Determinacy of a rational moment problem

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Abstract

We discuss a moment problem of Stieltjes type that is related to the theory of orthogonal rational functions. We obtain results which lead to a sufficient condition of Carleman type for determinacy of the moment problem.

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1 Introduction

The moments c_n of a positive measure μ on the non-negative real line $[0, \infty)$ are defined as

$$c_n = \int_0^\infty t^n \mathrm{d}\mu(t), \quad n = 0, 1, \ldots, \tag{1.1}$$

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if these integrals are finite. (See [1,26–28].) The *Stieltjes moment problem* associated with the sequence $\{c_n\}$ is called *determinate* if there is only one measure μ (with support in $[0,\infty)$) giving rise to the moments c_n through the formula (1.1). A well known criterion by Carleman gives a sufficient condition for determinacy: *If*

$$\sum_{n=1}^{\infty} c_n^{-\frac{1}{2n}} = \infty, \tag{1.2}$$

then the Stieltjes moment problem is determinate. See e.g. [10–12,20,26,27]. Our treatment in this paper of a rational Stieltjes problem is especially indebted to a method developed by Karlsson and von Sydow in [20]. By an argument in [1, p. 25-26, Problem 4] (building on results by Wouk [29]) for a similar situation, and also from results in [20], it can even be shown that *if the Stieltjes moment problem is solvable and (1.2) is satisfied, then the Hamburger moment problem is determinate* (i.e., there is only one measure with support in $(-\infty,\infty)$ giving rise to the moments c_n through formula $c_n = \int_{-\infty}^{\infty} t^n \mathrm{d}\mu(t), \quad n = 0,1,\ldots$). We shall discuss in detail the analogous situation for the rational moment problem treated in this paper in Section 5.

The *strong Stieltjes moment problem* arises when also moments with negative index are given:

$$c_n = \int_0^{\infty} t^n \mathrm{d}\mu(t), \quad n = 0, \pm 1, \pm 2, \ldots \tag{1.3}$$

(See e.g. [17,19,25].)

A result analogous to Carleman's criterion was proved by Aldén [2,3]: *If*

$$\sum_{n=0}^{\infty} \left[(c_n)^{-\frac{1}{2n}} + (c_{-n})^{-\frac{1}{2n}} \right] = \infty, \tag{1.4}$$

then the strong Stieltjes moment problem is determinate. (See also [15] and [24].) By an approach following that of Karlsson and von Sydow, two of the present authors [14] obtained essentially the same result. A condition equivalent to (1.4) is

$$\sum_{m=0}^{\infty} \left[(c_{2m})^{-\frac{1}{4m}} + (c_{-2m})^{-\frac{1}{4m-2}} \right] = \infty. \tag{1.5}$$

The classical moment problems are related to the theory of orthogonal polynomials. Similarly, the strong moment problems are related to the theory of orthogonal Laurent polynomials. (See [13,17,18].) In this paper, we discuss a moment problem of Stieltjes type that is related to the theory of orthogonal rational functions. The basic theory of this problem is developed in [9]. We obtain results which also in this situation lead to a sufficient condition of Carleman type for determinacy of the moment problem.

Related problems have been treated by different methods, and related Carleman type conditions obtained. See especially work by Lopez [21–23].

2 Orthogonal rational functions

The basic theory of orthogonal rational functions can be found in [8] and also in [4–6] for the situation discussed here where interpolation points may belong to the support of the measure.

Let $\{\alpha_k\}_{k=1}^{\infty}$ be a sequence of points on the real line $(-\infty,\infty)$. For some non-essential technical reasons, we assume that $\alpha_k \neq 0$ for all k. We might include the possibility that some or all of the points α_k could be the point at infinity (on the Riemann sphere), but that situation will not be considered in this paper.

We define

$$\omega_0 = 1, \quad \omega_n(z) = (z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n), \quad n = 1,2,\ldots \tag{2.1}$$

The space $\mathcal{L}$ is defined to be the linear span of the sequence $\{1/\omega_n\}_{n=0}^{\infty}$. We note that $\mathcal{L}$ consists of all functions of the form s_n/ω_n for some n, where s_n is a polynomial of degree at most n.

Other bases for $\mathcal{L}$ than $\{1/\omega_n\}_{n=0}^{\infty}$ may be useful. We shall in this paper mostly consider the situation when $\alpha_{2p} \neq \alpha_{2q+1}$ for all p and q. It is readily verified that the sequence $\{b_n\}_{n=0}^{\infty}$ is a basis for $\mathcal{L}$, where

$$b_0 = 1 \tag{2.2}$$

$$b_{2m}(z) = \frac{z^{2m}}{(z-\alpha_1)(1-\frac{z}{\alpha_2})\cdots(z-\alpha_{2m-1})(1-\frac{z}{\alpha_{2m}})} \tag{2.3}$$

$$b_{2m+1}(z) = \frac{1}{(z-\alpha_1)(1-\frac{z}{\alpha_2})\cdots(1-\frac{z}{\alpha_{2m}})(z-\alpha_{2m+1})}. \tag{2.4}$$

In the limiting situation $\alpha_{2m} \to \infty$, $\alpha_{2m+1} \to 0$, these functions become $b_0 = 1$, $b_{2m} = z^m$, $b_{2m+1} = 1/z^{m+1}$, which is a standard basis for the Laurent polynomials.

