1. The fast wavelet transform (FWT)

Suppose we are given a function on $[0,1]$:

$$f(t) = \sum_{k=0}^{2^J-1} s_{J,k}\varphi_{J,k}(t) \in V_J$$

Actually, this is a continuous representation of the discrete data set $\{s_{J,k}\}_{k=0\ldots 2^J-1}$. We want the wavelet decomposition:

$$f(t) = \sum_{j=L}^{J-1}\sum_{k=0}^{2^j-1} w_{j,k}\psi_{j,k}(t) + \sum_{k=0}^{2^L-1} s_{L,k}\varphi_{L,k}(t)$$

Obviously, in the orthonormal case:

$$w_{j,k} = \int_{-\infty}^{\infty} f(t)\overline{\psi}_{j,k}(t)dt$$

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But we can compute this recursively:
Because:

$$\psi_{j,k}(t) = \sum_{m\in\mathbb{Z}} g_m \, \varphi_{j+1,2k+m}(t) = \sum_{l\in\mathbb{Z}} g_{l-2k} \, \varphi_{j+1,l}(t)$$

we can write

$$\begin{aligned} w_{j,k} &= \int_{-\infty}^{\infty} f(t)\overline{\psi}_{j,k}(t)dt \\ &= \int_{-\infty}^{\infty} f(t) \sum_{l\in\mathbb{Z}} \overline{g}_{l-2k} \, \overline{\varphi}_{j+1,l}(t)dt \\ &= \sum_{l\in\mathbb{Z}} \overline{g}_{l-2k} \int_{-\infty}^{\infty} f(t) \, \overline{\varphi}_{j+1,l}(t)dt \end{aligned}$$

$$\boxed{w_{j,k} = \sum_{l\in\mathbb{Z}} \overline{g}_{l-2k} s_{j+1,l}}$$

and similarly we derive that:

$$\boxed{s_{j,k} = \sum_{l\in\mathbb{Z}} \overline{h}_{l-2k} s_{j+1,l}}$$

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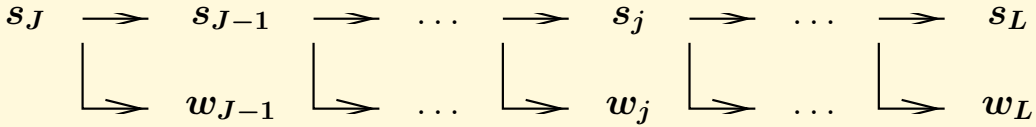
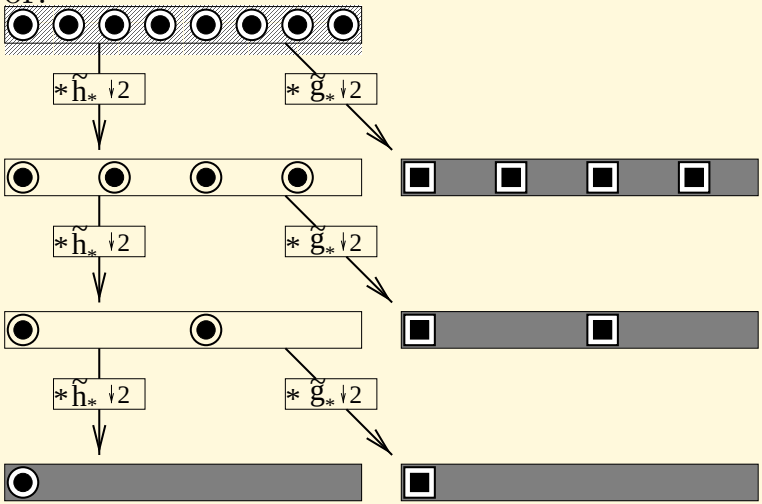
We have the following scheme:



or:



This is a recursive filterbank!!!
convolution with $\hat{h}_k = \bar{h}_{-k}$ and $\hat{g}_k = \bar{g}_{-k}$, and subsampling
Total complexity $= \mathcal{O}(N)$, where $N = 2^J = \#\{s_{J,k}\}$ is the input data size

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$$s_{j,k} = \sum_{l \in \mathbb{Z}} \overline{h}_{l-2k} s_{j+1,l}$$
$$w_{j,k} = \sum_{l \in \mathbb{Z}} \overline{g}_{l-2k} s_{j+1,l}$$

or

$$\boldsymbol{s}_j = (\downarrow 2)(\hat{\boldsymbol{h}} * \boldsymbol{s}_{j+1})$$
$$\boldsymbol{w}_j = (\downarrow 2)(\hat{\boldsymbol{g}} * \boldsymbol{s}_{j+1})$$

with $\hat{\boldsymbol{h}} = (\bar{h}_{-k})$ and $\hat{\boldsymbol{g}} = (\bar{g}_{-k})$

This is the FAST WAVELET TRANSFORM: it has linear complexity.

remark: The sum in these formulae are over $l \in \mathbb{Z}$. If the input $\boldsymbol{s}_J$ contains only $N = 2^J$ data, we extend it periodically.

note: transfer function of the low-pass filter is

$$\hat{H}(z) = \sum_k \hat{h}_k z^{-k} = \sum_k \bar{h}_{-k} z^{-k} = H_*(z)$$

with $H_*(z) = \overline{H(1/\overline{z})}$. Similarly $\hat{G}(z) = G_*(z)$.

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2. Inverse transform

Since

$$V_{j+1} = V_j \oplus W_j$$

we can write:

$$\varphi_{j+1,k} = \sum_{l \in \mathbb{Z}} a_l \varphi_{j,l} + \sum_{l \in \mathbb{Z}} b_l \psi_{j,l}$$

where:

$$\begin{aligned} a_l &= \langle \varphi_{j+1,k}, \varphi_{j,l} \rangle = \langle \varphi_{j+1,k}, \sum_{m \in \mathbb{Z}} h_{m-2l}\, \varphi_{j+1,m} \rangle \\ &= \sum_{m \in \mathbb{Z}} \overline{h}_{m-2l} \langle \varphi_{j+1,k}, \varphi_{j+1,m} \rangle = \overline{h}_{k-2l} \end{aligned}$$

and similarly $b_l = \overline{g}_{k-2l}$. So:

$$\varphi_{j+1,k} = \sum_{l \in \mathbb{Z}} \overline{h}_{k-2l} \varphi_{j,l} + \sum_{l \in \mathbb{Z}} \overline{g}_{k-2l} \psi_{j,l}$$

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This implies:

$$\begin{aligned} s_{j+1,k} &= \langle f, \varphi_{j+1,k} \rangle \\ &= \sum_{l \in \mathbb{Z}} h_{k-2l} \langle f, \varphi_{j,l} \rangle + \sum_{l \in \mathbb{Z}} g_{k-2l} \langle f, \psi_{j,l} \rangle \end{aligned}$$

$$\boxed{s_{j+1,k} = \sum_{l \in \mathbb{Z}} h_{k-2l} s_{j,l} + \sum_{l \in \mathbb{Z}} g_{k-2l} w_{j,l}}$$

Or, for short:

$$\boxed{\boldsymbol{s_{j+1}} = \boldsymbol{h} * (\uparrow 2)\boldsymbol{s_j} + \boldsymbol{g} * (\uparrow 2)\boldsymbol{w_j}}$$

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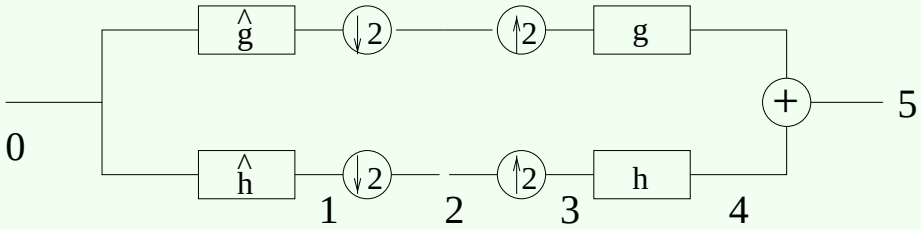
3. Orthogonal Filter Banks

We have found that an orthogonal wavelet transform is a filter-bank algorithm:



with:

$$\hat{g}_k = \overline{g}_{-k} \quad \text{or} \quad \hat{\mathsf{G}}(\omega) = \overline{\mathsf{G}(\omega)} \text{ or } \hat{G}(e^{i\omega}) = G_*(e^{i\omega})$$
$$\hat{h}_k = \overline{h}_{-k} \quad \text{or} \quad \hat{\mathsf{H}}(\omega) = \overline{\mathsf{H}(\omega)} \text{ or } \hat{H}(e^{i\omega}) = H_*(e^{i\omega})$$

Note that for real coefficients:

$$\overline{\mathsf{G}(\omega)} = \mathsf{G}(-\omega), \qquad \overline{\mathsf{H}(\omega)} = \mathsf{H}(-\omega)$$

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4. Perfect reconstruction

How about PR?

