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Two-dimensional descriptor systems ⋆

Bob Vergauwen * Bart De Moor, Fellow IEEE and SIAM *

* *Center for Dynamical Systems, Signal Processing, and Data Analytics (STADIUS), Dept. of Electrical Engineering (ESAT), KU Leuven, 3001 Leuven, Belgium (e-mail: {bob.vergauwen,bart.demoor}@esat.kuleuven.be)*

Abstract: Linear descriptor systems are governed by dynamical equations subject to algebraic constraints. In the one-dimensional case, where the systems only depend on a single index, usually time, the Weierstrass canonical form splits up the state vector in two parts, a causal part, running forward in time, and a non-causal part, running backward. In this paper linear time-invariant autonomous descriptor systems in two-dimensions are discussed and the condition on the existence of a non-trivial solution is derived, together with an explicit formula for the output of such systems. It is shown that the output of the model can be related to a causal and a non-causal part in each of the dimensions of the model, running forward and backward in the various dimensions respectively. The results are obtained by requiring that the solutions, for states and outputs, which are defined on a two-dimensional grid, are path invariant and unique.

Keywords: Descriptor systems, Singular systems, Differential algebraic equations, Weierstrass canonical form.

1. INTRODUCTION

Differential algebraic equations (DAE) (Brenan et al., 1996) are characterized by a dynamical part, together with an algebraic part. DAE's are also sometimes called descriptor system or singular systems, and are surveyed e.g. in (Lewis, 1986). It can be shown that the solution of a descriptor system consists of a causal and a non-causal part, running forward and backward in time respectively (see e.g. (Moonen et al., 1992)). The causal part is linked to the finite roots of the characteristic equation associated to the descriptor system and the non-causal part is linked to roots at infinity.

In this paper the properties of linear autonomous descriptor systems in two dimensions are analyzed. A natural question to ask is *under what condition are these system well-posed and what is the solution for the state sequence, given specified initial conditions?* We demonstrate that, in the same way as in the one-dimensional case, the output of a descriptor system in two dimensions can be decoupled in a causal and non-causal part. In the past, descriptor systems have been analyzed in multiple dimensions, but these models had a different model structure, for example in (Campbell, 1991; Kaczorek, 1988; Gregor, 1992).

This paper is structured as follows. In Section 2, an overview of linear time-invariant autonomous descriptor systems in one dimension is provided. In this section, we introduce the Weierstrass Canonical Form (WCF) of a matrix pencil and explain its relevance to linear systems. Secondly, multidimensional state space models are introduced, and we demonstrate that the system matrices must commute in order for the model to be well-posed. In Section 3, the Weierstrass Canonical Form is applied to a simplified model class, called semi-descriptor systems, and we prove that the state vector of this model class can be partitioned in a regular and a singular part. This simplified case demonstrates the general techniques that are used in the derivation of the main result of this paper, which is formulated in Section 4. In Section 5 a small numerical example is provided to clarify all the steps performed. Section 6 summarizes the main conclusions of this paper.

2. EXISTING STATE SPACE MODELS

Consider the following autonomous state space model

$$\begin{aligned} Ex[k+1] &= Ax[k], \\ y[k] &= Cx[k], \end{aligned} \tag{1}$$

where $y[k] \in \mathbb{R}^p$ and $x[k] \in \mathbb{R}^n$ are the output and the state vector of the system respectively. The matrices E and A are elements of $\mathbb{R}^{n\times n}$, and k is the (integer) discrete time index. A state space model in this form is called a descriptor system (Verghese et al., 1981). This state space model is unique modulo a left and right transformation with non-singular matrices. The properties of this system are determined by the generalized eigenvalues of the matrix pencil (E, A) (Golub and van Loan, 2013). We assume

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that $\det(Es - A)$, is not identically zero, such that there exists an s for which $Es - A$ is invertible. A matrix pencil that satisfies this condition is called a regular matrix pencil (Ikramov, 1993). The vector x_i is an eigenvector of the matrix pencil if and only if

$$(Es_i - A\gamma_i)x_i = 0$$

for some values of $(s_i, \gamma_i) \in \mathbb{C}^2$. This pair is the eigenvalue of the eigenvector x_i. If γ_i is equal to zero and $s_i \neq 0$, we say that the system has a pole at infinity.

2.1 Weierstrass canonical form

Closely related to the generalized eigenvalue problem is the Weierstrass Canonical Form of a matrix pencil (Gantmacher, 1960), which states that for every regular matrix pencil there exist matrices P, Q of full rank where

$$P(Es - A)Q = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & N \end{bmatrix} s - \begin{bmatrix} A_R & 0 \\ 0 & \mathbb{1} \end{bmatrix} = \bar{E}s - \bar{A},$$

with $\mathbb{1}$ the identity matrix. The matrix N is a nilpotent matrix, which can be further reduced to a Jordan form and the matrix A_R is a regular matrix which can also be put in a Jordan canonical form. Moonen et al. (1992) show that the model of Eqn. (1) can be put in the WCF such that

$$\begin{bmatrix} \mathbb{1} & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} x^R[k+1] \\ x^S[k+1] \end{bmatrix} = \begin{bmatrix} A_R & 0 \\ 0 & \mathbb{1} \end{bmatrix} \begin{bmatrix} x^R[k] \\ x^S[k] \end{bmatrix}.$$

$$y[k] = [C_R \; C_S] \begin{bmatrix} x^R[k] \\ x^S[k] \end{bmatrix}$$

By substituting $x^S[k] = \tilde{x}^S[k-1]$ the model is put in its final form

$$\begin{bmatrix} x^R[k+1] \\ \tilde{x}^S[k-1] \end{bmatrix} = \begin{bmatrix} A_R & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} x^R[k] \\ \tilde{x}^S[k] \end{bmatrix}$$

$$y[k] = [C_R \; C_S N] \begin{bmatrix} x^R[k] \\ \tilde{x}^S[k] \end{bmatrix}$$

For $0 \leq k \leq n$, the output of this model is equal to

$$y[k] = C_R A_R^k x^R[0] + C_R N^{n-k+1} \tilde{x}^S[n]$$

and consist of a causal part running forward, and a non-causal part running backward, where the causal part of the solution depends on the initial condition $x^R[0]$ and the non-causal part depends on the final state $\tilde{x}^S[n]$. As N is nilpotent, its higher powers will be zero. The matrices C_R and C_S correspond respectively to the regular and singular part of the output matrix.

2.2 Multidimensional state space models

Consider a linear autonomous two-dimensional state space model

$$\begin{aligned} x[k+1,l] &= Ax[k,l] \\ x[k,l+1] &= Bx[k,l] \\ y[k,l] &= Cx[k,l], \end{aligned}$$

where $y[k,l] \in \mathbb{R}^p$ and $x[k,l] \in \mathbb{R}^n$, the output and the state vector of the system respectively (Dreesen et al., 2018; Vergauwen et al., 2018). The system equations are characterized by two square matrices A and B. Obviously, both matrices must commute. This commutation constraint comes from the path invariance of the state. Take for example

$$\begin{aligned} x[k+1,l+1] &= A(x[k,l+1]) = ABx[k,l] \\ &= B(x[k+1,l]) = BAx[k,l]. \end{aligned}$$

This equation must hold true for all values of $x[k,l]$, from which it follows that

$$AB = BA.$$

The output of this model is given by

$$y[k,l] = CA^k B^l x[0,0].$$

So commutation is a necessary condition for the well-posedness of a system.