Let M be a positive linear functional on the product space $\mathcal{L}\cdot\mathcal{L}$ (consisting of products $f\cdot g$, with $f,g \in \mathcal{L}$). The *moments* $\mu_{m,n}$ are defined by

$$\mu_{m,n} = M[b_m \cdot b_n], \quad m,n = 0,1,2,\ldots. \tag{2.5}$$

For simplicity we normalize such that $\mu_{0,0} = 1$.

A positive measure with infinite support in $(-\infty,\infty)$ is said to solve the *rational (Hamburger) moment problem (RMP) on* $\mathcal{L}\cdot\mathcal{L}$ if

$$\mu_{m,n} = \int_{-\infty}^{\infty} b_m(t)b_n(t)\mathrm{d}\mu(t), \quad m,n = 0,1,2,\ldots, \tag{2.6}$$

and to solve the *(Hamburger) rational moment problem on* $\mathcal{L}$ if

$$\mu_{0,n} = \int_{-\infty}^{\infty} b_n(t)\mathrm{d}\mu(t), \quad n = 0,1,2,\ldots. \tag{2.7}$$

A measure which solves the moment problem on $\mathcal{L} \cdot \mathcal{L}$ obviously also solves the moment problem on $\mathcal{L}$.

We shall in the following assume that the moment problem on $\mathcal{L} \cdot \mathcal{L}$ has at least one solution μ. An inner product $\langle \cdot, \cdot \rangle$ is then defined on $\mathcal{L}$ by

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(t)\overline{g(t)}\mathrm{d}\nu(t), \quad f, g \in \mathcal{L}, \tag{2.8}$$

where ν is *any solution of the moment problem on* $\mathcal{L} \cdot \mathcal{L}$.

Let $\{\phi_n\}_{n=0}^{\infty}$ denote an orthonormal sequence in $\mathcal{L}$ obtained by orthonormalization with respect to $\langle \cdot, \cdot \rangle$ of the basis $\{1/\omega_n\}$, (or equivalently of the basis $\{b_n\}$). The associated orthogonal rational functions σ_n are defined by

$$\sigma_n(z) = M\left[\frac{\phi_n(t) - \phi_n(z)}{t - z}\right], \quad n = 0, 1, 2, \ldots \tag{2.9}$$

(the functional operating on its argument as a function of t). Equivalently,

$$\sigma_n(z) = \int_{-\infty}^{\infty} \frac{\phi_n(t) - \phi_n(z)}{t - z}\mathrm{d}\nu(t), \tag{2.10}$$

where ν is *any solution of the moment problem on* $\mathcal{L}$.

By the *Stieltjes transform* of a finite measure μ, we shall here mean the function

$$S(z, \mu) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\mu(t)}{z - t}. \tag{2.11}$$

The measure μ is determined by its Stieltjes transform $S(z, \mu)$, which means that if two measures have the same Stieltjes transform, then they are equal.

The rational function $\sigma_n(z)/\phi_n(z)$ is an $[n - 1/n]$ multi-point Padé approximant of $S(z, \mu)$ at the table $\{\infty, \alpha_1, \alpha_1, \ldots, \alpha_{n-1}, \alpha_{n-1}, \alpha_n\}$. See [7, Theorem 5.2].

We will need an estimate of the *error term* $\sigma_n(z)/\phi_n(z) - S(z, \mu)$. We have the following result.

Proposition 2.1 *Let μ be an arbitrary solution of the RMP on $\mathcal{L}$. Then the following formula is valid:*

$$\frac{\sigma_n(z)}{\phi_n(z)} - S(z, \mu) = \frac{1}{\phi_n(z)} \int_{-\infty}^{\infty} \frac{\phi_n(t)}{t - z}\mathrm{d}\mu(t). \tag{2.12}$$

Proof. This follows immediately from (2.10) and (2.11). □

3 A Stieltjes problem

We deal in this paper with a special situation, which may be considered as an extension of the Stieltjes situation for the classical (one-point) and strong (two-point) problems. Contributions to a theory of this rational Stieltjes problem are given in [9].

