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A NATURAL COMPANION TO CATALAN'S CONSTANT

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Abstract. We consider here a trilogarithmic expression that plays a similar role to Catalan's constant G in many ways:

$$\mathscr{G} := \Im\left(\mathrm{Li}_3\left(\frac{i+1}{2}\right)\right).$$

The main purpose of this article is to demonstrate how $\mathscr{G}$ is a naturally occurring and useful expression that deserves to be recognized as a mathematical constant and as a natural trilogarithmic "extension" of Catalan's constant G. Having identified this constant, we evaluate many new and non-trivial integrals, Euler-type sums, ${}_pF_q$ series, and binomial-harmonic series using $\mathscr{G}$, extending known results on the classical version of Catalan's constant.

1. Introduction

The term *constant*, as we mean it here, refers to some specific quantity whose value has been determined *a priori*; such a quantity should be considered to be of significant interest because of its repeated occurrence in various contexts. Of course, what is typically meant by the phrase "mathematical constant" also has much to do with *usefulness* and *applicability*. This dual notion, whereby numerical values that are of especial interest as well as of especial utility are worthy of the eminent title "mathematical constant", forms something of a recurring theme in our paper. We consider a natural analogue $\mathscr{G}$ of the constant

$$G := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \tag{1}$$

introduced by Eugène Catalan in 1865, and we demonstrate here how the value $\mathscr{G}$, as defined ahead, merits acceptance as a mathematical constant.

Euler discovered the formula

$$\beta(2m+1) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^{2m+1}} = \frac{(-1)^m E_{2m}}{2\cdot(2m)!}\left(\frac{\pi}{2}\right)^{2m+1},\ m \in \mathbb{N}_0,$$

where $\beta(s)$ is the Dirichlet beta function (also known as the Catalan beta function) and E_{2m} are the *Euler numbers*. No such formula exists for $\beta(2m)$. Catalan studied the sum

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$\beta(2)$, which he denoted by G, and, over the years, Catalan's constant G has become one of the most well-known and well-studied mathematical constants, and has found applications in diverse areas of mathematics. It is often useful to rewrite the defining series for this constant in (1) using the special function

$$\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n} \qquad (|z| \leq 1) \tag{2}$$

known as the polylogarithm function; we rewrite the series in (1) so that

$$G = \Im(\mathrm{Li}_2(i)), \tag{3}$$

since we are interested in a mathematical expression resembling the right-hand side of (3) that has arisen in the evaluation of many naturally occurring integrals and series but that does not seem to have been previously recognized as a "mathematical constant" per se. This expression is given as the right-hand side of the definition

$$\mathscr{G} := \Im\left(\mathrm{Li}_3\left(\frac{i+1}{2}\right)\right). \tag{4}$$

This value $\mathscr{G}$ is the subject of our present article. It appears that $\mathscr{G}$ does not admit any kind of meaningful evaluation in terms of any previously recognized constants [9, 10, 22, 24], and the decimal expansion of this constant $\mathscr{G} = 0.570077407088768...$ is not indexed in the On-line Encyclopedia of Integer Sequences [23], and is not recognized by the Inverse Symbolic Calculator[1]. The expression on the right-hand side of (4) has been used recently to symbolically express a number of interesting series, in 2020 in [24], and in 2019 in [8, 9, 10], and also in 2015 in [22]. The constant $\mathscr{G}$ was also used in a quantum field theory-based context in 2008 in [12], and was, in an equivalent form, used in the evaluation of Euler-type sums in [17].

Comparing (3) to (1), let us rewrite the right-hand side of (4) in the following manner, as in [24]:

$$\mathscr{G} = \frac{1}{16}\sum_{n=0}^{\infty} \frac{\left(-\frac{1}{4}\right)^n}{(2n+1)^3} + \frac{1}{2}\sum_{n=0}^{\infty} \frac{\left(-\frac{1}{4}\right)^n}{(4n+1)^3} + \frac{1}{4}\sum_{n=0}^{\infty} \frac{\left(-\frac{1}{4}\right)^n}{(4n+3)^3}. \tag{5}$$

Let us contrast this BBP-type series expansion with the remarkably simple and elegant integral formula for $\mathscr{G}$ given as follows:

$$\boxed{\mathscr{G} = \frac{1}{2}\int_0^1 \frac{\ln^2(1-x)}{1+x^2}\,dx.} \tag{6}$$

This nicely and closely reflects the following very similar formula for the classical version of Catalan's constant:

$$-\Im\left(\mathrm{Li}_2\left(\frac{1+i}{2}\right)\right) = \frac{\pi\ln(2)}{8} - G = \int_0^1 \frac{\ln(1-x)}{1+x^2}\,dx, \tag{7}$$

[1] See *URL:* `http://wayback.cecm.sfu.ca/projects/ISC/`.

recalling (4). With regard to the integrands shown in (6) and (7), we have something of a "rule of thumb": *Given a definite integral that produces an evaluation involving G, if we multiply the corresponding integrand by a logarithmic expression and integrate that over the same interval, the result often turns out to be a value involving $\mathscr{G}$.*

As noted in [1], we have that

$$\mathrm{Li}_2\left(\frac{1+i}{2}\right) = \frac{5\pi^2}{96} - \frac{\ln^2 2}{8} + i\left(G - \frac{\pi \ln 2}{8}\right) \tag{8}$$

as a consequence of Landen's identity for the dilogarithm (cf. equation (7)). Similarly, the real part of $\mathrm{Li}_3\left(\frac{i+1}{2}\right)$ is easily seen to admit a closed form, namely

$$\Re\left(\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right) = \frac{35\zeta(3)}{64} - \frac{5\pi^2 \ln(2)}{192} + \frac{\ln^3(2)}{48}, \tag{9}$$

a formula which follows from [20, p. 164, (6.54)] using the identity [20, p. 296, (5)]. Setting the parameter n in (2) equal to 2 or 3, we obtain, respectively, the *dilogarithm* and the *trilogarithm* functions. We remark that *Gieseking's constant* refers to a Catalan-like constant introduced in [4] that, as is the case with G, may be expressed as the imaginary part of the dilogarithm evaluated at a root of unity:

$$\mathfrak{G} = \Im\left(\mathrm{Li}_2\left(\sqrt[3]{-1}\right)\right) = \frac{3\sqrt{3}}{4}\sum_{n=0}^{\infty}\left(\frac{1}{(3n+1)^2} - \frac{1}{(3n+2)^2}\right).$$

One of the main ideas explored in this article is that the constant in (4) is a useful trilogarithmic "extension" of dilogarithm constants such as G and Gieseking's constant.