Recall perfect reconstruction (PR) says:

$$\begin{aligned} \hat{\mathsf{H}}(\omega)\mathsf{H}(\omega) + \hat{\mathsf{G}}(\omega)\mathsf{G}(\omega) &= 2 \\ \hat{\mathsf{H}}(\omega+\pi)\mathsf{H}(\omega) + \hat{\mathsf{G}}(\omega+\pi)\mathsf{G}(\omega) &= 0 \end{aligned}$$

In this case:

$$\begin{aligned} |\mathsf{H}(\omega)|^2 + |\mathsf{G}(\omega)|^2 &= 2 \\ \overline{\mathsf{H}(\omega+\pi)}\mathsf{H}(\omega) + \overline{\mathsf{G}(\omega+\pi)}\mathsf{G}(\omega) &= 0 \end{aligned}$$

Now if we have an orthogonal *wavelets*, then:

$$g_k = (-1)^k \overline{h}_{1-k} \quad \text{or} \quad \mathsf{G}(\omega) = -\mathrm{e}^{-i\omega}\overline{\mathsf{H}(\omega+\pi)}$$

Then we have:

$$\overline{G(\omega+\pi)} = -\overline{\mathrm{e}^{-i(\omega+\pi)}}\mathsf{H}(\omega+2\pi) = \mathrm{e}^{i\omega}\mathsf{H}(\omega)$$

and so the second PR-equation is automatically satisfied.

The first PR-equation reduces to:

$$|\mathsf{H}(\omega)|^2 + |\mathsf{H}(\omega+\pi)|^2 = 2$$

which was a property for orthogonal *scaling functions*.

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5. Analysis in the Z-domain

The discussion for bi-orthogonal wavelets will illustrate another way to go from (bi-)orthogonality to PR.

Recall: $\mathsf{X}(\omega) = X(z)|_{z=e^{i\omega}}$

PR-conditions in $\mathcal{Z}$-domain (exercise):

$$\begin{aligned} \hat{H}(z)H(z) + \hat{G}(z)G(z) &= 2 \\ \hat{H}(-z)H(z) + \hat{G}(-z)G(z) &= 0 \end{aligned}$$

For orthogonal filterbanks:

$$\hat{H}(z) = H_*(z) \text{ and } \hat{G}(z) = G_*(z)$$

where $X_*(z) = \overline{X(1/\overline{z})}$ is called the *para-hermitian conjugate* function of $X(z)$.

Perfect reconstruction now says:

$$\begin{aligned} H_*(z)H(z) + G_*(z)G(z) &= 2 \\ H_*(-z)H(z) + G_*(-z)G(z) &= 0 \end{aligned}$$

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In one matrix:

$$\begin{bmatrix} H(z) & G(z) \end{bmatrix} \cdot \begin{bmatrix} H_*(z) & H_*(-z) \\ G_*(z) & G_*(-z) \end{bmatrix} = \begin{bmatrix} 2 & 0 \end{bmatrix}$$

This is equivalent to the following, redundant equation:

$$\begin{bmatrix} H(z) & G(z) \\ H(-z) & G(-z) \end{bmatrix} \cdot \begin{bmatrix} H_*(z) & H_*(-z) \\ G_*(z) & G_*(-z) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

(The second row = the first with $z \leftrightarrow -z$).
Define the *modulation matrix*

$$M(z) = \begin{bmatrix} H(z) & G(z) \\ H(-z) & G(-z) \end{bmatrix}$$

and *para-hermitian conjugate* of a matrix-function

$$M_*(z) = \overline{M(1/\overline{z})}^T$$

Then we have:

$$M^T(z) M_*^T(z) = 2I$$

$M(z)/\sqrt{2}$ is called a *para-unitary matrix*

Para-unitary modulation matrices correspond to orthogonal filter banks

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6. The non-orthogonal case

Until now, we mainly worked with orthogonal bases.
This is a restrictive requirement.

Theorem: The only orthogonal basis with a symmetric, compactly supported father-function is the Haar-basis.

Proof: A symmetric father-function has symmetric coefficients in the dilation equation.
Now suppose
$\boldsymbol{h} = [\ldots, a_n, \ldots, a_1, a_0, a_0, a_1, \ldots, a_n, \ldots]$
ortogonality implies even shift orthogonality for $\boldsymbol{h}$:

$$\sum_{k\in\mathbb{Z}} h_k h_{k+2l} = 0, \qquad \text{if } l \neq 0$$

For $l = 2n$, this is:

$$a_n a_{n-1} + a_{n-1} a_n = 0;$$

For $l = 2n - 2$:

$$a_n a_{n-3} + a_{n-1} a_{n-2} + a_{n-2} a_{n-1} + a_{n-3} a_n = 0$$

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And so on. Solving these n equations with $n+1$ unknown variables would leave only one variable not equal to zero, so the only possible sequences are:

$$[\ldots, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, \ldots]$$

Among these possiblities only the Haar filter leads to convergence in the solution of the dilation equation.

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7. Bi-orthogonality

From filter banks we know that there is no reason why *decomposition* filters $\hat{\boldsymbol{g}}$ and $\hat{\boldsymbol{h}}$ and *reconstruction* filters $\boldsymbol{g}$ and $\boldsymbol{h}$ should satisfy:

$$\hat{h}_k = \bar{h}_{-k} \text{ and } \hat{g}_k = \bar{g}_{-k}$$

Now we relax this condition. We set

$$\tilde{h}_k = \bar{\hat{h}}_{-k}; \tilde{g}_k = \bar{\hat{g}}_{-k} \Leftrightarrow \tilde{H}(z) = \hat{H}_*(z);\ \tilde{G}(z) = \hat{G}_*(z)$$

and try to satisfy the PR-equations: (z-domain)

$$\begin{aligned} \tilde{H}_*(z)H(z) + \tilde{G}_*(z)G(z) &= 2 \\ \tilde{H}_*(-z)H(z) + \tilde{G}_*(-z)G(z) &= 0 \end{aligned}$$

Choose:

$$\begin{aligned} \bar{g}_k = (-1)^k \tilde{h}_{1-k} \quad &\text{or} \quad G_*(z) = -z\tilde{H}(-z) \\ \tilde{g}_k = (-1)^k \bar{h}_{1-k} \quad &\text{or} \quad \tilde{G}_*(z) = -zH(-z) \end{aligned}$$

then the second PR-condition is automatically satisfied.

The first reduces to:

$$\tilde{H}_*(z)H(z) + H(-z)\tilde{H}_*(-z) = 2$$

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In time-domain, this is: (recall multiplication $\leftrightarrow$ convolution)

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \tilde{h}_{-k}\bar{h}_{m-k} + \sum_{k\in\mathbb{Z}} (-1)^k \tilde{h}_{-k}(-1)^{m-k}\bar{h}_{m-k} &= 2\delta_{m,0} \\ \sum_{l\in\mathbb{Z}} \tilde{h}_l\bar{h}_{m+l} + \sum_{l\in\mathbb{Z}} (-1)^m \tilde{h}_l\bar{h}_{m+l} &= 2\delta_{m,0}\end{aligned}$$

For odd m, this reduces to $0 = 0$.
For even $m = 2n$, we have double shift orthogonality:

$$\sum_{l\in\mathbb{Z}} \tilde{h}_l\bar{h}_{l+2n} = \delta_{n,0}$$

Further suppose there exists functions $\varphi, \psi, \tilde{\varphi}, \tilde{\psi}$ such that:

$$\begin{aligned}\varphi(t) &= \sqrt{2}\sum_{k\in\mathbb{Z}} h_k\varphi(2t-k) \\ \psi(t) &= \sqrt{2}\sum_{k\in\mathbb{Z}} g_k\varphi(2t-k) \\ \tilde{\varphi}(t) &= \sqrt{2}\sum_{k\in\mathbb{Z}} \tilde{h}_k\tilde{\varphi}(2t-k) \\ \tilde{\psi}(t) &= \sqrt{2}\sum_{k\in\mathbb{Z}} \tilde{g}_k\tilde{\phi}(2t-k)\end{aligned}$$

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Then these functions constitute the basis for two multi-resolution analyses V_j and $\tilde{V}_j$:

$$V_{j+1} = V_j + W_j \quad \text{and} \quad \tilde{V}_{j+1} = \tilde{V}_j + \tilde{W}_j$$
$$V_j \perp \tilde{W}_j \quad \text{and} \quad \tilde{V}_j \perp W_j$$

From this, it follows immediately:

$$W_j \perp \tilde{W}_i \text{ for } i \neq j$$

which means for the basisfunctions:

$$\langle \tilde{\psi}(t), \psi(t-l) \rangle = \delta_l$$

For the father functions

$$\langle \tilde{\varphi}(t), \varphi(t-l) \rangle = \delta_l$$

follows from the double shift (bi)orthogonality:

$$\sum_{l \in \mathbb{Z}} \tilde{h}_l \bar{h}_{l+2n} = \delta_{n,0}$$

V_j is the *primal* and $\tilde{V}_j$ is the *dual* MRA.