We assume the existence of real numbers α and β such that

$$\alpha_{2m} \leq \alpha < \beta \leq \alpha_{2m-1} \leq 0, \quad m = 1, 2, \ldots \tag{3.1}$$

and such that for all measures μ considered the support is contained in the non-negative real line:

$$\mathrm{supp}(\mu) \subset [0, \infty). \tag{3.2}$$

Thus we assume the existence of at least one positive measure μ with support in $[0, \infty)$ which solves the moment problem on $\mathcal{L} \cdot \mathcal{L}$ for the sequence $\{\mu_{m,n}\}_{m,n}$, and we only consider solutions of the moment problem on $\mathcal{L}$ and on $\mathcal{L} \cdot \mathcal{L}$ with support in $[0, \infty)$. Such solutions are called solutions of the *rational Stieltjes moment problem (RSMP)* on $\mathcal{L}$ and $\mathcal{L} \cdot \mathcal{L}$ respectively. (In [9] we considered a slightly more general situation where 0 in (3.1) and (3.2) is replaced by an arbitrary real point γ.)

The orthogonal rational function ϕ_n may be written in the form

$$\phi_n(z) = \frac{p_n(z)}{\omega_n(z)} \tag{3.3}$$

where $p_n(z)$ is a polynomial of exact degree n. The polynomial p_n has n simple zeros in $(0, \infty)$. We assume that ϕ_n is normalized such that

$$\phi_{2m}(x) > 0, \quad \phi_{2m+1}(x) < 0, \quad x \in (\alpha, \beta). \tag{3.4}$$

(This determination is unambiguous, $\phi_n(x)$ having constant sign in (α, β).)

The function ϕ_n can be written as

$$\phi_n = v_n b_n + \cdots + u_n b_0. \tag{3.5}$$

(We may call v_n the *leading coefficient* and u_n the *trailing coefficient* with respect to the basis $\{b_n\}$.) It easily follows from the definitions (2.1),(2.3)–(2.4), and (3.4) and (3.5) that we have

$$p_{2m}(\alpha_{2m}) = (-1)^m \alpha_2 \alpha_4 \cdots \alpha_{2m} (\alpha_{2m})^{2m} v_{2m} \tag{3.6}$$

$$p_{2m+1}(\alpha_{2m+1}) = (-1)^m \alpha_2 \alpha_4 \cdots \alpha_{2m} v_{2m+1}. \tag{3.7}$$

The functions $\{\phi_m, \sigma_n\}$ satisfy a three-term recurrence relation of the following form (see [4,6–8]):

$$\begin{bmatrix}\sigma_n(z)\\ \phi_n(z)\end{bmatrix} = \frac{U_n(z-\alpha_{n-2})+V_n(z-\alpha_{n-1})}{z-\alpha_n}\begin{bmatrix}\sigma_{n-1}(z)\\ \phi_{n-1}(z)\end{bmatrix} + \frac{W_n(z-\alpha_{n-2})}{z-\alpha_n}\begin{bmatrix}\sigma_{n-2}(z)\\ \phi_{n-2}(z)\end{bmatrix}, \quad n=3,4,\ldots \tag{3.8}$$

$$\begin{bmatrix}\sigma_2(z)\\ \phi_2(z)\end{bmatrix} = \frac{U_2(z-\alpha_2)+V_2(z-\alpha_1)}{z-\alpha_2}\begin{bmatrix}\sigma_1(z)\\ \phi_1(z)\end{bmatrix} + \frac{W_2}{z-\alpha_2}\begin{bmatrix}\sigma_0(z)\\ \phi_0(z)\end{bmatrix} \tag{3.9}$$

$$\begin{bmatrix}\sigma_1(z)\\ \phi_1(z)\end{bmatrix} = \frac{U_1(z-\alpha_1)+V_1(z-\alpha_2)}{z-\alpha_1}\begin{bmatrix}\sigma_0(z)\\ \phi_0(z)\end{bmatrix} + \frac{W_1}{z-\alpha_1}\begin{bmatrix}\sigma_{-1}(z)\\ \phi_{-1}(z)\end{bmatrix} \tag{3.10}$$

$$\begin{bmatrix}\sigma_0(z)\\ \phi_0(z)\end{bmatrix} = \begin{bmatrix}0\\ 1\end{bmatrix}, \quad \begin{bmatrix}\sigma_{-1}(z)\\ \phi_{-1}(z)\end{bmatrix} = \begin{bmatrix}-\mu_{0,1}\\ 0\end{bmatrix}. \tag{3.11}$$

Here

$$U_n < 0 \text{ for } n = 1,2,3,\ldots \tag{3.12}$$
$$V_n > 0 \text{ for } n = 1,3,4,\ldots \tag{3.13}$$
$$W_n > 0 \text{ for } n = 1,2,3,\ldots. \tag{3.14}$$

We also find that (see [7, Proposition 3.4])

$$W_1 W_2 \cdots W_n = (-1)^n (\mu_{0,1})^{-1} (\alpha_{n-2} - \alpha_{n-1}) U_n. \tag{3.15}$$