Remarkably, considering the *many* elementary and very "natural-looking" integrals that we have evaluated using $\mathscr{G}$, it appears that the mathematical expression with which we have defined $\mathscr{G}$, as in the right-hand side of (4), does not seem to appear anywhere in any meaningful way throughout all the 1000+ pages of the classic text *Table of Integrals, Series, and Products* [18]. We note that equivalent expressions of $\mathscr{G}$ have been used a number of times in the Mathematics Stack Exchange online resource[2].

1.1. Organization of the article

We proceed according to what is outlined as follows:

- In Section 2, we offer motivating examples of particularly intriguing formulas and applications for our constant $\mathscr{G}$; previously known identities involving this constant are succinctly surveyed, and new results are also introduced in Section 2;
- In Section 3, we prove evaluations involving $\mathscr{G}$ for Euler-type sums, including an especially nontrivial evaluation for the particularly difficult series $\sum_{n=0}^{\infty} \frac{(-1)^{n+1} H_n^2}{2n+1}$, where $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ denotes the n^{th} harmonic number and H_n^2 denotes the square of this expression;

[2] See *URL:* `https://math.stackexchange.com/questions/918680`, in particular, and the links therein.

• In Section 4, we introduce a rigorous proof of an evaluation involving $\mathscr{G}$ for a ${}_4F_3\left(\frac{1}{2}\right)$ series, solving an open problem considered by Steven Finch as far back as 2007 [15];

• Finally, in Section 5, we make use of a ${}_5F_4(1)$ series for $\frac{\mathscr{G}}{\pi}$ due to Cantarini and D'Aurizio, as well as a $\frac{1}{\pi}$ series obtained recently via coefficient-extraction methods by Wang and Chu, to rigorously prove an explicit evaluation for the difficult series $\sum_n (\tilde{C}_n H_n)^2$, letting $(\tilde{C}_n : n \in \mathbb{N}_0)$ denote the normalized Catalan numbers.

2. Some interesting formulas and applications involving $\mathscr{G}$

One of the most important integral formulas for the classical version of Catalan's constant is as follows:

$$G = \int_0^1 \frac{\arctan(x)}{x}\, dx. \tag{10}$$

Again with regard to the "rule of thumb" that we have noted, and also in consideration of (10), we may obtain that:

$$\int_0^1 \frac{\arctan(x)\ln(1+x)}{x}\, dx = 3\mathscr{G} + \frac{3G\ln(2)}{2} - \frac{9\pi^3}{128} - \frac{3\pi\ln^2(2)}{32}, \tag{11}$$

$$\int_0^1 \frac{\arctan(x)\ln(1-x)}{x}\, dx = \mathscr{G} + \frac{G\ln(2)}{2} - \frac{7\pi^3}{128} - \frac{\pi\ln^2(2)}{32}. \tag{12}$$

Mathematica 12 is able to calculate both integrals if the fact that $\arctan(x) = \frac{1}{2i}(\ln(1+ix) - \ln(1-ix))$ for $0 \leq x \leq 1$ is used. Using the Maclaurin series for the initial factor in the numerator in the integrand in (11), we obtain an Euler-type sum evaluation involving $\mathscr{G}$; Section 3 below is devoted to such sums.

Comparing the very basic integral formula in (10) for G with its companions in (11) and (12), this reinforces the idea that $\mathscr{G}$ is a naturally occurring "extension" of G. Let us further reinforce this idea, along with the aforementioned "rule of thumb", in our providing the following computations:

$$\int_0^1 \frac{\arctan(x)\ln\left(1+x^2\right)}{x}\, dx = 2\mathscr{G} + G\ln(2) - \frac{3\pi^3}{64} - \frac{\pi\ln^2(2)}{16}, \tag{13}$$

$$\int_0^1 \frac{\arctan(x)\ln\left(1-x^2\right)}{x}\, dx = 4\mathscr{G} + 2G\ln(2) - \frac{\pi^3}{8} - \frac{\pi\ln^2(2)}{8}. \tag{14}$$

Another very basic way of expressing Catalan's constant as an integral is as follows: $-G = \int_0^1 \frac{\ln(x)}{1+x^2}\, dx$. With regard to this equality, let us further demonstrate our "rule of thumb" with the following computations:

$$\int_0^1 \frac{\ln(x)\ln(1-x)}{1+x^2}\,dx = \mathscr{G} - \frac{\pi^3}{128} - \frac{\pi\ln^2(2)}{32}, \tag{15}$$

$$\int_0^1 \frac{\ln(x)\ln(1+x)}{1+x^2}\,dx = -3\mathscr{G} - 2G\ln(2) + \frac{11\pi^3}{128} + \frac{3\pi\ln^2(2)}{32}, \tag{16}$$

$$\int_0^1 \frac{\ln(x)\ln\left(1-x^2\right)}{1+x^2}\,dx = -2\mathscr{G} - 2G\ln(2) + \frac{5\pi^3}{64} + \frac{\pi\ln^2(2)}{16}, \tag{17}$$

$$\int_0^1 \frac{\ln(x)\ln\left(1+x^2\right)}{1+x^2}\,dx = 2\mathscr{G} - G\ln(2) - \frac{\pi^3}{64} - \frac{\pi\ln^2(2)}{16}. \tag{18}$$

2.1. Results inspired by the recent work of Sofo and Nimbran

An integral identity equivalent to (11) and (12) was recently employed in [24] to obtain a number of results concerning $\mathscr{G}$, including the evaluation whereby:

$${}_4F_3\left[\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ \frac{3}{2},\frac{3}{2},\frac{3}{2}\end{matrix}\,\middle|\,\frac{1}{2}\right] = -\frac{\sqrt{2}\mathscr{G}}{2} + \frac{23\sqrt{2}\pi^3}{768} + \frac{3\sqrt{2}\pi\ln^2(2)}{64}. \tag{19}$$

The hypergeometric series in (19) is of considerable interest to us, for a variety of reasons, not the least of which being that: The problem of evaluating the equivalent binomial sum