The model that is considered in this paper is parameterized as

$$\begin{aligned} Ex[k+1,l] &= Ax[k,l], \\ Fx[k,l+1] &= Bx[k,l], \\ y[k,l] &= Cx[k,l], \end{aligned}$$

where $y[k,l] \in \mathbb{R}^p$ and $x[k,l] \in \mathbb{R}^n$, the output and the state vector of the system respectively. The system equations are characterized by four square matrices A, B, E and F. We refer to this this system with the shorthand notation (A, B, E, F).

3. SEMI-DESCRIPTOR SYSTEMS

3.1 Semi-descriptor systems with a strictly regular part

Before tackling the full problem of describing a two-dimensional descriptor system, we focus on a special case. We call this a semi-descriptor system in two dimensions. The model equations are now

$$\begin{aligned} Ex[k+1,l] &= Ax[k,l] \\ x[k,l+1] &= Bx[k,l] \\ y[k,l] &= Cx[k,l], \end{aligned} \tag{2}$$

where $y[k] \in \mathbb{R}^p$ and $x[k] \in \mathbb{R}^n$. The matrices A, B and E are square real system matrices of appropriate dimensions and the matrix pencil (E, A) is assumed to be regular.

Definition 1. We call the system of Eqn. (2) well-posed if a non-trivial state sequence $x[k,l]$ exists, that satisfies the model equations.

Lemma 2. If and only if the system is well-posed, there exists square matrices P, Q, of full rank, such that

$$(PEQ, PAQ) = \left(\begin{bmatrix} \mathbb{1} & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} A_R & 0 \\ 0 & \mathbb{1} \end{bmatrix} \right)$$

and applying Q to the state equation in l

$$(Q^{-1}Q, Q^{-1}BQ) = \left(\begin{bmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{bmatrix}, \begin{bmatrix} B_{1,1} & 0 \\ 0 & B_{2,2} \end{bmatrix} \right),$$

where the matrix N is nilpotent and

$$A_R B_{1,1} = B_{1,1} A_R, \quad N B_{2,2} = B_{2,2} N.$$

The commutation conditions place a restriction on the eigenvalue structure of matrices E, A and B.

It is important to observe that the matrix Q, from the WCF applied to the matrix pencil (E, A), should also block-diagonalize the system matrix B. This is not a trivial condition, and will only be true for certain values of E, A and B.

Proof. Without any loss of generality, there exist square matrices P and Q of full rank, such that the first equation of the system defined by $(PEQ, PAQ, Q^{-1}BQ, CQ)$ is put in the WCF (see (Moonen et al., 1992))

$$\begin{bmatrix} \mathbb{1} & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} x^R[k+1,l] \\ x^S[k+1,l] \end{bmatrix} = \begin{bmatrix} A_R & 0 \\ 0 & \mathbb{1} \end{bmatrix} \begin{bmatrix} x^R[k,l] \\ x^S[k,l] \end{bmatrix},$$

The state vector is transformed as

$$Q^{-1}x[k,l] = \begin{bmatrix} x^R[k,l] \\ x^S[k,l] \end{bmatrix}.$$

By applying the change of basis, $x^S[k,l] = \tilde{x}^S[k-1,l]$ (Moonen et al., 1992) the descriptor system is transformed to

$$\begin{bmatrix} x^R[k+1,l] \\ \tilde{x}^S[k-1,l] \end{bmatrix} = \begin{bmatrix} A_R & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} x^R[k,l] \\ \tilde{x}^S[k,l] \end{bmatrix}, \quad (3)$$

$$\begin{bmatrix} x^R[k,l+1] \\ \tilde{x}^S[k-1,l+1] \end{bmatrix} = \begin{bmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{bmatrix} \begin{bmatrix} x^R[k,l] \\ \tilde{x}^S[k-1,l] \end{bmatrix}, \quad (4)$$

where

$$Q^{-1}BQ = \begin{bmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{bmatrix}.$$

With the same reasoning as for non-descriptor system, we can reach the state $x[k,l]$ in multiple ways. This puts extra commutation constraints on the block matrices $B_{i,j}, A_R$ and N. The state vector at multi-index $[k+1,l]$ is calculated by using Eqn. (3)

$$\begin{bmatrix} x^R[k+1,l] \\ \tilde{x}^S[k+1,l] \end{bmatrix} = \begin{bmatrix} A_R x^R[k,l] \\ N\tilde{x}^S[k+2,l] \end{bmatrix}.$$

From Eqn. (4) it follows that the state vector at multi-index $[k,l+1]$ is given by

$$\begin{bmatrix} x^R[k,l+1] \\ \tilde{x}^S[k,l+1] \end{bmatrix} = \begin{bmatrix} B_{1,1}x^R[k,l] + B_{1,2}\tilde{x}^S[k-1,l] \\ B_{2,1}x^R[k+1,l] + B_{2,2}\tilde{x}^S[k,l] \end{bmatrix}.$$

The state vector at multi-index $[k+1,l+1]$ can be obtained in two possible ways. Firstly, Eqn. (4) is applied to the state vector $x[k+1,l]$, this results in

$$\begin{aligned} x^R[k+1,l+1] &= B_{1,1}x^R[k+1,l] + B_{1,2}\tilde{x}^S[k,l] \\ &= B_{1,1}A_R x^R[k,l] + B_{1,2}\tilde{x}^S[k,l], \\ x^S[k+1,l+1] &= B_{2,1}x^R[k+2,l] + B_{2,2}\tilde{x}^S[k+1,l] \\ &= B_{2,1}x^R[k+2,l] + B_{2,2}N\tilde{x}^S[k+2,l]. \end{aligned}$$

Secondly, starting from the state vector $x[k,l+1]$ and applying Eqn. (3) we obtain

$$\begin{aligned} x^R[k+1,l+1] &= A_R x^R[k,l+1] \\ &= A_R B_{1,1}x^R[k,l] + A_R B_{1,2}\tilde{x}^S[k-1,l], \\ x^S[k+1,l+1] &= N\tilde{x}^S[k+2,l+1] \\ &= NB_{2,1}x^R[k+3,l] + NB_{2,2}\tilde{x}^S[k+2,l]. \end{aligned}$$