The coefficients U_n, V_n can be expressed as follows:

$$\left.\begin{aligned} U_1 &= \frac{p_1(\alpha_2)}{\alpha_2-\alpha_1}, \quad U_2 = \frac{p_2(\alpha_1)}{p_1(\alpha_1)(\alpha_1-\alpha_2)},\\ U_n &= \frac{p_n(\alpha_{n-1})}{p_{n-1}(\alpha_{n-1})(\alpha_{n-1}-\alpha_{n-2})}, \quad n=3,4,\ldots \end{aligned}\right\} \tag{3.16}$$

$$V_1 = \frac{p_1(\alpha_1)}{\alpha_1-\alpha_2}, \quad V_n = \frac{p_n(\alpha_{n-1})}{p_{n-1}(\alpha_{n-2})(\alpha_{n-2}-\alpha_{n-1})}, \quad n=3,4,\ldots \tag{3.17}$$

For the approximants $\{\sigma_n(z)/\phi_n(z)\}$ of the continued fraction associated with the three-term recurrence relation given by (3.8)–(3.11) the following fundamental result holds (see [7]):

Theorem 3.1 *For $x \in (\alpha,\beta)$ the sequence $\{\sigma_{2m}(x)/\phi_{2m}(x)\}$ is decreasing and the sequence $\{\sigma_{2m+1}(x)/\phi_{2m+1}(x)\}$ is increasing. For $z \in \mathbb{C}\setminus[0,\infty)$, $\{\sigma_{2m}(z)/\phi_{2m}(z)\}$ converges to $S(z,\mu^{(0)})$ and $\{\sigma_{2m+1}(z)/\phi_{2m+1}(z)\}$ converges to $S(z,\mu^{(\infty)})$, where $\mu^{(0)}$ and $\mu^{(\infty)}$ are solutions of the RSMP on $\mathcal{L}$.*

4 Estimates on coefficients

We shall in the following restrict ourselves to consider the following situation:

The sequence $\{\alpha_{2m}\}_{m=1}^{\infty}$ *decreases to* $-\infty$, (4.1)
The sequence $\{\alpha_{2m+1}\}_{m=0}^{\infty}$ *increases to* 0. (4.2)

To discuss determinacy of the moment problem, we need estimates on the differences $\sigma_n(x)/\phi_n(x) - \sigma_{n-2}(x)/\phi_{n-2}(x)$. It follows from (3.15) and [7, formula (4.10)] that for $n = 3, 4, \ldots$ we may write

$$\frac{\sigma_n(z)}{\phi_n(z)} - \frac{\sigma_{n-2}(z)}{\phi_{n-2}(z)} = \Delta_n(z)\frac{1}{\phi_n(z)\phi_{n-2}(z)}, \tag{4.3}$$

where

$$\Delta_n(z) = \frac{U_{n-1}(\alpha_{n-3} - \alpha_{n-2})[U_n(z - \alpha_{n-2}) + V_n(z - \alpha_{n-1})]}{(z - \alpha_n)(z - \alpha_{n-1})(z - \alpha_{n-2})}. \tag{4.4}$$

Lemma 4.1 *For* $x \in (\alpha, \beta)$ *and* $n = 3, 4, \ldots$ *we have*

$$|\Delta_n(z)| > \frac{V_n|U_{n-1}(\alpha_{n-3} - \alpha_{n-2})|}{|(x - \alpha_n)(x - \alpha_{n-2})|}. \tag{4.5}$$

Proof. It follows from (3.1) and (3.12)–(3.13) that $U_n(x - \alpha_{n-2})$ and $V_n(x - \alpha_{n-1})$ have the same sign for $x \in (\alpha, \beta)$. Thus $|U_n(x - \alpha_{n-2}) + V_n(x - \alpha_{n-1})| > |V_n(x - \alpha_{n-1})|$. From this and the positivity of $\mu_{0,1}$ and V_n the inequality follows. □

Lemma 4.2 *The following results hold:*
A. The function $p_n(x)|$ *is decreasing with increasing* x *for* $x < 0$.
B. The function $|p_n(x)/x^n|$ *is increasing with increasing* x *for* $x < 0$.

Proof. The polynomial p_n has n simple zeros in $(0, \infty)$, hence p_n' has $n - 1$ zeros in $(0, \infty)$, and so p_n is monotonic for $x < 0$. Since $|p_n(x)| \to \infty$ as $x \to -\infty$, A. follows.

