On the Logical Geometry of Geometric Angles

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Abstract. In this paper we provide an analysis of the logical relations within the conceptual or lexical field of angles in 2D geometry. The basic tripartition into acute/right/obtuse angles is extended in two steps: first zero and straight angles are added, and secondly reflex and full angles are added, in both cases extending the logical space of angles. Within the framework of Logical Geometry, the resulting partitions of these logical spaces yield bitstring semantics of increasing complexity. These bitstring analyses allow a straightforward account of the Aristotelian relations between angular concepts. In addition, also relational concepts such as complementary and supplementary angles receive a natural bitstring analysis.

Mathematics Subject Classification (2010). Primary 03A10, 51M04; Secondary 03B80, 03G05, 51M20.

Keywords. Aristotelian relations, bitstring semantics, Logical Geometry, acute/right/obtuse angle, zero/straight angle, reflex/full angle, complementary/supplementary angles.

1. Introduction

In this paper we will investigate the conceptual or lexical field of angles in 2D geometry. A well-known mathematical reference work reports the following basic definitions and properties of angles:

> Given two intersecting lines or line segments, the amount of rotation about the point of intersection (the vertex) required to bring one into correspondence with the other is called the *angle* α between them. Angles are usually measured in degrees (denoted as $^\circ$), radians (denoted rad, or without a

This research is supported via the ID-N research project *Bitstring Semantics for Human and Artificial Reasoning* (BITSHARE). The second author holds a research professorship (BOFZAP) at KU Leuven. Thanks to Jens Lemanski, Andrew Schumann and two anonymous reviewers for their feedback on an earlier version of this paper.

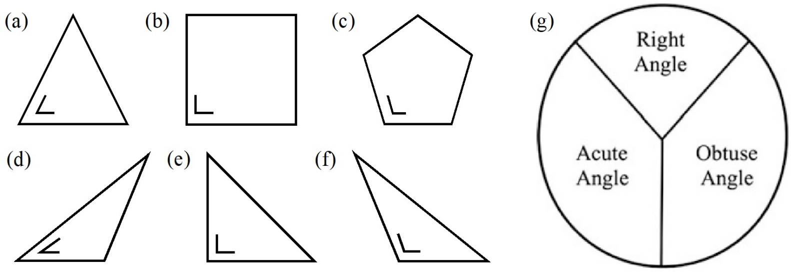

FIGURE 1. (a-b-c) Acute, right and obtuse angles in regular polygons (d-e-f) Acute, right and obtuse angles in triangles (g) Schopenhauer's Euler diagram for the basic tripartition of the logical space of angles.

> unit), or sometimes gradians (denoted grad).[1] One full rotation [...] corresponds to 360° [...]. Half a full rotation is called a *straight* angle, and a quarter of a full rotation is called a *right* angle. An angle less than a right angle is called an *acute* angle, and an angle greater than a right angle is called an *obtuse* angle. [21, p. 78, our emphases]

The acute, right and obtuse angles constitute a very natural basic constellation. First of all, they characterise the interior angles of the sequence of regular polygons in Figures 1(a-b-c): the angles of the triangle in Figure 1(a) are acute, those of the square in Figure 1(b) are right, and those of any other regular polygon with more than four angles — such as the pentagon in Figure 1(c) — are obtuse. Secondly, the acute, right and obtuse angles at the bottom left in Figures 1(d-e-f) also yield the three-way classification of triangles into acute, right and obtuse triangles respectively [21, p. 3023]. The basic intuition that these three types of angles constitute a natural tripartition of the 'logical space' of angles already inspired the 19th-century philosopher Arthur Schopenhauer to draw an Euler diagram as in Figure 1(g) [17, p. 66]. Schopenhauer's diagram has recently been discussed in more technical terms (cf. [4, p. 196, Figure 6] and [12, p. 118, Figure 9]), which provided the impetus for the more elaborate analysis of the logical space of geometric angles to be developed here. Finally, this fundamental tripartition of acute, right and obtuse angles also has natural links with topics from contemporary research on spatial logic, such as qualitative spatial reasoning [16] and the logical formalization of elementary geometry [1].

Our analysis is developed in the framework of Logical Geometry, and makes use of the so-called *Aristotelian relations*. In their most general form, these relations can be defined in the mathematical setting of Boolean algebra [3, 4]:

[1]In this paper, we systematically use degrees to measure angles.

Definition 1.1. Let $\mathbb{B} = \langle B, \wedge, \vee, \neg, \top, \bot \rangle$ be an arbitrary Boolean algebra [7]. Two elements $x, y \in B$ are said to be

$\mathbb{B}$-*contradictory*	iff	$x \wedge y = \bot$	and	$x \vee y = \top$,
$\mathbb{B}$-*contrary*	iff	$x \wedge y = \bot$	and	$x \vee y \neq \top$,
$\mathbb{B}$-*subcontrary*	iff	$x \wedge y \neq \bot$	and	$x \vee y = \top$,
in $\mathbb{B}$-*subalternation*	iff	$\neg x \vee y = \top$	and	$x \vee \neg y \neq \top$.

As a first special case of Definition 1.1, we take $\mathbb{B}$ to be a Boolean algebra of *statements*. The top and bottom elements of such a Boolean algebra are resp. the tautological and self-contradictory statements, while the algebraic operations of meet, join and complement correspond to the logical operations of resp. conjunction, disjunction and negation:

Definition 1.2. Let $\mathbb{B} = \langle B, \wedge, \vee, \neg, \top, \bot \rangle$ be a Boolean algebra of statements. Two statements $P, Q \in B$ are said to be

$\mathbb{B}$-*contradictory*	iff	$P \wedge Q = \bot$	and	$P \vee Q = \top$,
$\mathbb{B}$-*contrary*	iff	$P \wedge Q = \bot$	and	$P \vee Q \neq \top$,
$\mathbb{B}$-*subcontrary*	iff	$P \wedge Q \neq \bot$	and	$P \vee Q = \top$,
in $\mathbb{B}$-*subalternation*	iff	$\neg P \vee Q = \top$	and	$P \vee \neg Q \neq \top$.

We thus find that two statements P and Q are contradictory in this Boolean algebra iff $P \wedge Q = \bot$ and $P \vee Q = \top$, i.e. iff the conjunction of P and Q is self-contradictory, while the disjunction of P and Q is tautological. The first part means exactly that P and Q cannot be true together, while the second part means that P and Q cannot be false together. We have thus obtained the 'familiar' definition of contradiction for statements (in terms of being able to be true/false together). Similarly, P and Q are said to be (i) *contrary* iff they cannot be true together but can be false together, (ii) *subcontrary* iff they can be true together but cannot be false together, and (iii) in *subalternation* if P entails Q but Q does not entail P.

As a second special case of Definition 1.1, we take $\mathbb{B}$ to be a Boolean algebra of *sets*. The top and bottom elements of such a Boolean algebra are resp. the entire domain of discourse D and the empty set $\emptyset$, while the algebraic operations of meet, join and complement correspond to the set-theoretical operations of resp. intersection, union and complementation (with respect to D):

Definition 1.3. Let $\mathbb{B} = \langle B, \cap, \cup, \setminus, D, \emptyset \rangle$ be a Boolean algebra of sets. Two sets $X, Y \in B$ are said to be

$\mathbb{B}$-*contradictory*	iff	$X \cap Y = \emptyset$	and	$X \cup Y = D$,
$\mathbb{B}$-*contrary*	iff	$X \cap Y = \emptyset$	and	$X \cup Y \neq D$,
$\mathbb{B}$-*subcontrary*	iff	$X \cap Y \neq \emptyset$	and	$X \cup Y = D$,
in $\mathbb{B}$-*subalternation*	iff	$X \subseteq Y$	and	$X \not\supseteq Y$.