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8. The other way round

In the previous slides we examined how a PR-non-orthogonal filterbank with a certain choice of the filters *could* lead to bi-orthogonal wavelets.
Now, we *impose* the condition of biorthogonality:

$$V_j \perp \tilde{W}_j \text{ and } \tilde{V}_j \perp W_j$$

which, as before, implies:

$$W_j \perp \tilde{W}_i \text{ for } i \neq j$$

This is equivalent to orthogonality of the basis functions:

$$\langle \tilde{\varphi}(t), \psi(t-l) \rangle = 0 \text{ and } \langle \tilde{\psi}(t), \varphi(t-l) \rangle = 0$$

and

$$\langle \tilde{\psi}(t), \psi(t-l) \rangle = \delta_l$$

Furthermore, we impose that the father functions are bi-orthogonal:

$$\langle \varphi(t-l), \tilde{\varphi}(t) \rangle = \delta_l$$

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Use of dilation- and wavelet-equations results in

$$\sum_{k\in\mathbb{Z}} \tilde{h}_k \overline{h}_{k+2l} = \delta_l \quad \text{and} \quad \sum_{k\in\mathbb{Z}} \tilde{h}_k \overline{g}_{k+2l} = 0$$

$$\sum_{k\in\mathbb{Z}} \tilde{g}_k \overline{g}_{k+2l} = \delta_l \quad \text{and} \quad \sum_{k\in\mathbb{Z}} \tilde{g}_k \overline{h}_{k+2l} = 0$$

In $\mathcal{Z}$-domain, this corresponds to (exercise!):

$$\begin{aligned} H_*(z)\tilde{H}(z) + H_*(-z)\tilde{H}(-z) &= 2 \\ G_*(z)\tilde{H}(z) + G_*(-z)\tilde{H}(-z) &= 0 \\ H_*(z)\tilde{G}(z) + H_*(-z)\tilde{G}(-z) &= 0 \\ G_*(z)\tilde{G}(z) + G_*(-z)\tilde{G}(-z) &= 2 \end{aligned}$$

In matrix notation:

$$\tilde{M}(z)^T M_*(z)^T = 2I$$

with:

$$M(z) = \begin{bmatrix} H(z) & G(z) \\ H(-z) & G(-z) \end{bmatrix}; \ \& \ \tilde{M}(z) = \begin{bmatrix} \tilde{H}(z) & \tilde{G}(z) \\ \tilde{H}(-z) & \tilde{G}(-z) \end{bmatrix}$$

These are the PR-conditions!!!

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9. Smoothness

definition: A MRA has m *vanishing moments* if

$$t^l \in V_0, l = 0, \ldots, m-1.$$

This implies immediately that $t^l \in V_j, \forall j \in \mathbb{Z}$.

We can write:

$$\begin{aligned} t^l &= \sum_{k\in\mathbb{Z}} a_k \varphi(t-k) \\ \langle \tilde{\psi}, t^l \rangle &= \sum_{k\in\mathbb{Z}} a_k \langle \tilde{\psi}(t), \varphi(t-k) \rangle = 0 \end{aligned}$$

From

$$\int_{-\infty}^{\infty} t^l \tilde{\psi}(t) dt = 0, \ \forall l = 0, \ldots, m-1$$

we go to Fourier domain:

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If

$$\tilde{\Psi}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\psi}(t) \mathrm{e}^{-i\omega t} dt$$

then:

$$\tilde{\Psi}'(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\psi}(t)(-it) \mathrm{e}^{-i\omega t} dt$$

and:

$$\tilde{\Psi}^{(l)}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\psi}(t)(-i)^l t^l \mathrm{e}^{-i\omega t} dt$$

So

$$\int_{-\infty}^{\infty} t^l \tilde{\psi}(t) dt = 0 \quad \Leftrightarrow \quad \tilde{\Psi}^{(l)}(0) = 0, \ \forall l = 0, \ldots, m-1$$

The wavelet-equation

$$\tilde{\psi}(t) = \sqrt{2} \sum_{k \in \mathbb{Z}} \tilde{g}_k \tilde{\varphi}(2t - k)$$

in Fourier-domain:

$$\tilde{\Psi}(\omega) = \frac{1}{\sqrt{2}} \tilde{\mathsf{G}}(\frac{\omega}{2}) \tilde{\Phi}(\frac{\omega}{2})$$

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Now, since $\tilde{\Psi}(0) = 0$ and $\tilde{\Phi}(0) \neq 0$, we have:

$$\tilde{\Psi}(0) = \frac{1}{\sqrt{2}} \tilde{\mathsf{G}}(0)\tilde{\Phi}(0) \quad \Rightarrow \quad \tilde{\mathsf{G}}(0) = 0$$

Furthermore, we have:

$$\begin{aligned} \tilde{\Psi}'(\omega) &= \frac{1}{\sqrt{2}} \left[\frac{1}{2} \tilde{\mathsf{G}}'(\frac{\omega}{2}) \tilde{\Phi}(\frac{\omega}{2}) + \frac{1}{2} \tilde{\mathsf{G}}(\frac{\omega}{2}) \tilde{\Phi}'(\frac{\omega}{2}) \right] \\ \Downarrow & \\ \tilde{\Psi}'(0) &= \frac{1}{\sqrt{2}} \left[\frac{1}{2} \tilde{\mathsf{G}}'(0) \tilde{\Phi}(0) + \frac{1}{2} \tilde{\mathsf{G}}(0) \tilde{\Phi}'(0) \right] \end{aligned}$$

So, $\tilde{\Psi}'(0) = 0 = \tilde{\Psi}(0)$ implies:

$$\tilde{\mathsf{G}}(0) = 0 \text{ and } \tilde{\mathsf{G}}'(0) = 0$$

By induction: l vanishing moments imply:

$$\tilde{\mathsf{G}}^{(l)}(0) = 0, \forall l = 1, \ldots, m-1$$

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We know:

$$\tilde{\mathsf{G}}(\omega) = -\mathrm{e}^{-i\omega}\overline{\mathsf{H}(\omega+\pi)}$$

So

$$\tilde{\mathsf{G}}(0) = 0 \Leftrightarrow \mathsf{H}(\pi) = 0$$

and:

$$\tilde{\mathsf{G}}'(\omega) = i\mathrm{e}^{-i\omega}\overline{\mathsf{H}(\omega+\pi)} - \mathrm{e}^{-i\omega}\overline{\mathsf{H}'(\omega+\pi)}$$

So $\tilde{\mathsf{G}}'(0) = 0 = \tilde{\mathsf{G}}(0)$ implies: $\mathsf{H}(\pi) = 0$ and $\mathsf{H}'(\pi) = 0$
By induction, l vanishing moments imply:

$$\mathsf{H}^{(l)}(\pi) = 0, \forall l = 1, \ldots, m-1$$

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10. Projection

Call

$$f_j(t) = \mathcal{P}_j f(t) = \sum_{k\in\mathbb{Z}} \langle \tilde{\varphi}_{j,k}(t), f(t)\rangle \ \varphi_{j,k}(t)$$

This is a projection of $f(t) \in \mathrm{L}_2$ on $V_j \subset \mathrm{L}_2$.
If $\tilde{\varphi}_{j,k} = \varphi_{j,k}$, then we have orthogonal scaling functions, and this is the *orthogonal* projection on V_j.