We have $[p_n(x)/x^n]' = (xp_n'(x) - np_n(x))/x^{n+1}$. The numerator polynomial has degree at most $n - 1$. Since $p_n(x)/x^n$ has at least $n - 1$ extrema for $x > 0$, it follows that $p_n(x)/x^n$ is monotonic for $x < 0$. Since $|p_n(x)/x^n| \to \infty$ for $x \to 0^-$, we conclude that B. holds. □

It follows from (3.16)–(3.17) that we may write

$$V_nU_{n-1}(\alpha_{n-3} - \alpha_{n-2}) = \frac{p_n(\alpha_{n-2})}{p_{n-2}(\alpha_{n-2})(\alpha_{n-1} - \alpha_{n-2})}, \quad n = 3, 4, \ldots \tag{4.6}$$

Lemma 4.3 *For* $x \in (\alpha, \beta)$, *the following inequalities hold:*

$$A.\ |V_{2m}U_{2m-1}(\alpha_{2m-3} - \alpha_{2m-2})| \geq \frac{|\alpha_{2m-2}|^2|\alpha_{2m}|}{|\alpha_{2m-2} - \alpha_{2m-1}|}\frac{|v_{2m}|}{|v_{2m-2}|}, \quad m \geq 2 \tag{4.7}$$

$$B.\ |V_{2m+1}U_{2m}(\alpha_{2m-2} - \alpha_{2m-1})| \geq \frac{|\alpha_{2m}|}{|\alpha_{2m-1} - \alpha_{2m}|}\frac{|v_{2m+1}|}{|v_{2m-1}|}, \quad m \geq 1. \tag{4.8}$$

Proof. A. It follows from (3.6) and (4.6) that we may write

$$V_{2m}U_{2m-1}(\alpha_{2m-3}-\alpha_{2m-2}) = \frac{\alpha_2\alpha_4\cdots\alpha_{2m}(\alpha_{2m})^{2m}v_{2m}}{\alpha_2\alpha_4\cdots\alpha_{2m-2}(\alpha_{2m-2})^{2m-2}v_{2m-2}(\alpha_{2m-2}-\alpha_{2m-1})}\frac{p_{2m}(\alpha_{2m-2})}{p_{2m}(\alpha_{2m})}. \tag{4.9}$$

Lemma 4.2.B together with (4.1) implies that

$$\left|\frac{p_{2m}(\alpha_{2m-2})}{p_{2m}(\alpha_{2m})}\right| \geq \frac{|\alpha_{2m-2}|^{2m}}{|\alpha_{2m}|^{2m}}.$$

This together with (4.9) gives (4.7).

B. Similarly from (3.7) and (4.6) we find that

$$V_{2m+1}U_{2m}(\alpha_{2m-2}-\alpha_{2m-1}) = -\frac{\alpha_2\alpha_4\cdots\alpha_{2m}v_{2m+1}}{\alpha_2\alpha_4\cdots\alpha_{2m-2}v_{2m-1}(\alpha_{2m-1}-\alpha_{2m})}\frac{p_{2m+1}(\alpha_{2m-1})}{p_{2m+1}(\alpha_{2m+1})}. \tag{4.10}$$

Lemma 4.2.A together with (4.2) implies that $p_{2m+1}(\alpha_{2m-1}) > p_{2m+1}(\alpha_{2m+1})$. This together with (4.10) gives (4.8). □

Proposition 4.4 *For $x \in (\alpha,\beta)$, the following inequalities hold:*

$$A.\quad |\Delta_{2m}(x)| > \left|\frac{v_{2m}}{v_{2m-2}}\right|, \quad m = 2,3,\ldots \tag{4.11}$$

$$B.\quad |\Delta_{2m+1}(x)| > \frac{1}{\alpha^2}\left|\frac{v_{2m+1}}{v_{2m-1}}\right|, \quad m = 1,2,\ldots \tag{4.12}$$

with $\Delta_n(x)$ as given by (4.4).

Proof. Because of (3.1) we have $|\alpha_{2p}-\alpha_{2q+1}| < |\alpha_{2p}|$, $|\alpha_{2p}-x| < |\alpha_{2p}|$, and $|\alpha_{2q+1}-x| < |\alpha|$ for all p and q and for all $x \in (\alpha,\beta)$. We find from (4.5) and (4.7) that

$$|\Delta_{2m}(x)| > \frac{|\alpha_{2m-2}|^2|\alpha_{2m}|}{|\alpha_{2m-2}||\alpha_{2m}||\alpha_{2m-2}|}\left|\frac{v_{2m}}{v_{2m-2}}\right|,$$

which is (4.11). Similarly, we find from (4.5) and (4.8) that

$$|\Delta_{2m+1}(x)| > \frac{1}{\alpha^2}\frac{|\alpha_{2m}|}{|\alpha_{2m}|}\left|\frac{v_{2m+1}}{v_{2m-1}}\right|,$$

which is (4.12). □

5 Uniqueness of solutions

We shall in this section give sufficient conditions in terms of the leading coefficients v_n for determinacy of the RSMP on $\mathcal{L} \cdot \mathcal{L}$.