A central notion in the framework of Logical Geometry is that of a *bitstring*, i.e. a sequence of values 1 or 0, such as 100 or 01011 [6]. Bitstrings are said to belong to a *level* depending on the number of 1-bits they contain. For example, 100 is a level 1 (L1) bitstring, 01010 is a L2 bitstring, and so on. Bitstrings provide a compact way of representing the semantics of the expressions in a given logical fragment or lexical field, and allow us to study the logical relations holding between these expressions in terms of their bitstring representations. In particular, Aristotelian relations can straightforwardly be defined in terms of bitstrings. Hence, as a third and final special case of Definition 1.1, we can take $\mathbb{B}$ to be a Boolean algebra of *bitstrings* of length n, i.e. $\{0,1\}^n$. The top and bottom elements of such a Boolean algebra are resp. $\mathbf{1}_n$ and $\mathbf{0}_n$, i.e. the bitstrings $1\cdots1$ and $0\cdots0$, exclusively consisting of n values 1 or 0, respectively. The algebraic operations of meet, join and complement then correspond to the logical operations of resp. conjunction, disjunction and negation, applied bitwise, i.e. bit position by bit position:

Definition 1.4. Let $\mathbb{B} = \langle B, \wedge, \vee, \neg, \mathbf{1}_n, \mathbf{0}_n\rangle$ be a Boolean algebra $\{0,1\}^n$ of bitstrings of length n. Two bitstrings b_1 and $b_2 \in B$ are said to be

$\mathbb{B}$-*contradictory*	iff	$b_1 \wedge b_2 = \mathbf{0}_n$	and	$b_1 \vee b_2 = \mathbf{1}_n$,
$\mathbb{B}$-*contrary*	iff	$b_1 \wedge b_2 = \mathbf{0}_n$	and	$b_1 \vee b_2 \neq \mathbf{1}_n$,
$\mathbb{B}$-*subcontrary*	iff	$b_1 \wedge b_2 \neq \mathbf{0}_n$	and	$b_1 \vee b_2 = \mathbf{1}_n$,
in $\mathbb{B}$-*subalternation*	iff	$b_1 \wedge b_2 = b_1$	and	$b_1 \vee b_2 \neq b_1$.

In the present paper, Boolean algebras consisting of sets (cf. Definition 1.3) and of bitstrings (cf. Definition 1.4) will play a crucial role in the analysis of the logical space of geometric angles.

The paper consists of two main parts, followed by a brief conclusion. In Section 2 the basic classification of angles into acute, right and obtuse angles is given a bitstring analysis and the Aristotelian relations between these concepts are captured in a JSB hexagon (§ 2.1). Subsequently, the logical space of angles is extended in two steps, first by adding zero and straight angles (§ 2.2) and secondly by adding reflex and full angles (§ 2.3), thus stepwise increasing the complexity of the bitstring analysis as well. In Section 3, various logical relations between angles — such as complementarity (§ 3.1), supplementarity (§ 3.2) and counter-supplementarity (§ 3.3) — are discussed, and given an analysis in terms of mirroring or flipping operations on bitstrings. By way of conclusion, Section 4 recasts the results of this paper in terms of decreasing bitstring complexity, and briefly points to a possible connection with the notion of Duality in the realm of quantification.

2. Aristotelian relations between angular predicates

2.1. Acute, right and obtuse angles

In the most basic case, the logical space of angles is restricted to the open interval $\mathcal{A}_1 := \{\alpha \mid 0° < \alpha < 180°\}$. The language $\mathcal{L}_A$ to describe this logical space contains one-place predicates P — such as *acute*, *right*, *oblique*, *obtuse* — and individual constant symbols s such as a, b, ... as the labels for angles. These expressions are interpreted in $\mathcal{A}_1$ by means of the interpretation function $[\![\cdot]\!]_1 \colon \mathcal{L}_A \to \mathcal{A}_1$. In particular, one-place predicates P in $\mathcal{L}_A$ denote subsets of $\mathcal{A}_1$ — for all $P \in \mathcal{L}_A : [\![P]\!]_1 \subseteq \mathcal{A}_1$ — whereas the constant symbols

s in $\mathcal{L}_A$ denote elements of $\mathcal{A}_1$ — for all $s \in \mathcal{L}_A : \llbracket s \rrbracket_1 \in \mathcal{A}_1$. Well-formed formulae of $\mathcal{L}_A$ are of the form $P(s)$ where for all $P, s \in \mathcal{L}_A : \llbracket P(s) \rrbracket_1 = 1$ iff $\llbracket s \rrbracket_1 \in \llbracket P \rrbracket_1$. The three basic angular predicates are defined as follows [21, pp. 28, 2054, 2568]:

$$\begin{array}{lcl} \llbracket acute \rrbracket_1 & := & \{\alpha \in \mathcal{A}_1 \mid 0° < \alpha < 90°\} \\ \llbracket right \rrbracket_1 & := & \{90°\} \\ \llbracket obtuse \rrbracket_1 & := & \{\alpha \in \mathcal{A}_1 \mid 90° < \alpha < 180°\} \end{array}$$

The predicate *oblique* is then defined as *acute or obtuse*, in other words, as the negation of the predicate *right*. We can hence define the three negative counterparts of the above basic predicates as follows:

$$\begin{array}{lcl} \llbracket non\_acute \rrbracket_1 & := & \mathcal{A}_1 \setminus \llbracket acute \rrbracket_1 \\ & = & \{\alpha \in \mathcal{A}_1 \mid 90° \leq \alpha < 180°\} \\ \llbracket non\_right \rrbracket_1 = \llbracket oblique \rrbracket_1 & := & \mathcal{A}_1 \setminus \llbracket right \rrbracket_1 \\ & = & \{\alpha \in \mathcal{A}_1 \mid \alpha \neq 90°\} \\ \llbracket non\_obtuse \rrbracket_1 & := & \mathcal{A}_1 \setminus \llbracket obtuse \rrbracket_1 \\ & = & \{\alpha \in \mathcal{A}_1 \mid 0° < \alpha \leq 90°\} \end{array}$$

We can now take these six predicates together to define the fragment

$$\mathcal{F}_1 := \{acute,\ right,\ obtuse,\ non\_acute,\ oblique,\ non\_obtuse\}$$

In order to characterise the Aristotelian relations between the predicates in this fragment, we provide a bitstring analysis by defining the partition induced by the fragment [6, Definition 5].

Intuitively speaking, we get the "basic" tripartition —•— with the node for the central reference point of 90° and the two intervals to the left and to the right for $0° < \alpha < 90°$ and $90° < \alpha < 180°$ respectively.[2] Technically speaking, the partition induced by $\mathcal{F}_1$ is defined as

$$\Pi(\mathcal{F}_1) := \{acute,\ right,\ obtuse\}$$

and the corresponding bitstring mapping β_1 yields the following bitstrings of length three for the fragment $\mathcal{F}_1$:

$$\begin{array}{lcllcl} \beta_1(acute) & = & 100 & \beta_1(non\_acute) & = & 011 \\ \beta_1(right) & = & 010 & \beta_1(oblique) & = & 101 \\ \beta_1(obtuse) & = & 001 & \beta_1(non\_obtuse) & = & 110 \end{array}$$

Applying Definition 1.3 to the Boolean algebra of sets for (denotations of) angular predicates, or equivalently, Definition 1.4 to the Boolean algebra of bitstrings of length 3, gives rise to the standard hexagon in Figure 2(a), which belongs to the Aristotelian family of Jacoby-Sesmat-Blanché (JSB) hexagons [2, 9, 18]. More specifically, since the three pairwise contrary elements *acute* (100), *right* (010) and *obtuse* (001) are jointly exhaustive — i.e. $100 \vee 010 \vee 001 = 111$ —, this diagram belongs to the Boolean subfamily of *strong* JSB hexagons [15]. Using the coding conventions in Figure 2(c), we can observe (i) three diagonals for the contradiction relations, (ii) an upside down triangle for the contrariety relations between the three L1 predicates

[2] See [20] for the introduction of this diagrammatic representation format for scalar structures.

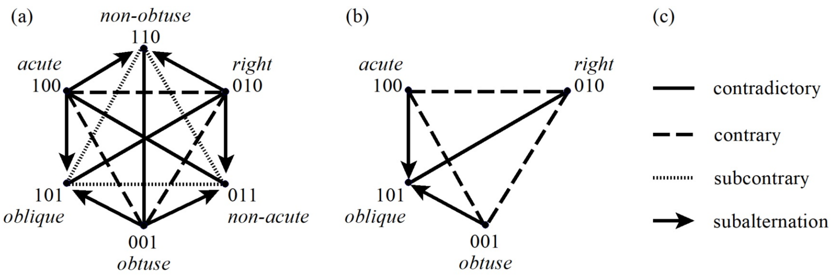

FIGURE 2. (a) Jacoby-Sesmat-Blanché hexagon for $\mathcal{F}_1$ (b) Kite structure for the primitively lexicalised subfragment of $\mathcal{F}_1$ (c) Coding conventions for the Aristotelian relations.

acute, *right* and *obtuse*, (iii) an upright triangle for the subcontrariety relations between the three L2 predicates *non_acute*, *oblique*, and *non_obtuse*, and (iv) six arrows along the edges of the hexagon for the subalternation relations from the L1 to the L2 elements.