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{2^{3n}(2n+1)^3} = {}_4F_3\left[\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ \frac{3}{2},\frac{3}{2},\frac{3}{2}\end{matrix}\,\middle|\,\frac{1}{2}\right] \tag{20}$$

was actually given as an open problem as far back as 2007, by Steven Finch [15], inspired by many closely related binomial summations that have been considered over the years (cf. equation (21)). The symbolic evaluation involving $\mathscr{G}$ for this difficult series was given *implicitly* in [24]. An alternate and *explicit* evaluation of this series is offered in Section 4. This new derivation improves upon the techniques used in [24], since it is unclear as to how, exactly, the value $\mathrm{Li}_3\left(\frac{i+1}{2}\right)$ "appears" (i.e., from other mathematical expressions) in [24]. The new proof of (19) is rigorous and shows precisely how the explicit evaluation in (19) is obtained. The series in (19) is a logical "next step" after the following hypergeometric series that we may evaluate using G:

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{2^{3n}(2n+1)^2} = {}_3F_2\left[\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ \frac{3}{2},\frac{3}{2}\end{matrix}\,\middle|\,\frac{1}{2}\right] = \frac{\sqrt{2}G}{2} + \frac{\sqrt{2}\pi\ln(2)}{8}. \tag{21}$$

To illustrate the usefulness of the hypergeometric formula for $\mathscr{G}$ given in [24], let us consider the following. Recalling our "rule of thumb" on finding integral formulas for $\mathscr{G}$, let us begin with the well-known evaluation

$$\int_0^{\frac{\pi}{4}} \ln\sin x\,dx = -\frac{\pi\ln 2}{4} - \frac{G}{2} \tag{22}$$

listed in [18, §4.224], which leads us to consider the following elementary and very "innocuous-looking" integral:

$$\int_0^{\frac{\pi}{4}} \ln^2 \sin x \, dx. \tag{23}$$

Mathematica actually evaluates this seemingly very simple integral in terms of the ${}_4F_3\left(\frac{1}{2}\right)$ series in (20). Later in this article, we offer a rigorous proof of the following evaluation for the remarkably simple integral in (23), making use of an integral formula for $\Re \mathrm{Li}_3(1-i)$ due to Lewin [20, p. 165]:

$$\boxed{\int_0^{\frac{\pi}{4}} \ln^2 \sin x \, dx = -\mathscr{G} + \frac{G \ln(2)}{2} + \frac{23\pi^3}{384} + \frac{9\pi \ln^2(2)}{32}.} \tag{24}$$

We offer a new proof of this result, as in the upcoming subsection.

2.2. An application of a complex trilogarithm formula due to Lewin

There is something of a "dual" version or variant of our Catalan-like constant $\mathscr{G}$ that we may define so that

$$\mathscr{G}^* = \frac{i}{2}\left(\mathrm{Li}_3(1-i) - \mathrm{Li}_3(1+i)\right) = \Im\left(\mathrm{Li}_3\left(1+i\right)\right). \tag{25}$$

We find that $\mathscr{G}$ and its cousin $\mathscr{G}^*$ are "equivalent" in the sense that

$$\mathscr{G} + \mathscr{G}^* = \frac{7\pi^3}{128} + \frac{3\pi \ln^2(2)}{32}. \tag{26}$$

To see this, we use the following identity for the trilogarithm [20, p. 296, (5)]: $\mathrm{Li}_3(-x) - \mathrm{Li}_3\left(-\frac{1}{x}\right) = -\frac{\pi^2 \ln(x)}{6} - \frac{1}{6}\ln^3(x)$, with $x = -1-i$. Much like with $\mathscr{G}$, the value $\mathscr{G}^*$ has frequently come up in our computations. As a way of illustrating this, let us consider the formula

$$\Im\left(\mathrm{Li}_3(1-i)\right) = \frac{\pi^3}{192} - \frac{\pi \ln^2(2)}{16} - \frac{G \ln(2)}{2} - \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln^2\left(2\sin\frac{\theta}{2}\right) d\theta \tag{27}$$

given in [20, p. 165]. This leads us toward constructing a rigorous proof of our symbolic form for the integral displayed in (23), which we had found, experimentally, to be equal to an expression involving Sofo and Nimbran's ${}_4F_3\left(\frac{1}{2}\right)$ series.

THEOREM 1. *The conjectured symbolic form involving* $\mathscr{G}$ *for* $\int_0^{\frac{\pi}{4}} \ln^2 \sin t \, dt$*, as given in* (24)*, holds true.*

Proof. We may rewrite Lewin's formula, as in (27), so that

$$\mathscr{G}^* = \int_0^{\frac{\pi}{4}} \ln^2(\sin t) \, dt - \frac{3\pi \ln^2(2)}{16} - \frac{G \ln(2)}{2} - \frac{\pi^3}{192}, \tag{28}$$

since the expression $-\frac{1}{2}\int_0^{\frac{\pi}{2}} \ln^2\left(2\sin\frac{\theta}{2}\right)d\theta$ must be equal to:

$$-\frac{1}{2}\ln^2(2)\int_0^{\frac{\pi}{2}} d\theta - \ln(2)\int_0^{\frac{\pi}{2}} \ln\left(\sin\frac{\theta}{2}\right)d\theta - \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln^2\left(\sin\frac{\theta}{2}\right)d\theta.$$

Thanks to the substitution $\frac{\theta}{2} = t$ that we apply to these latter two integrals, and recalling the well-known integral formula for Catalan's constant as displayed in (22) and recorded in [18, §4.224], we obtain the desired result.

2.3. A geometry-related application of our Catalan-inspired constant

An unexpected use of an equivalent form of $\mathscr{G}$ was considered in 2017 [16] in the context of a probabilistic problem in geometry. As in [16], we let α denote an angle in a random spherical triangle Δ, and we also let a be the side of this triangle that is opposite to α, letting the sphere under consideration be a unit sphere, and letting Δ be such that its vertices are independently and uniformly distributed. We are letting b and c denote the other sides of this spherical triangle. Again borrowing from [16], we set another side of Δ to be the fixed value $\frac{\pi}{2}$, and we consider the expected value $E(\alpha, a) = 3.05...$ This is inspired by a geometry problem dating back to the work of Miles in 1971 [21]. It is shown in [16] that $E(\alpha\, a | b = \pi/2)$ may be written as

$$\frac{1}{4}\int_0^{\pi}\left(2 - {}_2F_1\left[\begin{matrix}\frac{1}{2}, \frac{1}{2}\\ 2\end{matrix}\,\middle|\, \cos^2(t)\right]\cos(t)\right)t\,dt,$$

which, in turn, may be written as $\frac{\pi^2}{2} - \frac{4G}{\pi} - \frac{2}{\pi}{}_4F_3\left[\begin{matrix}\frac{1}{2}, \frac{1}{2}, 1, 1\\ \frac{3}{2}, \frac{3}{2}, \frac{3}{2}\end{matrix}\,\middle|\, 1\right]$. An evaluation using an equivalent form of $\mathscr{G}$ for this ${}_4F_3(1)$ series was actually recently proved using FL theory in [8]. This allows us to explicitly determine a symbolic form for the expected value under consideration: $E\left(\alpha\, a \,\middle|\, b = \frac{\pi}{2}\right) = -\frac{8\mathscr{G}}{\pi} - \frac{4G}{\pi} + \frac{9\pi^2}{16} + \frac{\ln^2(2)}{4}$.