Both expressions for the state must be the same in order for the model to be well-posed, comparing both for the regular state $x^R[k+1,l+1]$ we get

$$\begin{aligned} & B_{1,1}A_R x^R[k,l] + B_{1,2}\tilde{x}^S[k,l] = \\ & \qquad A_R B_{1,1}x^R[k,l] + A_R B_{1,2}\tilde{x}^S[k-1,l], \end{aligned} \quad (5)$$

this equation must hold for all values of $x^R[k,l]$ and $\tilde{x}^S[k,l]$, which implies that

$$B_{1,1}A_R = A_R B_{1,1}$$

and

$$B_{1,2}\tilde{x}^S[k,l] = A_R B_{1,2}\tilde{x}^S[k-1,l] = A_R B_{1,2}N\tilde{x}^S[k,l],$$

or

$$B_{1,2} = A_R B_{1,2} N. \quad (6)$$

From the nilpotency of the matrix N, it follows that $B_{1,2}$ is equal to zero. This can easily be demonstrated, assume the nilpotency index of N to be p, such that $N^p = 0 \neq N^{p-1}$, and multiply both sides of Eqn. (6) with N^{p-1}

$$B_{1,2}N^{p-1} = A_R B_{1,2} N^p = 0.$$

Because $B_{1,2}N^{p-1} = 0$ we can multiply both sides of Eqn. (6) with N^{p-2} and get

$$B_{1,2}N^{p-2} = A_R B_{1,2} N^{p-1} = 0.$$

Repeating this procedure p times proves that $B_{1,2} = 0$, in order for the system to be well-posed.

Comparing both expressions of the singular state $\tilde{x}^S[k+1,l+1]$ we get

$$\begin{aligned} & B_{2,1}x^R[k+2,l] + B_{2,2}\tilde{x}^S[k+1,l] = \\ & \qquad NB_{2,1}x^R[k+3,l] + NB_{2,2}\tilde{x}^S[k+2,l]. \end{aligned}$$

This equation can be rewritten as

$$\begin{aligned} & B_{2,1}x^R[k+2,l] + B_{2,2}N\tilde{x}^S[k+2,l] = \\ & \qquad NB_{2,1}Ax^R[k+2,l] + NB_{2,2}\tilde{x}^S[k+2,l]. \end{aligned}$$

Analogous as the reasoning for Eqn. (5), it implies that N and $B_{2,2}$ must commute and $B_{2,1} = 0$. We can therefore conclude that the state space model of Eqn. (2) is only well-posed when it can be transformed to

$$\begin{bmatrix} x^R[k+1,l] \\ \tilde{x}^S[k-1,l] \end{bmatrix} = \begin{bmatrix} A_R & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} x^R[k,l] \\ \tilde{x}^S[k,l] \end{bmatrix}$$

$$\begin{bmatrix} x^R[k,l+1] \\ \tilde{x}^S[k-1,l+1] \end{bmatrix} = \begin{bmatrix} B_{1,1} & 0 \\ 0 & B_{2,2} \end{bmatrix} \begin{bmatrix} x^R[k,l] \\ \tilde{x}^S[k-1,l] \end{bmatrix}$$

$$y[k,l] = [C_R \ C_S N] \begin{bmatrix} x^R[k,l] \\ \tilde{x}^S[k,l] \end{bmatrix},$$

with $A_R B_{1,1} = B_{1,1}A_R$ and $NB_{2,2} = B_{2,2}N$. This is a necessary condition that needs to be satisfied for the matrices E, A and B. The matrices $B_{1,1}$ and $B_{2,2}$ do not have to be nilpotent. This demonstrates that, under the condition that the model equations of Eqn. (2) are well-posed and a non-trivial state sequence exists, it is always possible to find a linear transformation described by P and Q that separates the state vector in a regular and a singular part and both parts are completely decoupled from each other. When the system is put in the form described in Lemma 2, the well-posed is demonstrated in Section 3.2 by calculating the solution of the state sequence. □

3.2 Solution of the semi-descriptor system

When the first model equation of the semi-descriptor system is put in the WCF via the transformation

$$(PEQ, PAQ, Q^{-1}BQ, CQ),$$

the expression of the output as a function of the system matrices can be explicitly derived. The dynamic equations describing the regular part of the state vector are given by,

$$\begin{aligned} x^R[k+1,l] &= A_R x^R[k,l], \\ x^R[k,l+1] &= B_{1,1}x^R[k,l]. \end{aligned}$$

The state sequence that satisfies both equations is

$$x^R[k,l] = A_R^k B_{1,1}^l x^R[0,0].$$

The dynamics of the singular part are described by

$$\begin{aligned} \tilde{x}^S[k-1,l] &= N\tilde{x}^S[k,l] \\ \tilde{x}^S[k,l+1] &= B_{2,2}\tilde{x}^S[k,l]. \end{aligned}$$

The solution to this state sequence for $0 \leq k \leq n$ and $0 \leq l$ is

$$\tilde{x}^S[k,l] = N^{n-k} B_{2,2}^l \tilde{x}^S[0,0].$$

Therefore the model output is

$$y[k,l] = C_R A_R^k B_{1,1}^l x^R[0,0] + C_S N^{n-k+1} B_{2,2}^l \tilde{x}^S[n,0],$$

with

$$CQ = [C_R \; C_S] \, .$$

Because of the nilpotency of N, the singular, anti-causal part, will only have a finite memory and propagate anti-causal over a time window given by the nilpotency index of N.

3.3 Semi-descriptor system with no causal part

A second simplified case is

$$\begin{aligned} Ex[k+1,l] =& Ax[k,l] \\ x[k,l-1] =& Mx[k,l] \\ y[k,l] =& Cx[k,l], \end{aligned} \tag{7}$$

which differs from Eqn. (2) by the second equation, which runs backward and the matrix M is nilpotent. The properties of this state space model are used to derive the main result, formulated in Conjecture 4.

Lemma 3. The system of Eqn. (7) is well-posed and a non-trivial state sequence $x[k,l]$ exist, that satisfies the model equations, if and only if there exists square matrices P, Q, of full rank, such that

$$(PEQ, PAQ) = \left(\begin{bmatrix} \mathbb{1} & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} A_R & 0 \\ 0 & \mathbb{1} \end{bmatrix} \right)$$

and

$$(Q^{-1}Q, Q^{-1}MQ) = \left(\begin{bmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{bmatrix}, \begin{bmatrix} M_{1,1} & 0 \\ 0 & M_{2,2} \end{bmatrix} \right),$$

where the matrices N, $M_{1,1}$ and $M_{2,2}$ are nilpotent and

$$A_R M_{1,1} = M_{1,1} A_R, \quad N M_{2,2} = M_{2,2} N.$$

The nilpotency of N follows from the properties of the WCF. The matrices $M_{1,1}$ and $M_{2,2}$ are nilpotent because the matrix M is assumed to be nilpotent.

The assumption that M is nilpontent is part of the model description, in Section 4 this assumption will be dropped.

