On the Aristotelian Roots of the Modal Square of Opposition

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Abstract. The modal square of opposition is a diagram visualizing the opposition and entailment relations holding between the modalities of possibility, impossibility, necessity, and contingency. A time-honored claim in the scholarship holds that the modal square was first described by Aristotle at *De int.* 12-13. This paper establishes two theses: (1) the notion that the modal square is explicitly contained in *De int.* 12-13 is not supported by the textual evidence; (2) the modal square can be obtained from *De int.* 12-13 under the assumption of a contraposition law that Aristotle establishes elsewhere, at *An. pr.* I.46. The conclusion is that Aristotle cannot be seen as the father of the modal square, but that the square can be derived from his logic.

1. Introduction

Consider Fig. 1 and 2. These figures display two of the best-known diagrams from the history of logic. Fig. 1 displays a categorical square of opposition, and Fig. 2 a modal square of opposition. Fig. 1 charts the opposition and entailment relations between propositions containing the quantifiers “all”, “some”, “no”, and “not all”, which are traditionally labelled the *A*-, *I*-, *E*- and *O*-propositions (after the first two vowels of the Latin verb forms “*affirmo*” and “*nego*”, respectively). Fig. 2 charts the same relations, but between propositions containing the modalities of necessity, possibility, impossibility and contingency. These relations are the following:

- *contradiction*: the *relata* cannot be true together, and they cannot be false together
- *contrariety*: the *relata* cannot be true together, but they can be false together
- *subcontrariety*: the *relata* can be true together, but they cannot be false together
- *subalternation*: the truth of the first *relatum* entails the truth of the second *relatum*, though not *vice versa* (i.e., the second *relatum* may be true even though the first *relatum* is false)

In both diagrams, solid, dashed and dotted lines represent the relations of contradiction, contrariety and subcontrariety, respectively, while arrows represent the relation of subalternation. Each diagram has certain peculiar features. As is well known, the relations in the categorical square are sensitive to existential import. For instance, they hold under the assumption that the *A*- and *I*-proposition have an import, while the *E*- and *O*-proposition do not; but they do not hold if the *I*- and *O*-proposition are assumed to have an import, while the *A*- and *E*-proposition do not, as is common in contemporary predicate logic. The *relata* in the modal square, moreover, are not individual propositions, as in the categorical square, but equivalence

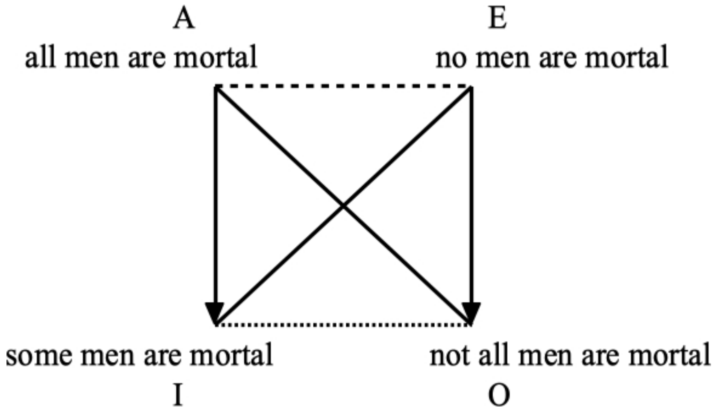

Fig. 1: A categorical square of opposition

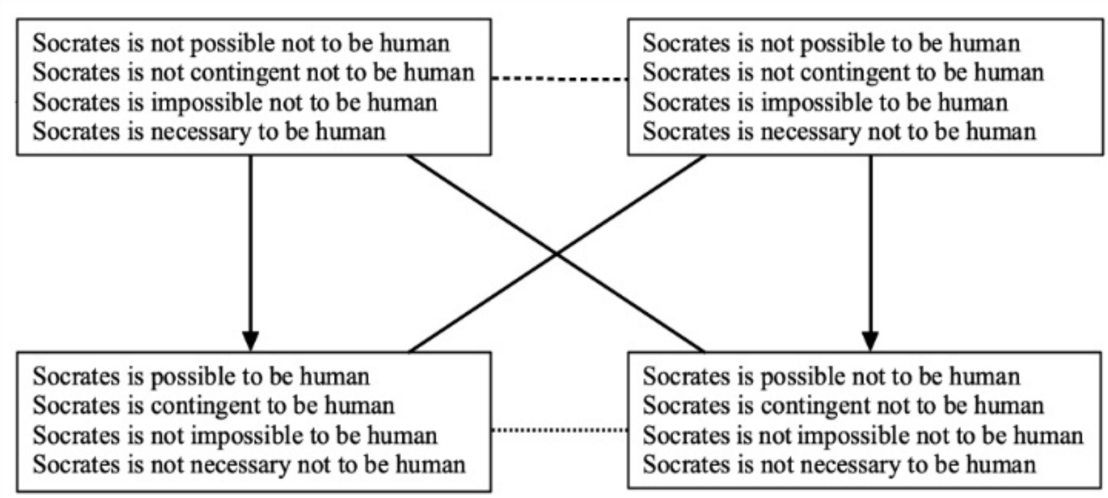

Fig. 2: A modal square of opposition

classes of propositions.[1] Each and every proposition in the upper-left corner is contrary to each and every proposition in the upper-right corner, and contradictory to each and every proposition in the lower-right corner, for instance. If defined up to logical equivalence, then each vertex collapses into a single (equivalence class of) proposition(s). Moreover, contrary to common usage in contemporary modal logic, the contingent in the square signifies the one-sided possible in each equivalence class, and thus has the same meaning as the possible.

Both the categorical and the modal square of opposition have a long history. The intricate details of the historical development of the categorical square are relatively well known.[2] The categorical square goes back to Aristotle's *De int.* 5-7. Aristotle there treats the properties of the categorical proposition (ἀπόφανσις). His main aim is to get clear on the notions of

[1] One can also view the *relata* of the categorical square as equivalence classes, rather than individual propositions. For example, the upper-left corner of such a categorical square would then include the propositions of the form "all *S* are *P*", "no *S* are not *P*" and "not some *S* are not *P*". Such squares were not unknown in the history of logic; see, e.g., De Rijk (1962-1967-ii.2: 474). However, the 'prototypical' (and historically most frequent) appearance of the categorical square contains individual propositions as *relata*, whereas that of the modal square contains equivalence classes of multiple propositions, as is shown in Figs. 1 and 2.

[2] See esp. Correia (2012, 2017a, 2017b); Gombocz (1990); Lemaire (2017); Londey and Johanson (1984, 1987); Parsons (2017); Pozzi (1974: 39-61); Sullivan (1967: 63-75).

affirmation (κατάφασις) and denial (ἀπόφασις).[3] Following *De int.* 5-6, an affirmation is a sentence of the subject-predicate form, usually with a form of "to be" (εἶναι) as its copula, in either the present, past, or future tense. A denial is a sentence of the same form, but with a negated copula. Thus for every affirmation there is precisely one (syntactic) denial. Aristotle first establishes this at *De int.* 6, 17a32-35, calling a pair of an affirmation and its denial a contradictory pair (αἱ ἀντικείμεναι; see also *De int.* 7, 18a1-7). At *De int.* 7, 17b27-29, Aristotle introduces the semantics of contradictory pairs, and stipulates as working hypothesis that such pairs satisfy both the Law of Non-Contradiction (**LNC**) and the Law of Excluded Middle (**LEM**); that is, for every contradictory pair, necessarily, one member is true, and the other is false. After Whitaker, we call this the Rule of Contradictory Pairs, **RCDP** for short.

> **RCDP**: For every pair of contradictory propositions $\{\varphi, \psi\}$, the conjunction 'φ and ψ' is necessarily false, and the disjunction 'φ or ψ' is necessarily true.

Much of Aristotle's work in the remainder of *De interpretatione* is about delimiting the scope of **RCDP**, and identifying cases where propositions are each other's contradictories syntactically but not semantically. The best-known exception in present-day scholarship are singular propositions of future tense, which Aristotle discusses at length at *De int.* 9. The exception that is most relevant to our purposes are pairs of affirmations and negations where both members "are universal in form", or universally quantified, "with a universal for subject" (καθόλου ἀποφαίνηται ἐπὶ τοῦ καθόλου: *De int.* 7, 17b4-5). Such pairs satisfy **LNC**, but they do not satisfy **LEM**, and are hence not contradictory pairs proper (*De int.* 7, 17b24; *De int.* 10, 20a16-30). Aristotle calls such propositions contraries (αἱ ἐναντίαι). The Rule of Contrary Pairs (**RCP**) is thus as follows.

> **RCP**: For every pair of contrary propositions $\{\varphi, \psi\}$, the conjunction 'φ and ψ' is necessarily false, and the disjunction 'φ or ψ' is not necessarily true.

Following *De int.* 7, 17b17-23, the pairs $\{A, O\}$ and $\{E, I\}$ are contradictory pairs, while the pair $\{A, E\}$ is a contrary pair.[4] Those are the only two relations that Aristotle describes. He does not recognize that the pairs $\{A, I\}$ and $\{E, O\}$ are subaltern pairs, or that the pair $\{I, O\}$ is a subcontrary pair. *That* these pairs satisfy the subalternation and subcontrariety relation, respectively, can be shown by means of two straightforward arguments that rely on **RCDP** and **RCP** only.[5]

- (*Subalternation*) Assume that the A-proposition is true. Then the E-proposition is false, by **RCP**, and thus the I-proposition is true, by **RCDP**. Hence the A-proposition entails the I-proposition. But the I-proposition does not entail the A-proposition. After all, since the A- and E-propositions are contraries, it is possible that they are both false, and hence, by **RCDP**,

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3 See esp. Jones (2010); Whitaker (1996), who we follow in this and the next paragraph.

4 It is not fully accurate to commit Aristotle to the claim that the contrariety relation holds between the A- and E-templates, and thus between each and every $\{A, E\}$-instance. A more faithful characterization would be that only a subset of all $\{A, E\}$-instances satisfies **RCP**, namely those pairs where the predicate does not necessarily nor impossibly inhere in the subject. The characterization given in the running text however suffices for our purposes; see Bogen (1991) for further details.

5 Both arguments are well known in the scholarship; see, e.g., Corkum (2018: 104); Parsons (2017).

it is possible that the *A*-proposition is false while the *I*-proposition is true. Analogously, the *E*-proposition entails the *O*-proposition, but not *vice versa*.[6]

- (*Subcontrariety*) Aristotle admits on at least two occasions that the *I*- and *O*-propositions can be true together (*De int.* 7, 17b25-26; 10, 20a19-20). That they cannot be false together is seen as follows. Assume that the *I*-proposition is false. Then the *E*-proposition is true, by **RCDP**; so the *A*-proposition is false, by **RCP**; and the *O*-proposition is true, again by **RCDP**. Analogously, if the *O*-proposition is assumed to be false, then the *I*-proposition is true.

In short, while it is not true to say that Aristotle accepts *all* relations occurring in the square, it is true to say that, for each relation in the square, either Aristotle accepts it or it follows from relations that Aristotle accepts. Furthermore, there is no evidence that Aristotle can be credited with the square *diagram*, or the geometrical representation of the different opposition and entailment relations in the form of a squared figure. The square diagram is presumably an innovation of ancient scholastic logic, occurring for the first time among the preserved sources in *De interpretatione* of Apuleius of Madaura (c. 124 – c. 170 AD). The diagram was later incorporated in the *De nuptiis Mercurii et Philologiae* of Martianus Capella (fl. 410 – 420 AD), and also in the commentaries on Aristotle's *De interpretatione* of Boethius (c. 475 – 526 AD) and Ammonius Hermiae (c. 440 – 520 AD).[7] Apuleius and Boethius transmitted the square to the Middle Ages, and it formed an integral part of logical theory well into the early twentieth century and the rise of symbolic logic following the publication of Frege's *Begriffschrift* (1879).