Notice that the L2 angular predicate *oblique* is the only negative predicate which itself receives a simple, non-compound lexicalisation. On the basis of similar linguistic observations of lexicalisation patterns, Seuren and Jaspers [10, 19] have proposed a so-called 'kite' structure, which reduces the JSB hexagon by eliminating the 'non-natural' elements at the top and the bottom right of the hexagon. Applying this strategy to the hexagon of angular predicates in Figure 2(a) yields the kite structure in Figure 2(b).[3]

2.2. Adding zero and straight angles

In a first, minimal move of extending the logical space of angles, we close the interval $\mathcal{A}_1$ by including its extreme values 0° and 180°. We thus define the new logical space as the closed interval $\mathcal{A}_2 := \{\alpha \mid 0^\circ \leq \alpha \leq 180^\circ\}$ and the interpretation function $[\![\cdot]\!]_2 \colon \mathcal{L}_A \to \mathcal{A}_2$. Two extra one-place predicates are added to the language $\mathcal{L}_A$, namely *zero* and *straight* [21, p. 2869], thus yielding the following five basic angular predicates:

$$\begin{array}{lcl}
[\![\mathit{zero}]\!]_2 & := & \{0^\circ\} \\
[\![\mathit{acute}]\!]_2 & := & \{\alpha \in \mathcal{A}_2 \mid 0^\circ < \alpha < 90^\circ\} \\
[\![\mathit{right}]\!]_2 & := & \{90^\circ\} \\
[\![\mathit{obtuse}]\!]_2 & := & \{\alpha \in \mathcal{A}_2 \mid 90^\circ < \alpha < 180^\circ\} \\
[\![\mathit{straight}]\!]_2 & := & \{180^\circ\}
\end{array}$$

We can then again define their negative counterparts as follows:

[3]See [3, Section 3] for an analogous kite-based analysis of the mathematical terminology of *compatibility* and strong/weak *contrariety*.

$$
\begin{array}{lcl}
[\![non\_zero]\!]_2 & := & \mathcal{A}_2 \setminus [\![zero]\!]_2 \\
 & = & \{\alpha \in \mathcal{A}_2 \mid 0^\circ < \alpha \leq 180^\circ\} \\
[\![non\_acute]\!]_2 & := & \mathcal{A}_2 \setminus [\![acute]\!]_2 \\
 & = & \{\alpha \in \mathcal{A}_2 \mid \alpha = 0^\circ \textit{ or } 90^\circ \leq \alpha \leq 180^\circ\} \\
[\![non\_right]\!]_2 & := & \mathcal{A}_2 \setminus [\![right]\!]_2 \\
 & = & \{\alpha \in \mathcal{A}_2 \mid \alpha \neq 90^\circ\} \\
[\![non\_obtuse]\!]_2 & := & \mathcal{A}_2 \setminus [\![obtuse]\!]_2 \\
 & = & \{\alpha \in \mathcal{A}_2 \mid 0^\circ \leq \alpha \leq 90^\circ \textit{ or } \alpha = 180^\circ\} \\
[\![non\_straight]\!]_2 & := & \mathcal{A}_2 \setminus [\![straight]\!]_2 \\
 & = & \{\alpha \in \mathcal{A}_2 \mid \alpha < 180^\circ\}
\end{array}
$$

We can now take these ten predicates together to define the fragment

$$
\mathcal{F}_2 \;:=\; \{ \;\; \textit{zero, acute, right, obtuse, straight, non\_zero, non\_acute, non\_right, non\_obtuse, non\_straight} \;\}
$$

In order to provide a bitstring analysis, we define the partition induced by the fragment. Intuitively speaking, the original basic tripartition —•— is turned into a five-partition •—•—• by adding two nodes at the left and right extremes for the extreme values of 0° and 180°. Technically speaking, the partition induced by $\mathcal{F}_2$ is defined as

$$
\Pi(\mathcal{F}_2) := \{\textit{zero, acute, right, obtuse, straight}\}
$$

and the corresponding bitstring mapping β_2 yields the following bitstrings of length five for the fragment $\mathcal{F}_2$:

$$
\begin{array}{lcl@{\qquad\qquad}lcl}
\beta_2(\textit{zero}) & = & 10000 & \beta_2(\textit{non\_zero}) & = & 01111 \\
\beta_2(\textit{acute}) & = & 01000 & \beta_2(\textit{non\_acute}) & = & 10111 \\
\beta_2(\textit{right}) & = & 00100 & \beta_2(\textit{non\_right}) & = & 11011 \\
\beta_2(\textit{obtuse}) & = & 00010 & \beta_2(\textit{non\_obtuse}) & = & 11101 \\
\beta_2(\textit{straight}) & = & 00001 & \beta_2(\textit{non\_straight}) & = & 11110
\end{array}
$$

A first remark concerns the interpretation of the predicate *acute*, which is often used in a broader way than defined above, namely as smaller than 90°, but possibly 0°. This situation obviously resembles the distinction drawn between so-called one-sided and two-sided readings of quantifiers and modal operators [8], for instance one-sided *some and perhaps even all* versus two-sided *some but not all*, or similarly one-sided *possible and perhaps even necessary* versus two-sided *possible but not necessary* (i.e. *contingent*). In other words, we should distinguish between one-sided $acute_1$ and two-sided $acute_2$, which receive distinct bitstring representations, resp. as L2 and L1 elements:

$$
\beta_2(acute_1) \;=\; 11000 \qquad\qquad \beta_2(acute_2) \;=\; 01000
$$

A second question concerns the interpretation of the predicate *oblique*. In the basic logical space $\mathcal{A}_1$ of § 2.1 it was equal to the disjunction of *acute* and *obtuse*, and also to the negation of *right*. In the more elaborate logical space $\mathcal{A}_2$ that we are considering now, the former, disjunctive characterisation — namely *acute or obtuse* — still seems to make sense, whereas the latter, negative characterisation does not. In particular, the two extreme predicates *zero* and *straight* are typically understood as falling under *non_oblique*. Put

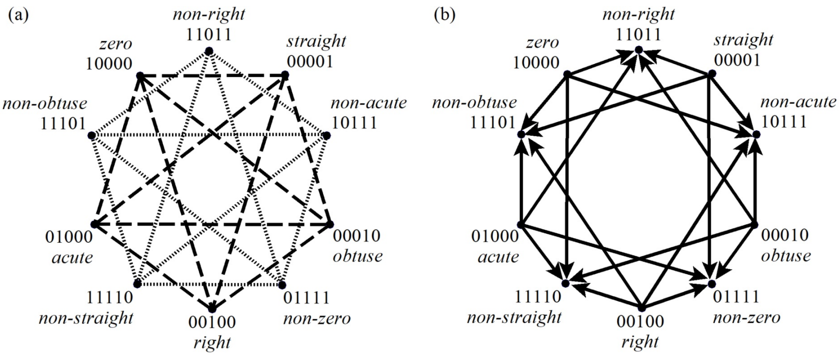

FIGURE 3. (a) Contrariety and subcontrariety relations in $\mathcal{F}_2$ (b) Subalternation relations in $\mathcal{F}_2$.

differently, the predicate *non_oblique* is equivalent to the three-way disjunction *zero or right or straight* and thus gets represented as a L3 bitstring:

$$\beta_2(\textit{oblique}) \;=\; 01010 \qquad\qquad \beta_2(\textit{non\_oblique}) \;=\; 10101$$

Notice that *oblique* is still a L2 predicate, but now relative to a more fine-grained logical space, i.e. with respect to bitstrings of length 5 instead of length 3.

At this point, we can again apply Definitions 1.3 and 1.4 to the angular predicates in the fragment $\mathcal{F}_2$ and their corresponding bitstrings of length 5 to build the decagonal diagrams for the Aristotelian relations in Figure 3. For the sake of visual clarity, we omit the five diagonals for the five contradiction relations. In Figure 3(a), both the ten relations of contrariety between the five L1 predicates and the ten relations of subcontrariety between the five L4 predicates are visualised by means of a pentagon with an inscribed pentagonal star.[4] In Figure 3(b), by contrast, the twenty arrows represent the subalternation relations from each of the five L1 predicates to the four L4 predicates that it is not contradictory with.