Proof. By introducing the matrices $PEQ, PAQ, Q^{-1}MQ$ and CQ, the model is transformed in the same way as before, to obtain

$$\begin{bmatrix} x^R[k+1,l] \\ \tilde{x}^S[k-1,l] \end{bmatrix} = \begin{bmatrix} A_R & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} x^R[k,l] \\ \tilde{x}^S[k,l] \end{bmatrix},$$

$$\begin{bmatrix} x^R[k,l-1] \\ \tilde{x}^S[k-1,l-1] \end{bmatrix} = \begin{bmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{bmatrix} \begin{bmatrix} x^R[k,l] \\ \tilde{x}^S[k-1,l] \end{bmatrix}.$$

The further analysis of the system goes in the same line as before where we calculate the state vector at several locations and require the equations to be consistent. In exactly the same way we find that $M_{1,2} = 0 = M_{2,1}$ in order for the model to be well-posed, this results in a canonical form given by

$$\begin{bmatrix} x^R[k+1,l] \\ \tilde{x}^S[k-1,l] \end{bmatrix} = \begin{bmatrix} A_R & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} x^R[k,l] \\ \tilde{x}^S[k,l] \end{bmatrix},$$

$$\begin{bmatrix} x^R[k,l-1] \\ \tilde{x}^S[k-1,l-1] \end{bmatrix} = \begin{bmatrix} M_{1,1} & 0 \\ 0 & M_{2,2} \end{bmatrix} \begin{bmatrix} x^R[k,l] \\ \tilde{x}^S[k-1,l] \end{bmatrix}.$$

where the matrices $M_{1,1}, M_{2,2}$ and N are nilpotent and $A_R M_{1,1} = M_{1,1} A_R$, $N M_{2,2} = M_{2,2} N$. By re-substituting $\tilde{x}^S[k,l] = x^S[k,l]$, we retrieve the matrix pencils representing the system dynamics

$$(PEQ, PAQ) = \left(\begin{bmatrix} \mathbb{1} & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} A_R & 0 \\ 0 & \mathbb{1} \end{bmatrix} \right)$$

and

$$(Q^{-1}MQ, Q^{-1}Q) = \left(\begin{bmatrix} M_{1,1} & 0 \\ 0 & M_{2,2} \end{bmatrix}, \begin{bmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{bmatrix} \right).$$

□

The output of this model is

$$y[k,l] = C_R A_R^k M_{1,1}^{m-l} x^R[0,m] + C_S N^{n-k+1} M_{2,2}^{l-m} \tilde{x}^S[n,m],$$

for $0 \leq k \leq n$ and $0 \leq l \leq m$.

Both results derived in this section will be used to analyze the main result in Section 4.

3.4 Example: Semi-descriptor system

To illustrate the obtained results so far, a small example is provided. Take the system

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x[k+1,l] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} x[k,l]$$

$$x[k,l+1] = \begin{bmatrix} 1 & 0 \\ 0.75 & 2 \end{bmatrix} x[k,l]$$

$$y[k,l] = [1 \;\; 1] \, x[k,l].$$

Note that the system matrices as such, do not commute. The two matrices

$$P = \begin{bmatrix} 1 & -0.5 \\ 0 & 0.25 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ -0.75 & 1 \end{bmatrix},$$

put the first equation in its WCF and the system is transformed to

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x^R[k+1,l] \\ x^S[k+1,l] \end{bmatrix} \begin{bmatrix} .5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x^R[k,l] \\ x^S[k,l] \end{bmatrix}$$

$$\begin{bmatrix} x^R[k,l+1] \\ x^S[k,l+1] \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x^R[k,l] \\ x^S[k,l] \end{bmatrix}$$

$$y[k,l] = [.25 \;\; 1] \begin{bmatrix} x^R[k,l] \\ x^S[k,l] \end{bmatrix}$$

For $0 \leq k \leq m$ and $0 \leq l$, the output of this model is

$$y[k,l] = \frac{1}{4} \left(\frac{1}{2^k} x^R[0,0] \right) + 2^l 0^{m-k} x^S[m,0].$$

Note that the vector $\tilde{x}^S[k,l]$ is indeed non-causal in k but causal in l. The vector $x^S[k,l]$ is zero for all $k < m$, when $k = m$ the singular state satisfies

$$x^S[m,l] = 2^l x^S[m,0].$$

In this particular case, the nilpotency index of the non-causal part is 1. The difference equation of the system is calculated by using the $z-$transformation. We have

$$\det(z_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}) = -4z_1 - 2$$

$$\det(z_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0.75 & 2 \end{bmatrix}) = (z_2 - 1)(z_2 - 2)$$

The difference equations related to the semi-descriptor system are thus,

$$y[k+1,l] - \frac{1}{2} y[k,l] = 0, \quad y[k,l+2] - 3y[k,l+1] + 2y[k,l] = 0$$

4. MULTIDIMENSIONAL DESCRIPTOR SYSTEMS

The linear autonomous descriptor systems in two dimensions that we will now consider are described by

$$\begin{aligned} Ex[k+1,l] &= Ax[k,l] \\ Fx[k,l+1] &= Bx[k,l] \\ y[k,l] &= Cx[k,l], \end{aligned} \tag{8}$$

where $y[k,l] \in \mathbb{R}^p$ and $x[k,l] \in \mathbb{R}^n$, the output and the state vector of the system respectively. The system equations are characterized by four square matrices A, B, E and F. Although up to now we have no complete proof for the generalized canonical form for E, A, F, B, we could try to find necessary conditions under which such a reduction to a canonical form would be possible. A natural question to ask is, *under what conditions is this model well-posed and does there exist a canonical form*?

We conjecture that a potential generalization of the WCF to 2 pairs of matrices, describing a 2-dimensional descriptor system, could have a canonical form as follows:

Conjecture 4. The two-dimensional system described in Eqn. (8), with (E,A) and (F,B), two regular pencils, is well-posed if and only if there exist square matrices P, Q, and U of full rank, such that the equivalent model

$$\begin{aligned} PEQx[k+1,l] &= PAQx[k,l] \\ UFQx[k,l+1] &= UBQx[k,l] \\ y[k,l] &= CQx[k,l], \end{aligned}$$

exists with

$$PEQ = \begin{bmatrix} \mathbb{1} & & & \\ & \mathbb{1} & & \\ & & E_1 & \\ & & & E_2 \end{bmatrix}, PAQ = \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \mathbb{1} & \\ & & & \mathbb{1} \end{bmatrix},$$

$$UFQ = \begin{bmatrix} \mathbb{1} & & & \\ & F_1 & & \\ & & \mathbb{1} & \\ & & & F_2 \end{bmatrix}, UBQ = \begin{bmatrix} B_1 & & & \\ & \mathbb{1} & & \\ & & B_2 & \\ & & & \mathbb{1} \end{bmatrix},$$

and

$$CQ = [C_{RR}\ C_{RS}\ C_{SR}\ C_{SS}],$$

where the matrices E_1, E_2, F_1, and F_2 are nilpotent. In this basis, the state vector has the form

$$Q^{-1}x[k,l] = \begin{bmatrix} x^{RR}[k,l] \\ x^{RS}[k,l] \\ x^{SR}[k,l] \\ x^{SS}[k,l] \end{bmatrix}$$

and the following additional equations must hold

$$\begin{aligned} A_1B_1 = B_1A_1,\ A_2F_1 = F_1A_2, \\ E_1B_2 = B_2E_1,\ E_2F_2 = F_2E_2. \end{aligned}$$

Furthermore, in this basis, the output of the model is equal to

$$\begin{aligned} y[k,l] = C_{RR}A_1^kB_1^lx^{RR}[0,0] + C_{RS}A_2^kF_1^{m-l}x^{RS}[0,m] + \\ C_{SR}E_1^{n-k}B_2^lx^{SR}[n,0] + C_{SS}E_2^{n-k}F_2^{m-l}x^{SS}[n,m] \end{aligned} \tag{9}$$

for some integer values m and n and $0 \le k \le m, 0 \le l \le n$.