In contrast to the categorical square, the history of the modal square is less well documented. We know that the modal square starts to regularly occur in texts on modal logic from the later twelfth century onwards, during the transition period between the nominalist-realist debates surrounding authors such as Peter Abelard (1079 – 1142) and William of Champeaux (c. 1070 – 1122), and the heyday of terminist logic, with authors like Peter of Spain (fl. c. 1220 – 1250) and William of Sherwood (c. 1200 – c. 1270).[8] We also know that the fate of the modal square

6 This not to say that Aristotle was unaware of the fact that the *A*-proposition entails the *I*-proposition, or that the *E*-proposition entails the *O*-proposition. Aristotle uses these forms of immediate inference on a number of occasions in his logical works (see, e.g., *Top*. II.1, 108b34-109a7; *Top*. III.6 (119a34-36). The claim here is simply that Aristotle does not describe the subalternation relation at *De int*. 6-7.

7 This, at least, is the received view in the scholarship; see e.g. Bobzien (2020); Bocheński (1951: 37); Kahane (2021); Pozzi (1974: 39-61), along with the authors mentioned in note 2. The ascription of *De interpretatione* to Apuleius is a matter of debate, however. There is an alternative view, argued by e.g. Lumpe (1982) and Ramsey (2017: 69-73), which holds that *De interpretatione* stems from a much later period in time (presumably the 5th century AD), and was even based on Capella. On this interpretation, Capella's *De nuptiis Mercurii et Philologiae* is the oldest preserved text that contains the square diagram. We wish to thank an anonymous reviewer for pointing us to this issue.

8 See Knuuttila (2017); Uckelman (2008: 395-396). Among the oldest texts to contain a modal square are William of Lucca's *Summa dialetice artis*, along with the anonymous *Logica* "*Cum sit nostra*", *Logica* "*Ut dicit*", *Summe Metenses*, and the *Dialectica Monacensis*, all of which date from the late twelfth or early thirteenth century; see De Rijk (1962-1967-ii.2: 392-394, 431, 484); Pozzi (1975: 109). The *terminus a quo* of the late twelfth century can be moved downwards to the late eleventh century if one accepts the so-called "Byzantine thesis" put forward by Carl Prantl (1855-1870-ii: 264-293) in the late nineteenth century. This thesis says, *inter cetera*, that the notorious *Synopsis of Aristotle's Science of Logic*, which Prantl claims played a decisive role in the development of medieval Latin terminist logic and, importantly, contains a modal square, was authored by the Byzantine scholar Michael Psellos (1018 – 1078). Thus if one accepts the Byzantine thesis, like Rescher (1969: 66), then it follows that the modal square originated not in the Latin West in the late twelfth century, but rather in the Greek East in the late eleventh century. Grabmann and others have however refuted the Byzantine thesis, by convincingly

is similar to that of the categorical square, in the sense that the modal square remained an often-discussed diagram in logic texts well into the early twentieth century, occurring in such texts as the *Institutiones logicae* (1626) of Franco Burgersdijk (1590 – 1635), the *Logique de Port-Royal* (1662) of Antoine Arnauld (1612 – 1694) and Pierre Nicole (1625 – 1695), and the *Petite logique* (1923) of Jacques Maritain (1882 – 1973).[9] What is less clear are the roots of the modal square, and, in particular, whether the modal square should be fathered on Aristotle. It is common knowledge that the modal square is somehow related with Aristotle's modal logic at *De int*. 12-13, and in particular with a diagram that Aristotle first introduces at *De int.* 13, 22a23-33 and later revises at *De int*. 13, 22b10-35 (printed as Tables 1 and 2 below). This diagram consists of four quadrants, and it involves the exact same modalities as the modal square. So *prima facie* it is tempting to say that this diagram constitutes the first occurrence of the modal square.

But can Aristotle's diagram *really* be viewed as the first occurrence of the modal square? Or, to put the same question differently, can this diagram, given Aristotle's surrounding discussion at *De int.* 12-13, be considered a geometrical representation of the same logical relations, between the same *relata*, as the ones occurring in the modal square? The relation between the modal square and the diagram at *De int*. 13, 22a23-33 has not yet been studied in detail – the present paper is, to our knowledge, the first one to address this topic in detail. But it has been touched upon in passing in a number of studies, on both the history of modal logic and systematic modal logic. A look at the scarce comments on the Aristotelian roots of the modal square in these studies reveals that scholars disagree on the issue.

The vast majority of past studies appear to be committed to one of two theses, which for ease of reference we shall call the "Weak and Strong Inheritance Theses". The two theses are of our own coinage, and their intricate details, which are discussed shortly, have not been described in past scholarship. What we find in past studies are a number of remarks and comments indicating that their authors have certain assumptions about the way in which the modal square is related to Aristotle's logic. These assumptions are explicated in our two theses, which should thus be regarded as *developing* the two predominant positions that have been taken in past scholarship, rather than *summarizing* these positions. The assessment of the two theses is the main focus of this paper. The next paragraphs introduce Weak and Strong Inheritance, and outline the goals and structure of the paper.

Some researchers explicitly father the modal square on Aristotle on the grounds of the diagram at *De int.* 13, 22a23-33. Among them are Carnielli and Pizzi (2008: 25), who call the modal square "Aristotle's square" (see also Pizzi 2017: 201); Fitting and Mendelsohn (1998: 7), who say that "the modal square is due to Aristotle"; Koslow (2010: 587), who dubs the modal square "Aristotle's modal square of opposition"; Rocci (2017: 18), who writes that the diagram at *De int.* 13, 22a23-33 can be "beautifully mapped onto" the modal square; and Sallantin, Dartnell and Afshar (2006: 232), who refer to the modal square as "the Aristotelian square of modalities". These remarks commit their authors to the view that the diagram at *De int.* 13, 22a23-33 amounts to the first occurrence of the modal square. They also commit them to the view that the discussions of the modal square found in medieval and post-medieval

arguing that the *Synopsis* is authored not by Psellos but by the fifteenth-century scholar Georgios Scholarios (c. 1400 – c. 1473); see, e.g., De Rijk (1972: lxi-lxvii); Grabmann (1937: 7-8).

9 Specifically on the *Logique de Port-Royal*, see Grey (2017).

sources are of limited originality, as they are derived from *De int.* 12-13. Such discussions are all cases of "old wine in new bottles". This is the Strong Inheritance Thesis.[10] It should be noted that most studies that are committed to Strong Inheritance are not historically oriented, but rather stem from systematic research on modal logic and argumentation theory.

Other researchers acknowledge the close ties between the modal square and the diagram at *De int.* 13, 22a23-33, but reject that the modal square should be fathered on Aristotle. Among them are Aho and Yrjönsuuri (2009: 49), who write that the modal square "follows Aristotle's presentation" of the relations between the modalities at *De int.* 12-13; Horn (2018) and Horn and Wansing (2015), who state that the modal square is "based on" *De int.* 12-13 (see also Horn 2001: 11-12); and Knuuttila (2017), who writes that the modal square is "modified from" *De int.* 12-13. Similarly, Kneale and Kneale (1962: 86) and Weidemann (2015: 262) point out that the relations between the modalities discussed at *De int.* 12-13 "may be set out simply in a square of opposition" (the Kneales) or can be "visualized with the aid of a modal square" (*mit Hilfe eines modallogischen Quadrats veranschaulichen*; Weidemann), yet do not suggest that the modal square is explicitly contained in Aristotle's text. These researchers are committed to accepting the view that the modal square historically developed from the diagram at *De int.* 13, 22a23-33, and they are committed to denying the view that Aristotle's diagram constitutes the first occurrence of the modal square. The modal square can at most be said to be inspired by Aristotle, just as the categorical square. Medieval and post-medieval discussions of the modal square go back to *De int.* 12-13, but they are not simply cases of "old wine in new bottles". This is the Weak Inheritance Thesis. It should be noted that most studies that are committed to Weak Inheritance are not systematically but historically oriented, and stem from research in ancient and medieval logic in particular.

As mentioned, neither Weak nor Strong Inheritance have been developed in depth in the scholarship, and even the very fact that there *is* a lack of consensus regarding the roots of the modal square has thus far almost entirely remained under the radar. This is almost certainly due to the fact that these theses find support in different research communities, each with a different literature. As mentioned, advocates of Strong Inheritance mostly work in systematic modal logic, while advocates of Weak Inheritance mostly work in the history of logic. Since only few systematic logicians are thoroughly familiar with the work of their historically-oriented colleagues and *vice versa*, it is little wonder that the advocates of Weak and Strong Inheritance are largely oblivious of each other. Yet as both theses are mutually incompatible, the nature of the relation between the modal square and *De int.* 12-13 is an *explanandum* – and an important one, too, since the properties and the history of logical diagrams have recently enjoyed a surge of interest among scholars, and their study is slowly crystallizing as a research program of its own (viz., logical geometry; see note 10). This *explanandum* is the topic of the present paper. The paper gives a rebuttal of Strong Inheritance, and provides a textually grounded account in favor of Weak Inheritance. The paper establishes two theses, one negative and one positive in nature:

[10] Using terminology from the contemporary research program of logical geometry, Strong Inheritance holds that Aristotle's own diagram and the modal square are *Aristotelian isomorphic* to each other. See Demey and Smessaert (2018) and Demey (2018) for more details on the notion of Aristotelian isomorphism.

(1) (*Negative Thesis*) At *De int*. 12-13, Aristotle describes only part of the relations occurring in the modal square.
(2) (*Positive Thesis*) All relations occurring in the modal square can be obtained from the logic at *De int.* 12-13 if this logic is further developed by means of an instance of a contraposition law that Aristotle proves elsewhere, at *An. pr.* I.46, 52a40-52b9.

The Negative Thesis implies that the diagram at *De int.* 13, 22a23-33 cannot be considered as the first occurrence of the modal square. Aristotle's diagram and the modal square may involve the same modalities, but the relations that Aristotle takes to hold between the modalities in his diagram are only a subset of the relations that hold between the modalities in the modal square. Since Strong Inheritance postulates an identity between these two sets of relations, it follows that once the Negative Thesis is established, then Strong Inheritance is refuted. The Positive Thesis holds that there is a logical principle that is valid in Aristotle's own logic and that allows for the completion of the diagram at *De int.* 13, 22a23-33 into the modal square. Thus the Negative and Positive Theses jointly provide strong evidence in support of the main claim of Weak Inheritance, viz. that the modal square is not identical with, yet historically developed from, the diagram at *De int*. 13, 22a23-33.

The structure of the paper is as follows. Sect. 2-4 engage in a close reading of those passages from *De int.* 12-13 where Aristotle sets out the relations that hold between the modalities that he includes in the diagram at *De int.* 13, 22a23-33. Sect. 5 wraps up the first part of our discussion, and establishes the Negative Thesis. We then move on to the Positive Thesis. Sect. 6 switches focus to *An. pr*. I.46, 52a40-52b9, and introduces the contraposition law by means of which the logic that is reconstructed in Sect. 2-4 can be further developed. Sect. 7 establishes the Positive Thesis. Sect. 8 concludes our discussion.