Observe, finally, that the original fragment

$$\mathcal{F}_1 = \{\textit{acute, right, obtuse, non\_acute, non\_right, non\_obtuse}\,\}$$

is a subfragment of $\mathcal{F}_2$, and still yields a JSB hexagon inside the decagon. However, since the three pairwise contrary elements *acute* (01000), *right* (00100) and *obtuse* (00010) are no longer jointly exhaustive, we now get a *weak* JSB hexagon, rather than a strong one [15].[5]

[4]Aristotelian diagrams like these — as well as the JSB hexagon discussed above — are called *α-structures* by Moretti [13]. See [4] for some further theoretical results on (the Boolean properties of) α-structures, and [11] for another example of a decagonal α-structure, which once again derives from the works of Schopenhauer.

[5]See [6, Section 4.3] for a more detailed analysis of the strong/weak Boolean subfamilies of the Aristotelian family of JSB hexagons.

2.3. Adding reflex and full angles

In a second move, the logical space of angles is extended more radically, i.e. it is ‘doubled’ by including the angles between 180° and 360°. We thus define the new logical space $\mathcal{A}_3 := \{\alpha \mid 0° \leq \alpha \leq 360°\}$ and the interpretation function $[\![\cdot]\!]_3 \colon \mathcal{L}_A \to \mathcal{A}_3$. Again two extra one-place predicates are added to the language $\mathcal{L}_A$, namely *reflex* and *full* [21, pp. 1117, 2514] — see Figures 3(a-b-c-d) below — thus yielding the following seven angular predicates:

$$\begin{array}{lcl} [\![\mathit{zero}]\!]_3 & := & \{0°\} \\ [\![\mathit{acute}]\!]_3 & := & \{\alpha \in \mathcal{A}_3 \mid 0° < \alpha < 90°\} \\ [\![\mathit{right}]\!]_3 & := & \{90°\} \\ [\![\mathit{obtuse}]\!]_3 & := & \{\alpha \in \mathcal{A}_3 \mid 90° < \alpha < 180°\} \\ [\![\mathit{straight}]\!]_3 & := & \{180°\} \\ [\![\mathit{reflex}]\!]_3 & := & \{\alpha \in \mathcal{A}_3 \mid 180° < \alpha < 360°\} \\ [\![\mathit{full}]\!]_3 & := & \{360°\} \end{array}$$

We can now take these seven predicates together with their seven negative counterparts to define the fragment

$$\begin{array}{lcll} \mathcal{F}_3 & := & \{ & \textit{zero, acute, right, obtuse, straight, reflex, full,} \\ & & & \textit{non\_zero, non\_acute, non\_right, non\_obtuse,} \\ & & & \textit{non\_straight, non\_reflex, non\_full } \} \end{array}$$

In order to provide a bitstring analysis, we define the partition induced by the fragment. Intuitively speaking, the five-partition •—•—• from § 2.2 is turned into a seven-partition by adding an interval and a node to the right: •—•—•—•. Technically speaking, the partition induced by $\mathcal{F}_3$ is defined as

$$\Pi(\mathcal{F}_3) := \{\textit{zero, acute, right, obtuse, straight, reflex, full}\}$$

and the corresponding bitstring mapping β_3 yields the following bitstrings of length seven for the fragment $\mathcal{F}_3$:

$\beta_3(\mathit{zero})$	=	1000000	$\beta_3(\mathit{non\_zero})$	=	0111111
$\beta_3(\mathit{acute})$	=	0100000	$\beta_3(\mathit{non\_acute})$	=	1011111
$\beta_3(\mathit{right})$	=	0010000	$\beta_3(\mathit{non\_right})$	=	1101111
$\beta_3(\mathit{obtuse})$	=	0001000	$\beta_3(\mathit{non\_obtuse})$	=	1110111
$\beta_3(\mathit{straight})$	=	0000100	$\beta_3(\mathit{non\_straight})$	=	1111011
$\beta_3(\mathit{reflex})$	=	0000010	$\beta_3(\mathit{non\_reflex})$	=	1111101
$\beta_3(\mathit{full})$	=	0000001	$\beta_3(\mathit{non\_full})$	=	1111110

At first sight, the above analysis also allows a straightforward generalisation for the two higher-level predicates *oblique* and *non_oblique*. In particular, *oblique* would correspond to the three-way disjunction *acute or obtuse or reflex* and its negation *non_oblique* to the four-way disjunction *zero or right or straight or full*:

$$\beta_3(\mathit{oblique}) = 0101010 \qquad \beta_3(\mathit{non\_oblique}) = 1010101$$

However, an angle of 270° — as represented in Figure 4(b) — classifies as *reflex* according to the above definition, and hence as *oblique*. This seems to run into conflict with the fundamental connection between the angles of 90° and 270°, which we will turn to in more detail in Section 3. We therefore

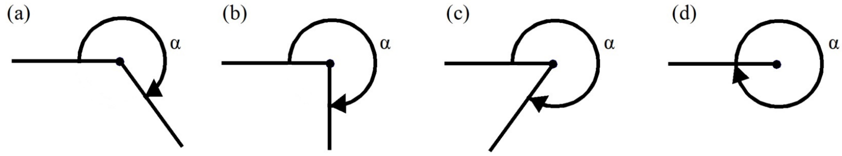

FIGURE 4. (a) Reflex, counter-obtuse angle (b) Reflex, counter-right angle (c) Reflex, counter-acute angle (d) Full angle.

propose an alternative analysis, which turns the five-partition •—•—• from § 2.2 into a nine-partition •—•—•—•—• by copying the four left-most components of the former, i.e. •—•—, and rotating them around the 180° node, i.e. —•—•. In this way, two nodes are added, namely for 270° and for 360°, and two intervals for $180° < \alpha < 270°$ and $270° < \alpha < 360°$ respectively. In other words, the denotation of the predicate *reflex* is split into three subareas, for which we introduce the new names *counter_obtuse*, *counter_right* and *counter_acute* — or *c_obtuse*, *c_right* and *c_acute* for short — which are represented in Figures 4(a-b-c) respectively, and which are defined as follows:

$$
\begin{array}{lcl}
[\![zero]\!]_3 & = & \{0°\} \\
[\![acute]\!]_3 & = & \{\alpha \in \mathcal{A}_3 \mid 0° < \alpha < 90°\} \\
[\![right]\!]_3 & = & \{90°\} \\
[\![obtuse]\!]_3 & = & \{\alpha \in \mathcal{A}_3 \mid 90° < \alpha < 180°\} \\
[\![straight]\!]_3 & = & \{180°\} \\
[\![c\_obtuse]\!]_3 & := & \{\alpha \in \mathcal{A}_3 \mid 180° < \alpha < 270°\} \\
[\![c\_right]\!]_3 & := & \{270°\} \\
[\![c\_acute]\!]_3 & := & \{\alpha \in \mathcal{A}_3 \mid 270° < \alpha < 360°\} \\
[\![full]\!]_3 & = & \{360°\}
\end{array}
$$

The alternative fragment

$$
\begin{array}{lcll}
\mathcal{F}'_3 & := & \{ & zero,\ acute,\ right,\ obtuse,\ straight,\ c\_obtuse,\ c\_right, \\
 & & & c\_acute,\ full,\ non\_zero,\ non\_acute,\ non\_right, \\
 & & & non\_obtuse,\ non\_straight,\ non\_c\_obtuse, \\
 & & & non\_c\_right,\ non\_c\_acute,\ non\_full\ \}
\end{array}
$$

induces the partition

$$
\begin{array}{lcll}
\Pi(\mathcal{F}'_3) & := & \{ & zero,\ acute,\ right,\ obtuse,\ straight, \\
 & & & c\_obtuse,\ c\_right,\ c\_acute,\ full\ \}
\end{array}
$$

and the corresponding bitstring mapping β'_3 yields the following bitstrings of length nine for the fragment $\mathcal{F}'_3$:

$\beta_3'(zero)$	$=$	100000000	$\beta_3'(non\_zero)$	$=$	011111111
$\beta_3'(acute)$	$=$	010000000	$\beta_3'(non\_acute)$	$=$	101111111
$\beta_3'(right)$	$=$	001000000	$\beta_3'(non\_right)$	$=$	110111111
$\beta_3'(obtuse)$	$=$	000100000	$\beta_3'(non\_obtuse)$	$=$	111011111
$\beta_3'(straight)$	$=$	000010000	$\beta_3'(non\_straight)$	$=$	111101111
$\beta_3'(c\_obtuse)$	$=$	000001000	$\beta_3'(non\_c\_obtuse)$	$=$	111110111
$\beta_3'(c\_right)$	$=$	000000100	$\beta_3'(non\_c\_right)$	$=$	111111011
$\beta_3'(c\_acute)$	$=$	000000010	$\beta_3'(non\_c\_acute)$	$=$	111111101
$\beta_3'(full)$	$=$	000000001	$\beta_3'(non\_full)$	$=$	111111110

On this more fine-grained analysis, *reflex* corresponds to the three-way disjunction *c_obtuse or c_right or c_acute*:

$$\beta_3'(reflex) = 000001110 \qquad \beta_3'(non\_reflex) = 111110001$$

More importantly, *oblique* now gets a very natural interpretation as the four-way disjunction *acute or obtuse or c_obtuse or c_acute* and its negation *non_oblique* the equally straightforward analysis as the five-way disjunction *zero or right or straight or c_right or full*:

$$\beta_3'(oblique) = 010101010 \qquad \beta_3'(non\_oblique) = 101010101$$

At this point, we could, once again, apply Definitions 1.3 and 1.4 to the angular predicates in the fragments $\mathcal{F}_3$ and $\mathcal{F}_3'$ and their corresponding bitstrings of lengths 7 and 9, in order to build the Aristotelian diagrams that extend the hexagon and decagon in Figures 2 and 3. However, in view of the considerable graphical complexity of these tetra-decagonal and octa-decagonal diagrams, we will refrain from doing so here.