Theorem: Suppose a function $f(t)$ in L_2 with a square integrable mth order derivative, and suppose $\mathsf{H}(\omega)$ has m zeros in 0, then:

$$\|f(t) - f_j(t)\| \leq C 2^{-jm} \|f^{(m)}(t)\|$$

Piecewise smooth functions:

A wavelet analysis is especially useful for piecewise smooth functions

This is because singularities (discontinuities) only have local impact: wavelets have compact support ($\leftrightarrow$ sine/cosine). Only a few of them "meet" the singularity.

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11. Sample values as coefficients

In practice we are given a finite set of *samples*, not wavelet coefficients!
The input for the filterbank = *wavelet coefficients*, not samples.
Using samples = wavelet crime.
Transforming samples in wavelet coefficients = extra computation.
If not, only blurred approximation.

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12. Overview of wavelet design properties

1. orthogonality
2. compact support
   the support is determined by filter length
3. rational coefficients
   for efficient computer implementation
4. symmetry
   suppose we have two mirror images, it is reasonable that both have a mirrored wavelet decomposition. However, this is only true for symmetric basis functions. This assumes symmetric filters (generalised linear phase)
5. smoothness of primal wavelet
   for smooth reconstruction from compressed data

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6. dual vanishing moments

   - determines convergence rate of MRA projections
   - necessary for singularity detection
   - smoothness of dual wavelet

7. interpolation

   if $\varphi(k) = \delta_k$, it is easy to find the interpolating approximation, i.e., the function $f_J^{[i]}(t)$ that interpolates $f_k = f(k2^{-J})$. It is simply:

$$f_J^{[i]}(t) = \sum_{k \in \mathbb{Z}} f_k \varphi(2^J t - k)$$

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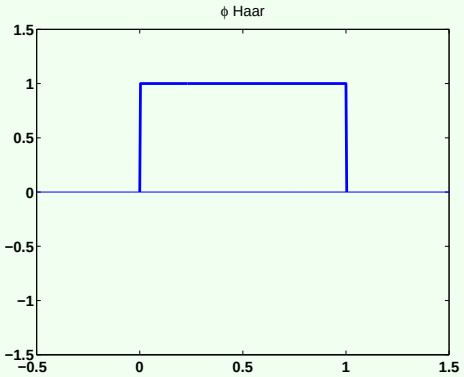
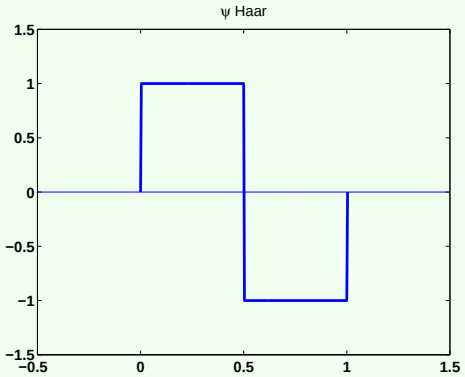
13. Types of wavelets

13.1. The Haar wavelet



Haar scaling function

$\boldsymbol{h} = (1/\sqrt{2}, 1/\sqrt{2})$



Haar wavelet function

$\boldsymbol{g} = (1/\sqrt{2}, -1/\sqrt{2})$

- orthogonal
- compact support
- scaling function is symmetric, wavelets are anti-symmetric

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it is the only compactly supported, symmetric and orthogonal wavelet basis.

- one vanishing moment (the absolute minimum)
- If we define the scaling function as:

$$\varphi(t) = \chi_{[0,1)}$$

then this is an interpolating basis

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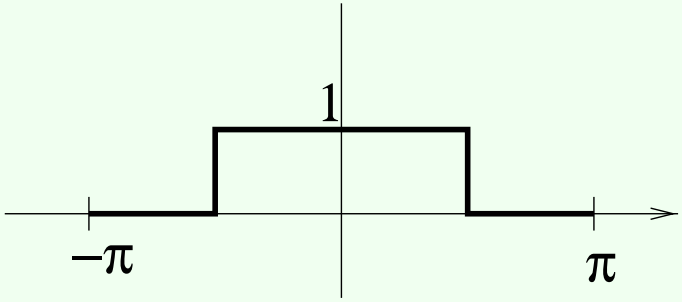
13.2. The shannon/sinc wavelet

We know that the ideal low-pass filter



$\mathsf{H}(\omega) = \sqrt{(2)} \times$ corresponds to the shannon-scaling function:

$$\varphi(t) = \frac{\sin \pi t}{\pi t}$$

We have:

$$\varphi(t) = \sqrt{2} \sum_{k \in \mathbb{Z}} h_k \varphi(2t - k)$$

The ideal high-pass filter defines the wavelet:

$$\psi(t) = \sqrt{2} \sum_{k \in \mathbb{Z}} g_k \varphi(2t - k)$$

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Remember that:

$$\begin{aligned} h_0 &= \sqrt{2}/2 = g_0 \\ h_{2n} &= 0 = g_{2n} \\ h_{2n+1} &= (-1)^n \sqrt{2}/(2n+1)\pi \\ g_{2n+1} &= (-1)^{n+1} \sqrt{2}/(2n+1)\pi \end{aligned}$$

This means that

$$\begin{aligned} \psi(t) &= \sqrt{2} \sum_{k \in \mathbb{Z}} g_k \varphi(2t-k) = \sqrt{2} \sum_{k \in \mathbb{Z}_0} -h_k \varphi(2t-k) + \sqrt{2}\, h_0 \varphi(2t) \\ &= \sqrt{2} \sum_{k \in \mathbb{Z}} -h_k \varphi(2t-k) + \sqrt{2}\, 2h_0 \varphi(2t) \\ &= -\varphi(t) + \sqrt{2}\, 2h_0 \varphi(2t) = -\varphi(t) + 2\varphi(2t) \\ &= 2\, \frac{\sin 2\pi t}{2\pi t} - \frac{\sin \pi t}{\pi t} = \frac{\sin 2\pi t - \sin \pi t}{\pi t} \end{aligned}$$

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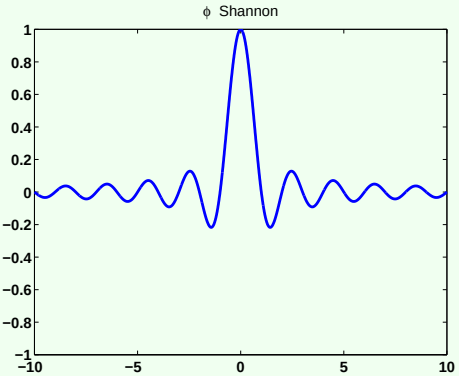
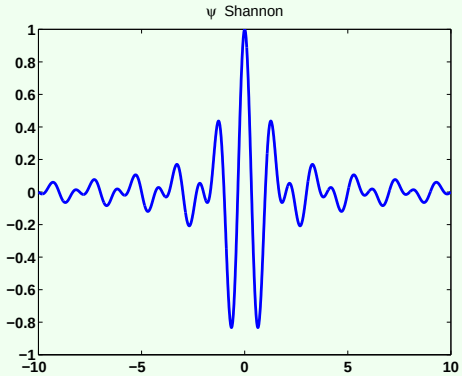


Sinc scaling function



Sinc wavelet function

- orthogonal
- infinite number of vanishing moments
- interpolating scaling function
- infinite support and very slow decay IIR-filters (non-causal)

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13.3. Meyer wavelets

"Smoothing" the frequency plot of the sinc wavelet and scaling function yields Meyer wavelets. (for details, see Daubechies, 10 Lectures on Wavelets)

- orthogonal
- band-limited frequency characteristics
- faster decay than sinc, but still infinite support IIR-filters.