2. Internal and external negation of modal predicates: *De int.* 12

De int. 12-13 contains a modal logic that is based on the modalities of possibility (τὸ δυνατόν), contingency (τὸ ἐνδεχόμενον), necessity (τὸ ἀναγκαῖον) and impossibility (τὸ ἀδύνατον). In these passages, Aristotle does not consider these modalities as proposition-forming operators on propositions, as most contemporary logicians do, but rather as predicate-forming operators on predicates. Throughout *De int.* 12-13, Aristotle uses expressions of the form "*M* to be" (*M* εἶναι), where *M* is a schematic variable that is to be substituted by any of the four modalities just mentioned. Such expressions should be attached to a predicate, say *P*, to form complex predicates of the form "*M* to be *P*" (*M* εἶναι *P*). Once such complex predicates are said of a subject *S*, we obtain modal propositions of the form "*S* is *M* to be *P*" (*S* ἔστι *M* εἶναι *P*).[11] Considering modalities as predicate-forming operators on predicates allows Aristotle to reduce modal propositions to instances of the subject-copula-predicate template that underpins the discussion of the proposition throughout *De interpretatione*. Modal propositions are simply

[11] See, e.g., Ackrill (1963: 150); Seel (1982: 135-145); Weidemann (2015: 248-249); Whitaker (1996: 159-160); and esp. Weidemann (2012: 100): "[a]ccording to the position adopted by Aristotle ..., modal statements are formed out of assertoric statements neither by attaching the modal expressions "possible", "necessary", etc. as *semantic* (or *metalinguistic*) *predicates* to the names of assertoric statements, nor by attaching them as *statement-forming operators on statements* to assertoric statements themselves, but rather by attaching them as *predicate-forming operators on predicates* to the predicates of assertoric statements" (italics in the original).

complex versions of categorical propositions, with the additional complexity residing entirely inside the predicate. In the following, we write modal predicates using the prefix notation $M(P)$, where the letter M is substituted by either p (for possibility), c (for contingency), n (for necessity), or i (for impossibility). We write $M(\neg P)$ to indicate the internal negation of a modal predicate, which reads "M not to be P", and $\neg M(P)$ to indicate its external negation, which reads "not M to be P". A modal predicate that is negated both internally and externally is thus written as $\neg M(\neg P)$, which reads "not M not to be P".

The focus of *De int.* 12 is on the negation of modal predicates. The notion of contradiction can be lifted from propositions to predicates: a pair of predicates is a *contradictory* pair just in case **RCDP** is satisfied by the two propositions that are obtained by predicating the two predicates of that pair of one and the same subject. At *De int.* 12, Aristotle asks which modal predicates constitute contradictory pairs. In particular, this chapter addresses the question whether the contradictory negation of a modal predicate is obtained by negating the predicate internally or externally. Aristotle first considers internal negation, and focuses on the possible as a test case. Consider T1, which gives the passage at *De int*. 12, 21b10-20.

> T1: [i] So then, if this [viz., the idea that a modality's contradictory negation is its internal negation] holds good everywhere, then the negation of "possible to be" (τοῦ δυνατὸν εἶναι) is "possible not to be" (τὸ δυνατὸν μὴ εἶναι), and not "not possible to be" (τὸ μὴ δυνατὸν εἶναι). [ii] Yet it seems that for the same thing it is possible both to be and not to be (τὸ αὐτὸ δύνασθαι καὶ εἶναι καὶ μὴ εἶναι). [iii] For everything possible of being cut or of walking is possible also of not walking or of not being cut. The reason is that whatever is possible in this way is not always actual, so that the negation too will hold of it: what can walk is capable also of not walking, and what can be seen of not being seen. [iv] But it is impossible for opposite expressions to be true of the same thing (ἀδύνατον κατὰ τοῦ αὐτοῦ ἀληθεύεσθαι τὰς ἀντικειμένας φάσεις). [v] "Possible not to be", then, is not the negation of "possible to be". [vi] For it follows from these things that (1) either the same thing is said and denied of the same thing at the same time (τὸ αὐτὸ φάναι καὶ ἀποφάναι ἅμα καὶ κατὰ τοῦ αὐτοῦ), (2) or it is not by "to be" and "not to be" being added that affirmations and negations are produced. [vii] So if the former is impossible, we must choose the latter.[12]

A reconstruction of the argument in T1 is given here below.

[i]	The contradictory negation of $p(P)$ is $p(\neg P)$	Assumption, towards a contradiction
[iv]	If the pair $\{p(P), p(\neg P)\}$ is a contradictory pair, then, by **RCDP**, the conjunction "$p(P)$ and $p(\neg P)$" is a contradiction, meaning it cannot be truly predicated of one and the same subject	Assumption
[iii]	(Counterexamples) (1) "for every S, S is p(*walking*) and $p(\neg$*walking*$)$" is not a contradiction; (2) "for every S, S is p(*to be cut*) and $p(\neg$*to be cut*$)$" is not a contradiction	
[ii]	(Generalization) The conjunction "$p(P)$ and $p(\neg P)$" is not a contradiction	

[12] Translation by Barnes (1984-i: 34), with modifications.

[v] (Intermediary conclusion, regarding "possible to be P") the pair $\{p(P), p(\neg P)\}$ is not a contradictory pair — *Modus tollens*; [ii], [iv]

[vi] (Generalization) (1) either the internal negation is the contradictory negation, and then modal contradictories do not satisfy **RCDP** (following [ii]), (2) or the internal negation is not the contradictory negation

[vii] (General conclusion) (1) is to be rejected. Thus (2) follows.[13]

In T1, Aristotle shows that his first working hypothesis cannot be maintained: the contradictory negation of a modal predicate is not established by inserting an internal negation (if the predicate was not already internally negated) or removing an internal negation (if the predicate was already internally negated). Elsewhere, at *De int.* 13, 22b1-2 (T2), he briefly considers a second hypothesis, and asks whether the contradictory negation of an internally negated modal predicate can perhaps be obtained not by removing its internal negation, but rather by replacing this negation with an external negation. Here the case of "necessary not to be P" ($n(\neg P)$) provides a counterexample. The contradictory negation of $n(\neg P)$ would then be $\neg n(P)$. But, so Aristotle says, $n(\neg P)$ entails $\neg n(P)$. So $\neg n(P)$ is true of any subject of which $n(\neg P)$ is true; and thus, in particular, propositions of the form "S is $n(\neg P)$" and "S is $\neg n(P)$" can be true together. It follows that such propositions do not satisfy **LNC**, and thus neither **RCDP**. Hence $\{n(\neg P), \neg n(P)\}$ is not a contradictory pair, and the second hypothesis should also be rejected.

T2: For the negation of "necessary not to be" (τοῦ ἀνάγκη μὴ εἶναι) is not "not necessary to be" (τὸ οὐκ ἀνάγκη εἶναι). For both may be true of the same thing, since the necessary not to be is not necessary to be (τὸ γὰρ ἀναγκαῖον μὴ εἶναι οὐκ ἀναγκαῖον εἶναι).[14]

So T1 shows that the contradictory negation of a modal predicate is not obtained by adding or removing an internal negation; and T2 shows that the contradictory negation of an internally negated modal predicate is not obtained by replacing the internal negation with an external negation. At *De int*. 12, 21b23-26 (T3), Aristotle concludes that internal negation is irrelevant to the question of modal contradiction, and lays down that a modal predicate's contradictory negation is obtained by either adding or removing an external negation, again depending on whether the predicate was already externally negated.

T3: The negation of "possible to be" (τοῦ δυνατὸν εἶναι), therefore, is "not possible to be" (τὸ μὴ δυνατὸν εἶναι). The same account holds for "contingent to be" (τοῦ ἐνδεχόμενον εἶναι). Its negation is "not contingent to be" (τὸ μὴ ἐνδεχόμενον εἶναι). Similarly with the others, "necessary" and "impossible".[15]

At *De int.* 12, 22a5-14, Aristotle accepts that the following pairs are all contradictory pairs. In agreement with T3, one member of each contradictory pair is the external negation of that pair's other member.

- $\{p(P), \neg p(P)\}$
- $\{c(P), \neg c(P)\}$
- $\{p(\neg P), \neg p(\neg P)\}$
- $\{c(\neg P), \neg c(\neg P)\}$

13 See also Weidemann (2015: 249-257); Whitaker (1996: 157-158).
14 Translation by Barnes (1984-i: 35).
15 Translation by Barnes (1984-i: 34), with modifications.

- $\{i(P), \neg i(P)\}$
- $\{i(\neg P), \neg i(\neg P)\}$
- $\{n(P), \neg n(P)\}$
- $\{n(\neg P), \neg n(\neg P)\}$

3. Aristotle's modal diagram (*De int.* 13, 22a14-22b9)

At *De int.* 13, 22a14-22b9, Aristotle moves on to discuss entailment relations holding between the modal predicates which he introduced at *De int.* 12. At *De int.* 13, 22a22-32, he gives the diagram that adherents of Strong Inheritance take to amount to the first occurrence of the modal square of opposition (see Sect. 1). This diagram is printed as Table 1 below. Just as in T1, the possible is taken as the point of reference. It is affirmed in (I), negated externally in (II), negated internally in (III), and negated both internally and externally in (IV).

(I)	Possible to be	Not possible to be	(II)
	Contingent to be	Not contingent to be	
	Not impossible to be	Impossible to be	
	Not necessary to be	Necessary not to be	
(III)	Possible not to be	Not possible not to be	(IV)
	Contingent not to be	Not contingent not to be	
	Not impossible not to be	Impossible not to be	
	Not necessary not to be	Necessary to be	

Table 1: Aristotle's modal diagram at *De int.* 13, 22a14-22b28

The predicates of possibility occur horizontally adjacently to their contradictories in all cases. The predicate $p(P)$, in (I), is horizontally adjacent to $\neg p(P)$ in (II); and $p(\neg P)$, in (III), is horizontally adjacent to $\neg p(\neg P)$ in (IV). The same holds of the predicates of contingency and impossibility, as can be seen, but not of those of necessity. Aristotle remarks on the disposition of the predicates of necessity at *De int.* 13, 22a38-39 (T4), which is the passage immediately preceding T2 printed above.

T4: How things stand with the necessary should now be discussed. Evidently things are different here: it is contraries which follow, and the contradictories are separated (αἱ ἐναντίαι ἕπονται· αἱ δ᾽ἀντιφάσεις χωρίς).[16]

T4 is among the most obscure passages of *De int.* 12-13. Following Weidemann, who takes his cue from Boethius, Aristotle's point in T4 is presumably that the contradiction relations between predicates of necessity are not placed *horizontally* adjacently, as is the case with all other predicates, but rather *diagonally*: the contradictory of $\neg n(P)$ in (I) does not occur in (II), as was to be expected, but rather in (IV); and the contradictory of $\neg n(\neg P)$, in (III), does not occur in (IV), but rather in (II). The predicate occurring horizontally adjacently to $\neg n(P)$ in (I) is $n(\neg P)$; and $n(\neg P)$ is contrary to $n(P)$, which itself is the contradictory of $\neg n(P)$. So $\neg n(P)$, in (I), occurs horizontally adjacently to the contrary of its contradictory. The same holds of $\neg n(\neg P)$ in (III). It is placed horizontally adjacently to $n(P)$; and $n(P)$, as we have just seen, is the contrary of $n(\neg P)$, which itself is the contradictory of $\neg n(\neg P)$. So $\neg n(\neg P)$ also occurs horizontally

[16] Translation by Barnes (1984-i: 35), with modifications.

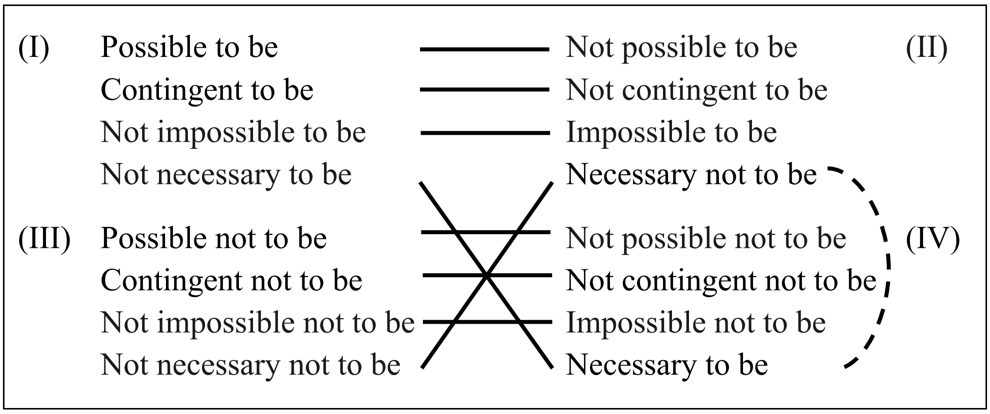

Fig. 3: A visualization of T4

adjacently to the contrary of its contradictory.[17] This is visualized in Fig. 3. As before, solid lines indicate contradiction, and dashed lines contrariety. What is most important to our purposes is not the disposition of the formulas of necessity, but rather the assumption underlying the argument in T4, at least on the Weidemann-Boethius reading of this passage: Aristotle holds that $n(P)$ is contrary to $n(\neg P)$.