2.4. Connection to quantum field theory

Let the Clausen function Cl_2 be defined so that

$$\mathrm{Cl}_2(\theta) = \sum_{n=1}^{\infty}\frac{\sin n\theta}{n^2}. \tag{29}$$

In [11, 12], evaluations for integrals as in $\int_{\pi/3}^{\pi/2}\ln\left|\frac{\tan t+\sqrt{7}}{\tan t-\sqrt{7}}\right|dt$ are considered, and, as noted in [11, 12], such expressions are of use in the disciplines of hyperbolic geometry and quantum field theory. The techniques applied in [12] in the evaluation of this "ln tan" integral led to the following BBP-type formula:

$$\sum_{j=0}^{\infty}\left(\frac{1}{16}\right)^j\left(\frac{4}{(8j+1)^3} - \frac{2}{(8j+4)^3} - \frac{1}{(8j+5)^3} - \frac{1}{(8j+6)^3}\right)$$
$$= 4\mathscr{G} + \frac{7\zeta(3)}{4} - \frac{\pi^2\ln(2)}{16},$$

noting the resemblance to (5). The following series expansion for $\mathscr{G}$ was also introduced in [12]:

$$\mathscr{G} = \sum_{n=1}^{\infty} \sum_{m=0}^{n} \frac{1}{2^n n^3} \left(\binom{n}{4m+1} - \binom{n}{4m+3} \right),$$

which is easily seen to be equivalent to the formula

$$\mathscr{G} = \sum_{n=1}^{\infty} \frac{\sin\left(\frac{\pi n}{4}\right)}{2^{n/2} n^3} \tag{30}$$

given in [24], which is used in [24] to obtain an equivalent form of the ${}_4F_3\left(\frac{1}{2}\right)$ evaluation displayed in (19).

2.5. An integral due to Huvent

In [19], Huvent discusses the derivation of BBP formulas for various constants and states that the proofs for these results are based on transformations of the series to elementary integrals that may be evaluated with polylogarithmic functions. The equalities

$$\int_0^1 \frac{(2y-2)\ln^k(y)}{y^2+2y+2}\, dy = (-1)^{k+1} k! \left(\mathrm{Li}_{k+1}\left(\frac{1+i}{2}\right) + \mathrm{Li}_{k+1}\left(\frac{1-i}{2}\right) \right)$$

and

$$\int_0^1 \frac{2\ln^k(y)}{y^2-2y+2}\, dy = i(-1)^{k+1} k! \left(\mathrm{Li}_{k+1}\left(\frac{1+i}{2}\right) - \mathrm{Li}_{k+1}\left(\frac{1-i}{2}\right) \right) \tag{31}$$

are given in [19], along with:

$$\mathrm{Li}_3\left(\frac{1+i}{2}\right) + \mathrm{Li}_3\left(\frac{1-i}{2}\right) = \frac{35\zeta(3)}{32} - \frac{5\pi^2 \ln(2)}{96} + \frac{\ln^3(2)}{24}.$$

The two polylogarithms in the left-hand side of this equation are complex conjugates, which gives us (9). We also have that $\mathscr{G} = -\Im\left(\mathrm{Li}_3\left(\frac{1-i}{2}\right)\right)$. The integral in (31) yields the following for $k = 2$: $\mathscr{G} = \frac{1}{2}\int_0^1 \frac{\ln^2(y)}{y^2-2y+2}\, dy$. We record two remarkably similar results for our "dual" version $\mathscr{G}^*$:

$$\mathscr{G}^* = \frac{1}{2}\int_0^1 \frac{\ln^2(y)}{2y^2-2y+1}\, dy = \frac{1}{2}\int_1^{\infty} \frac{\ln^2(y)}{y^2-2y+2}\, dy.$$

2.6. Fourier–Legendre-related results

We briefly review some FL-related results concerning $\mathscr{G}$ presented in [8, 9, 10, 22]. For the sake of brevity, we assume familiarity with FL theory, notation for hypergeometric series, elliptic integrals, beta integrals, etc. The problem of evaluating the hypergeometric series

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+1)^2} = {}_4F_3\left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, \frac{3}{2}, \frac{3}{2} \end{matrix} \,\middle|\, 1\right] \tag{32}$$

was considered by Nimbran in 2015 in [22], and an evaluation was given for this series involving the value $\mathrm{Li}_3\left(\frac{i+1}{2}\right)$. Comparing the summand shown in (32) to the summand in the known equality

$$G = -\frac{\pi}{4}\ln(2) + \sqrt{2}\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{2^{3n}(2n+1)^2} \tag{33}$$

noted in [3] and [5, p. 264], this further demonstrates one of the main "theses" of our paper, i.e., that $\mathscr{G}$ is a natural analogue to and extension of G. In [10], the FL-based integration method that was introduced recently in [9] was used to construct a proof sketch for Nimbran's ${}_4F_3(1)$ evaluation, giving us that the Ramanujan-like series in (32) in an equivalent form as

$$-\frac{16\mathscr{G}}{\pi} + \frac{3\pi^2}{8} + \frac{\ln^2(2)}{2} \tag{34}$$

holds [10, p. 630–631].