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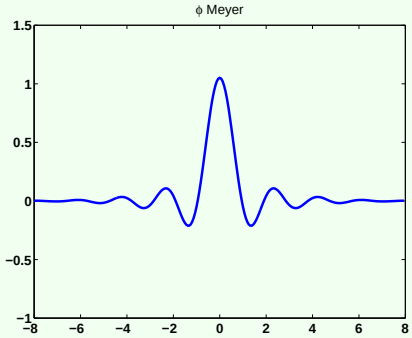
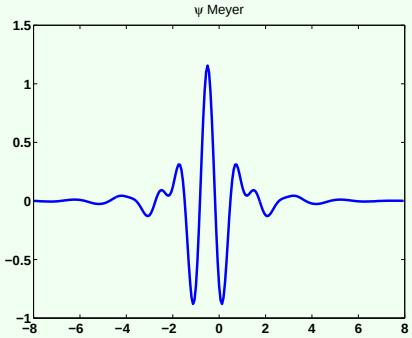
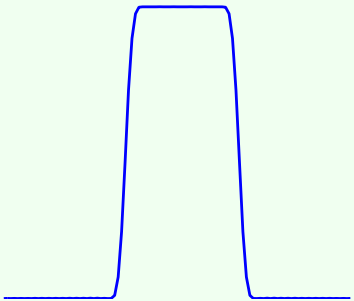


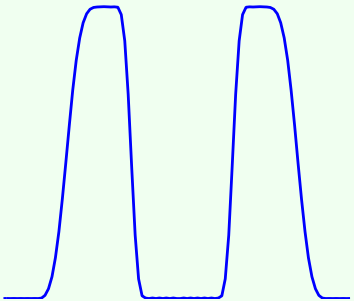
Meyer scaling function



Meyer wavelet function



Fourier transform scaling function



Fourier transform wavelet function

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13.4. Daubechies' maxflat wavelets

We look for orthogonal wavelets with compact support. orthogonality requires:

$$\sum_{n\in\mathbb{Z}} h_n h_{n+2k} = \delta_k$$

Suppose FIR-filter: $h_n = 0$ unless $n = 0, \ldots, n^+$, $n^+ = 2p-1$ is odd, i.e. there are $2p$ non-zero coefficients.

Then this condition is non-trivial for

$$k = 0, 1, \ldots, p-1$$

We have p orthogonality conditions, leaving us with p degrees of freedom.

We need one degree of freedom to write:

$$\sum_{k\in\mathbb{Z}} h_k = \sqrt{2}$$

This is a necessary condition for the dilation equation.

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For maxflat wavelets, we use *all* remaining degrees of freedom to maximise the number of vanishing moments:

$$\mathsf{H}^{(i)}(\pi) = 0, \qquad i = 0, \ldots, p$$

The first vanishing moment follows from the orthogonality. It is the partition of unity property.
For instance: $p = 2$,

$$\mathsf{H}(\omega) = \sum_{k=0}^{3} h_k \mathrm{e}^{-i\omega k}, \quad \mathsf{H}'(\omega) = \sum_{k=0}^{3} h_k (-ik) \mathrm{e}^{-i\omega k}$$

So:

$$\mathsf{H}'(\pi) = 0 \Leftrightarrow \sum_{k=0}^{3} (-1)^k k h_k = 0$$

Together with the orthogonality conditions, this gives the following set of equations:

$$\begin{cases} h_0^2 + h_1^2 + h_2^2 + h_3^2 = 1 \\ h_0 h_2 + h_1 h_3 = 0 \\ h_0 + h_1 + h_2 + h_3 = \sqrt{2} \\ -h_1 + 2h_2 - 3h_3 = 0 \end{cases}$$

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with solution:

$$h_0 = \frac{1+\sqrt{3}}{4\sqrt{2}}, \quad h_1 = \frac{3+\sqrt{3}}{4\sqrt{2}}$$

$$h_2 = \frac{3-\sqrt{3}}{4\sqrt{2}}, \quad h_3 = \frac{1-\sqrt{3}}{4\sqrt{2}}$$

The corresponding wavelets are called "Daubechies 2"-wavelets, in general, "Daubechies p" wavelets.
Sometimes, the filter length is used as index, so the filter have computed is refered to as the "Daubechies 4" filter.

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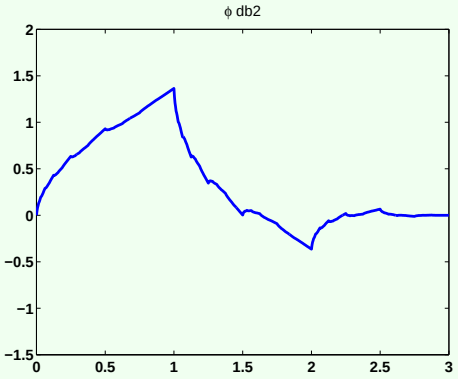
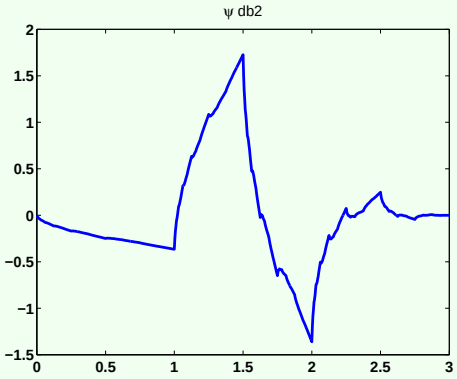


Daubechies 2
scaling function



Daubechies 2
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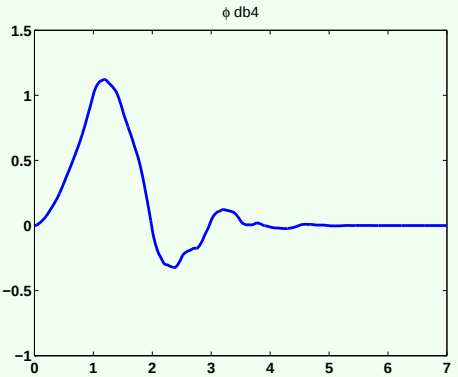
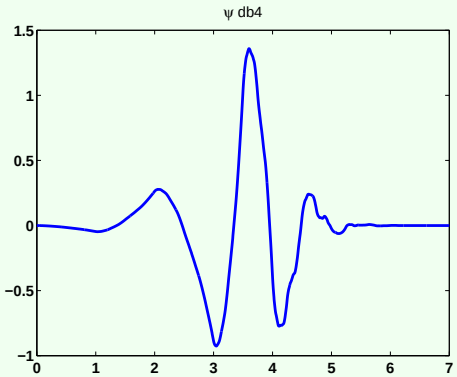
- Daubechies 1 = Haar
- for $p \geq 2$, the functions have no symmetry
- compact support on $[0, p-1]$
- orthogonal
- p vanishing moments

For higher p, the functions get smoother:



Daubechies 4 scaling function



Daubechies 4 wavelet function

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13.5. Symlets (Daubechies)

(for reference only)

For higher p, the "maxflat problem" is mostly solved in z-domain. The solution is not unique. The "classical" Daubechies maxflat filters correspond to a "minimum phase choice",i.e. all zeros of $\mathsf{H}(z)$ are inside the unit-circle. Other choices could lead to more (although not completely) symmetric basis functions. These are called symlets.

Symlets are not symmetric.

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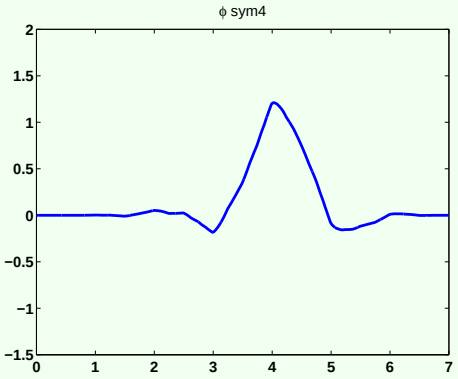
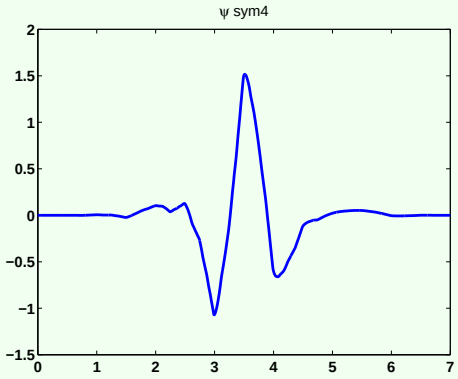


Symlet 4
scaling function



Symlet 4
wavelet function

note: symlet 2 = Daubechies 2.