At *De int.* 13, 22a15-21 (T5), Aristotle states that for each quadrant, the predicate of possibility in that quadrant entails all other predicates in that quadrant. With regard to (I) he lays down that the predicate of contingency also reverse-entails, and is thus equivalent to, the predicate of possibility. He does not say this about the predicates of possibility and contingency in (II), (III) and (IV). But he does say that *both* the predicate of possibility *and* the predicate of contingency in each of these quadrants entail the predicates of impossibility and necessity in that quadrant. This suggests that he is assuming the equivalence of the possible and the contingent also in quadrants (II), (III) and (IV).[18] We will see below that these equivalences are indeed implied in the text, even though they are not explicitly described by Aristotle.[19]

T5: With this treatment the relations of following (αἱ ἀκολουθήσεις) work out in a reasonable way. From "possible to be" follow "contingent to be" (and, reciprocally, the former from the latter [καὶ τοῦτο ἐκείνῳ ἀντιστρέφει]) and "not impossible to be" and "not necessary to be". From "possible not to be" and "contingent not to be" follow both "not necessary not to be" and "not impossible not to be". From "not possible to be" and "not contingent to be" follow "necessary not to be" and "impossible to be". From "not possible not to be" and "not contingent not to be" follow "necessary to be" and "impossible not to be".[20]

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17 Weidemann (2012: 103; 2015: 260-262).

18 See also Weidemann (2015: 258); Malink (2016: 39 [n. 26]).

19 Aristotle discusses the relation between the possible and the contingent in several places, but these different discussions cannot be unified into a single, coherent account. At *An. pr.* I.13, 32a18-20, for instance, he says that the contingent signifies "that which is not necessary but, being assumed, results in nothing impossible" (translation by Barnes 1984-i: 51), i.e., the two-sided possible; while at *Met.* IX.3, 1047a24-26, he says that something is possible "if there is nothing impossible in its having the actuality of that of which it is said to have the capacity" (translation by Barnes 1984-ii: 1653), from which it follows that the possible is the one-sided possible; see esp. Mignucci (2002). Putting these two passages together, we obtain that the contingent entails the possible, though not *vice versa*. This view agrees with the common usage in both medieval and contemporary modal logic (see, e.g., Cook 2009: 66, 224; Knuuttila 2017), but since it does not take the contingent and the possible to be equivalent, it is clearly at odds with Aristotle's account in the passage at *De int*. 13, 22a15-21.

20 Arist., *De int*. 13, 22a14-16. Translation by Barnes (1984-i: 35), with modifications.

The predicate of impossibility, in each quadrant, also entails the predicate of necessity in that quadrant. This is laid down, for (II) and (IV), at *De int.* 13, 22b5-7 (T6). With regard to (I), Aristotle assumes that $\neg i(P)$ entails $\neg n(P)$ at *De int.* 13, 22b15-17, which is discussed as T9 below. With regard to (III), Aristotle does not explicitly say that $\neg i(\neg P)$ entails $\neg n(\neg P)$, but we shall see shortly that this, too, is implied in the text.

T6: For if it is impossible to be, it is necessary for this (not, to be, but) not to be (ἀναγκαῖον τοῦτο οὐκ εἶναι ἀλλὰ μὴ εἶναι); and if it is impossible not to be, it is necessary for this to be.[21]

4. The modal quadrants revised: on the semantics of the possible (*De int.* 13, 22b10-35)

As the attentive reader may have noticed, Aristotle uses the term "possible" in the sense of both the one-sided possible and the two-sided possible throughout T1-6. Take the entailment from $p(P)$ to $\neg n(P)$ in (I), for instance. This entailment does not hold on the one-sided reading of "possible", but only on the two-sided reading. For then it amounts to applying simplification: on the two-sided reading of the possible, $p(P)$ is equivalent to the conjunction of $p(P)$ and $\neg n(P)$, and thus it entails either conjunct. But consider the right-hand side of Table 1. There the term "possible" is used in its one-sided sense. In (II), for instance, $\neg p(P)$ is said to entail $n(\neg P)$. This holds on the one-sided reading of "possible", but not on its two-sided reading. For on the two-sided reading, $\neg p(P)$ merely entails the disjunction of $n(P)$ and $n(\neg P)$, rather than $n(\neg P)$ in specific.

In other words, Aristotle's use of the term "possible" is incoherent. If not negated externally, then this term signifies the two-sided possible; and if negated externally, then it signifies the one-sided possible. This ambiguity also affects "contingent", due to its equivalence with "possible", as well as "impossible", which occurs as the negation of the one-sided possible on the right-hand side of Table 1, and as the negation of the two-sided possible on the left-hand side of Table 1. Take the entailment from $\neg i(P)$ to $\neg n(P)$ in (I). If "impossible" signifies the negation of the one-sided possible, then, assuming double negation elimination, $\neg i(P)$ is equivalent to the one-sided possible, and thus does not entail $\neg n(P)$. But the entailment does hold if "impossible" signifies the negation of the two-sided possible, as can be seen as follows:

(1)	$\neg i(P)$	$\equiv$	$\neg(\neg(p \text{ and } \neg n))(P)$	
(2)	$\neg i(P)$	$\equiv$	$(p \text{ and } \neg n)(P)$	double negation elimination, (1)
(3)	$\neg i(P)$	$\Rightarrow$	$\neg n(P)$	simplification, (2)

Likewise, in (III), $\neg i(\neg P)$ entails $\neg n(\neg P)$ if "impossible" is the negation of two-sided possible:

(1)	$\neg i(\neg P)$	$\equiv$	$\neg(\neg(p \text{ and } \neg n))(\neg P)$	
(2)	$\neg i(\neg P)$	$\equiv$	$(p \text{ and } \neg n)(\neg P)$	double negation elimination, (1)
(3)	$\neg i(\neg P)$	$\Rightarrow$	$\neg n(\neg P)$	simplification, (2)

[21] Translation by Barnes (1984-i: 35). The passage at *De int*. 13, 22b5-7 does not establish that the predicates of necessity and impossibility in (II) and (IV) are equivalent, *pace* Weidemann (2015: 258-259).

But $\neg i(\neg P)$ does not entail $\neg n(\neg P)$ if "impossible" is the negation of the one-sided possible. Let's now move from the left- to the right-hand-side of Table 1. In (II), $i(P)$ entails $n(\neg P)$ only if "impossible" signifies the negation of the one-sided possible:

(1)	$i(P)$	$\equiv$	$\neg p(P)$
(2)	$i(P)$	$\Rightarrow$	$n(\neg P)$

The same holds for the entailment from $i(\neg P)$ to $n(P)$ in (IV):

(1)	$i(\neg P)$	$\equiv$	$\neg p(\neg P)$
(2)	$i(\neg P)$	$\Rightarrow$	$n(P)$

It is likely that the lack of coherency in the use of "possible" is due to Aristotle's taking his cue from everyday speech. As several scholars have pointed out, his wavering between the two senses of the possible is understandable from the perspective of Gricean conversational implicature.[22] Under the assumption that any speaker tries to be as informative as possible (i.e., the maxim of quantity), someone saying that x is possible is, in many cases, not only asserting that x is one-sided possible, but is additionally implying that, for all he knows, x is not necessary. By combining the semantic meaning of "possible" (one-sided possibility) and its pragmatic implicature (non-necessity), we obtain the two-sided interpretation. Negation in natural language however does not usually cause the addition of an implicature to an expression's meaning. Thus when a speaker says that x is not possible, then he is most probably simply denying that x is one-sided possible, without implying anything else. The situation is different with doubly negated expressions like "x is not impossible", of course, which usually do trigger the extra implicature.

The equivocality of "possible" is however highly problematic. It threatens to undermine the entire system contained in *De int.* 12-13. Consider T7, which gives the passage at *De int.* 13, 22b10-19. Aristotle there establishes by means of a *reductio* proof that $n(P)$ entails $p(P)$. He adds that the reverse-entailment is not valid: $p(P)$ does not entail $n(P)$, nor does it entail $n(\neg P)$.

T7: [i] The necessary to be is possible to be (τὸ μὲν γὰρ ἀναγκαῖον εἶναι δυνατὸν εἶναι). [ii] Otherwise the negation will follow, since [iii] it is necessary either to affirm or to deny something of something (ἀνάγκη γὰρ ἢ φάναι ἢ ἀποφάναι). And then, [iv] if it is not possible to be, it is impossible to be (εἰ μὴ δυνατὸν εἶναι, ἀδύνατον εἶναι). So [v] the necessary to be is impossible to be, which is absurd (ἀδύνατον ἄρα εἶναι τὸ ἀναγκαῖον εἶναι, ὅπερ ἄτοπον). However, it is not "necessary to be" nor yet "necessary not to be" that follows from "possible to be".[23]

Sentence [i], which is the principle that T7 sets out to prove, is a simple categorical proposition containing a modal predicate as both its subject term ("necessary to be P") and predicate term ("possible to be P"). It is syntactically an indefinite proposition, lacking any quantifier, but it is equivalent in meaning to the (slightly awkward) *A*-proposition "Every 'necessary to be P' is 'possible to be P'" – or, more fluently, "Every necessity is a possibility" –, which says that every predicate that can be necessarily said of a subject can also be possibly said of that same

[22] See, e.g., Horn (1973: 205-208); Malink (2016: 47).
[23] Translation by Barnes (1984-i: 35-36), with modifications.

subject. Bearing in mind that affirmative propositions in Aristotle have existential import, we can formalize this proposition as follows:

$$\exists P(n(P)) \wedge \forall P(n(P) \rightarrow p(P))$$

where the first conjunct says that *n* is non-empty: there is at least one predicate *P* that can be necessarily said of an assumed subject *S*. So because Aristotle interprets modal expressions as predicates, he simply has to establish the truth of an *A*-proposition, which is one of the proposition types that he had introduced at *De int.* 6. This means that he can rely on the usual machinery for subject-copula-predicate propositions. In particular, the proof in T7 makes use of the following principles:

- The result obtained in T3: the contradictory negation of a modal predicate is its external negation (step 2)
- The pair $\{A, O\}$ satisfies **RCDP** (step 2)
- **LEM** (step 5)
- The entailment from $\neg p(P)$ to $i(P)$, which is one of the relations recognized in (II) in Table 1 (step 3)
- The *reductio* proof method, familiar from the categorical syllogistic (see esp. *An. pr.* I.5-6) (step 4*bis*)
- The disjunctive syllogism, which is mainly known from the *Topica* (see esp. *Top*. II.6, 112a24-31): $(\varphi \vee \psi)$ and $\neg\psi$ jointly entail φ (step 6)

Here is a reconstruction. For reasons of brevity, the existence claim $\exists P(n(P))$ has been left out; but the non-emptiness of *n* is still being assumed.