In what follows, we first define the well-posedness of state equations for a 2D system, and then proceed by carefully investigating the necessary conditions that could lead to a canonical form for 2D systems in which one could partition the state space in pure regular (RR), pure singular (SS) and mixed regular-singular (RS and SR) complementary parts. The RR part corresponds to states that, in the canonical basis, propagate causally on a 2D grid in both directions. The RS part corresponds to states that propagate causally in one direction and anti-causally in the other, in the SR part corresponds to states that propagate the other way around. The SS part represents states that propagate purely anti-causally. The notion of well-posedness we propose includes

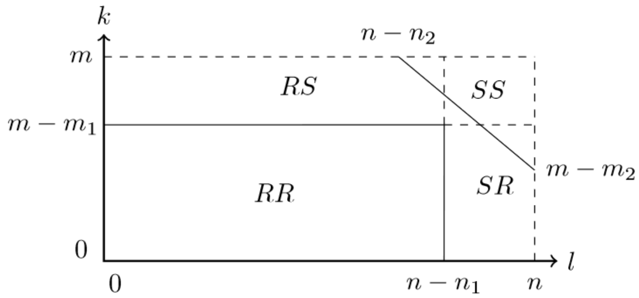

Fig. 1. Schematic overview of the regular and singular parts of the state vector on a two-dimensional grid, with the multi-index $0 \le k \le m$ and $0 \le l \le n$. The nilpotency of the singular matrices is denoted by n_i and m_i (with $i = 1$ and $i = 2$). The state vector of a two-dimensional descriptor system can be split up in 4 parts, a strictly regular part that is causal in both dimensions, a part that is regular in one equation and singular in the other and vise versa. And lastly, a part that is singular in both dimensions.

(1) the uniqueness of the state vector as it propagates on a 2D 'equidistant' discrete grid, after providing proper initial states for the 4 complementary parts of the state vector (RR,RS,SR,SS) and
(2) a consistency condition, that guarantees the uniqueness of the state vector as different paths can be followed in order to get to a specified end-state. As we will see, this imposes certain conditions of commutativity between matrices in the canonical state space basis.

The four partitions of the state vector are graphically represented in Fig. 1. Lets now elaborate on the necessary condition proposed in Conjecture 4.

Consider the two-dimensional descriptor system presented in Eqn. (8), where (E,A) and (F,B) are both regular pencils. Using the same techniques as for the semi-descriptor system, an equivalent model is constructed defined by the matrices (PEQ, PAQ, FQ, BQ) where the first equation is put in the WCF. In this form, the system equations are given by

$$\begin{bmatrix} x^R[k+1,l] \\ \tilde{x}^S[k-1,l] \end{bmatrix} = \begin{bmatrix} A_R & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} x^R[k,l] \\ \tilde{x}^S[k,l] \end{bmatrix},$$

$$\begin{bmatrix} F_{1,1} & F_{1,2} \\ F_{2,1} & F_{2,2} \end{bmatrix} \begin{bmatrix} x^R[k,l+1] \\ \tilde{x}^S[k-1,l+1] \end{bmatrix} = \begin{bmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{bmatrix} \begin{bmatrix} x^R[k,l] \\ \tilde{x}^S[k-1,l] \end{bmatrix}.$$

The solution evolving in k is equal to

$$\begin{aligned} x^R[k,l] &= A_R^kx^R[0,l] \\ \tilde{x}^S[k,l] &= N^{n-k}\tilde{x}^S[n,l], \end{aligned} \tag{10}$$

for all $0 \le k \le n$ and $0 \le l$, it consists of a causal part, running forward in the coordinate k, and a non-causal part, running backward. Substituting Eqn. (10) in the second model equation of Eqn. (8) gives

$$\begin{bmatrix} F_{1,1} & F_{1,2} \\ F_{2,1} & F_{2,2} \end{bmatrix} \begin{bmatrix} A_R^k x^R[0,l+1] \\ N^{n-k+1}\tilde{x}^S[0,l+1] \end{bmatrix} = \begin{bmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{bmatrix} \begin{bmatrix} A_R^k x^R[0,l] \\ N^{n-k+1}\tilde{x}^S[n,l] \end{bmatrix}. \quad (11)$$

We first consider the dynamic-modes of this system in detail. These are the modes associated to the regular part of the state vector. The matrix N, in the Weierstrass canonical form, is nilpotent, such that there exist a value m where $N^m = 0$. Assume there exists an index $k \geq 0$ with $n - k + 1 > m$, such that $\tilde{x}^S[k,l] = 0$. In this case Eqn. (11) reduces to

$$\begin{aligned} F_{1,1}A_R^k x^R[0,l+1] &= B_{1,1}A_R^k x^R[0,l] \\ F_{2,1}A_R^k x^R[0,l+1] &= B_{2,1}A_R^k x^R[0,l] \end{aligned}, \quad (12)$$

and the system is purely dictated by the dynamic modes. For $k = 0$ and $A_R^k = \mathbb{1}$, Eqn. (12) is reduced to

$$\begin{bmatrix} F_{1,1} \\ F_{2,1} \end{bmatrix} x^R[0,l+1] = \begin{bmatrix} B_{1,1} \\ B_{2,1} \end{bmatrix} x^R[0,l].$$

If the system is well-posed, this equation must be valid for every value of the initial state $x^R[0,l]$, which is free to choose. This is only the case when

$$\text{range}\left(\begin{bmatrix} F_{1,1} \\ F_{2,1} \end{bmatrix}\right) \supseteq \text{range}\left(\begin{bmatrix} B_{1,1} \\ B_{2,1} \end{bmatrix}\right).$$

This condition can be formulated as a rank constraint

$$\text{rank}\left(\begin{bmatrix} F_{1,1} \\ F_{2,1} \end{bmatrix}\right) = \text{rank}\left(\begin{bmatrix} F_{1,1} & B_{1,1} \\ F_{2,1} & B_{2,1} \end{bmatrix}\right) \leq \dim(x^R) \quad (13)$$

or said in words, if the number of linearly independent equations in Eqn. (12) is less than or equal to, dimension of the state vector [1]. Under this rank condition it is possible to reduce the matrix in Eqn. (13) by means of elementary row operations, described by a partitioned matrix U, to the form

$$\begin{bmatrix} U_{1,1} & U_{1,2} \\ U_{2,1} & U_{2,2} \end{bmatrix} \begin{bmatrix} F_{1,1} & B_{1,1} \\ F_{2,1} & B_{2,1} \end{bmatrix} = \begin{bmatrix} F'_{1,1} & B'_{1,1} \\ 0 & 0 \end{bmatrix}, \quad (14)$$