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13.6. Coiflets (Daubechies)

Suppose we have orthogonal wavelets with p vanishing moments:

$$\int_{-\infty}^{\infty} t^k \psi(t) dt = 0, \qquad k = 0, \ldots, p$$

Of course:

$$\int_{-\infty}^{\infty} \varphi(t) dt = \int_{-\infty}^{\infty} |\varphi(t)|^2 dt = 1$$

Now, we also want:

$$\int_{-\infty}^{\infty} t^k \varphi(t) dt = 0, \qquad k = 1, \ldots, p$$

In that case, we have for a polynomial P of degree at most p that:

$$a_{0,k} = \int_{-\infty}^{\infty} P(t)\varphi(t-k)dt = \int_{-\infty}^{\infty} P(u+k)\varphi(u)du$$

Hence $a_{0,k} = P(0+k) = P(k)$

This means that the projection

$$\hat{f}_J(t) = \sum_{k \in \mathbb{Z}} f(2^{-J}k)\varphi(2^J t - k)$$

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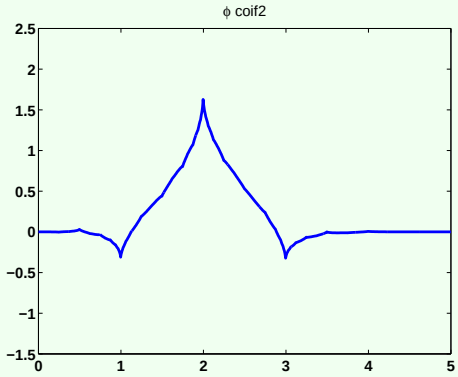
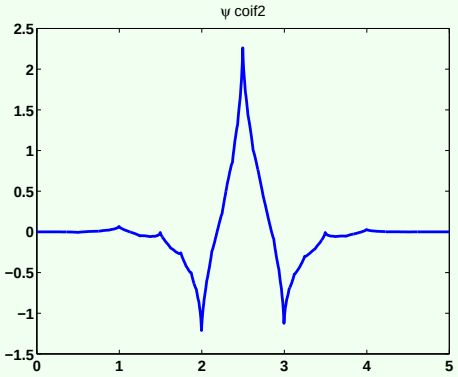
converges as fast as $\mathcal{O}(2^{-Jp})$ to $f(t)$. Remember that in general, this convergence is only $\mathcal{O}(2^{-J})$. These extra conditions require more non-zero coefficients. This causes a larger support.



Coiflet 2
scaling function



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13.7. Intermezzo: spline functions and B-spline basis

A spline of degree n (order $n+1$) and with knots $x_k, k = 0, \ldots, N$ is a function with $n-1$ continuous derivatives. On each interval $[x_{k-1}, x_k], k = 1, \ldots, N$ it is a polynomial of degree n.

A first order spline is a piecewise constant function.
A second order spline (degree 1) is a continuous, piecewise linear function
In general, it is a piecewise polynomial.

One application of these functions is *interpolating*: they are much better than polynomial interpolation, since a high polynomial degree is necessary to match all data points. This polynomial therefore oscillates between the data points. Spline functions are much smoother.

A second application is *approximation*, e.g. least squares approximation.

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Now consider splines with equidistant knots, and suppose

$$x_k = k$$

These functions are called *cardinal splines*
Then it one can prove that the space of cardinal splines of order n is spanned by translates of one father function $\varphi_n(x)$, where

$$\begin{aligned} \varphi_1(x) &= \chi_{[0,1)} \\ \varphi_n(x) &= \chi_{[0,1)} * \varphi_{n-1}(x) \end{aligned}$$

These basis functions are called (cardinal) *B-splines*. For $n = 1$ (first order, degree 0), this reduces to the Haar basis.
For higher order, the basis remains symmetric, which means that it can no longer be orthogonal.

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The recursion in Fourier domain becomes:

$$\Phi_1(\omega) = \frac{1}{\sqrt{2\pi}} \frac{1 - \mathrm{e}^{-i\omega}}{i\omega}$$

$$\Phi_n(\omega) = (2\pi)^{\frac{n-1}{2}} [\Phi_1(\omega)]^n = (2\pi)^{\frac{n}{2}-1} \left(\frac{1 - \mathrm{e}^{-i\omega}}{i\omega} \right)^n$$

It is obvious that:

$$\left(\frac{1 - \mathrm{e}^{-i\omega}}{i\omega} \right)^n = \left(\frac{1 + \mathrm{e}^{-i\omega/2}}{2} \right)^n \left(\frac{1 - \mathrm{e}^{-i\omega/2}}{i\omega/2} \right)^n$$

This is of the form: $\Phi(\omega) = C(\frac{\omega}{2})\Phi(\frac{\omega}{2})$ with

$$C(\omega) = \left(\frac{1 + \mathrm{e}^{-i\omega}}{2} \right)^n = 2^{-n} \sum_{k=0}^{n} \binom{n}{k} \mathrm{e}^{-i\omega k}$$

So, we have:

$$\varphi_n(x) = 2^{-n} \sum_{k=0}^{n} \binom{n}{k} \varphi_n(2x - k)$$

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13.8. Orthogonal spline wavelets (Battle-Lemarié)

Because a B-spline basis $\varphi(x-k)$ is not orthogonal,

$$\mathsf{R}(\omega) = 2\pi \sum_{n\in\mathbb{Z}} |\Phi(\omega+2n\pi)|^2 \neq 1$$

But if we consider:

$$\Phi_\perp(\omega) = \frac{\Phi(\omega)}{\sqrt{\mathsf{R}(\omega)}}$$

then we do have:

$$\mathsf{R}_\perp(\omega) = 2\pi \sum_{n\in\mathbb{Z}} |\Phi_\perp(\omega+2n\pi)|^2 = 1$$

This *orthogonalisation procedure* leads to *Battle-Lemarié*-wavelets. Since

$$\frac{1}{\sqrt{\mathsf{R}(\omega)}}$$

is not a polynomial in $\mathrm{e}^{-i\omega}$, but an infinite series, we get IIR-filters and wavelets with infinite support. However, these have fast decay and they are symmetric.

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13.9. Bi-orthogonal spline wavelets (CDF)(Cohen, Daubechies, Feauveau)

- coefficients are dyadic rationals (up to a $\sqrt{2}$ factor)
- compact support
- symmetric

example 1: CDF 2,2: two primal vanishing moments and two dual vanishing moments

$$\begin{aligned} \boldsymbol{h} &= \frac{1}{\sqrt{2}}\left(\frac{1}{2}, 1, \frac{1}{2}\right) \\ \tilde{\boldsymbol{h}} &= \frac{1}{\sqrt{2}}\left(-\frac{1}{4}, \frac{1}{2}, 1, \frac{1}{2}, -\frac{1}{4}\right) \end{aligned}$$

The synthesis scaling function is called the hat function. It has the interpolating property.

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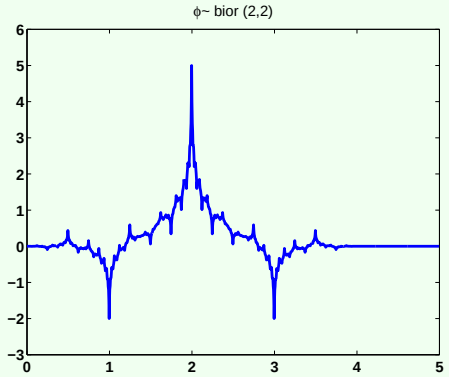
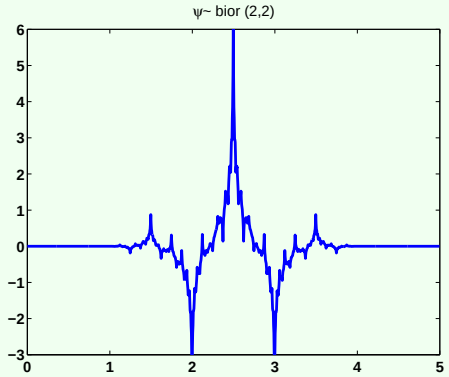
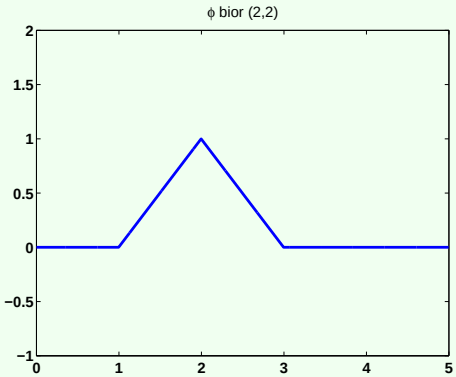


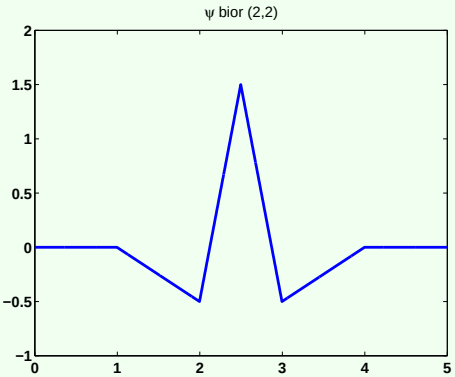
CDF 2,2 $\tilde{\varphi}(x)$
analysis scaling function



CDF 2,2 $\tilde{\psi}(x)$
analysis wavelet function



CDF 2,2 $\varphi(x)$
synthesis scaling function



CDF 2,2 $\psi(x)$
synthesis wavelet function

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14. The Continuous Wavelet Transform

- Historically, the continuous wavelet transform came first.
- it is completely different from the discrete wavelet transform
- it is popular among physicists, whereas the DWT is more common in numerical analysis, signal- and image-processing.