(1)	$\neg\forall P(n(P) \rightarrow p(P))$	Assumption, towards a contradiction	[ii]
(2)	$\exists P(n(P) \wedge \neg p(P))$	Equivalent to (1), following T3 and **RCDP**	
(3)	$\exists P(n(P) \wedge i(P))$	Following (II), Table 1	[iv]
(4)	$\bot$	(3) is impossible	[v]
(4*bis*)	$\neg(\neg\forall P(n(P) \rightarrow p(P)))$	*reductio*, (1), (4)	
(5)	$\forall P(n(P) \rightarrow p(P))$ or $\neg\forall P(n(P) \rightarrow p(P))$	**LEM**	[iii]
(6)	$\forall P(n(P) \rightarrow p(P))$	Disjunctive syllogism, (4*bis*), (5)	[i][24]

[24] There has been some debate about Aristotle's argument at T7, and, in particular, about what Aristotle means by "the negation" when he says that "otherwise the negation will follow" ([ii]). According to *De int*. 6, the (syntactic) negation of a non-modal proposition is established by negating the copula. Thus the negation of "the necessary to be is possible to be" is "the necessary to be is not possible to be". But there are some doubts about the correct interpretation of the latter proposition. Scholars have interpreted it as either "Some 'necessary to be *P*' is not 'possible to be *P*'" ($\exists P(n(P) \wedge \neg p(P))$ in symbols); or "Every 'necessary to be *P*' is not 'possible to be *P*'" ($\forall P(n(P) \rightarrow \neg p(P))$ in symbols). On the first reading, which is adopted in the above reconstruction and is also accepted by Malink (2016: 42-44), Weidemann (2015: 263-264; 2017: 192-194) and Whitaker (1996: 165-167), among others, Aristotle's argument is valid. On the second reading, which is imputed to Aristotle by Ackrill (1963: 152), Bluck (1963: 218-220), Brandon (1978: 174-175) and Rehder (1980: 60-61), among others, the argument is invalid. The second reading is however unlikely. As pointed out, Aristotle considers the indefinite proposition "the necessary to be is possible to be" as being equivalent to an *A*-proposition. Thus, by

The result that $n(P)$ entails $p(P)$ lays bare the problems arising from the equivocality of the term "possible" in T1-6, and is thus of considerable importance. It is therefore unsurprising that Aristotle goes to some lengths to defend it. He gives a second proof in a parallel passage at *De int*. 13, 22b29-33 (T8), apparently in order to counter the objection that the entailment from $n(P)$ to $p(P)$ purportedly requires that we accept the result obtained in T3, which together with **RCDP** licenses the transition from step (1) to step (2) in the above argument. Even if we reject that result and prefer to stick with the principle that the contradictory negation of a modal predicate is its internal negation, which Aristotle proposed as the first working hypothesis in T1, then, still, we should endorse that $n(P)$ entails $p(P)$: the internal negation of the possible suffices to derive an impossibility.

T8: Someone might ask whether "possible to be" follows from (ἕπεται) "necessary to be". For [i] if it does not follow, the contradictory will follow, "not possible to be" (ἡ ἀντίφασις ἀκολουθήσει, τὸ μὴ δυνατὸν εἶναι). And if someone were to say that this is not the contradictory, [ii] then he should say that "possible not to be" is (ἀνάγκη λέγειν τὸ δυνατὸν μὴ εἶναι); both of which are false of "necessary to be".[25]

The proof suggested by T8 runs parallel to the one suggested by T7. Here is a reconstruction.

(1)	$\neg\forall P(n(P) \to p(P))$	Assumption, towards a contradiction	[i]
(2)	$\exists P(n(P) \wedge p(\neg P))$	Equivalent to (1), following T1 and **RCDP**	[ii]
(3)	$\exists P(n(P) \wedge ((p \text{ and } \neg n)(\neg P)))$	Equivalent to (2), following the semantics of $p(\neg P)$ in (III)	
(3*bis*)	$\exists P(n(P) \wedge ((p \text{ and } \neg n)(P)))$	Equivalent to (3), following the equivalence of the two-sided possible and its internal negation[26]	
(4)	$\exists P(n(P) \wedge \neg n(P))$	Simplification, (3*bis*)	
(5)	$\bot$	(4) is impossible, $\{n(P), \neg n(P)\}$ is a contradictory pair following T3	
(5*bis*)	$\neg(\neg\forall P(n(P) \to p(P)))$	*reductio*, (1), (5)	
(6)	$\forall P(n(P) \to p(P))$ or $\neg\forall P(n(P) \to p(P))$	**LEM**	
(7)	$\forall P(n(P) \to p(P))$	Disjunctive syllogism, (5*bis*), (6)[27]	

Now consider T7-8's theorem, which is a forerunner of the present-day modal axiom D (i.e., $\Box\varphi \to \Diamond\varphi$), in light of the diagram in Table 1. Following T5, $p(P)$ in (I) entails $\neg n(P)$. So by the entailment from *n(P)* to *p(P)* established in T7-8 and the transitivity of entailment, we arrive at the conclusion that $n(P)$ entails $\neg n(P)$. Thus $n(P)$ and $\neg n(P)$ can both be true of one and the same subject. But this as an impossible position following T3, where it is laid down that $\{n(P), \neg n(P)\}$ is a contradictory pair, which means, in particular, that there is no predicate that can be

maintaining the second reading, we hold Aristotle to the claim that an *A*- and *E*-proposition satisfy **LEM**. Aristotle however expressly rejects this at *De int.* 7, 17b17-23.

25 Translation by Barnes (1984-i: 36), with modifications.

26 Aristotle states this equivalence on several occasions, including *De int.* 12, 21b35-37, as part of his discussion of the contradictory negation of modal predicates, and *An. pr.* I.13, 32a30-32b4, in the context of the modal syllogistic.

27 Notice that **LEM** is not expressly mentioned in T8. It is added in analogy with the argument in T7.

both necessarily and not necessarily said of the same subject. The passage at *De int.* 13, 22b15-17 (T9) shows that Aristotle was aware of this inconsistency.

T9: However, from "possible to be" follows "not impossible to be", and from this follows "not necessary to be"; with the result that the necessary to be is not necessary to be, which is absurd.[28]

Aristotle also realizes that the root of the problem lies with the equivocality of the term "possible". At *De int.* 13, 22b21-28 (T10), he proposes a revised version of Table 1. This version is printed in Table 2.

T10: [i] It remains, therefore, for "not necessary not to be" to follow "possible to be" (λείπεται τοίνυν τὸ οὐκ ἀναγκαῖον μὴ εἶναι ἀκολουθεῖν τῷ δυνατὸν εἶναι) ... Moreover, this [i.e., "not necessary not to be"] proves to be contradictory to what follows from "not possible to be"; for on that follow both "impossible to be" and "necessary not to be", whose negation is "not necessary not to be". [ii] So these contradictories also follow in the aforesaid manner (ἀκολουθοῦσί τε ἄρα καὶ αὗται αἱ ἀντιφάσεις κατὰ τὸν εἰρημένον τρόπον), and [iii] nothing impossible results when they are so placed.[29]

At [i], Aristotle points out that $\neg n(P)$ in (I) should be replaced by $\neg n(\neg P)$. It follows that $\neg n(\neg P)$ in (III) should be replaced by $\neg n(P)$, although Aristotle does not expressly say this. The predicate $p(P)$ entails $\neg n(\neg P)$ on both the one- and the two-sided interpretation of "possible" (I). Likewise, $p(\neg P)$ entails $\neg n(P)$ on both interpretations of "possible" (III). Remember that "possible" already signified the one-sided possible in (II) and (IV), i.e., in the right-hand side of Table 1. It continues to do so in the right-hand side of Table 2. Thus all entailments in Table

(I)	Possible to be	Not possible to be	(II)
	Contingent to be	Not contingent to be	
	Not impossible to be	Impossible to be	
	Not necessary not to be	Necessary not to be	
(III)	Possible not to be	Not possible not to be	(IV)
	Contingent not to be	Not contingent not to be	
	Not impossible not to be	Impossible not to be	
	Not necessary to be	Necessary to be	

Table 2: Aristotle's modal diagram at *De int.* 13, 22a14-22b28

2 hold under the one-sided interpretation of "possible". It follows that unlike Table 1, Table 2 is coherent. Also notice that the disposition of the contradictory predicates of necessity now neatly parallels the disposition of the contradictory predicates of possibility, contingency, and impossibility (see Fig. 4). Aristotle points out this parallelism at [ii]: $\neg n(\neg P)$ and $n(\neg P)$ now "follow in the aforesaid manner", in the sense that they now occur horizontally adjacently, like the contradictory predicates of possibility, contingency and impossibility.[30] At [iii], Aristotle

28 Translation by Barnes (1984-i: 36).

29 Translation by Barnes (1984-i: 36), with modifications.

30 The verb ἀκολουθεῖν thus carries a rather different meaning in [i] and [ii]: in [i], it indicates a relation of following *qua* entailment, and in [ii] it indicates a relation of following *qua* disposition, like the verb

states that these revisions render the theory coherent. This is correct, as we have just seen. But Aristotle seems to be merely *postulating* this coherence, without actually *proving* it; and this is questionable. That is, Aristotle seems to be concerned only with the operation of swapping the predicates $\neg n(P)$ and $\neg n(\neg P)$. He seems to be assuming that the entailments that were stated as part of the original theory continue to hold in the revised theory, i.e., after $\neg n(P)$ and $\neg n(\neg P)$ have switched places. Entailments that do not involve either of these two predicates are obviously unaffected by the swapping operation, and thus straightforwardly continue to hold. Consequently, there is no issue with quadrants (II) and (IV) in Table 2. But matters are not so straightforward with entailments involving the predicates of necessity in quadrants (I) and (III). In the original theory, stated in T5, there are entailments from $p(P)$ to $\neg n(P)$ (I), and from $p(\neg P)$ to $\neg n(\neg P)$ (III). These entailments only hold under the two-sided interpretation of "possible", as we have already seen. In the revised version of the theory, they become entailments from $p(P)$ and $p(\neg P)$ (now under a *one-sided* interpretation of "possible") to resp. $\neg n(\neg P)$ (now *with* an internal negation, because of swapping) and $\neg n(P)$ (now *without* an internal negation, because of swapping). The validity of these revised entailments, it seems, needs to be proven independently; it cannot simply be assumed that swapping is entailment-preserving. The same goes for $\neg i(P)$ in (I), which entails $\neg n(P)$ on the original theory and $\neg n(\neg P)$ on the revised theory; and for $\neg i(\neg P)$ in (III), which entails $\neg n(\neg P)$ on the original theory and $\neg n(P)$ on the revised theory.[31]

(I)	Possible to be	Not possible to be	(II)
	Contingent to be	Not contingent to be	
	Not impossible to be	Impossible to be	
	Not necessary not to be	Necessary not to be	
(III)	Possible not to be	Not possible not to be	(IV)
	Contingent not to be	Not contingent not to be	
	Not impossible not to be	Impossible not to be	
	Not necessary to be	Necessary to be	

Fig. 4: Revised version of Fig. 3

5. The Negative Thesis

ἕπεσθαι in T4; see Seel (1982: 152). The properties of ἀκολουθεῖν as indicating entailment are discussed in more detail shortly.

31 Maybe Aristotle provides a partial proof of the coherence of his revised theory at *De int.* 13, 22b17-22, where he argues that $p(P)$ entails $\neg n(\neg P)$. The passage is however notoriously obscure, which is largely due to Aristotle's frequent use of anaphoric pronouns, and it is also rather unclear whether $p(P)$ in this passage should be interpreted as the one-sided possible or rather as the two-sided possible. *De int.* 13, 22b17-22 is the passage immediately following T9 and preceding T10, and thus in light of the development of Aristotle's argument, from T7-8 over T9 to T10, it is more natural to adopt the one-sided reading; but the passage also makes sense on the two-sided reading. Weidemann, following Ammonius, adopts the one-sided reading; see Busse (1897: 236 [ll. 20-34]); Weidemann (2015: 264). Boethius adopts the two-sided reading in his first commentary on *De interpretatione*; see Meiser (1877: 193-195). Whatever the case, the fact remains that Aristotle does not try to prove the other entailments in (I) and (III) that are affected by swapping.