3. Complementarity and supplementarity relations between angles and angular predicates

In Section 2, we progressively introduced a whole series of one-place predicates of the language $\mathcal{L}_A$, describing properties of individual angles, and yielding an ever more refined classification of angles. The bitstring semantics assigned to fragments of these predicates naturally leads to the characterisation of the Aristotelian relations between these angular properties. In this section, we move from properties of individual angles to relations between angles. Hence, we will add several two-place predicates to the language $\mathcal{L}_A$.

3.1. Complementary angles

The first angular relation to be analysed is that of *complementarity*: two angles are said to be *complementary* iff they add up to 90° [21, p. 482]. More formally, for $i = 1, 2, 3$:

$$[\![complementary]\!]_i := \{(\alpha, \beta) \in \mathcal{A}_i \times \mathcal{A}_i \mid \alpha + \beta = 90^\circ\}$$

In other words, for $i = 1, 2, 3$ and for all $a, b \in \mathcal{L}_A$:

$$[\![complementary(a, b)]\!]_i = 1 \textit{ iff } [\![a]\!]_i + [\![b]\!]_i = 90^\circ$$

This *complementarity* relation is symmetric, i.e. for $i = 1, 2, 3$ and for all $\alpha, \beta \in \mathcal{A}_i$:

$$(\alpha, \beta) \in [\![complementary]\!]_i \textit{ iff } (\beta, \alpha) \in [\![complementary]\!]_i$$

We can now type-shift [14] the *complementarity* relation from a two-place first-order predicate — i.e. a relation between two angles $\alpha, \beta \in \mathcal{A}_i$ — to a two-place second-order predicate *complementary*^{2} — i.e. a relation between two sets of angles $\Gamma, \Delta \subseteq \mathcal{A}_i$. For $i = 1, 2, 3$, $[\![complementary^2]\!]_i :=$

$$\{(\Gamma, \Delta) \in \wp(\mathcal{A}_i) \times \wp(\mathcal{A}_i) \mid \exists\alpha \in \Gamma, \exists\beta \in \Delta : (\alpha, \beta) \in [\![complementary]\!]_i\}$$

Thus, for $i = 1, 2, 3$ and for all $P, Q \in \mathcal{L}_A$: $[\![complementary^2(P, Q)]\!]_i = 1$ iff

$$\exists[\![a]\!]_i \in [\![P]\!]_i, \exists[\![b]\!]_i \in [\![Q]\!]_i : [\![complementary(a, b)]\!]_i = 1$$

In virtue of the symmetry of the first-order *complementary* relation, also the second-order *complementary*^{2} relation is symmetric, i.e. for $i = 1, 2, 3$ and for all $\Gamma, \Delta \subseteq \mathcal{A}_i$:

$$(\Gamma, \Delta) \in [\![complementary^2]\!]_i \textit{ iff } (\Delta, \Gamma) \in [\![complementary^2]\!]_i$$

In the most basic case — described in § 2.1 — the logical space of angles is restricted to $\mathcal{A}_1 = \{\alpha \mid 0° < \alpha < 180°\}$. In this scenario, there is only one type of complementarity, namely that between two acute angles. In other words, $[\![complementary(a, b)]\!]_1 = 1$ only if $[\![acute(a)]\!]_1 = 1$ and $[\![acute(b)]\!]_1 = 1$. Or put in second-order terms:

$$[\![complementary^2]\!]_1 = \{([\![acute]\!]_1, [\![acute]\!]_1)\}$$

Extending the logical space of angles to $\mathcal{A}_2 = \{\alpha \mid 0° \leq \alpha \leq 180°\}$, as in § 2.2, however, three scenarios arise: if $[\![complementary(a, b)]\!]_2 = 1$, then:

$$\begin{array}{rlcl} \text{either} & [\![zero(a)]\!]_2 = 1 & \text{and} & [\![right(b)]\!]_2 = 1, \\ \text{or} & [\![acute(a)]\!]_2 = 1 & \text{and} & [\![acute(b)]\!]_2 = 1, \\ \text{or} & [\![right(a)]\!]_2 = 1 & \text{and} & [\![zero(b)]\!]_2 = 1. \end{array}$$

Or equivalently, in second-order terms:

$$\begin{array}{rcl} [\![complementary^2]\!]_2 & = & \{ \ ([\![zero]\!]_2, [\![right]\!]_2), ([\![acute]\!]_2, [\![acute]\!]_2), \\ & & \quad ([\![right]\!]_2, [\![zero]\!]_2) \ \} \end{array}$$

Notice that extending the logical space of angles to $\mathcal{A}_3 = \{\alpha \mid 0° \leq \alpha \leq 360°\}$, as in § 2.3, does not add any extra complexity. In other words, exactly the same three scenarios arise as with $\mathcal{A}_2$:

$$\begin{array}{rcl} [\![complementary^2]\!]_3 & = & \{ \ ([\![zero]\!]_3, [\![right]\!]_3), ([\![acute]\!]_3, [\![acute]\!]_3), \\ & & \quad ([\![right]\!]_3, [\![zero]\!]_3) \ \} \end{array}$$

3.2. Supplementary angles

The second angular relation to be analysed is that of *supplementarity*: two angles are said to be *supplementary* iff they add up to 180° [21, p. 2897]. More formally, for $i = 1, 2, 3$:

$$[\![supplementary]\!]_i := \{(\alpha, \beta) \in \mathcal{A}_i \times \mathcal{A}_i \mid \alpha + \beta = 180°\}$$

In other words, for $i = 1, 2, 3$ and for all $a, b \in \mathcal{L}_A$:

$$[\![supplementary(a, b)]\!]_i = 1 \textit{ iff } [\![a]\!]_i + [\![b]\!]_i = 180°$$

This *supplementarity* relation is again symmetric, i.e. for $i = 1, 2, 3$ and for all $\alpha, \beta \in \mathcal{A}_i$:

$$(\alpha, \beta) \in [\![supplementary]\!]_i \text{ iff } (\beta, \alpha) \in [\![supplementary]\!]_i$$

As with the *complementarity* relation in § 3.1, we can now type-shift the *supplementarity* relation from a two-place first-order relation between two angles $\alpha, \beta \in \mathcal{A}_i$ to a two-place second-order *supplementary*^{2} relation between two sets of angles $\Gamma, \Delta \subseteq \mathcal{A}_i$. For $i = 1, 2, 3$, $[\![supplementary^2]\!]_i :=$

$$\{(\Gamma, \Delta) \in \wp(\mathcal{A}_i) \times \wp(\mathcal{A}_i) \mid \exists\alpha \in \Gamma, \exists\beta \in \Delta : (\alpha, \beta) \in [\![supplementary]\!]_i\}$$

Thus, for $i = 1, 2, 3$ and for all $P, Q \in \mathcal{L}_A$: $[\![supplementary^2(P, Q)]\!]_i = 1$ iff

$$\exists[\![a]\!]_i \in [\![P]\!]_i, \exists[\![b]\!]_i \in [\![Q]\!]_i : [\![supplementary(a, b)]\!]_i = 1$$

In virtue of the symmetry of the first-order *supplementary* relation, also the second-order *supplementary*^{2} relation is symmetric, i.e. for $i = 1, 2, 3$ and for all $\Gamma, \Delta \subseteq \mathcal{A}_i$:

$$(\Gamma, \Delta) \in [\![supplementary^2]\!]_i \text{ iff } (\Delta, \Gamma) \in [\![supplementary^2]\!]_i$$

