulation with auxiliary variables converges for all instances and networks.

This behavior confirms the analysis in Section III. Furthermore, as stated in (1), the equivalence of ACR and IVR only holds when voltages are nonzero. In deterministic OPF, this is guaranteed by the lower bound on the voltage magnitude, independent of whether polar, rectangular of lifted voltage variables are used, e.g., (10d). In the stochastic extension, the voltage limit (10d) is formulated as the chance constraint (23f).



Fig. 2. Computation time of the gPC-CC-OPF with polynomial degree one for (a) 14-bus, (b) 30-bus, (c) 57-bus, and (d) 118-bus system, respectively. Each subfigure shows the computation time related to formulations: 1) IVR w. aux, 2) IVR w/o aux, 3) ACR w. aux, and 4) ACR w/o aux, respectively. The numbers of simulations out of six that reached the iteration limit is denoted by (*), where * denotes the number.

 TABLE IV

 EXPECTED GENERATION COST AND COMPUTATION TIME OF THE GPC-CC-OPF USING THE IVR FORMULATION AND AUX VARIABLES.

		14-BUS SYSTEM				30-BUS SYSTEM				57-BUS SYSTEM				118-BUS SYSTEM			
		$N_{d} = 1$		$N_d = 2$		$N_d = 1$		$N_d = 2$		$N_d = 1$		$N_d = 2$		$N_d = 1$		$N_d = 2$	
σ	ε	cost [\$]	time [s]	cost [\$]	time [s]	cost [\$]	time [s]	cost [\$]	time [s]	cost [\$]	time [s]	cost [\$]	time [s]	cost [\$]	time [s]	cost [\$]	time [s]
0.10	0.05	2195.71	0.78026	2195.71	6.87220	599.245	1.54441	599.245	11.6154	37614.2	3.09476	-	-	97177.4	133.214	-	-
	0.10	2195.70	0.56104	2195.70	7.26801	599.240	1.15217	599.240	25.4086	37612.5	5.78977	37612.5	193.025	97156.7	180.280	-	-
	0.15	2195.70	0.63611	2195.70	6.79953	599.236	1.08340	599.237	15.4373	37611.3	25.8857	-	-	97143.1	196.203	-	-
0.15	0.05	2271.08	0.68390	2268.68	9.82704	599.369	1.22294	599.369	17.2999	37626.7	4.13501	37626.7	127.463	97279.1	269.288	-	-
	0.10	2196.50	0.63158	2196.50	7.01366	599.358	0.93242	599.358	13.0235	37622.9	4.17222	37623.0	42.4937	97237.6	265.499	-	-
	0.15	2196.47	0.60114	2196.47	6.07373	599.347	1.08706	599.347	8.67785	37620.0	3.71313	-	-	97210.7	146.090	-	-

Consequently, the individual $U_{i,k}^{\text{re}}$, $U_{i,k}^{\text{im}}$ variables only have box bounds,

$$-U_i^{\max} \le \mathsf{U}_{i,k}^{\mathrm{re}}, \mathsf{U}_{i,k}^{\mathrm{im}} \le U_i^{\max}, \tag{25}$$

while (23e) is used to derive bounds on the lifted voltage,

$$-2U_i^{\max} \le \mathsf{W}_{i,k} \le 2U_i^{\max},\tag{26}$$

and therefore 0 + j0 is in the feasible set of voltage variables for individual k. Effectively, the ACR KCL is now a relaxation of the original IVR:

$$ACR: \sum S = 0 \xrightarrow{U=0} \sum I \neq 0: IVR.$$
 (27)

The absence of voltage lower bounds introduces additional, potentially non-physical solutions to the feasible set of ACR that violate the true KCL. These additional solutions are likely to impede convergence of the interior-point method. It is observed that virtually all voltages $U_{i,k}^{re}, U_{i,k}^{im}, \forall k \in \mathcal{K}_0$ are close to zero, consequently resulting in a significant speed-up for IVR. This suggests a significant limitation of the lifting to power variables using the natural voltage variables in the absence of voltage magnitude lower bounds. Nevertheless, Tellegen's theorem states it is possible to pick alternative voltage variables for the lifting and still obtain results consistent with KCL. Whether similar opportunities for variable elimination remain is a topic for future work.

Lastly, Table IV shows the objective values and computation times for the IVR formulation of the gPC-CC-OPF with auxiliary variable for polynomial degrees one and two. The results of the 30-bus case are consistent with [23], validating the implementation. Note that simulations for the 118-bus system with polynomial degree two did not converge within the set iteration limit.

V. CONCLUSION

A novel tractable reformulation of chance-constrained OPF is proposed using general polynomial chaos expansion in the as-yet unexplored current-voltage variable space. The implementation is validated against the more common rectangular power-voltage form. In the proposed IVR formulation, chance constraints are second-order cones, and the Galerkin projection simplifies to constraints of order two due to the introduction of auxiliary variables and constraints. Consequently, the problem remains a quadratically constrained program similar to its deterministic counterpart. The numerical illustration shows the advantages of the IVR formulation of the one-shot gPC-CC-OPF using auxiliary variables with respect to computation time, showing on average an order of magnitude speed-up with respect to ACR and variants without auxiliary variables.

However, computation times are still significant, especially for the larger networks and higher polynomial degrees. Consequently, future work includes investigation of voltage variable bounds, exploiting the complex nature of the variables, and applying techniques set out in [19] in an optimization context.

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