The distribution of the modalities in Table 2 is identical to the distribution of the modalities in Fig. 2's modal square of opposition, in the sense that each of the four quadrants in Table 2 precisely parallels a corner in the square from Fig. 2. Thus the basic structure of the modal square is obtained from Table 2 by simply rearranging the sequence of its quadrants. This is done in Fig. 5. Moreover, T1-10 discuss *all* relations between modal predicates that Aristotle gives at *De int*. 12-13 and occur in the modal square. For the sake of clarity, these relations are listed again in Table 3, and they are mapped onto the rearranged version of Table 2 in Fig. 6. This figure gives us a litmus test regarding the relation between the modal square and *De int.* 12-13. Strong Inheritance is true only if the relations in Fig. 2 *coincide* with the relations in Fig. 6; while Weak Inheritance is true only if the relations in these two figures *do not* coincide. Hence, in order to solve the issue regarding the Aristotelian roots of the modal square, settling the relation between Fig. 2 and Fig. 6 is an important first step.

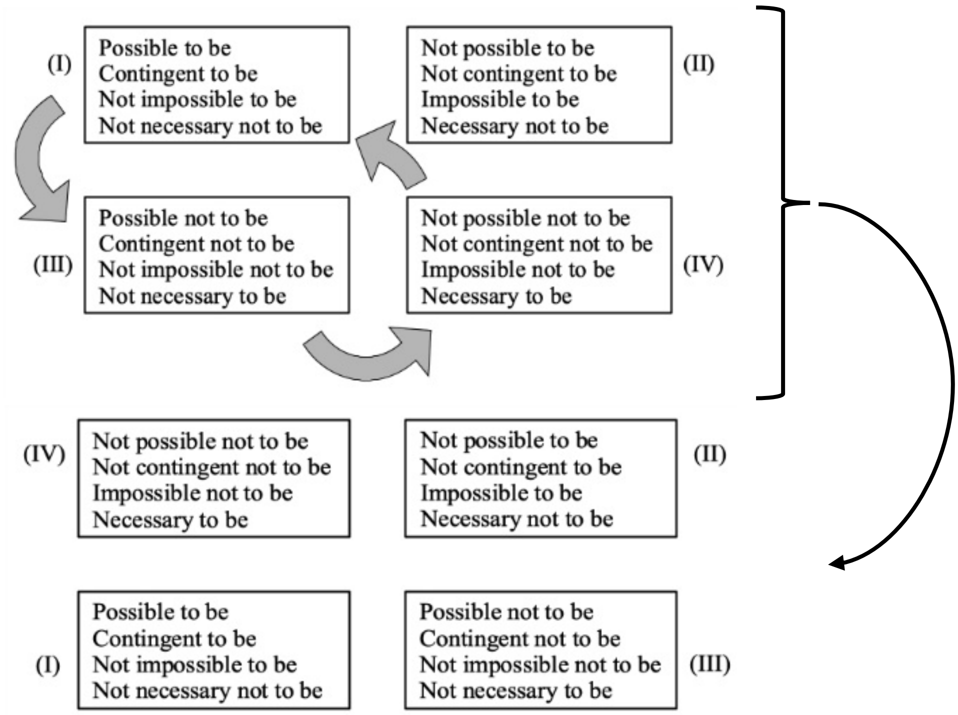

Fig. 5: The sequence of the modal quadrants rearranged

The comparison of Fig. 2 and Fig. 6 shows that these figures differ in a number of respects. For a start, the vertices in Fig. 2 are all equivalence classes; that is, the four formulas occurring inside each vertex are all logically equivalent to each other. By contrast, this is not the case for the quadrants in Fig. 6. Aristotle determines some entailment relations inside each quadrant, but these relations fall short of establishing that the four formulas in each quadrant are mutually logically equivalent. There is nothing in *De int.* 12-13 suggesting that Aristotle was committed to the view that these quadrants are equivalence classes, although some studies postulate that he was.[32] The matter is, admittedly, not entirely straightforward, especially in view of *An. pr.* I.13, 32a23-27. This is a parallel passage to *De int.* 13, 22a14-36, and Aristotle there states that "not contingent to belong", "impossible to belong" and "necessary not to belong" are "either

[32] See, e.g., Bocheński (1951: 59); Correia (2017a: 11); Kneale and Kneale (1962: 84); Seel (1982: 154, 157); Thom (2003: 6). See also note 38.

the same or follow one another" (ταὐτά ἐστιν ἢ ἀκολουθεῖ ἀλλήλοις), by which he means that these expressions are equivalent.[33] He adds that the same goes for their contradictories "contingent to belong", "not impossible to belong" and "not necessary not to belong". The first three expressions ("not contingent to belong", "impossible to belong" and "necessary not to belong") all occur in quadrant (II) in Fig. 6; their contradictories all occur in (I). Moreover, we know from T5 that Aristotle took the predicate of possibility in each quadrant to be equivalent to the predicate of contingency in that quadrant. Thus if we combine *An. pr.* I.13, 32a23-27 with *De int.* 12-13, then we obtain that Aristotle did accept that (I) and (II) in Fig. 6 are equivalence classes. That he also accepted that (III) and (IV) are equivalence classes then follows by analogy. However, this argument is unconvincing, mainly because the authenticity of *An. pr.* I.13, 32a23-27 is open to doubt.[34] Moreover, even if the passage would turn out to be authentic, as has recently been argued by Malink,[35] then it still remains true to say (i) that there is no textual evidence that Aristotle took *all* quadrants in Fig. 6 as equivalence classes, and, most importantly, (ii) that *De int.* 12-13 by itself certainly does not contain any clue in that direction.

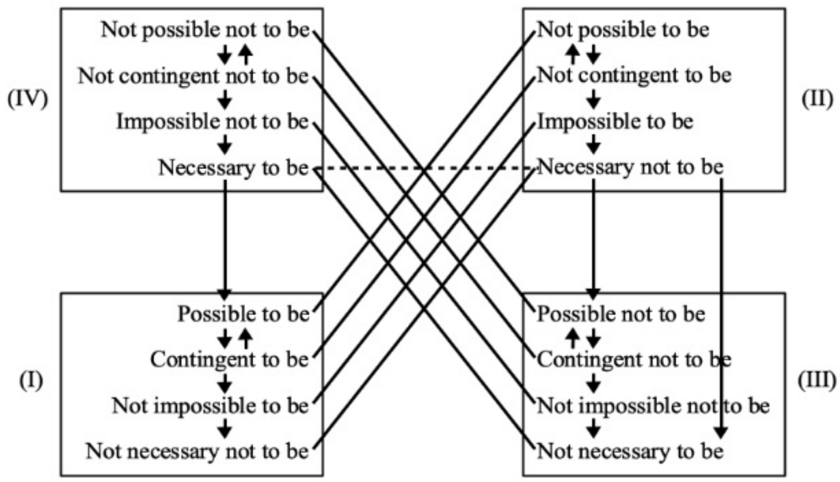

Fig. 6: The relations from T1-10 mapped onto the modal quadrants (rearranged)[36]

There is also a further issue. For Fig. 2 and Fig. 6 to involve the same relations, it is required that there be a subcontrariety relation between $p(P)$ and $p(\neg P)$ (or, rather, between (I) and (III)), and a subalternation relation between $n(\neg P)$ and $\neg n(P)$ (or, rather, between (II) and (III)). Neither relation however occurs in Fig. 6. Subcontrariety, as we have seen, involves two components: the *relata* should satisfy **LEM**, and they should not satisfy **LNC**. Aristotle does not address either condition. True, he accepts, in T1, that $p(P)$ and $p(\neg P)$ can be true together

[33] Notice that Aristotle again refers to the modalities by using elliptical expressions, though here not involving the verb "to be" (εἶναι), but rather the verb "to belong" (ὑπάρχειν). These elliptical expressions are equivalent in meaning to the complex modal predicates from *De int.* 12-13.

[34] The passage is considered spurious by, e.g., Ross (1957: 327-328); Seel (1982: 163); Smith (1989: 125-126); Striker (2009: 128-129).

[35] See Malink (2016).

[36] As opposed to Fig. 1 and Fig. 2, arrows in Fig. 6 indicate entailment relations, not subalternation relations. Also note that within each quadrant, we do not display those entailments that follow (by transitivity) from the entailments that are displayed, in order to optimize visual clarity. This concretely means that (i) the entailment from the predicate of possibility to the predicate of impossibility and (ii) the entailments from the predicates of possibility and contingency to the predicate of necessity in each quadrant are not displayed.

• T1	$p(P)$ and $p(\neg P)$ can be true together			
• T2	$n(\neg P)$ entails $\neg n(P)$			
• T3	$M(P)$ is contradictory to $\neg M(P)$, for any modality $M \in \{p, c, i, n\}$			
• T4	$n(\neg P)$ is contrary to $n(P)$			
• T5 (rev. in T10)	(1)	$p(P)$ is equivalent to $c(P)$	(7)	$\neg p(P)$ is equivalent to $\neg c(P)$
	(2)	$p(P)$ entails $\neg i(P)$	(8)	$\neg p(P)$ entails $i(P)$
	(3)	$p(P)$ entails $\neg n(\neg P)$	(9)	$\neg p(P)$ entails $n(\neg P)$
	(4)	$p(\neg P)$ is equivalent to $c(\neg P)$	(10)	$\neg p(\neg P)$ is equivalent to $\neg c(\neg P)$
	(5)	$p(\neg P)$ entails $\neg i(\neg P)$	(11)	$\neg p(\neg P)$ entails $i(\neg P)$
	(6)	$p(\neg P)$ entails $\neg n(P)$	(12)	$\neg p(\neg P)$ entails $n(P)$
• T6	(1)	$i(P)$ entails $n(\neg P)$	(2)	$i(\neg P)$ entails $n(P)$
• T7-8	(1)	$n(P)$ entails $p(P)$	(2)	$p(P)$ does not entail $n(P)$ or $n(\neg P)$
• T9 (rev. in T10)	$\neg i(P)$ entails $\neg n(\neg P)$			

Table 3: A summary of the entailment and opposition relations established in T2-10

of the same subject, which amounts to a formulation that the resulting propositions do not satisfy **LNC**; but if we want this statement to be relevant for our current purposes, then we should be certain that the term "possible" is used in T1 in the one-sided rather than the two-sided sense of the possible, and it is unclear whether it does, as it is only much further on in the text, at *De int.* 13, 22b10-19 (T7), that Aristotle lays down that non-externally negated occurrences of "possible" signify the one-sided possible. Moreover, even under the assumption that both $p(P)$ and $p(\neg P)$ involve the one-sided possible, Aristotle still fails to point out that these predicates cannot be false together of the same subject (**LEM**); and this alone is sufficient to deny him the credit of describing subcontrariety. Likewise, while he does accept the entailment from $n(\neg P)$ to $\neg n(P)$, he remains silent about whether the converse entailment also holds, and hence he cannot be credited with this subalternation relation either. That Aristotle does not fully develop the subcontrariety and subalternation relations at *De int.* 12-13 is not entirely surprising; remember that he did not do so either in his discussion of the categorical proposition at *De int*. 6-7.

The above observations clearly show that the relations in Fig. 2 do not coincide with those in Fig. 6, and thus establish the Negative Thesis. This means that Strong Inheritance is refuted: the modal square is *not* explicitly contained in *De int*. 12-13. It is simply wrong to claim that the modal square can already be found in Aristotle. We now move on to the Positive Thesis, and argue that, under the assumption of an instance of a contraposition law that Aristotle proves at *An. pr.* I.46, 52a40-52b9, it can be shown (1) that all predicates in each of the quadrants are logically equivalent, such that the relations in each quadrant can be lifted from the level of the individual predicates to the level of the quadrants;[37] and (2) that the subcontrariety and

[37] Because of their semantic definitions in terms of **LNC** and **LEM**, the opposition relations can easily be 'lifted' from individual formulas to equivalence classes of formulas. For example, we say that two equivalence classes X and Y are contradictory to each other if, and only if, for every formula $\varphi \in X$ and for every formula $\psi \in Y$, it holds that φ and ψ are contradictory to each other (see Demey 2019). It are precisely these 'lifted' versions of the opposition relations that are at play in the modal square of opposition in Fig. 2.

subalternation relations follow from the relations occurring in Fig. 6. That is, Fig. 6 can be completed into Fig. 2 if the background logical system of Fig. 6 is further developed.