14.1. The continuous wavelet transform (CWT)

Recall the CWT

$$\mathcal{W}_\psi f = S(a,b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(t) \overline{\psi\left(\frac{t-b}{a}\right)} dt$$

This is an *overcomplete* function representation. (from one dimension, we transform into a 2D space!!). We want to be able to reconstruct $f(t)$ from this representation. This is possible if the *admissibility condition* is satisfied:

$$C_\psi = \int_{-\infty}^{\infty} \Psi(\omega) \frac{d\omega}{|\omega|} < \infty$$

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A necessary (and normally also sufficient) condition is: $\Psi(0) = 0$ which means:

$$\int_{-\infty}^{\infty} \psi(t)dt = 0$$

All functions with zero integral are *wavelets*
There are:

- no multi-resolution analysis
- no father/scaling function

All wavelets in the DWT-theory remain wavelets here, but these functions are mostly too irregular for proper use in a CWT-analysis.

The CWT is often used to characterize singularities in functions, and from this it can distinguish between noise and signal (Jaffard, Mallat, Hwang, Zhong)

It can be used to study fractals, etc.

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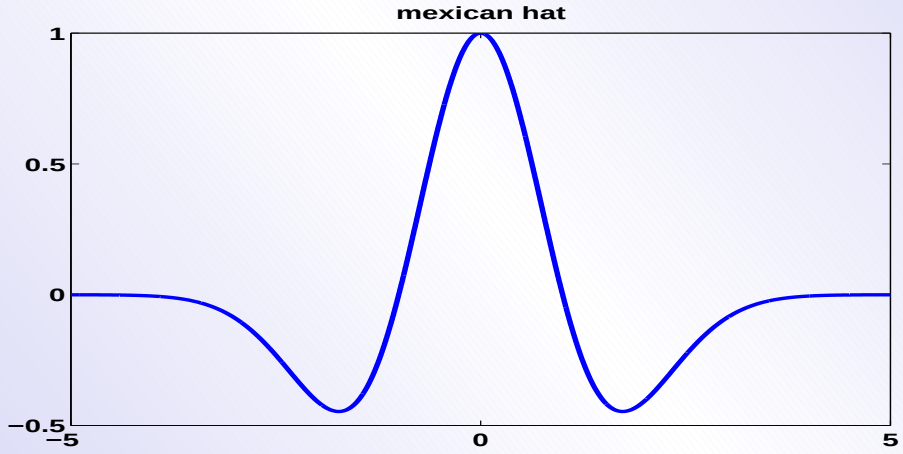
Two functions are popular for CWT-analysis. These functions do *not fit into a MRA*.

1. Mexican hat wavelet

$$\psi(x) = (1 - x^2)\mathrm{e}^{-x^2/2}$$

This is the second derivative of a Gaussian.

$$\Psi(\omega) = \omega^2 \mathrm{e}^{-\omega^2}$$



2. The Morlet wavelet

$$\psi(x) = \mathrm{e}^{i\omega_0 x}\mathrm{e}^{-x^2/2\sigma_0^2}$$

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This function does not satisfy the admissibility condition exactly. It therefore has to be corrected a bit.

14.2. Discretisation of the CWT

Another difference between CWT and DWT:
DWT starts with a discrete set of data and considers a dyadic set of scales.
Discretisation comes first! (Hence Discrete WT!!)

A CWT must be decretised to be implemented on a computer. If $\psi(t)$ fits into a MRA, then a discretisation like in the discrete case is of course possible, but in general, the representation remains redundant.

Such overcomplete function expansions are called *frames*. Loosely spoken, a frame is a family of functions ψ_j, such that a function f can be reconstructed from

$$\langle f(t), \psi_j(t) \rangle$$

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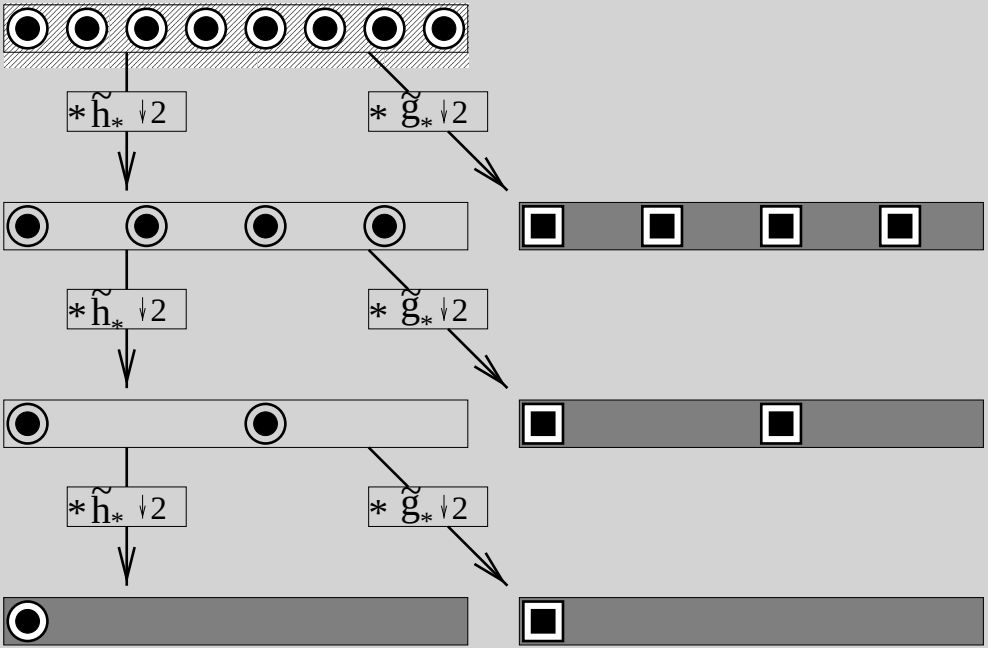
15. Stationary wavelet transform or redundant wavelet transform or a trous algorithm

Suppose we have a FWT:



Because of the subsampling step, we have less coefficients at coarse scales.

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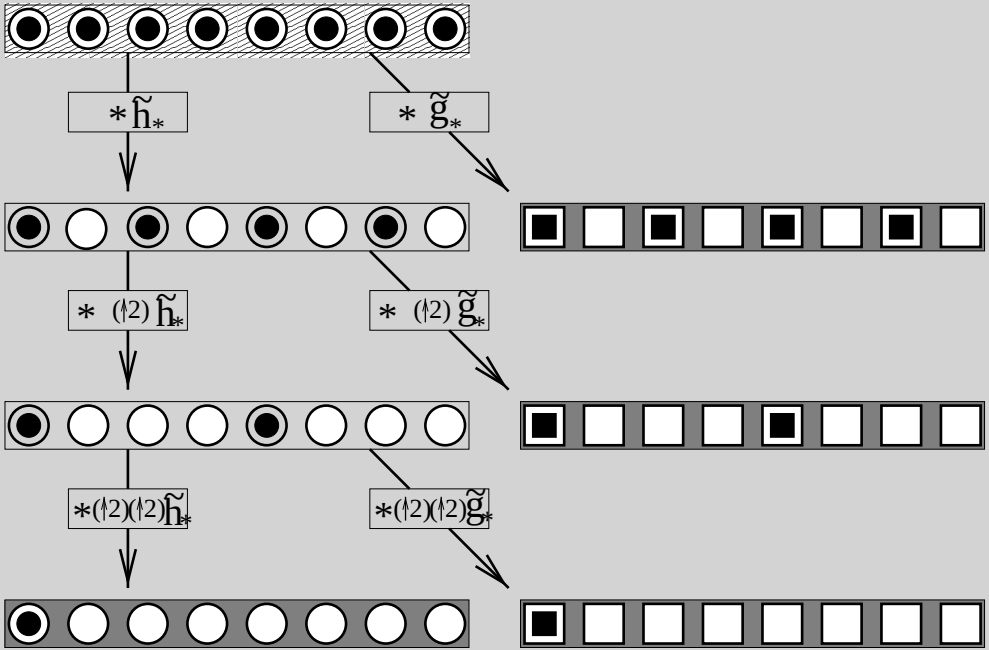
If we omit the subsampling step, we get a redundant wavelet transform. (RWT)



The subsampling is compensated by upsampling the filters.
$\rightarrow$ the transform is consistent with the FWT.