As in [2], Catalan's constant is sometimes defined so that

$$G = \frac{1}{2}\int_0^1 \mathbf{K}(x)\,dx; \tag{35}$$

(with $\mathbf{K}(x)$ the complete elliptic integral of the first kind), so let us consider determining similar formulas for $\mathscr{G}$. An equivalent form of the following natural and elegant variant of the defining integral for G in (35) is given in [10]:

$$\boxed{\int_0^1 \mathbf{K}(x)\ln(x)\,dx \;=\; 8\mathscr{G} - \frac{3\pi^3}{16} - \frac{\pi\ln^2(2)}{4}.} \tag{36}$$

The evaluation

$${}_5F_4\left[\begin{matrix}\frac{1}{2},\frac{1}{2},1,1,1\\ 2,2,2,2\end{matrix}\,\middle|\,1\right] = \frac{-256\mathscr{G}-128G+128}{\pi} + 6\pi^2 - 48 - 24\ln^2(2) + 64\ln(2), \tag{37}$$

an equivalent form of which was introduced and proved via FL theory in [9], will be of particular importance later in our present article, in our making use of results from [26] to prove new evaluations involving $\frac{\mathscr{G}}{\pi}$. The FL-based technique from [9] is also used in [8] to provide an evaluation involving $\mathscr{G}$ for

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3\left(\frac{1}{4^n}\binom{2n}{n}\right)} = {}_4F_3\left[\begin{matrix}\frac{1}{2},\frac{1}{2},1,1\\ \frac{3}{2},\frac{3}{2},\frac{3}{2}\end{matrix}\,\middle|\,1\right] = 4\mathscr{G} - \frac{\pi^3}{32} - \frac{\pi\ln^2(2)}{8}. \tag{38}$$

using integrals as in $\int_0^1 \frac{\arctan(x)\ln(x)}{1\pm x}\,dx$; we note that (38) nicely reflects the formula involving G for $\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2\binom{2n}{n}}$ indicated in [1].

3. Euler-type sum evaluations involving $\mathscr{G}$

The correspondence between Euler and Goldbach during 1742–3 led to the seminal article [13] by Euler in 1776, in which the following fundamental recursion is given (§22, p. 165):

$$\sum_{n=1}^{\infty} \frac{H_n}{n^m} = \frac{m+2}{2}\zeta(m+1) - \frac{1}{2}\sum_{k=1}^{m-2} \zeta(m-k)\zeta(k+1),\ m = 2,3,\dots \tag{39}$$

Considering the rich history surrounding the study of Euler sums and generalizations, and in consideration of Euler-like sum evaluations involving G such as

$$\sum_{n=0}^{\infty} (-1)^{n+1} \frac{H_n}{2n+1} = \frac{\pi}{2}\ln(2) - G,$$

we are led to consider the determination of infinite series formulas for $\mathscr{G}$ inspired by sums of the form in (39).

EXAMPLE 1. The evaluation

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1} H_{2n}}{(2n+1)^2} = \mathscr{G} + \frac{G\ln(2)}{2} - \frac{3\pi^3}{128} - \frac{\pi \ln^2(2)}{32} \tag{40}$$

given in [10, 17] was used directly in the proof in [10] for Nimbran's Ramanujan-like ${}_4F_3(1)$-series. We proceed to offer a proof of this evaluation. The series expansion in [18, p. 55, 1.517.3] whereby

$$\arctan(x)\,\ln\left(1+x^2\right) = 2\sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{2k+1}}{2k+1} \sum_{n=1}^{2k} \frac{1}{n}, \quad x^2 \leq 1$$

may be easily verified, using the Cauchy product of the factors on the left-hand side of this equation. Dividing both sides of this equality by x, we are led to find that: The problem of analytically evaluating the harmonic sum in (40) reduces to the problem of determining such an evaluation for the definite integral displayed in (13), which was done in paragraph 2. Along the way we have to use the Dominated Convergence Theorem to justify the interchange of sum and integral.

Since the Euler-type sum evaluation in (40) was employed to evaluate the Ramanujan-like series in (32), this inspires us to consider variants and generalizations of this series.

EXAMPLE 2. Let $H_n^{(2)} = 1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$. From the generating function (g.f.) identity whereby

$$\sum_{n=0}^{\infty} (-1)^{n+1} H_n^{(2)} x^{2n} = -\frac{\mathrm{Li}_2\left(-x^2\right)}{x^2+1},$$

we find that *Mathematica* is easily able to evaluate the antiderivative of the right-hand side of this equality, to give us that:

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1} H_n^{(2)}}{2n+1} = -4\mathscr{G} - 2G\ln(2) + \frac{11\pi^3}{96} + \frac{\pi \ln^2(2)}{8}. \tag{41}$$

The evaluation in (41) seems to beg the question as to how we may evaluate the corresponding series obtained by replacing the quadratic harmonic number $H_n^{(2)}$ with H_n^2. This leads us to the following proof, in which the log-sine integral for $\mathscr{G}$ given in Section 2.2 plays an important role.

The evaluation highlighted as a theorem as follows is remarkable and nontrivial for a variety of reasons; on this note, we claim that it is not feasible to use direct applications of standard integral formulas for the classical harmonic sequence to prove this evaluation. In particular, recalling the usual integral formula $H_n = \int_0^1 -nx^{n-1}\ln(1-x)\,dx$ for the sequence of harmonic numbers, we may express

$$\sum_{n=0}^{\infty} \frac{(-1)^n H_n \left(-nx^{n-1}\ln(1-x)\right)}{2n+1}$$

using dilogarithms, but current CAS software cannot compute the required antiderivative, in this case. Our evaluation in (42) is also remarkable in that it is not possible to use the g.f. for the sequence of squared harmonic numbers to prove this evaluation. In particular, using the g.f. for the sequence of harmonic numbers along with the aforementioned integral formula for H_n, we may establish that

$$\sum_{n=0}^{\infty} (-1)^n H_n^2 y^{2n} = \frac{\ln^2\left(y^2+1\right) - 2\,\mathrm{Li}_2\left(\frac{y^2}{y^2+1}\right)}{2y^2+2},$$

so that it remains to evaluate the antiderivative of the right-hand side, but this is not possible with today's CAS software. As one might expect, such software cannot evaluate the series given in the following theorem. It is also not possible to use (41) in conjunction with the identity whereby $\int_0^1 nx^{n-1}\ln^2(1-x)\,dx$ equals $H_n^2 + H_n^{(2)}$.