6. The relation of following and contraposition

Again consider T5 and T10. In these passages, Aristotle uses the verb “to follow” to indicate that one predicate is entailed by another predicate.[38] For instance, $i(P)$ is entailed by $\neg p(P)$, and Aristotle expresses this by saying that $i(P)$ “follows on” (ἀκολουθεῖ) $\neg p(P)$. The use of the verb “to follow” (ἀκολουθεῖν, ἕπεσθαι) to indicate entailment relations between predicates is quite common in Aristotle, and it occurs both in the *Organon* and elsewhere (see, e.g., *De caelo* I.12, 282a27-b1). Importantly, Aristotle sometimes uses the same terminology in the context of *A*-propositions. On some occasions he says that *P* “follows on” *S* when he wants to indicate that *P* can be universally predicated of *S* (see, e.g., *An. pr.* I.4, 26b6; I.27, 43b4).[39] Moreover, compare T7 and T8. In both passages, it is proven that $n(P)$ entails $p(P)$; but this entailment is formulated differently in each passage: T7 proves that “(Every) ‘necessary to be’ is ‘possible to be’”, while T8 establishes that “‘possible to be’ follows from ‘necessary to be’”. This suggests that the relation of following between predicates – which we will denote with ‘$\vdash$’ – and the universal predication relation are extensionally equivalent in Aristotle’s logic. Thus Aristotle is presumably committed to the following principle.

$\boldsymbol{P_1}$: For any two predicates *A* and *B*, *B* follows on *A* (abbreviated $A \vdash B$) if, and only if, *B* can be universally predicated of *A*.

Since Aristotle takes *A*-propositions to have existential import, we obtain from $\boldsymbol{P_1}$ that $A \vdash B$ if, and only if, $A \neq \{\}$ and $A \subseteq B$. Translated into the language of first-order logic, this means that $A \vdash B$ if, and only if, $\exists x Ax \wedge \forall x(Ax \rightarrow Bx)$. The reader trained in modern logic will notice that the equivalence between the $\vdash$-relation and the universal predication relation that is postulated by $\boldsymbol{P_1}$ is similar in spirit to the equivalence between the semantic entailment relation and the material implication that is described by the Tarski-Herbrand Deduction Theorem (i.e., $\varphi \vDash \psi$ if, and only if, $\vDash \varphi \rightarrow \psi$). $\boldsymbol{P_1}$ thus suggests that Aristotle already had some notion of the correspondence between meta-logical and object-logical implication that is valid in many present-day logical systems.[40]

Aristotle recognizes that one of the properties of the $\vdash$-relation is that of contraposition. He uses this property on a number of occasions in the *Organon* (see, e.g., *An. pr.* II.2, 53b12-13; *Top.* II.8, 113b15-19; *Top.* VIII.14, 163a30-37), and he proves its validity at *An. pr.* I.46, 52a40-52b9. This passage is printed as T11.

T11: In general, [i] whenever *A* and *B* are so related that they cannot belong to the same thing (τῷ αὐτῷ μὴ ἐνδέχεσθαι), and one of the two necessarily belongs to everything (παντὶ δὲ

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38 Hintikka (1973: 45-47) argues that the verb ἀκολουθεῖν signifies logical equivalence in these passages, and not one-way entailment. On that reading, the equivalences of all predicates in each quadrant are already given by Aristotle himself. But Hintikka’s interpretation of ἀκολουθεῖν is hardly likely. Aristotle usually indicates equivalence either by the verb ἀντιστρέφειν (see T5), or by the expression ἀκολουθεῖν ἀλλήλοις (see, e.g., *An. pr.* I.13, 32a25), but not by ἀκολουθεῖν by itself. See also Bluck (1963); Brandon (1978); Rehder (1980); Seel (1982: 148-149).

39 See, e.g., Malink (2016: 34-35); Patzig (1968: 9 [n. 18]); Striker (2009: 192).

40 This correspondence can be generalized to several other types of relation; see, e.g., Keynes (1906: 117-119, 170-174).

> ἐξ ἀνάγκης θάτερον), and again [ii] whenever *C* and *D* are related in the same way, then [iii] if *A* follows *C* but not the reverse, then *D* will follow *B* but not the reverse ... First it is clear from the following that *D* follows *B*. For [iv] since either *C* or *D* necessarily belongs to everything; and [v] since *C* cannot belong to that to which *B* belongs, because it carries *A* along with it, and [vi] *A* and *B* cannot belong to the same thing; [vii] it is clear that *D* must follow *B*.[41]

In this passage, Aristotle introduces four predicates: *A*, *B*, *C* and *D*. Following [i], *A* and *B* cannot both be truly predicated of one and the same subject, but one of them should be so predicated. Thus the pair $\{A, B\}$ is a contradictory pair, and *B* is equivalent to $\neg A$, by T3. Following [ii], the same holds of the predicates *C* and *D*: $\{C, D\}$ is a contradictory pair, and *D* is equivalent to $\neg C$. At [iii], Aristotle states that if *A* follows on *C*, but not *vice versa*, then *D* follows on *B*, but not *vice versa*.[42] This implication can be restated as follows:

$$\text{If } C \vdash A\text{, then } \neg A \vdash \neg C$$

which by $\boldsymbol{P_1}$ is equivalent to the following object-logical implication:

$$(\exists x Cx \land \exists y \neg Ay) \rightarrow (\forall x(Cx \rightarrow Ax) \rightarrow \forall y(\neg Ay \rightarrow \neg Cy))^{43}$$

Aristotle proves this implication at [iv]-[vii] by assuming the antecedent – that *A* follows on *C* – and by showing that it implies the consequent – that $\neg C$ follows on $\neg A$. Given the equivalence between the $\vdash$-relation and the universal predication relation, we might reconstruct the proof as follows. The existence claims have been left out, but the non-emptiness of *C* and $\neg A$ is being assumed.

(1)		$\forall x(Cx \rightarrow Ax)$	Assumption
(2)		$\forall x(\neg Ax \rightarrow \neg Cx)$ or $\neg\forall x(\neg Ax \rightarrow \neg Cx)$	**LEM**
(3)		$\neg\forall x(\neg Ax \rightarrow \neg Cx)$	assumption, towards a contradiction
(4)		$\exists x(\neg Ax \land \neg\neg Cx)$	equivalent to (3) by **RCDP**
(5)	[iv]	$\forall x(Cx \lor \neg Cx)$	**LEM**
(6)		$\exists x(\neg Ax \land Cx)$	(4), (5)
(7)	[v]	$\exists x(\neg Ax \land Ax)$	(1), (6)
(7*bis*)	[vi]	$\bot$	(7) is impossible
(8)		$\neg(\neg\forall x(\neg Ax \rightarrow \neg Cx))$	*reductio*, (3), (7*bis*)
(9)	[vii]	$\forall x(\neg Ax \rightarrow \neg Cx)$	disjunctive syllogism, (2), (8)

41 Translation by Barnes (1984-i: 84).

42 Notice that Aristotle's account of contraposition is semantic in nature, rather than syntactic. In particular, he assumes that *A* and *B*, and also *C* and *D*, satisfy **LNC** and **LEM**, which is a semantic characterization of the contradictory pairs $\{A, B\}$ and $\{C, D\}$. In our (informal and symbolic) restatements, we write $\neg A$ instead of *B*, and likewise $\neg C$ instead of *D*, thus making use of a negation sign. However, Aristotle himself does not explicitly talk about negation in T11, let alone about the syntactic operations of inserting/deleting a negation sign.

43 Notice that abstraction is made of the notion that the reverse statements "*C* follows on *A*" and "*B* follows on *D*" are invalid. The theorem Aristotle is actually proving is this: $(\exists x Cx \land \exists y \neg Ay) \rightarrow ((\forall x(Cx \rightarrow Ax) \land \neg\forall x(Ax \rightarrow Cx)) \rightarrow (\forall y(\neg Ay \rightarrow \neg Cy) \land \neg\forall y(\neg Cy \rightarrow \neg Ay)))$. The conjuncts $\neg\forall x(Ax \rightarrow Cx)$ and $\neg\forall y(\neg Cy \rightarrow \neg Ay)$ are not taken into account in what follows.

The proof is valid. Aristotle does not prove the reverse-implication, but it can be proven analogously (by means of double negation elimination). Thus T11 shows that the following principle is valid in Aristotle's logic.

***P*$_2$**: For any two predicates A and B, $A \vdash B$ if, and only if, $\neg B \vdash \neg A$.

The schematic variables A and B in ***P*$_2$** can be substituted by different kinds of predicates. For instance, A and B might be substituted by unary predicates like "donkey" or "tree". Then the negations in $\neg A$ and $\neg B$ are metathetic negations, and $\neg A$ and $\neg B$ are what Aristotle calls indefinite names (ὀνόματα ἀόριστα) at *De int.* 1, 16a30-31, or terms ranging over all entities – existent and, as some scholars argue, also non-existent[44] – that are not included within the range of resp. A and B. An example of this interpretation occurs at *Top*. II.8, 113b15-25. There Aristotle says that "animal" (τὸ ζῷον) follows on "man" (ὁ ἄνθρωπος), and that the negation of the latter, "not-man" (τὸ μὴ ἄνθρωπος), follows on the negation of the former, "not-animal" (τὸ μὴ ζῷον). But ***P*$_2$** (or T11) does not require that A and B be substituted by such simple predicates. In particular, we might as well take both A and B as schematic variables for more complex predicates, i.e., unary predicate variables to which the verb "to be" is added. Earlier in *An. pr.* I.46, at 51b36-52a5, Aristotle discussed entailment relations between the predicates "to be good" (τὸ εἶναι ἀγαθὸν), "to be not-good" (τὸ εἶναι μὴ ἀγαθὸν), "not to be good" (τὸ μὴ εἶναι ἀγαθὸν), and "not to be not-good" (τὸ δὲ μὴ εἶναι μὴ ἀγαθὸν), suggesting that ***P*$_2$** was first and foremost intended to apply to such complex predicates. Aristotle nowhere uses a contraposition law when discussing complex *modal* predicates except in the aforementioned passage at *An. pr.* I.13, 32a23-27, which many scholars consider spurious (see note 34). There is thus no compelling textual evidence that Aristotle applied ***P*$_2$** to the kind of predicates with which *De int.* 12-13 is concerned. Yet it seems reasonable to assume that he would be willing to do so, especially in view of the fact that there *is* conclusive evidence that ***P*$_2$** was designed to cover more than one kind of predicate, as the passages at *Top*. II.8, 113b15-25 and *An. pr.* I.46, 51b36-52a5 show. Thus we may suppose that ***P*$_3$** is an instance of ***P*$_2$**, and is hence also valid in Aristotle's logic.

***P*$_3$**: For any two modal predicates $M_1(P)$ and $M_2(P)$, $M_1(P) \vdash M_2(P)$ if, and only if, $\neg M_2(P) \vdash \neg M_1(P)$.

By way of an example, let us apply ***P*$_3$** to the theorem from T7-T8. These passages prove that whenever P can be necessarily said of a subject S, then P can also be possibly said of S. By ***P*$_3$** we obtain that this theorem is logically equivalent to the theorem that whenever P cannot be possibly said of S, then P cannot be necessarily said of S either.

7. The Positive Thesis

The principle P_3 is all that is needed to bridge the gap between Fig. 2 and Fig. 6. As indicated in Sect. 5, Fig. 6 lacks all the equivalence and subcontrariety relations as well as some of the subalternation relations that occur in Fig. 2. We will now show that, given ***P*$_3$**, all these missing relations can be obtained after all. We start with the equivalence relations. Take quadrant (II) from Fig. 6. Following T5 and the revisions introduced in T10, Aristotle accepts (*a*), (*b*) and (*c*)

[44] See, e.g., Bäck (2000: 202).

as valid (see (7), (8) and (9) in Table 3). By $\boldsymbol{P}_3$ (and double negation elimination), (*a*), (*b*) and (*c*) yield (*a'*), (*b'*) and (*c'*), respectively.