In the logical space $\mathcal{A}_1 = \{\alpha \mid 0° < \alpha < 180°\}$, three types of supplementarity arise: if $[\![supplementary(a, b)]\!]_1 = 1$ then:

$$\begin{array}{rrcr} \text{either} & [\![acute(a)]\!]_1 = 1 & \text{and} & [\![obtuse(b)]\!]_1 = 1, \\ \text{or} & [\![right(a)]\!]_1 = 1 & \text{and} & [\![right(b)]\!]_1 = 1, \\ \text{or} & [\![obtuse(a)]\!]_1 = 1 & \text{and} & [\![acute(b)]\!]_1 = 1. \end{array}$$

In terms of the second-order *supplementary*^{2} relation this yields:

$$\begin{array}{rcll} [\![supplementary^2]\!]_1 & = & \{ & ([\![acute]\!]_1, [\![obtuse]\!]_1), ([\![right]\!]_1, [\![right]\!]_1), \\ & & & ([\![obtuse]\!]_1, [\![acute]\!]_1) \ \} \end{array}$$

Moving to the logical space $\mathcal{A}_2 = \{\alpha \mid 0° \leq \alpha \leq 180°\}$, however, supplementarity shows up in five different shapes: if $[\![supplementary(a, b)]\!]_2 = 1$ then:

$$\begin{array}{rrcr} \text{either} & [\![zero(a)]\!]_2 = 1 & \text{and} & [\![straight(b)]\!]_2 = 1, \\ \text{or} & [\![acute(a)]\!]_2 = 1 & \text{and} & [\![obtuse(b)]\!]_2 = 1, \\ \text{or} & [\![right(a)]\!]_2 = 1 & \text{and} & [\![right(b)]\!]_2 = 1, \\ \text{or} & [\![obtuse(a)]\!]_2 = 1 & \text{and} & [\![acute(b)]\!]_2 = 1, \\ \text{or} & [\![straight(a)]\!]_2 = 1 & \text{and} & [\![zero(b)]\!]_2 = 1. \end{array}$$

These five scenarios straightforwardly correspond to the five pairs of angular predicates in the denotation of the second-order *supplementary*^{2} relation:

$$\begin{array}{rclll} [\![supplementary^2]\!]_2 & = & \{ & ([\![zero]\!]_2, [\![straight]\!]_2), & ([\![acute]\!]_2, [\![obtuse]\!]_2), \\ & & & ([\![straight]\!]_2, [\![zero]\!]_2), & ([\![obtuse]\!]_2, [\![acute]\!]_2), \\ & & & ([\![right]\!]_2, [\![right]\!]_2), & \} \end{array}$$

As was the case with the *complementary*^{2} relation at the end of § 3.1, extending the logical space of angles from $\mathcal{A}_2$ to $\mathcal{A}_3 = \{\alpha \mid 0° \leq \alpha \leq 360°\}$ does not add any extra complexity. In other words, exactly the same five pairs of angular predicates constitute the denotation of the *supplementary*^{2} relation as with $\mathcal{A}_2$:

$$\begin{array}{lcll} [\![supplementary^2]\!]_3 & = & \{ & ([\![zero]\!]_3, [\![straight]\!]_3), \quad ([\![acute]\!]_3, [\![obtuse]\!]_3), \\ & & & ([\![straight]\!]_3, [\![zero]\!]_3), \quad ([\![obtuse]\!]_3, [\![acute]\!]_3), \\ & & & ([\![right]\!]_3, [\![right]\!]_3) \qquad \} \end{array}$$

This second-order relationship of *supplementarity* can now be connected to the bitstring analysis provided for the one-place predicates in Section 2. The basic intuition is that any bitstring can be 'flipped', i.e. reversed from left to right: flipping 100, for instance, yields 001, and flipping 1100 yields 0011. Using $[\beta(X)]_i$ to denote the i-th position in the bitstring $\beta(X)$ of first-order predicates X in $\mathcal{L}_A$, we can define ϕ as the *flipping* operation on bitstrings in the following way: for all bitstrings $\beta(X)$ and $\beta(Y)$ of length n: $\phi(\beta(X)) = \beta(Y)$ iff for all $1 \leq i \leq n$, $[\beta(Y)]_i = [\beta(X)]_{n+1-i}$. Applying the flipping operation ϕ to the six non-trivial bitstrings of length 3, for instance, yields the following picture:

$$\begin{array}{lcl@{\qquad\qquad}lcl} \phi(100) & = & 001 & \phi(110) & = & 011 \\ \phi(010) & = & 010 & \phi(101) & = & 101 \\ \phi(001) & = & 100 & \phi(011) & = & 110 \end{array}$$

Observe that the flipping operation ϕ is an involution, or self-inverse function: for all bitstrings $\beta(X)$ and $\beta(Y)$ of length n: $\phi(\phi(\beta(X))) = \beta(X)$, or equivalently $\phi(\beta(X)) = \beta(Y) \Leftrightarrow \phi(\beta(Y)) = \beta(X)$. Observe, furthermore, that in the case of symmetric bitstrings — i.e. bitstrings where $[\beta(X)]_i = [\beta(X)]_{n+1-i}$ for all $1 \leq i \leq n$ — the ϕ-operation maps a bitstring onto itself. With the above bitstrings of length 3, for instance, this is the case for $\phi(010) = 010$ and $\phi(101) = 101$.

More importantly, a straightforward connection can now be established between the second-order *supplementary*^{2}-relation between two angular predicates X and Y and the flipping operation on their bitstrings counterparts $\beta(X)$ and $\beta(Y)$. More precisely, for $i = 1, 2$ we have:[6]

$$[\![supplementary^2(X, Y)]\!]_i = 1 \Leftrightarrow \phi(\beta_i(X)) = \beta_i(Y)$$

In the smallest logical space $\mathcal{A}_1 = \{\alpha \mid 0° < \alpha < 180°\}$, this equivalence first of all holds for the *supplementary*^{2} relations — henceforth abbreviated as *suppl*^{2} — between the L1 predicates described above:

$$\begin{array}{rcrcl} [\![suppl^2(acute, obtuse)]\!]_1 = 1 & \Leftrightarrow & \phi(\beta_1(acute)) & = & \beta_1(obtuse) \\ & \Leftrightarrow & \phi(100) & = & 001 \\ [\![suppl^2(right, right)]\!]_1 = 1 & \Leftrightarrow & \phi(\beta_1(right)) & = & \beta_1(right) \\ & \Leftrightarrow & \phi(010) & = & 010 \end{array}$$

Completely analogously, we get the equivalence for the extra *supplementary*^{2} relations between the negative L2 counterpart predicates:

$$\begin{array}{rcrcl} [\![suppl^2(n\_acute,\ n\_obtuse)]\!]_1 = 1 & \Leftrightarrow & \phi(\beta_1(n\_acute)) & = & \beta_1(n\_obtuse) \\ & \Leftrightarrow & \phi(011) & = & 110 \\ [\![suppl^2(n\_right,\ n\_right)]\!]_1 = 1 & \Leftrightarrow & \phi(\beta_1(n\_right)) & = & \beta_1(n\_right) \\ & \Leftrightarrow & \phi(101) & = & 101 \end{array}$$

[6] The case $i = 3$ will be discussed in more detail later.