Under this transformation, the system matrices become

$$UFQ = \begin{bmatrix} F'_{1,1} & F'_{1,2} \\ 0 & F'_{2,2} \end{bmatrix}, \quad UBQ = \begin{bmatrix} B'_{1,1} & B'_{1,2} \\ 0 & B'_{2,2} \end{bmatrix}.$$

and the second model equation is reduced to

$$\begin{bmatrix} F'_{1,1} & F'_{1,2} \\ 0 & F'_{2,2} \end{bmatrix} \begin{bmatrix} x^R[k,l+1] \\ \tilde{x}^S[k-1,l+1] \end{bmatrix} = \begin{bmatrix} B'_{1,1} & B'_{1,2} \\ 0 & B'_{2,2} \end{bmatrix} \begin{bmatrix} x^R[k,l] \\ \tilde{x}^S[k-1,l] \end{bmatrix} \quad (15)$$

The singular part of the vector, denoted by $\tilde{x}^S$, is now fully decoupled from the regular part and its dynamics are governed by the subsystem

$$\begin{aligned} \tilde{x}^S[k-1,l] &= N\tilde{x}^S[k,l] \\ F'_{2,2}\tilde{x}^S[k-1,l+1] &= B'_{2,2}\tilde{x}^S[k-1,l], \end{aligned} \quad (16)$$

which has the form of a semi-descriptor system and has been analyzed in detail in Section 3. However, the regular part is still coupled with the singular part via the system matrices $F'_{1,2}$ and $B'_{1,2}$.

We demonstrate that under certain conditions the vector $x^R[0,0]$ and $x^R[0,1]$ can both be assumed to be zero. This assumption will allow us to reduce the problem to a dynamic system that is only governed by the singular vector, from which we can derive further conditions on the system matrices. This is not a trivial assumption, because $x^R[0,0]$ can always be chosen freely, but $x^R[0,1]$ is determined by a dynamic equation.

Consider the regular matrix pencil $(F'_{1,1}, B'_{1,1})$, its regularity follows directly from the fact that the applied transformation to the regular matrix pencil (F, B) preserves its regularity and that (F, B) is put in a block diagonal form with $(F'_{1,1}, B'_{1,1})$ being one of the blocks on the diagonal. A consequence of the regularity of the pencil is that the matrices $F'_{1,1}$ and $B'_{1,1}$ share no vector in both null spaces. If such a vector x existed, that lies in the null space of $F'_{1,1}$ and $B'_{1,1}$, we would have that

$$(sF'_{1,1} - B'_{1,1})x = 0 \qquad \forall s,$$

which would imply that the pencil is singular. Because both matrices do not share a common vector in the null space, the matrix $[F_{1,1}^T, B_{1,1}^T]^T$ is of full rank. By noticing that the matrix U in Eqn (14) is of full rank, it follows that the inequality in Eqn. (13) is in fact an equality. As before, the pencil $(F'_{1,1}, B'_{1,1})$, can be put in the WCF. Therefore, there exist matrices P_1 and Q_1 such that

$$(P_1F'_{1,1}Q_1, P_1B'_{1,1}Q_1) = \left(\begin{bmatrix} \mathbb{1} & 0 \\ 0 & N_1 \end{bmatrix}, \begin{bmatrix} B_1 & 0 \\ 0 & \mathbb{1} \end{bmatrix}\right).$$

We can also calculate the WCF of the regular pencil $(F'_{2,2}, B'_{2,2})$, which is regular by the same reasoning as for $(F'_{1,1}, B'_{1,1})$, such that

$$(P_2F'_{2,2}Q_2, P_2B'_{2,2}Q_2) = \left(\begin{bmatrix} \mathbb{1} & 0 \\ 0 & N_2 \end{bmatrix}, \begin{bmatrix} B_2 & 0 \\ 0 & \mathbb{1} \end{bmatrix}\right).$$

Combining both transformations yields

$$\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} F'_{1,1} & F'_{1,2} \\ 0 & F'_{2,2} \end{bmatrix} \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} = \begin{bmatrix} P_1F'_{1,1}Q_1 & P_1F'_{1,2}Q_2 \\ 0 & P_2F'_{2,2}Q_2 \end{bmatrix}$$

and

$$\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} B'_{1,1} & B'_{1,2} \\ 0 & B'_{2,2} \end{bmatrix} \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} = \begin{bmatrix} P_1B'_{1,1}Q_1 & P_1B'_{1,2}Q_2 \\ 0 & P_2B'_{2,2}Q_2 \end{bmatrix}.$$

By introducing a new state vector

$$\begin{bmatrix} Q_1^{-1} & 0 \\ 0 & Q_2^{-1} \end{bmatrix} \begin{bmatrix} x^R[k,l] \\ \tilde{x}^S[k,l] \end{bmatrix} = \begin{bmatrix} x^{RR}[k,l] \\ x^{RS}[k,l] \\ x^{SR}[k,l] \\ x^{SS}[k,l] \end{bmatrix}. \quad (17)$$

the descriptor system has the following form

$$\begin{bmatrix} x^{RR}[k+1,l] \\ x^{RS}[k+1,l] \\ x^{SR}[k-1,l] \\ x^{SS}[k-1,l] \end{bmatrix} = \begin{bmatrix} Q_1^{-1}A_RQ_1 & 0 \\ 0 & Q_2^{-1}NQ_2 \end{bmatrix} \begin{bmatrix} x^{RR}[k,l] \\ x^{RS}[k,l] \\ x^{SR}[k,l] \\ x^{SS}[k,l] \end{bmatrix}$$

$$\left[\begin{array}{cc|c} \mathbb{1} & 0 & \multirow{2}{*}{$P_1F'_{1,2}Q_2$} \\ 0 & N_1 & \\ \hline \multicolumn{2}{c|}{0} & \begin{matrix} \mathbb{1} & 0 \\ 0 & N_2 \end{matrix} \end{array}\right] \begin{bmatrix} x^{RR}[k,l+1] \\ x^{RS}[k,l+1] \\ x^{SR}[k-1,l+1] \\ x^{SS}[k-1,l+1] \end{bmatrix} = \left[\begin{array}{cc|c} B_1 & 0 & \multirow{2}{*}{$P_1B'_{1,2}Q_2$} \\ 0 & \mathbb{1} & \\ \hline \multicolumn{2}{c|}{0} & \begin{matrix} B_2 & 0 \\ 0 & \mathbb{1} \end{matrix} \end{array}\right] \begin{bmatrix} x^{RR}[k,l] \\ x^{RS}[k,l] \\ x^{SR}[k-1,l] \\ x^{SS}[k-1,l] \end{bmatrix} \quad (18)$$

[1] In a later step in the derivation, we will show that the inequality of Eqn. (13) is in fact an equality, which follows from the regularity of the pencil $(F_{1,1}, B_{1,1})$, however, this is far from trivial at this point in the proof.