Proposition 5.1 *The following implications hold:*

$$A.\ \textit{If} \sum_{m=1}^{\infty} \left| \frac{v_{2m}}{v_{2m-2}} \right|^{1/2} = \infty,\ \textit{then} \sum_{m=0}^{\infty} |\phi_{2m}(x)|^2 = \infty \ \textit{for}\ x \in (\alpha, \beta), \tag{5.1}$$

$$B.\ \textit{If} \sum_{m=1}^{\infty} \left| \frac{v_{2m+1}}{v_{2m-1}} \right|^{1/2} = \infty,\ \textit{then} \sum_{m=0}^{\infty} |\phi_{2m+1}(x)|^2 = \infty \ \textit{for}\ x \in (\alpha, \beta). \tag{5.2}$$

Proof. It follows from Proposition 4.4 that there exists a constant K independent of n and x such that

$$\left| \frac{v_n}{v_{n-2}} \right| \leq K |\Delta_n(x)|. \tag{5.3}$$

Recall from (4.3) that

$$\Delta_n(x) = \left[\frac{\sigma_n(x)}{\phi_n(x)} - \frac{\sigma_{n-2}(x)}{\phi_{n-2}(x)} \right] \phi_n(x) \phi_{n-2}(x). \tag{5.4}$$

If

$$\sum_{m=1}^{\infty} \left| \frac{v_{2m}}{v_{2m-2}} \right|^{1/2} = \infty \quad \text{then} \quad \sum_{m=1}^{\infty} |\Delta_{2m}(x)|^{1/2} = \infty$$

and hence by (5.4), Schwartz' inequality and Theorem 3.1

$$\left\{ \sum_{m=1}^{\infty} \left[\frac{\sigma_{2m-2}(x)}{\phi_{2m-2}(x)} - \frac{\sigma_{2m}(x)}{\phi_{2m}(x)} \right] \right\} \left\{ \sum_{m=1}^{\infty} |\phi_{2m}(x) \phi_{2m-2}(x)| \right\} = \infty. \tag{5.5}$$

Similarly we find that $\sum_{m=1}^{\infty} |v_{2m+1}/v_{2m-1}|^{1/2} = \infty$ implies

$$\left\{ \sum_{m=1}^{\infty} \left[\frac{\sigma_{2m+1}(x)}{\phi_{2m+1}(x)} - \frac{\sigma_{2m-1}(x)}{\phi_{2m-1}(x)} \right] \right\} \left\{ \sum_{m=1}^{\infty} |\phi_{2m+1}(x) \phi_{2m-1}(x)| \right\} = \infty. \tag{5.6}$$

Again by Theorem 3.1 we conclude that the series to the left in (5.5) and (5.6) converge to finite values, hence $\sum_{m=1}^{\infty} |\phi_{2m}(x)\phi_{2m-2}(x)| = \infty$ holds if (5.5) holds and $\sum_{m=1}^{\infty} |\phi_{2m+1}(x)\phi_{2m-1}(x)| = \infty$ if (5.6) holds. Divergence of $\sum_{m=0}^{\infty} |\phi_{2m}(x)|^2$ and $\sum_{m=0}^{\infty} |\phi_{2m+1}(x)|^2$ respectively now easily follows by Schwartz' inequality. □

Let ν be an arbitrary solution of the RSMP on $\mathcal{L}\cdot\mathcal{L}$. Recall that the *error term* $E_n(z,\nu)$ is defined to be

$$E_n(z,\nu) = \frac{\sigma_n(x)}{\phi_n(x)} - S(z,\nu), \tag{5.7}$$

(cf. Section 2).

Proposition 5.2 *Let ν be an arbitrary solution of the RSMP on $\mathcal{L}\cdot\mathcal{L}$, and let $x \in (\alpha,\beta)$.*

A. If $\sum_{m=0}^{\infty} |\phi_{2m}(x)|^2 = \infty$, then $E_{2m}(x,\nu)$ tends to zero as $m \to \infty$,

B. If $\sum_{m=0}^{\infty} |\phi_{2m+1}(x)|^2 = \infty$, then $E_{2m+1}(x,\nu)$ tends to zero as $m \to \infty$,

Proof. According to Proposition 2.1 we have

$$E_n(z,\nu) = \frac{1}{\phi_n(x)} \int_0^{\infty} \frac{\phi_n(t)}{t-z} \mathrm{d}\nu(t). \tag{5.8}$$

Let $x \in (\alpha,\beta)$. Then the function $t \to 1/(t-x)$ is square integrable with respect to ν. Its Fourier coefficients with respect to the system $\{\phi_n\}$ are $E_n(x,\nu)\phi_n(x)$ according to (5.8), hence by Bessel's inequality

$$\sum_{n=0}^{\infty} |E_n(z,\nu)|^2 |\phi_n(x)|^2 < \infty. \tag{5.9}$$

(Note that the use of Bessel's inequality requires that the system $\{\phi_n\}$ is orthogonal with respect to ν, and hence that ν is a solution of the RSMP on $\mathcal{L}\cdot\mathcal{L}$, not only on $\mathcal{L}$.)