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15.1. Reconstruction

In each step (i.e. scale) the number of data doubles. Hence at each step the input can be reconstructed from the output in 2 independent ways.

15.2. properties

1. the number of coefficients is equal at all scales. It equals the number of input samples.
2. the transform is translation-invariant ($\leftrightarrow$ FWT)
3. the transform is immediately extensible for non-dyadic inputs.
4. This is an overcomplete data representation.
   reconstruction is not unique: if the coefficients are manipulated, the result is probably not the RWT of any function. Averaging the results from different reconstructions has smoothing effect.
5. Computational complexity and storage is $\mathcal{O}(N \log N)$

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15.3. RWT and frames

Suppose:

$$f(t) = \sum_{k \in \mathbb{Z}} s_{J,k} \varphi(2^J - k)$$

then we know:

$$w_{j,k}^{\text{FWT}} = 2^{j/2} \int_{-\infty}^{\infty} f(t) \, \overline{\tilde{\psi}(2^j t - k)} \, dt$$

And similarly:

$$w_{j,k}^{\text{RWT}} = 2^{j/2} \int_{-\infty}^{\infty} f(t) \, \overline{\tilde{\psi}(2^j t - 2^{j-J} k)} \, dt$$

This can be interpreted as a dyadic discretisation of the CWT:

$$(\mathcal{W}_\psi f)(a,b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(t) \, \overline{\tilde{\psi}\left(\frac{t-b}{a}\right)} \, dt$$

with:

$$a_j = 2^{-j} \text{ and } b_k = 2^{-J} k$$

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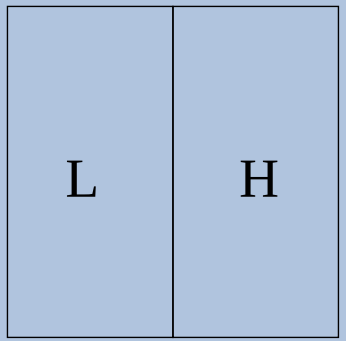
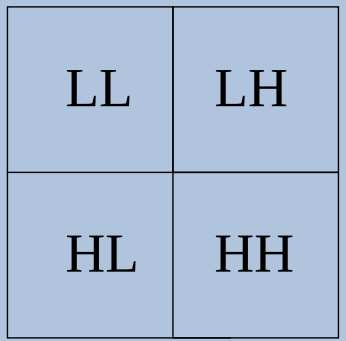
16. 2D Wavelet transform

1. the square wavelet transform

We first perform one step of the transform on all rows:
The left side of the matrix contains downsampled lowpass coefficients of each row, the right contains the highpass coefficients



Next, we apply one step to all columns:



This results in four types of coefficients:

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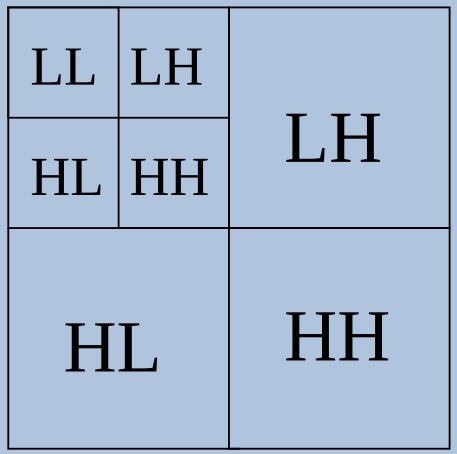
(a) coefficients that result from a convolution with $\boldsymbol{g}$ in both directions (HH) represent diagonal features of the image.

(b) coefficients that result from a convolution with $\boldsymbol{g}$ on the columns after a convolution with $\boldsymbol{h}$ on the rows (HL) correspond to horizontal structures.

(c) coefficients from highpass filtering on the rows, followed by lowpass filtering of the columns (LH) reflect vertical information.

(d) the coefficients from lowpass filtering in both directions are further processed in the next step.



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At each level, we have three *components* or *subbands*. If we start with:

$$f(t) = \sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} s_{J,k,l}\varphi_{J,k}(x)\varphi_{J,l}(y)$$

we end up with:

$$\begin{aligned} f(t) \;=\; & \sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} s_{L,k,l}\varphi_{L,k}(x)\varphi_{L,l}(y) \quad \in V_L \otimes V_L \\ + \sum_{j=L}^{J-1} \Bigg[& \sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} w^{\mathrm{HH}}_{j,k,l}\psi_{j,k}(x)\psi_{j,l}(y) \qquad \bigoplus_{j=L}^{J-1} \; [\quad W_j \otimes W_j \\ & + w^{\mathrm{LH}}_{j,k,l}\psi_{j,k}(x)\varphi_{j,l}(y) \qquad\qquad \oplus W_j \otimes V_j \\ & + w^{\mathrm{HL}}_{j,k,l}\varphi_{j,k}(x)\psi_{j,l}(y) \Bigg] \qquad\qquad \oplus V_j \otimes W_j \;] \end{aligned}$$

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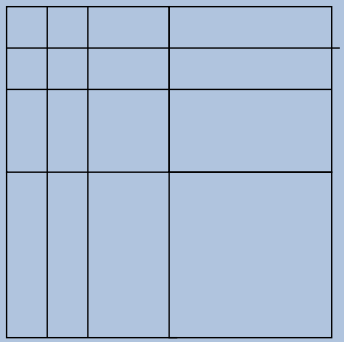
2. The rectangular wavelet transform

We could also transform in each step *all* rows and columns (and not only those that are in the LL-subband) This is the following scheme:



In this case, we can change order of operations: we could for instance, perform the complete transform on the columns, before transforming the rows.

If $\mathbf{w} = \mathbf{W}\mathbf{y}$ is the matrix representation of a 1D wavelet transform, then the rectangular transform, applied to an image $\mathbf{M}$ is:

$$\mathbf{W}\mathbf{M}\mathbf{W}^T$$

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The basis corresponding to this decomposition contains functions that are tensor products of wavelets at *different* scales:

$$\psi_{j,k}(x)\psi_{i,l}(y)$$

Such functions do not appear in the basis of a square wavelet transform.

For image processing, we prefer the *square* transform:

- less computations
- more specific info on vertical, diagonal, horizontal features at *each scale*

Both rectangular and square wavelet transform are *separable* transforms.

example

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$s_{L,k,l}$ and $255 - |w^c_{j,k,l}|$,
$c = \mathrm{HH}, \mathrm{HL}, \mathrm{LH}$

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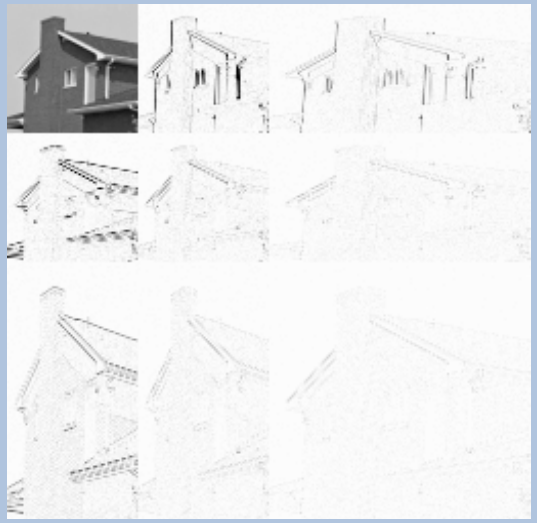

comparison of rectangular



and square transform

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