Note that the WCF of the first equation is preserved and the matrix $Q_2^{-1}NQ_2$ is still nilpotent [2]. However, it may no longer be in a Jordan form.

The next step is to proof that the matrices $P_1F'_{1,2}Q_2$ and $P_1B'_{1,2}Q_2$ can be further eliminated in the block structure. This is done by demonstrating that under certain conditions, we can assume that $x^R[0,0] = 0 = x^R[0,1]$, which leads to a rank condition on the system matrices. The singular vector $\tilde{x}^S[k,l]$ satisfies the dynamic equation presented in Eqn. (16), where the matrix N is nilpotent, with nilpotency index p, and its solution is

$$\tilde{x}^S[k,l] = N^{n-k}\tilde{x}^S[n,l].$$

Assume that $n > p$, this implies that $\tilde{x}^S[0,l] = 0$, (for all l) we also have

$$Q_2^{-1}\tilde{x}^S[0,0] = 0 = \left[x^{SR}[0,0]^T \; x^{SS}[0,0]^T\right]^T.$$

Under this condition, we can eliminate the singular part of the state vector and Eqn. (18) reduces to

$$\begin{bmatrix} \mathbb{1} & 0 \\ 0 & N_1 \end{bmatrix}\begin{bmatrix} x^{RR}[k,l+1] \\ x^{RS}[k,l+1] \end{bmatrix} = \begin{bmatrix} B_1 & 0 \\ 0 & \mathbb{1} \end{bmatrix}\begin{bmatrix} x^{RR}[k,l] \\ x^{RS}[k,l] \end{bmatrix}.$$

The state sequence that satisfies these equations is

$$\begin{array}{c} x^{RR}[k,l] = B_1^l x^{RR}[k,0] \\ x^{RS}[k,l] = N_1^{n-l}x^{RS}[k,n] \end{array} \quad \forall k.$$

Due to the nilpotency of the matrix N_1 there exists a value n_1 such that $N_1^{n_1} = 0$. Assume $n - l - 1 > n_1$, and we have $x^{RS}[k,0] = x^{RS}[k,1] = 0$. Furthermore, the value of $x^{RR}[0,0]$ can be freely chosen, and is assumed to be zero. The value of the state vector $x^{RR}[0,1]$ is now equal to

$$x^{RR}[0,1] = B_1x^{RR}[0,0] = 0$$

Under these two conditions we have that,

$$\begin{bmatrix} x^{RR}[0,0] \\ x^{RS}[0,0] \end{bmatrix} = 0 = \begin{bmatrix} x^{RR}[0,1] \\ x^{RS}[0,1] \end{bmatrix}.$$

Because x^{RR} and x^{RS} are related to the vector x^R via a non-singular transformation as described in Eqn. (17), we have proven that we can always assume

$$x^R[0,0] = 0 = x^R[0,1],$$

without loss of generality. The vector $x^R[k,0]$ satisfies the regular equation in Eqn. (10), such that

$$x^R[k,0] = A_R^k x^R[0,0] = 0,$$
$$x^R[k,1] = A_R^k x^R[0,1] = 0,$$

for all values of k. Applying this to Eqn. (15) yields,

$$\begin{array}{c} F'_{1,2}\tilde{x}^S[k,l+1] = B'_{1,2}\tilde{x}^S[k,l] \\ F'_{2,2}\tilde{x}^S[k,l+1] = B'_{2,2}\tilde{x}^S[k,l]. \end{array}$$

This system of equations is over determined and only has a solution for all values of $\tilde{x}^S[k,l]$ when

$$\text{range}\left(\begin{bmatrix} F'_{1,2} \\ F'_{2,2} \end{bmatrix}\right) \supseteq \text{range}\left(\begin{bmatrix} B'_{1,2} \\ B'_{2,2} \end{bmatrix}\right).$$

This implies that

$$\text{rank}\left(\begin{bmatrix} F'_{1,2} & B'_{1,2} \\ F'_{2,2} & B'_{2,2} \end{bmatrix}\right) \leq \dim(\tilde{x}^S). \quad (19)$$

Furthermore, from the regularity of the matrix pencil $(F'_{2,2}, B'_{2,2})$ it follows that the pencil $(F'^T_{2,2}, B'^T_{2,2})$ is also regular and that there does not exist a vector x for which

$$F'^T_{2,2}x = 0 = B'^T_{2,2}x \Rightarrow (sF'^T_{2,2} - B'^T_{2,2})x = 0 \quad \forall s,$$

[2] It is clear that $(Q_2^{-1}NQ_2)^p = Q_2^{-1}N^pQ_2$, which proves that the transformed matrix has the same nilpotency index as N.

otherwise the pencil would be singular. This implies that

$$\text{rank}\left(\begin{bmatrix} F'^T_{2,2} \\ B'^T_{2,2} \end{bmatrix}\right) = \dim(\tilde{x}^S).$$

Note that the columns of this matrix are the bottom rows in the matrix present in Eqn. (19). And we have that

$$\text{rank}\left(\begin{bmatrix} 0 & 0 \\ F'_{2,2} & B'_{2,2} \end{bmatrix}\right) = \dim(\tilde{x}^S).$$

This demonstrates that the inequality in Eqn. (19) is an equality and that the rows span by $[F'_{1,2}, B'_{1,2}]$ must be linearly depended on $[F'_{2,2}, B'_{2,2}]$ in order for the rank condition to be satisfied. We thus have

$$\text{range}\left(\begin{bmatrix} F'_{2,2} \\ B'_{2,2} \end{bmatrix}\right) \supseteq \text{range}\left(\begin{bmatrix} F'_{1,2} \\ B'_{1,2} \end{bmatrix}\right),$$

and that there exists a matrix $V_{1,2}$, which can be singular, for which

$$\begin{bmatrix} F'_{2,2} \\ B'_{2,2} \end{bmatrix} V_{1,2} = \begin{bmatrix} F'_{1,2} \\ B'_{1,2} \end{bmatrix}.$$

The matrix in Eqn. (11) can thus further be reduced by means of elementary row operations to the form

$$\begin{bmatrix} \mathbb{1} & -V_{1,2} \\ 0 & \mathbb{1} \end{bmatrix}\begin{bmatrix} F'_{1,2} & B'_{1,2} \\ F'_{2,2} & B'_{2,2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ F'_{2,2} & B'_{2,2} \end{bmatrix}.$$

with

$$V = \begin{bmatrix} \mathbb{1} & -V_{1,2} \\ 0 & \mathbb{1} \end{bmatrix}, \quad (20)$$

note that the matrix V is of full rank, despite that $V_{1,2}$ can be singular. Applying this transformation to Eqn. (11) results in

$$\begin{bmatrix} F'_{1,1} & 0 \\ 0 & F'_{2,2} \end{bmatrix}\begin{bmatrix} x^R[k,l+1] \\ x^S[k-1,l+1] \end{bmatrix} = \begin{bmatrix} B'_{1,1} & 0 \\ 0 & B'_{2,2} \end{bmatrix}\begin{bmatrix} x^R[k,l] \\ x^S[k-1,l] \end{bmatrix}$$

which decouples the regular and the singular part of the state vector. Note that it was important that the transformation matrix in Eqn. (20) is upper triangular in order to preserve the existing zero block under the block diagonal.