THEOREM 2. *The following Euler-type sum evaluation must hold:*

$$\boxed{\sum_{n=0}^{\infty} \frac{(-1)^{n+1} H_n^2}{2n+1} = -8\mathscr{G} + \frac{3\pi^3}{16} - \frac{3\pi\ln^2(2)}{4}.} \tag{42}$$

Proof. From the g.f. for $(H_n)_{n\in\mathbb{N}_0}$, we have that

$$\frac{1}{2}\ln^2(1-u) = u\sum_{n=0}^{\infty} \frac{H_n}{n+1} u^n,$$

and by applying the partial sum operator $\frac{1}{1-u}$, we have that:

$$\frac{1}{2}\frac{\ln^2(1-u)}{1-u} = u\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{H_k}{k+1}\right)u^n. \tag{43}$$

We also find that [25, Theorem 16]: $\sum_{m=1}^{n} \frac{H_m}{m+1} = \frac{1}{2}\left(H_{n+1}^2 - H_{n+1}^{(2)}\right)$. This gives us a way of evaluating the sum of $(-1)^n \frac{H_n^2 - H_n^{(2)}}{2n+1}$ for $n \in \mathbb{N}_0$, as follows:

$$\sum_{n=0}^{\infty} (-1)^{n+1} \frac{H_n^2 - H_n^{(2)}}{2n+1} = \frac{7\pi^3}{48} + \frac{5\pi \ln^2(2)}{4} - 2G\ln(2) - 4\mathscr{G}^*. \tag{44}$$

To prove this, we calculate $\int_0^{\pi/4} \ln^2(\sin t)\, dt$ in a special way:

$$\begin{aligned}
&2\int_0^{\pi/4} \ln^2(\sin t)\, dt + 2\int_0^{\pi/4} \ln^2(\cos t)\, dt \\
&= \int_0^{\pi/4} (\ln(\sin t) + \ln(\cos t))^2\, dt + \int_0^{\pi/4} (\ln(\sin t) - \ln(\cos t))^2\, dt \\
&= \int_0^{\pi/4} \ln^2(\sin t \cos t)\, dt + \int_0^{\pi/4} \ln^2(\tan t)\, dt \\
&= \frac{1}{2}\int_0^{\pi/2} \ln^2\left(\frac{\sin\theta}{2}\right) d\theta + \frac{\pi^3}{16} \\
&= \frac{1}{2}\int_0^{\pi/2} \ln^2(\sin\theta)\, d\theta - \ln 2 \int_0^{\pi/2} \ln(\sin\theta)\, d\theta + \frac{\pi}{4}\ln^2 2 + \frac{\pi^3}{16} \\
&= \frac{1}{2}\left(\frac{\pi^3}{24} + \frac{\pi}{2}\ln^2(2)\right) - \ln(2)\left(-\frac{\pi}{2}\ln(2)\right) + \frac{\pi}{4}\ln^2(2) + \frac{\pi^3}{16} = \frac{\pi^3}{12} + \pi\ln^2(2)
\end{aligned}$$

using [18, §4.261]. Furthermore, we have that:

$$\int_0^{\pi/4} \ln^2(\cos t)\, dt = \frac{1}{4}\int_0^{\pi/4} \ln^2(\cos^2 t)\, dt = \frac{1}{4}\int_0^1 \frac{\ln^2(1+x^2)}{1+x^2}\, dx, \tag{45}$$

and hence:

$$\int_0^1 \frac{\ln^2(1+x^2)}{1+x^2}\, dx = \frac{\pi^3}{6} + 2\pi\ln^2(2) - 4\int_0^{\pi/4} \ln^2(\sin t)\, dt, \tag{46}$$

where this log-sine integral is as in (28). Recalling (43), we set $u = -x^2$, and this gives us that:

$$\frac{\ln^2(1+x^2)}{1+x^2} = \sum_{n=0}^{\infty} (-1)^{n+1} (H_{n+1}^2 - H_{n+1}^{(2)}) x^{2n+2} = \sum_{n=1}^{\infty} (-1)^n (H_n^2 - H_n^{(2)}) x^{2n}.$$

Applying the operator $\int_0^1 \cdot\, dx$, we again encounter the integral $\int_0^1 \frac{\ln^2(1+x^2)}{1+x^2}\, dx$, and from (46), this gives us the equation displayed in (44). So, the desired result comes from combining the Euler sums in (41) and (44).

We may also[3] determine that:

$$\sum_{n=0}^{\infty} (-1)^{n+1} \frac{H_n}{(2n+1)^2} = -2\mathscr{G} + G\ln(2) + \frac{\pi^3}{64} + \frac{\pi\ln^2(2)}{16}.$$

[3] See *URL:* `https://math.stackexchange.com/questions/918135`.

Also, many similar series as in $\sum_{n=0}^{\infty} \frac{(-1)^{n+1} H_{2n}^{(2)}}{2n+1}$ may be evaluated using our integral evaluations as in Section 2, but, for the sake of brevity, we leave this for a future research endeavor.

4. A proof for Finch's ${}_4F_3\left(\frac{1}{2}\right)$ series

In this section, we provide an alternate proof of the remarkable evaluation displayed in (19) that was given quite recently, in 2020, by Sofo and Nimbran [24]. The approach used in [24] to establish the symbolic form for the infinite series

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{2^{3n}(2n+1)^3} \tag{47}$$

in (20) is as outlined in Section 2.1. Since it is not shown in [24] how the value $\Im\left(\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right)$ may be obtained from $Q(-1,1,1,1)$, this motivates the rigorous approach we employ to solving the open problem of evaluating (47), as given by Finch in 2007 [15].

To begin with, let us recall our definition for $\mathscr{G}^*$, as in (25), and let us consider the following result given by Lewin [20, p. 165, (6.56); p. 298, (49)], with $\theta = \pi/2$:

$$\Im\left(\mathrm{Li}_3(1-\mathrm{e}^{i\theta})\right) = \frac{\theta^3}{24} - \frac{\theta}{2}\ln^2\left|2\sin\frac{\theta}{2}\right| - \mathrm{Cl}_2(\theta)\ln\left|2\sin\frac{\theta}{2}\right| + \mathrm{Ls}_3(\theta),$$

recalling (27). We recall the definition of the Clausen function Cl_2 shown in (29), noting that $\mathrm{Cl}_2(\pi/2) = G$. We employ the following notation:

$$\mathrm{Ls}_n(\sigma) := -\int_0^{\sigma} \ln^{n-1}\left(2\sin\frac{\theta}{2}\right) d\theta. \tag{48}$$

THEOREM 3. *The explicit symbolic evaluation involving $\mathscr{G}$ for Finch's ${}_4F_3\left(\frac{1}{2}\right)$ series, as in* (19) *and as introduced in an equivalent form in* [24]*, must hold.*