Assume e.g., that $\sum_{m=0}^{\infty} |\phi_{2m}(x)|^2 = \infty$. It then follows from (5.9) that at least a subsequence of $\{E_{2m}(x,\nu)\}$ must tend to zero, and hence the whole sequence tends to zero since it is monotonic by Theorem 3.1. Similarly we find that $\{E_{2m+1}(x,\nu)\}$ tends to zero if $\sum_{m=0}^{\infty} |\phi_{2m+1}(x)|^2 = \infty$. □

Theorem 5.3 *Assume that $\sum_{n=2}^{\infty} |v_n/v_{n-2}|^{1/2} = \infty$. Then the RSMP on $\mathcal{L}\cdot\mathcal{L}$ is determinate.*

Proof. If $\sum_{n=2}^{\infty} |v_n/v_{n-2}|^{1/2} = \infty$, then at least one of the series

$$\sum_{m=1}^{\infty} |v_{2m}/v_{2m-2}|^{1/2} \quad \text{and} \quad \sum_{m=1}^{\infty} |v_{2m+1}/v_{2m-1}|^{1/2}$$

diverges. It follows from Proposition 5.1 that at least one of the series

$$\sum_{m=0}^{\infty} |\phi_{2m}(x)|^2 \quad \text{and} \quad \sum_{m=0}^{\infty} |\phi_{2m+1}(x)|^2$$

diverges for all $x \in (\alpha, \beta)$. Proposition 5.2 then implies that at least one of the sequences $\{E_{2m}(x,\nu)\}$ and $\{E_{2m+1}(x,\nu)\}$ tend to zero for all solutions ν of the RSMP on $\mathcal{L}\cdot\mathcal{L}$. Consequently at least one of the sequences $\{\sigma_{2m}(x)/\phi_{2m}(x)\}$ and $\{\sigma_{2m+1}(x)/\phi_{2m+1}(x)\}$ tends to $S(x,\nu)$ for all solutions of the RSMP on $\mathcal{L}\cdot\mathcal{L}$ and all $x \in (\alpha,\beta)$. In any case, $S(x,\nu)$ is then the same function on (α,β) for all solutions ν of the RSMP on $\mathcal{L}\cdot\mathcal{L}$, which implies that the solution is unique. □

Remark 5.4 It follows from Lemma 11.7.3 and Theorem 11.7.5 of [8] that if $\sum_{n=0}^{\infty}|\phi_n(z)|^2$ converges for some $z \in \mathbb{C}\setminus\{\mathbb{R}\cup\{\mathbf{i}\}\cup\{-\mathbf{i}\}\}$, then $\sum_{n=0}^{\infty}|\phi_n(z)|^2$ converges for every $z \in \mathbb{C}\setminus\{\mathbb{R}\cup\{\mathbf{i}\}\cup\{-\mathbf{i}\}\}$. Actually the proof shows that if $\sum_{n=0}^{\infty}|\phi_n(z)|^2$ converges for some $z \in \mathbb{C}\setminus\{\mathbb{R}\cup\{\mathbf{i}\}\cup\{-\mathbf{i}\}\}$, then $\sum_{n=0}^{\infty}|\phi_n(z)|^2$ converges for every $z \in \mathbb{C}_\alpha := \mathbb{C}\setminus\{\hat{\mathbb{A}}\cup\{\mathbf{i}\}\cup\{-\mathbf{i}\}\}$ where $\hat{\mathbb{A}}$ is the closure of the set $\mathbb{A}$ of interpolation points α_n. Furthermore it follows from Lemma 11.7.3 and Corollary 11.8.3 of [8] that if $\sum_{n=0}^{\infty}|\phi_n(z)|^2$ diverges for some (or all) $z \in \mathbb{C}\setminus\{\mathbb{R}\cup\{\mathbf{i}\}\cup\{-\mathbf{i}\}\}$, then the rational (Hamburger) moment problem on $\mathcal{L}\cdot\mathcal{L}$ (see Section 1) is determinate. Thus we may conclude from Proposition 5.1that the following stronger version of Theorem 5.3 holds.
Assume that (4.1)–(4.2) hold, that the RSMP on $\mathcal{L}\cdot\mathcal{L}$ is solvable and that

$$\sum_{n=2}^{\infty}|v_n/v_{n-2}|^{1/2} = \infty.$$

Then the rational (Hamburger) moment problem on $\mathcal{L}\cdot\mathcal{L}$ is determinate.