This suggests that the state space model of Eqn. (8) can always be reduced to the form

$$\begin{bmatrix} \mathbb{1} & 0 \\ 0 & N \end{bmatrix}\begin{bmatrix} x^R[k+1,l] \\ x^S[k+1,l] \end{bmatrix} = \begin{bmatrix} A_R & 0 \\ 0 & \mathbb{1} \end{bmatrix}\begin{bmatrix} x^R[k,l] \\ x^S[k,l] \end{bmatrix}$$
$$\begin{bmatrix} F'_1 & 0 \\ 0 & F'_2 \end{bmatrix}\begin{bmatrix} x^R[k,l+1] \\ x^S[k,l+1] \end{bmatrix} = \begin{bmatrix} B'_1 & 0 \\ 0 & B'_2 \end{bmatrix}\begin{bmatrix} x^R[k,l] \\ x^S[k,l] \end{bmatrix}$$

This model consist of two separate semi-descriptor systems for which canonical forms have been derived in Section 3.

The other way around, if the descriptor system is put in the described form of Conjecture 4, it is clear that the descriptor system is well-posed and that the output is given by Eqn. (9).

The reduction of the two-dimensional descriptor system to the form described in Conjecture 4 has some further implications.

Corollary 5. When the two-dimensional descriptor system is transformed to the form described in Conjecture 4, all four transformed system matrices E, A, F, B commute. As a consequence, the state sequence $x[k,l]$ satisfies the following equation

$$E^nF^mx[n,m] = A^nB^mx[0,0].$$

Proof. The commutation of the matrices A, B, E and F, presented in Conjecture 4, follows directly from the block structure of the matrices and the commutation constraints on the blocks. Consider the first model equation

$$Ex[k,l] = Ax[k-1,l].$$

Both sides are multiplied by the square matrix E

$$E^2x[k,l] = EAx[k-1,l] = AEx[k-1,l] = A^2x[k-2,l].$$

By repeating this step n-times and setting $k = n$ we get

$$E^nx[n,l] = A^nx[0,l]. \tag{21}$$

Next, we multiply both sides of Eqn. (21) by F to get

$$FE^nx[n,l] = FA^nx[n,l] = A^nFx[n,l] = A^nBx[n,l-1].$$

We repeat this step $m-$times and put $m = l$ such that

$$E^nF^mx[n,m] = A^nB^mx[0,0].$$

This proves the result from Corollary 5. □

5. EXAMPLE

To demonstrate the above results, we take a descriptor system defined by the following matrices,

$$E = \begin{bmatrix}1&1&0\\1&1&0\\0&0&0\end{bmatrix}, A = \begin{bmatrix}1&1&0\\2&3&2\\0&1&1\end{bmatrix}, F = \begin{bmatrix}0&1&1\\1&2&2\\1&2&1\end{bmatrix}, B = \begin{bmatrix}0&1&1\\2&3&3\\1&2&1\end{bmatrix}$$

$$C = [2\ 3\ 2]$$

Notice that the four matrices do not commute and that the matrices E and F are singular. Both matrix pencils (E, A) and (F, B) are regular, which can be demonstrated by calculating the associated characteristic equations

$$\det(Es - A) = -s + 1,$$

$$\det(Fs - B) = -2s^2 + 4s - 2.$$

The matrix pencil (E, A) can be put in the WCF via the transformation matrices

$$P = \begin{bmatrix}1&0&0\\-1&1&-1\\0&0&1\end{bmatrix}, \quad Q = \begin{bmatrix}0&1&-1\\1&-1&1\\-1&1&0\end{bmatrix}$$

The four system matrices (PEQ, PAQ, FQ, BQ) become

$$PEQ = \begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \quad PAQ = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},$$

$$FQ = \begin{bmatrix}0&0&1\\0&0&1\\1&0&1\end{bmatrix}, \quad BQ = \begin{bmatrix}0&0&1\\0&2&1\\1&0&1\end{bmatrix},$$

$$CQ = [1\ 1\ 1].$$

In the next step, we calculate the matrix U which generates a first row compression to the matrices (FQ, BQ), in practice the row compression can be computed via the singular value decomposition. In this example we have

$$U \left[\begin{array}{c|c}F_{1,1}&B_{1,1}\\F_{2,1}&B_{2,1}\end{array}\right] = \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix} \left[\begin{array}{cc|cc}0&0&0&0\\0&0&0&2\\\hline 1&0&1&0\end{array}\right] = \left[\begin{array}{cc|cc}1&0&1&0\\0&0&0&2\\\hline 0&0&0&0\end{array}\right]$$

Such that the partially reduced matrices are

$$UFQ = \begin{bmatrix}1&0&1\\0&0&1\\0&0&1\end{bmatrix}, \quad UBQ = \begin{bmatrix}1&0&1\\0&2&1\\0&0&1\end{bmatrix}.$$

In this form it is clear that the matrix pencil (UFQ, UBQ) is still regular. In the next step we calculate the matrix V to further compress the rows

$$V \left[\begin{array}{c|c}F_{1,2}&B_{1,2}\\F_{2,2}&B_{2,2}\end{array}\right] = \begin{bmatrix}1&0&-1\\0&1&-1\\0&0&1\end{bmatrix} \left[\begin{array}{c|c}1&1\\1&1\\\hline 1&1\end{array}\right] = \left[\begin{array}{c|c}0&0\\0&0\\\hline 1&1\end{array}\right].$$

The second model equation is put in its final form

$$VUFQ = \begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}, \quad VUBQ = \begin{bmatrix}1&0&0\\0&2&0\\0&0&1\end{bmatrix}.$$

And the system is transformed to its final form (EQ, AQ, $VUFQ$, $VUBQ$, CQ). For the multi-index $[k, l]$ with $0 \leq k \leq n$, $0 \leq l \leq m$ the output of the model is

$$y[k,l] = x^{RR}[0,0] + 0^{n-k}x^{RS}[0,0] + 0^{n-k}0^{m-l}x^{SS}[n,m].$$

6. CONCLUSION

In this paper, some necessary conditions for the existence of a non-trivial state sequence $x[k, l]$ of a two-dimensional descriptor system has been derived. The derivation is based on the classical results of the Weierstrass canonical form and the final result could be considered as an extension of this canonical form to multiple matrix pencils. As a consequence of this conjecture it is shown that the descriptor system can always be transformed to a form where the system matrices commute.

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