Proof. We begin by expanding the expression $\frac{1}{\sqrt{1-t^2}}$ with its Maclaurin series; multiplying both sides of this equality by $\ln^k(t)$, and, for a variable x, applying $\int_0^x \cdot\, dt$ to both sides of this resultant equation, we obtain the following:

$$\int_0^x \frac{\ln^k(t)}{\sqrt{1-t^2}}\, dt = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{2^{2n}} \int_0^x t^{2n} \ln^k(t)\, dt.$$

If we use the substitution $t = \sin u$ in this integral, we find that:

$$\int_0^{\arcsin(x)} \ln^k(\sin u)\, du = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{2^{2n}} \int_0^x t^{2n} \ln^k(t)\, dt. \tag{49}$$

For the integral on the right-hand side, we have the following recurrence relation, which can be obtained using partial integration:

$$I_{k,n}(x) := \int_0^x t^{2n} \ln^k(t)\, dt, \text{then}\, I_{k,n}(x) = \frac{1}{2n+1}\left(x^{2n+1}\ln^k(x) - k\, I_{k-1,n}(x)\right),$$

with $I_{0,n}(x) = \frac{1}{2n+1}x^{2n+1}$, leading to: $I_{k,n}(x) = \sum_{i=0}^{k}(-1)^i i!\binom{k}{i}\ln^{k-i}(x)\frac{x^{2n+1}}{(2n+1)^{i+1}}$. Hence, from (49), we may obtain that:

$$\int_0^{\arcsin(x)} \ln^k(\sin u)\, du = \sum_{i=0}^{k}\left((-1)^i i!\binom{k}{i}\ln^{k-i}(x)\sum_{n=0}^{\infty}\frac{\binom{2n}{n}x^{2n+1}}{(2n+1)^{i+1}2^{2n}}\right). \tag{50}$$

So, we can evaluate sums of the form $\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{(2n+1)^{i+1}2^{2n}}x^{2n+1}$ recursively, and through the use of known values of log-sine integrals. Note that the right-hand side of (48) is related to the following integral, via the substitution $u = \theta/2$:

$$\mathrm{Ls}_n(\sigma) = -2\int_0^{\sigma/2}(\ln(2) + \ln(\sin u))^{n-1}\, du.$$

In particular, to prove (19), we make use of use (50): For for $x = \frac{\sqrt{2}}{2}$, this takes on the form

$$\int_0^{\pi/4} \ln^k(\sin u)\, du = (-1)^k\sum_{j=0}^{k}\left(j!\binom{k}{j}\frac{\ln^{k-j}(2)}{2^{k-j}\sqrt{2}}\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{(2n+1)^{j+1}2^{3n}}\right),$$

so that

$$\sqrt{2}\int_0^{\frac{\pi}{4}} \ln^2(\sin u)\, du = \frac{1}{4}\ln^2(2)\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{(2n+1)\,2^{3n}}$$
$$+\ln(2)\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{(2n+1)^2\,2^{3n}} + 2\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{(2n+1)^3\,2^{3n}}.$$

Setting $x = \frac{1}{2\sqrt{2}}$ in the expansion $\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{2n+1}x^{2n} = \frac{\arcsin 2x}{2x}$, this allows us to evaluate

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{(2n+1)2^{3n}}.$$

We recall the well-known Catalan number formula given in (33), and we find that:

$$\int_0^{\frac{\pi}{4}} \ln^2(\sin u)\, du = \frac{3\pi\ln^2(2)}{16} + \frac{G\ln(2)}{2} + \sqrt{2}\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{(2n+1)^3\,2^{3n}};$$

this, together with (27), proves (19).

5. Evaluations involving $\mathscr{G}$ for new harmonic sums involving squared central binomial coefficients

We let $(C_n)_{n\in\mathbb{N}_0}$ denote the famous, ubiquitous sequence of Catalan numbers, with $C_n = \binom{2n}{n}/(n+1)$ for all $n \in \mathbb{N}_0$. As in [8], we "normalize" this sequence, so that

$$\tilde{C}_n = \frac{\binom{2n}{n}}{4^n}\frac{1}{n+1},$$

recalling the g.f. evaluations given as follows, along with the property whereby $\frac{\binom{2n}{n}}{4^n} \sim \frac{1}{\sqrt{\pi n}}$:

$$\frac{1}{\sqrt{1-x}} = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n} x^n \quad \text{and} \quad \frac{1-\sqrt{1-x}}{x} = \frac{1}{2}\sum_{n=0}^{\infty} \tilde{C}_n x^n.$$

Let us also recall how Cantarini and D'Aurizio's evaluation involving $\frac{\mathscr{G}}{\pi}$ for Nimbran's series

$$\sum_{n=0}^{\infty} \left(\frac{1}{4^n}\binom{2n}{n}\right)^2 \frac{1}{(2n+1)^2}$$

is such a natural extension of Ramanujan's evaluation of the corresponding ${}_3F_2(1)$ series with a linear denominator, recalling the value of this latter sum as $\frac{4G}{\pi}$. Inspired by this, we want to further explore the idea of "extending" series evaluations involving $\frac{G}{\pi}$ to new infinite sum that may be expressed in terms of $\frac{\mathscr{G}}{\pi}$. Since we have, so far, considered series for $\mathscr{G}$ involving nonlinear central binomial coefficients, as well as harmonic sums that may be evaluated using $\mathscr{G}$, it is natural to think of "combinations" of these two kinds of series that admit expressions involving $\mathscr{G}$, i.e., series involving summand factors of the form

$$\left(\frac{1}{16}\right)^n \binom{2n}{n}^2 \Delta(n) \tag{51}$$

with symbolic expressions that involve $\mathscr{G}$, letting $(\Delta(n) : n \in \mathbb{N}_0)$ denote a sequence of harmonic-type numbers. In this regard, a number of series containing such summand factors, as in (51), have been recently shown to be evaluable in terms of $\frac{G}{\pi}$; we make particular note of the following results from [6, 7]:

$$\sum_{n=1}^{\infty} \tilde{C}_n^2 H_n = 16 + \frac{32G - 64\ln 2}{\pi} - 16\ln 2, \tag{52}$$

$$\sum_{n=1}^{\infty} \tilde{C}_n^2 H_{2n} = 4 + \frac{16G + 24 - 48\ln 2}{\pi} - 8\ln 2.$$

