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The Toda lattice and universality for the computation of the eigenvalues of a random matrix.

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These lecture notes summarize the work of the authors in estimating the time needed to compute the eigenvalues of (large) random matrices. Universality refers here to the fact that the halting times (until convergence or deflation up to ϵ-accuracy) are distributed independently of the distribution of the entries of the matrix as well as independent of the iterative procedure used. The Toda lattice refers to the mathematical model describing the isospectral flow of the matrix during the solution process from the original matrix until its diagonal form is obtained or until it can be deflated. This flow is a continuous analogue of the QR method and is described as an integrable Hamiltonian system in the form of a Lax equation.

After an introductory chapter, the authors give a survey of Hamiltonian mechanics and integrable systems, involving Hamiltonians on symplectic manifolds with differential 2-forms ω, Lie calculus, orbits, integrability and how it relates to the existence of Lax pairs.

Chapter 3 is the longest and forms the core of this monograph. It deals with the Toda system $\partial_t(x,y) = (\partial_x, -\partial_y)H(x,y)$ where $x, y \in \mathbb{R}^N$ and $H(x,y) = 2\sum_{i=1}^N a_i^2 + 4\sum_{i=1}^{N-1} b_i^2$ with $a_i = -\frac{1}{2}y_i$ and $b_i = \frac{1}{2}e^{(x_i - x_{i+1})/2} > 0$ is the Hamiltonian. Then (x,y) will solve the system if it holds that $\partial_t X = [X, B(X)]$ (Lie brackets) associated with a (symmetric tridiagonal) Jacobi matrix $X(0) = J = \text{tridiag}(\mathbf{b}, \mathbf{a}, \mathbf{b})$, $\mathbf{a} = (a_i)_{i=1}^N$, $\mathbf{b} = (b_i)_{i=1}^{N-1}$ and $B(X) = \text{tridiag}(-\mathbf{b}, \mathbf{0}, \mathbf{b})$. This describes an isospectral flow $X(t)$ towards the diagonal matrix with the ordered the eigenvalues of $X(0)$. Convergence and error estimates are proved for $t \to \pm\infty$, the relation with the QR algorithm is shown and the system is a completely integrable system in the sense of Liouville. This implies that the original variables (x,y) can be transformed in angle-action variables (θ, λ), with simple dynamics (θ is linear in t and λ is constant). All these results are obtained first for the real tridiagonal Jacobi matrix and are then lifted to a full Hermitean matrix.

The 3 remaining chapters are much shorter. Chapter 4 deals with the introduction of new variables associated with a class of real symmetric matrices with simple spectrum that allows simple explicit Toda flows without relying on the Hamiltonian structure. See e.g. [*R. Leite* et al., Linear Algebra Appl. 429, No. 1, 387–402 (2008; Zbl 1144.65025)]. This extends the scope of the classical approach to the Toda algorithm. Finally chapter 5 introduces some properties and conditions imposed on Wigner or invariant ensembles of real symmetric or Hermitian random matrices,which culminates in chapter 6 with a proof of the universality property of the Toda algorithm for these ensembles to deflate their largest eigenvalue.

Although all the concepts are introduced, it is advisable that the reader is somewhat familiar with Lie algebras and Hamiltonian systems. There is some advise on existing literature in the first chapter. There are no dedicated sections with exercises, but several exercises are interlaced with the text, often to verify some extra property or part of a long proof.

Reviewer: Adhemar Bultheel (Leuven)

MSC:

65-02 Research exposition (monographs, survey articles) pertaining to numerical analysis
65F10 Iterative numerical methods for linear systems
60B20 Random matrices (probabilistic aspects)
15B52 Random matrices (algebraic aspects)

Keywords:

Hamiltonian system; Toda flow; random matrix; eigenvalue; deflation; halting time; universality principle;

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