Moving from the logical space $\mathcal{A}_1$ to $\mathcal{A}_2 = \{\alpha \mid 0° \leq \alpha \leq 180°\}$, the complexity of the bitstring semantics increases from length 3 to length 5. Again, the equivalence between the *supplementary*^{2} relation between angular predicates and the flipping operation ϕ on the corresponding bitstrings holds, first of all, for the L1 predicates described above:

$$\begin{array}{rcrcl}
[\![suppl^2(zero,\ straight)]\!]_2 = 1 & \Leftrightarrow & \phi(\beta_2(zero)) & = & \beta_2(straight) \\
& \Leftrightarrow & \phi(10000) & = & 00001 \\
[\![suppl^2(acute,\ obtuse)]\!]_2 = 1 & \Leftrightarrow & \phi(\beta_2(acute)) & = & \beta_2(obtuse) \\
& \Leftrightarrow & \phi(01000) & = & 00010 \\
[\![suppl^2(right,\ right)]\!]_2 = 1 & \Leftrightarrow & \phi(\beta_2(right)) & = & \beta_2(right) \\
& \Leftrightarrow & \phi(00100) & = & 00100
\end{array}$$

Secondly, we get the equivalence for the extra *supplementary*^{2} relations between the negative counterpart predicates. However, with bitstrings of length 5, the latter are L4 predicates:

$$\begin{array}{rcrcl}
[\![suppl^2(n_zero,\ n_straight)]\!]_2 = 1 & \Leftrightarrow & \phi(\beta_2(n_zero)) & = & \beta_2(n_straight) \\
& \Leftrightarrow & \phi(01111) & = & 11110 \\
[\![suppl^2(n_acute,\ n_obtuse)]\!]_2 = 1 & \Leftrightarrow & \phi(\beta_2(n_acute)) & = & \beta_2(n_obtuse) \\
& \Leftrightarrow & \phi(10111) & = & 11101 \\
[\![suppl^2(n_right,\ n_right)]\!]_2 = 1 & \Leftrightarrow & \phi(\beta_2(n_right)) & = & \beta_2(n_right) \\
& \Leftrightarrow & \phi(11011) & = & 11011
\end{array}$$

Notice that the identical pairs $(right, right)$ and (non_right, non_right) of the *supplementary*^{2} relation perfectly match the fact that the ϕ-operation maps the respective symmetric bitstrings onto themselves. Exactly the same situation holds in the case of the disjunctive L2 predicate *obliq(ue)* and its negative L3 counterpart *n(on)_obliq(ue)*:

$$\begin{array}{rcrcl}
[\![suppl^2(obliq,\ obliq)]\!]_2 = 1 & \Leftrightarrow & \phi(\beta_2(obliq)) & = & \beta_2(obliq) \\
& \Leftrightarrow & \phi(01010) & = & 01010 \\
[\![suppl^2(n_obliq,\ n_obliq)]\!]_2 = 1 & \Leftrightarrow & \phi(\beta_2(n_obliq)) & = & \beta_2(n_obliq) \\
& \Leftrightarrow & \phi(10101) & = & 10101
\end{array}$$

It is important to stress that the equivalence between the *supplementary*^{2} relation between angular predicates and the flipping operation ϕ on the corresponding bitstrings no longer holds when moving to the most complex logical space $\mathcal{A}_3 = \{\alpha \mid 0° \leq \alpha \leq 360°\}$. In the latter case, the complexity of the bitstring semantics increases to length 7 or 9, and the corresponding flipping operation ϕ concerns all seven or nine bit positions. The range of application of the *supplementary*^{2} relation, by contrast, is restricted to $0° \leq \alpha \leq 180°$, i.e. to the first *five* bit positions in bitstrings of length 7 or 9. In the next subsection, however, we introduce a new relation between angular predicates which allows us to restore the equivalence with the flipping operation on bitstrings in $\mathcal{A}_3$.

3.3. Counter-supplementary angles

Remember from § 2.3 that the semantic analysis of the most complex logical space $\mathcal{A}_3 = \{\alpha \mid 0° \leq \alpha \leq 360°\}$ was presented in two steps. In particular, we argued in favour of replacing the original seven-partition $\Pi(\mathcal{F}_3)$ — which naturally arises from the standard definitions of *reflex* and *full* angles — with the nine-partition $\Pi(\mathcal{F}_3')$. The crucial modification involved the subdivision of the denotation of the predicate *reflex* into three subareas: (i) *c(ounter)_obtuse* for $180° < \alpha < 270°$, (ii) *c(ounter)_right* for $\alpha = 270°$, and (iii) *c(ounter)_acute* for $270° < \alpha < 360°$. The primary motivation for this modification concerned the problematic position of an angle of 270° with respect to the entailment relation between the predicates *reflex* and *oblique*.

In this subsection we will provide a second argument, by presenting an analysis which aims to do justice to the fundamental connection between the angles of 90° and 270°. The key purpose of this proposal is precisely to generalise the concept of *supplementarity* defined in § 3.2 for the logical space $\mathcal{A}_2$ to the most complex logical space $\mathcal{A}_3$. In order to do so, we define the new relation of *counter-supplementarity* or *c-supplementarity*. Two angles are said to be *counter-supplementary*, or *c-supplementary* for short, iff they add up to 360°. More formally, for $i = 1, 2, 3$:

$$[\![\textit{c-supplementary}]\!]_i := \{(\alpha, \beta) \in \mathcal{A}_i \times \mathcal{A}_i \mid \alpha + \beta = 360°\}$$

In other words, for $i = 1, 2, 3$ and for all $a, b \in \mathcal{L}_A$:

$$[\![\textit{c-supplementary(a,b)}]\!]_i = 1 \textit{ iff } [\![a]\!]_i + [\![b]\!]_i = 360°$$

This *c-supplementarity* relation is again symmetric, i.e. for $i = 1, 2, 3$ and for all $\alpha, \beta \in \mathcal{A}_i$:

$$(\alpha, \beta) \in [\![\textit{c-supplementary}]\!]_i \textit{ iff } (\beta, \alpha) \in [\![\textit{c-supplementary}]\!]_i$$

As with the *complementarity* relation in § 3.1 and the *supplementarity* relation in § 3.2, we can now type-shift the *c-supplementarity* relation from a two-place first-order relation between two angles $\alpha, \beta \in \mathcal{A}_i$ to a two-place second-order *c-suppl(ementary)*^{2} relation between two sets of angles $\Gamma, \Delta \subseteq \mathcal{A}_i$. For $i = 1, 2, 3$, $[\![\textit{c-supplementary}^2]\!]_i :=$

$$\{(\Gamma, \Delta) \in \wp(\mathcal{A}_i) \times \wp(\mathcal{A}_i) \mid \exists \alpha \in \Gamma, \exists \beta \in \Delta : (\alpha, \beta) \in [\![\textit{c-supplementary}]\!]_i\}$$

Thus, for $i = 1, 2, 3$ and for all $P, Q \in \mathcal{L}_A$: $[\![\textit{c-supplementary}^2(P, Q)]\!]_i = 1$ iff

$$\exists [\![a]\!]_i \in [\![P]\!]_i, \exists [\![b]\!]_i \in [\![Q]\!]_i : [\![\textit{c-supplementary(a,b)}]\!]_i = 1$$

In virtue of the symmetry of the first-order *c-supplementary* relation, also the second-order *c-supplementary*^{2} relation is symmetric, i.e. for $i = 1, 2, 3$ and for all $\Gamma, \Delta \subseteq \mathcal{A}_i$:

$$(\Gamma, \Delta) \in [\![\textit{c-supplementary}^2]\!]_i \textit{ iff } (\Delta, \Gamma) \in [\![\textit{c-supplementary}^2]\!]_i$$

It is easy to see that, relative to the logical spaces $\mathcal{A}_1 = \{\alpha \mid 0° < \alpha < 180°\}$ and $\mathcal{A}_2 = \{\alpha \mid 0° \leq \alpha \leq 180°\}$, the relation of *c-supplementarity* is not particularly interesting. In the former case, the extreme value of 180° is excluded, so it is impossible in principle to have two angles from $\mathcal{A}_1$ add up to 360°. In the latter case, the only pair of angles that does stand in the relation of *c-supplementarity* is that of two straight angles (180°, 180°). In terms of the second-order *c-suppl(ementary)*^{2} relation this yields:

$$\begin{array}{lcl} [\![c\text{-}suppl^2]\!]_1 & = & \{\ \} \\ [\![c\text{-}suppl^2]\!]_2 & = & \{([\![straight]\!]_2, [\![straight]\!]_2)\ \} \end{array}$$

By contrast, in the logical space $\mathcal{A}_3 = \{\alpha \mid 0° \leq \alpha \leq 360°\}$, c-supplementarity shows up in nine different shapes: if $[\![c\text{-}supplementary(a, b)]\!]_3 = 1$ then:

$$\begin{array}{lrcr} \text{either} & [\![zero(a)]\!]_3 = 1 & \text{and} & [\![full(b)]\!]_3 = 1, \\ \text{or} & [\![acute(a)]\!]_3 = 1 & \text{and} & [\![c\_acute(b)]\!]_3 = 1, \\ \text{or} & [\![right(a)]\!]_3 = 1 & \text{and} & [\![c\_right(b)]\!]_3 = 1, \\ \text{or} & [\![obtuse(a)]\!]_3 = 1 & \text{and} & [\![c\_obtuse(b)]\!]_3 = 1, \\ \text{or} & [\![straight(a)]\!]_3 = 1 & \text{and} & [\![straight(b)]\!]_3 = 1, \\ \text{or} & [\![c\_obtuse(a)]\!]_3 = 1 & \text{and} & [\![obtuse(b)]\!]_3 = 1, \\ \text{or} & [\![c\_right(a)]\!]_3 = 1 & \text{and} & [\![right(b)]\!]_3 = 1, \\ \text{or} & [\![c\_acute(a)]\!]_3 = 1 & \text{and} & [\![acute(b)]\!]_3 = 1, \\ \text{or} & [\![full(a)]\!]_3 = 1 & \text{and} & [\![zero(b)]\!]_3 = 1. \end{array}$$