6 A Carleman condition

We shall briefly point out how we can obtain from Theorem 5.3 a sufficient condition of Carleman type for the determinacy of the RSMP on $\mathcal{L}\cdot\mathcal{L}$. Let μ be any solution of the RSMP on $\mathcal{L}\cdot\mathcal{L}$. We recall the definition (2.5), which means that we may write the moments $\mu_{m,n}$ as

$$\mu_{m,n} = \int_0^{\infty} b_m(t)b_n(t)\mathrm{d}\mu(t), \quad m,n = 0,1,2,\ldots \tag{6.1}$$

Lemma 6.1 *For $n = 0,1,2,\ldots$ we have*

$$|v_n| \geq \frac{1}{\sqrt{\mu_{n,n}}}. \tag{6.2}$$

Proof. We find from the orthogonality of $\{\phi_n\}$ and the representation (3.5) that we may write

$$1 = v_n\int_0^{\infty} b_n(t)\phi_n(t)\mathrm{d}\mu(t). \tag{6.3}$$

By using Schwartz' inequality in the integral we obtain

$$1 \leq |v_n|\sqrt{\mu_{n,n}}$$

which is (6.2). □

Theorem 6.2 *Assume that*

$$\sum_{n=0}^{\infty} \frac{1}{(\mu_{n,n})^{\frac{1}{2n}}} = \infty. \tag{6.4}$$

Then the RSMP on $\mathcal{L} \cdot \mathcal{L}$ is determinate.

Proof. Carleman's inequality for non-negative numbers $a_1, a_2, \ldots, a_k, \ldots$ may be written

$$\sum_{k=1}^{\infty} a_k \geq \frac{1}{\mathrm{e}} \sum_{m=1}^{\infty} (a_1 a_2 \cdots a_m)^{\frac{1}{m}} \tag{6.5}$$

(see e.g., [16]). Applying this inequality to the situation $a_k = |v_{2k}/v_{2k-2}|^{1/2}$ we obtain

$$\sum_{k=1}^{\infty} \left| \frac{v_{2k}}{v_{2k-2}} \right|^{1/2} \geq \frac{1}{\mathrm{e}} \sum_{m=1}^{\infty} |v_{2m}|^{\frac{1}{2m}}. \tag{6.6}$$

Similarly by applying the inequality to the situation $a_k = |v_{2k-1}/v_{2k-3}|^{1/2}$ (with $v_{-1} = 1$) we obtain (with m replaced by $m+1$)

$$\sum_{k=1}^{\infty} \left| \frac{v_{2k-1}}{v_{2k-3}} \right|^{1/2} \geq \frac{1}{\mathrm{e}} \sum_{m=1}^{\infty} |v_{2m+1}|^{\frac{1}{2m+2}}. \tag{6.7}$$

If (6.4) holds, then at least one of the series

$$\sum_{m=1}^{\infty} \frac{1}{(\mu_{2m,2m})^{\frac{1}{4m}}} \quad \text{and} \quad \sum_{m=1}^{\infty} \frac{1}{(\mu_{2m+1,2m+1})^{\frac{1}{4m+4}}}$$

diverges. (Note that $(\mu_{2m+1,2m+1})^{\frac{1}{4m+2}} > (\mu_{2m+1,2m+1})^{\frac{1}{4m+4}}$ if $\mu_{2m+1,2m+1} > 1$. If $\mu_{2m+1,2m+1} \leq 1$ for infinitely many m, then the series

$$\sum_{m=1}^{\infty} \frac{1}{(\mu_{2m+1,2m+1})^{\frac{1}{4m+4}}}$$

trivially diverges.) It follows from (6.2) and (6.6)–(6.7) that also at least one of the series

$$\sum_{k=1}^{\infty} \left| \frac{v_{2k}}{v_{2k-2}} \right|^{1/2} \quad \text{and} \quad \sum_{k=1}^{\infty} \left| \frac{v_{2k+1}}{v_{2k-1}} \right|^{1/2}$$

diverges, which means that the series $\sum_{n=1}^{\infty} \left| \frac{v_n}{v_{n-2}} \right|^{1/2}$ diverges. The result now follows from Theorem 5.3. □

Remark 6.3 In the limiting situation $\alpha_{2m} = -\infty$, $\alpha_{2m+1} = 0$ for all m, the moments $\mu_{n,n}$ become $\mu_{2m-1,2m-1} = c_{-2m}$ and $\mu_{2m,2m} = c_{2m}$. The condition $\sum_{n=0}^{\infty} \frac{1}{(\mu_{n,n})^{\frac{1}{2n}}}$ becomes

$$\sum_{m=0}^{\infty} \frac{1}{(c_{-2m})^{\frac{1}{4m-2}}} + \sum_{m=0}^{\infty} \frac{1}{(c_{2m})^{\frac{1}{4m}}} = \infty.$$

(See Section 1.)

Remark 6.4 Taking into account the Remark 5.4, we find that Theorem 6.2 can be strengthened as follows.
Assume that (4.1)–(4.2) hold, that the RSMP on $\mathcal{L} \cdot \mathcal{L}$ is solvable and that

$$\sum_{n=0}^{\infty} \frac{1}{(\mu_{n,n})^{\frac{1}{2n}}} = \infty.$$

Then the rational (Hamburger) moment problem on $\mathcal{L} \cdot \mathcal{L}$ is determinate.

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