In this section, we extend (52), by providing a full proof of a new evaluation involving $\frac{\mathscr{G}}{\pi}$ for the very natural companion to (52) shown as follows:

$$\sum_{n=1}^{\infty} \tilde{C}_n^2 H_n^2. \tag{53}$$

In particular, by directly making use of Cantarini's and D'Aurizio's ${}_5F_4(1)$ series from (37) and a series for $\frac{1}{\pi}$ recently introduced in [26], along with reindexing arguments as in [6], we can show that the series in (53) must be equal to:

$$\frac{256\ln^2(2)-128G-768\mathscr{G}-256G\ln(2)}{\pi}-\frac{40\pi}{3}+\frac{58\pi^2}{3}+64\ln(2)-8\ln^2(2). \quad (54)$$

This new and very non-trivial evaluation, together with our proof of this result, is one of the main results in our article. We note that the series in (53) may also be thought of as a "next step" after the series

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2 H_n^2}{16^n(n+1)}=\frac{64\ln^2(2)-32G}{\pi}-\frac{10\pi}{3}+16\ln(2) \quad (55)$$

evaluated via the FL method in [9], again reinforcing the idea that $\mathscr{G}$ is a natural extension of G, with regard to our analytic form for (53) involving $\frac{\mathscr{G}}{\pi}$ compared with the evaluation with $\frac{G}{\pi}$ for the sum in (55).

THEOREM 4. *The explicit evaluation for $\sum_n(\tilde{C}_nH_n)^2$ indicated in* (54) *holds.*

Proof. We begin as follows:

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2 H_n^2}{16^n(n+1)}=\frac{64\ln^2(2)-32G}{\pi}-\frac{10\pi}{3}+16\ln(2).$$

This is proved in [9], and this leads to:

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2 H_n^2}{16^n}\left(\frac{1}{(n+1)^2}-\frac{4}{n+1}\right)=-\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2 H_n^2}{16^n}\frac{4n+3}{(n+1)^2}$$
$$=-\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2 H_n^2}{16^n}\left(\frac{2n+1}{(n+1)^2}+\frac{2}{n+1}\right).$$

We have the following, noting that we may start from $n=0$:

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2 H_n^2}{16^n}\frac{2n+1}{(n+1)^2}=\sum_{n=0}^{\infty}\frac{\binom{2n}{n}^2 H_n^2}{16^n}\frac{(2n+2)^2(2n+1)^2}{(n+1)^2(2n+2)^2(2n+1)}$$
$$=\frac{1}{4}\sum_{n=0}^{\infty}\frac{\binom{2n+2}{n+1}^2 H_n^2}{16^n}\frac{1}{2n+1}=4\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2 H_{n-1}^2}{16^n}\frac{1}{2n-1}=4\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2}{16^n}\left(H_n-\frac{1}{n}\right)^2\frac{1}{2n-1}.$$

By expanding the square, we get a sum of the following 3 terms:

$$4\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2}{16^n}H_n^2\frac{1}{2n-1},$$

which can be found in [26],

$$-8\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2}{16^n}\frac{H_n}{n(2n-1)} = -8\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2}{16^n}H_n\left(\frac{2}{2n-1}-\frac{1}{n}\right),$$

where the first series can be found in [9] and the second one in [10], and

$$4\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2}{16^n}\frac{1}{n^2(2n-1)} = 4\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2}{16^n}\left(\frac{4}{2n-1}-\frac{2}{n}-\frac{1}{n^2}\right),$$

where the first series, i.e., this Wallis series, is classical, and the second and third one can be rewritten using reindexing:

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2}{16^n}\frac{1}{n} = \ldots = \frac{1}{4}\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2}{16^n}\frac{(2n+2-1)^2}{(n+1)^3}$$
$$= \sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2}{16^n}\frac{1}{n+1} - \sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2}{16^n}\frac{1}{(n+1)^2} + \frac{1}{4}\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2}{16^n}\frac{1}{(n+1)^3},$$

where the first series is due to Glaisher, the second one is in Nimbran's paper [22], and the third one is proved in [6, 10, 14], and

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2}{16^n}\frac{1}{n^2} = \ldots = \frac{1}{4}\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2}{16^n}\frac{(2n+2-1)^2}{(n+1)^4}$$
$$= \sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2}{16^n}\frac{1}{(n+1)^2} - \sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2}{16^n}\frac{1}{(n+1)^3} + \frac{1}{4}\sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2}{16^n}\frac{1}{(n+1)^4},$$

where the third series is formula (37), giving us our desired evaluation involving $\mathscr{G}$.

Using reindexing arguments, we may generalize our evaluation for the series in Theorem 4, to obtain analytic forms involving $\frac{\mathscr{G}}{\pi}$ for

$$\sum_{n=0}^{\infty}\left(\frac{1}{4^n}\binom{2n}{n}\right)^2\frac{H_n^2}{n+z} \tag{56}$$

for $z\in\mathbb{Z}_{>0}$. We may also apply series for $\frac{1}{\pi}$ introduced in [26] involving quadratic harmonic numbers to obtain new formulas for $\mathscr{G}$. Following the reindexing approach as in our proof of Theorem 4, but as applied to series due to Wang and Chu with expressions as in $H_n^{(2)}$ as summand factors, we can show that

$$\sum_{n=0}^{\infty}\tilde{C}_n^2H_n^{(2)} = \frac{128}{\pi}+\frac{8\pi}{3} - {}_5F_4\left[\begin{matrix}\frac{1}{2},\frac{1}{2},1,1,1\\ 2,2,2,2\end{matrix}\,\middle|\,1\right]-48,$$

so, from the formula in (37) due to Campbell, D'Aurizio, and Sondow, we obtain the elegant formula whereby

$$\sum_{n}\tilde{C}_n^2H_n^{(2)} = \frac{128G}{\pi}+\frac{256\mathscr{G}}{\pi}+\frac{8\pi}{3}-6\pi^2-64\ln(2)+24\ln^2(2),$$

again nicely reflecting how the constants G and $\mathscr{G}$ are natural "companions". Again employing reindexing recursions in the vein of [6], we may evaluate the series obtained from (56) by replacing H_n^2 with $H_n^{(2)}$ in terms of $\frac{\mathscr{G}}{\pi}$ for $z \in \mathbb{Z}_{>0}$; for the sake of brevity, we leave this as an exercise.

From the results given in this article, it should be clear that the value $\mathscr{G}$ is worthy of consideration as an important and useful mathematical constant, and we strongly encourage further investigations into series and integral formulas for $\mathscr{G}$.

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