(*a*)	$\neg p(P) \vdash \neg c(P)$	(*a'*)	$c(P) \vdash p(P)$
(*b*)	$\neg p(P) \vdash i(P)$	(*b'*)	$\neg i(P) \vdash p(P)$
(*c*)	$\neg p(P) \vdash n(\neg P)$	(*c'*)	$\neg n(\neg P) \vdash p(P)$

Now consider quadrant (I). Again by T5 and T10, Aristotle accepts (*d*), (*e*) and (*f*) as valid (see (1), (2) and (3) in Table 3). By $\boldsymbol{P}_3$ (and double negation elimination), (*d*), (*e*), and (*f*) yield (*d'*), (*e'*) and (*f'*), respectively.

(*d*)	$p(P) \vdash c(P)$	(*d'*)	$\neg c(P) \vdash \neg p(P)$
(*e*)	$p(P) \vdash \neg i(P)$	(*e'*)	$i(P) \vdash \neg p(P)$
(*f*)	$p(P) \vdash \neg n(\neg P)$	(*f'*)	$n(\neg P) \vdash \neg p(P)$

By (*d*) and (*a'*), $p(P)$ and $c(P)$ are equivalent (as Aristotle already pointed out in T5); by (*e*) and (*b'*), $p(P)$ and $\neg i(P)$ are equivalent; and by (*f*) and (*c'*), $p(P)$ and $\neg n(\neg P)$ are also equivalent. Thus all predicates in (I) are equivalent. Likewise for (II): by (*a*) and (*d'*) $\neg p(P)$ and $\neg c(P)$ are equivalent; by (*b*) and (*e'*) $i(P)$ and $\neg p(P)$ are equivalent; and by (*c*) and (*f'*) $n(\neg P)$ and $\neg p(P)$ are also equivalent. Thus all predicates in (II) are equivalent. Remember, the equivalences of the predicates of contingency, necessity and impossibility in (I) and (II) are also mentioned in the dubious passage at *An. pr.* I.13, 32a23-27. We can now see that they are also provable from the account at *De int.* 12-13 under the assumption of $\boldsymbol{P}_3$.[45]

Now consider quadrants (III) and (IV). By T5 and T10, Aristotle accepts (*g*)-(*l*) as valid (see (4), (5), (6), (10), (11) and (12) in Table 3). By $\boldsymbol{P}_3$ (and double negation elimination), (*g*)-(*l*) yield (*g'*)-(*l'*), respectively.

(*g*)	$p(\neg P) \vdash c(\neg P)$	(*g'*)	$\neg c(\neg P) \vdash \neg p(\neg P)$
(*h*)	$p(\neg P) \vdash \neg i(\neg P)$	(*h'*)	$i(\neg P) \vdash \neg p(\neg P)$
(*i*)	$p(\neg P) \vdash \neg n(P)$	(*i'*)	$n(P) \vdash \neg p(\neg P)$
(*j*)	$\neg p(\neg P) \vdash \neg c(\neg P)$	(*j'*)	$c(\neg P) \vdash p(\neg P)$
(*k*)	$\neg p(\neg P) \vdash i(\neg P)$	(*k'*)	$\neg i(\neg P) \vdash p(\neg P)$
(*l*)	$\neg p(\neg P) \vdash n(P)$	(*l'*)	$\neg n(P) \vdash p(\neg P)$

By (*g*) and (*j'*), $p(\neg P)$ and $c(\neg P)$ are equivalent; by (*h*) and (*k'*), $p(\neg P)$ and $\neg i(\neg P)$ are equivalent; and by (*i*) and (*l'*) $p(\neg P)$ and $\neg n(P)$ are also equivalent. Thus all formulas in (III) are equivalent. Likewise for those in (IV): $\neg c(\neg P)$ and $\neg p(\neg P)$ are equivalent by (*j*) and (*g'*); $i(\neg P)$ and $\neg p(\neg P)$ are equivalent by (*k*) and (*h'*); and $n(P)$ and $\neg p(\neg P)$ are equivalent by (*l*) an (*i'*). We obtain that all quadrants in Fig. 6 are equivalence classes under the assumption that $\boldsymbol{P}_3$ is available. It follows that the relations of contradiction and contrariety in Fig. 6 can now be characterized up to logical equivalence (recall note 37); and once these relations are so characterized, then the subcontrariety and subalternation relations can be easily derived. Here is how.

[45] That a contraposition law suffices to turn the different quadrants in the diagram into equivalence classes is also pointed out by Weidemann (2002: 446). However, Weidemann does not link *De int.* 12-13 with *An. pr.* I.46, 52a40-52b9, nor is he concerned with bridging the gap between Aristotle's diagram at *De int.* 12-13 and the actual modal square.

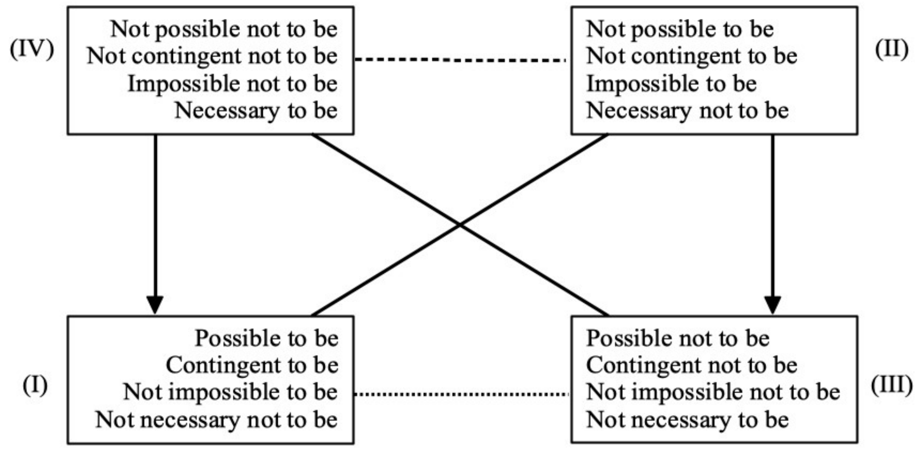

Fig. 7: Revised version of Fig. 6

- (*Contrariety*) By T4, $n(P)$ (IV) is contrary to $n(\neg P)$ (II). But given $\boldsymbol{P_3}$, (IV) and (II) are both equivalence classes. Thus (IV) and (II) are contrary to each other.
- (*Contradiction*) By T3, $\neg p(\neg P)$ (IV) is contradictory to $p(\neg P)$ (III). But under the assumption of $\boldsymbol{P_3}$, (IV) and (III) are both equivalence classes. Thus (IV) and (III) are contradictory to each other. Likewise, $\neg p(P)$ (II) is contradictory to $p(P)$ (I) by T3. Thus under the assumption of $\boldsymbol{P_3}$, (I) and (II) are contradictory to each other.
- (*Subcontrariety*) Assume that any predicate in (I) is false of a subject S. Then all predicates in (II) are true of S, by **RCDP** (and $\boldsymbol{P_3}$). Thus all predicates in (IV) are false of S, by **RCP**; and all predicates in (III) are true of S, by **RCDP**. Analogously, when a predicate in (III) is assumed to be false of S, it follows that those in (I) are true of S. Thus the predicates in (I) and (III) satisfy **LEM**. But they do not satisfy **LNC**; for it is possible for the predicates in (I) and (III) to be true together. So (I) and (III) are subcontraries.
- (*Subalternation*) The predicate $n(P)$ (IV) entails $p(P)$ (I), but $p(P)$ does not entail $n(P)$, by T7-8. Thus $p(P)$ is subaltern to $n(P)$; and under the assumption of $\boldsymbol{P_3}$, (I) is subaltern to (IV). Furthermore, $n(\neg P)$ (II) entails $\neg n(P)$ (III) by T2. Thus under the assumption of $\boldsymbol{P_3}$, (II) entails (III). By T4, we obtain that the reverse entailment is invalid. For assume that (III) *did* entail (II). Then (II) and (III) would constitute one equivalence class. But (I) is contradictory to (II). So now (I) would also be contradictory to (III), while, as we have just seen, (I) and (III) are contraries, rather than contradictories. Thus it is not the case that (III) entails (II); and (III) is subaltern to (II).

The relations that are proven in this section are displayed in the diagram in Fig. 7, which is a revision of Fig. 6 in light of $\boldsymbol{P_3}$. They are precisely those relations that occur in Fig. 2, as the reader can check. This establishes the Positive Thesis: for each relation in the modal square, either Aristotle describes it, or it follows from principles that Aristotle endorses. Add to this the Negative Thesis from Section 5, and we obtain the conclusion that while the system from *De int.* 12-13 is too weak to justify the claim that the diagram at *De int.* 13 is the first occurrence of the modal square, strengthening the system with a contraposition law of the kind that Aristotle proves at *An. pr.* I.46, 52a40-52b9 is all that is required to complete the diagram into the modal square. This conclusion, in turn, is a strong argument in favor of the notion that the modal square, while not identical with the diagram at *De int.* 13, historically developed from this diagram, which is the main claim of Weak Inheritance.

8. Conclusion

The past decade has witnessed a renewed and intensified interest in logic diagrams among logicians, philosophers, linguists and computer scientists (see, e.g., Béziau and Jacquette 2012; Béziau and Read 2014). In light of this recent surge in research, it has become all the more important to obtain a firm understanding of the intricate historical development of these diagrams. This paper has focused on the roots of the so-called "modal square of opposition", which enjoyed a great popularity in modal logic between the twelfth and twentieth centuries. The paper has argued that past discussion of this topic, which has not been explored in depth in the existing scholarship, can be framed in terms of the opposition between two positions, dubbed "Strong and Weak Inheritance". Both positions accept some kind of dependence of the modal square on Aristotle's modal logic at *De int*. 12-13, and the diagram at *De int*. 13, 22a23-33 in specific. Strong Inheritance, which mainly finds support in the literature on systematic modal logic, ascribes the modal square to Aristotle, and claims that the diagram at *De int*. 13, 22a23-33 amounts to the first occurrence of the modal square. Weak Inheritance, which mainly finds support in the literature on the history of logic, maintains that the modal square historically developed from Aristotle's discussion at *De int*. 12-13, yet does not claim that the diagram at *De int*. 13, 22a23-33 is the first occurrence of the modal square.

By means of a careful reconstruction of the logic from *De int.* 12-13, this paper has argued against Strong Inheritance, and has provided a textually grounded defense of Weak Inheritance. It was shown that while the modalities occurring in the modal square and the diagram at *De int*. 13, 22a23-33 are identical, the relations that Aristotle recognizes to hold between the modalities in the diagram constitute a proper subset of the relations occurring in the modal square. It was argued that this establishes the falsity of Strong Inheritance, which requires that the relation between these two sets be one of identity. The paper has further argued that this identity relation can be established by strengthening the system that is described in *De int.* 12-13 by means of a contraposition law that Aristotle proves elsewhere, at *An. pr*. I.46, 52a40-52b9. The upshot is that while the diagram at *De int*. 13, 22a23-33 cannot be considered the first occurrence of modal square, Aristotle's logic contains all the ingredients that are needed to complete the diagram into the modal square. This provides a strong argument in favor of Weak Inheritance. It was further suggested that, in this sense, the earliest history of the modal square resembles the earliest history of the categorical square of opposition, which previous studies have shown to have developed from, yet not be contained in, Aristotle's *De int*. 6-7.

It goes without saying that the paper has only taken the first steps toward a better understanding of the development of the modal square, whose history remains woefully understudied. While it provided compelling evidence that the modal square is not Aristotle's invention, it did not address the thorny question of where and when the modal square *did* originate, and whether it is an innovation of medieval logicians (recall note 8) or perhaps already occurs in the late-ancient sources. Formulating an answer to this question is an important task for future research into the history of logical geometry.

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