These nine scenarios straightforwardly correspond to the nine pairs of angular predicates in the denotation of the second-order *c-suppl(ementary)*^{2} relation:

$$\begin{array}{lcll} [\![c\text{-}suppl^2]\!]_3 & = & \{ & ([\![zero]\!]_3, [\![full]\!]_3), \quad ([\![acute]\!]_3, [\![c\_acute]\!]_3), \\ & & & ([\![full]\!]_3, [\![zero]\!]_3), \quad ([\![c\_acute]\!]_3, [\![acute]\!]_3), \\ & & & ([\![right]\!]_3, [\![c\_right]\!]_3), \quad ([\![obtuse]\!]_3, [\![c\_obtuse]\!]_3), \\ & & & ([\![c\_right]\!]_3, [\![right]\!]_3), \quad ([\![c\_obtuse]\!]_3, [\![obtuse]\!]_3), \\ & & & ([\![straight]\!]_3, [\![straight]\!]_3) \quad \} \end{array}$$

Remember from the end of § 3.2 that the equivalence between the *supplementary*^{2} relation between angular predicates and the flipping operation ϕ on the corresponding bitstrings no longer holds when moving to the most complex logical space $\mathcal{A}_3 = \{\alpha \mid 0° \leq \alpha \leq 360°\}$. This logical space is in a sense 'too big' to have the *supplementary*^{2} relation correspond to a flipping operation on bitstrings of length nine. In this subsection, by contrast, we encounter the mirror-image constellation, so to speak: the logical spaces $\mathcal{A}_1$ and $\mathcal{A}_2$ are 'too small' to have the *c-supplementary*^{2} relation correspond to a flipping operation on bitstrings of length 3 or 5 only. This obviously relates to the fact observed above that the denotations of the $[\![c\text{-}suppl^2]\!]_1$ and $[\![c\text{-}suppl^2]\!]_2$ relations are hardly interesting.

Furthermore, as for $\mathcal{A}_3$, we have replaced the bitstring mapping β_3 corresponding to the seven-partition $\Pi(\mathcal{F}_3)$ with the β'_3 mapping corresponding to the nine-partition $\Pi(\mathcal{F}'_3)$. On the face of it, both partitions are 'balanced' and 'symmetric'. Nevertheless, there is a clear intuition of a discrepancy between the 'sizes' of the subareas. In particular, in β_3, bit position 6 (for *reflex*) has the same 'size' as the combination of bit positions 2 (*acute*), 3 (*right*) and 4 (*obtuse*). In β'_3, by contrast, this discrepancy is resolved, since there is a perfect match between bit positions 2 (*acute*), 3 (*right*) and 4 (*obtuse*) on the one hand, and positions 8 (*c_acute*), 7 (*c_right*) and 6 (*c_obtuse*) on the other hand. As a consequence, the only kind of equivalence that can be established in $\mathcal{A}_3$ is that between the *c-supplementary*^{2} relation and the flipping operation ϕ on bitstrings of length 9. More formally, we have:

$$[\![\textit{c-supplementary}^2(X, Y)]\!]_3 = 1 \Leftrightarrow \phi(\beta'_3(X)) = \beta'_3(Y)$$

This equivalence holds, first of all, for the L1 predicates described above:

$$\begin{array}{rcrcl}
[\![\textit{c-suppl}^2(\textit{zero, full})]\!]_3 = 1 & \Leftrightarrow & \phi(\beta'_3(\textit{zero})) & = & \beta'_3(\textit{full}) \\
 & \Leftrightarrow & \phi(100000000) & = & 000000001 \\
[\![\textit{c-suppl}^2(\textit{acute, c_acute})]\!]_3 = 1 & \Leftrightarrow & \phi(\beta'_3(\textit{acute})) & = & \beta'_3(\textit{c_acute}) \\
 & \Leftrightarrow & \phi(010000000) & = & 000000010 \\
[\![\textit{c-suppl}^2(\textit{right, c_right})]\!]_3 = 1 & \Leftrightarrow & \phi(\beta'_3(\textit{right})) & = & \beta'_3(\textit{c_right}) \\
 & \Leftrightarrow & \phi(001000000) & = & 000000100 \\
[\![\textit{c-suppl}^2(\textit{obtuse, c_obtuse})]\!]_3 = 1 & \Leftrightarrow & \phi(\beta'_3(\textit{obtuse})) & = & \beta'_3(\textit{c_obtuse}) \\
 & \Leftrightarrow & \phi(000100000) & = & 000001000 \\
[\![\textit{c-suppl}^2(\textit{straight, straight})]\!]_3 = 1 & \Leftrightarrow & \phi(\beta'_3(\textit{straight})) & = & \beta'_3(\textit{straight}) \\
 & \Leftrightarrow & \phi(000010000) & = & 000010000
\end{array}$$

Secondly, we again get the equivalences for the extra *c-supplementary*^{2}-relations between the negative L8 counterpart predicates:

$$\begin{array}{rlcl}
[\![\textit{c-suppl}^2 & (\textit{n_zero, n_full})]\!]_3 & = & 1 \\
\Leftrightarrow & \phi(\beta'_3(\textit{n_zero})) & = & \beta'_3(\textit{n_full}) \\
\Leftrightarrow & \phi(011111111) & = & 111111110 \\
[\![\textit{c-suppl}^2 & (\textit{n_acute, n_c_acute})]\!]_3 & = & 1 \\
\Leftrightarrow & \phi(\beta'_3(\textit{n_acute})) & = & \beta'_3(\textit{n_c_acute}) \\
\Leftrightarrow & \phi(101111111) & = & 111111101
\end{array}$$

$$\begin{array}{rlcl}
[\![\textit{c-suppl}^2 & (\textit{n_right, n_c_right})]\!]_3 & = & 1 \\
\Leftrightarrow & \phi(\beta'_3(\textit{n_right})) & = & \beta'_3(\textit{n_c_right}) \\
\Leftrightarrow & \phi(110111111) & = & 111111011 \\
[\![\textit{c-suppl}^2 & (\textit{n_obtuse, n_c_obtuse})]\!]_3 & = & 1 \\
\Leftrightarrow & \phi(\beta'_3(\textit{n_obtuse})) & = & \beta'_3(\textit{n_c_obtuse}) \\
\Leftrightarrow & \phi(111011111) & = & 111110111 \\
[\![\textit{c-suppl}^2 & (\textit{n_straight, n_straight})]\!]_3 & = & 1 \\
\Leftrightarrow & \phi(\beta'_3(\textit{n_straight})) & = & \beta'_3(\textit{n_straight}) \\
\Leftrightarrow & \phi(111101111) & = & 111101111
\end{array}$$

And finally, the observed equivalence between the reflexivity of the *c-supplementary*^{2}-relation with the predicates *straight* and *non_straight* and the mapping of the respective symmetric bitstrings onto themselves by the ϕ-operation straightforwardly carries over to the disjunctive L4 predicate *oblique* and its negative L5 counterpart *non_oblique*, :

$$\begin{array}{rlcl}
[\![\textit{c-suppl}^2 & (\textit{oblique, oblique})]\!]_3 & = & 1 \\
\Leftrightarrow & \phi(\beta'_3(\textit{oblique})) & = & \beta'_3(\textit{oblique}) \\
\Leftrightarrow & \phi(010101010) & = & 010101010 \\
[\![\textit{c-suppl}^2 & (\textit{n_oblique, n_oblique})]\!]_3 & = & 1 \\
\Leftrightarrow & \phi(\beta'_3(\textit{n_oblique})) & = & \beta'_3(\textit{n_oblique}) \\
\Leftrightarrow & \phi(101010101) & = & 101010101
\end{array}$$

4. Conclusion and prospects

In this paper we have provided an analysis of the logical relations within the conceptual or lexical field of angles in 2D geometry. In Section 2, the basic set of angular predicates *acute*, *right*, and *obtuse* was extended in three steps, by adding (i) *zero* and *straight*, (ii) *reflex* and *full* and (iii) *counter_acute*, *counter_right* and *counter_obtuse* respectively. The former two extensions correspond to extending the logical space of angles from $\mathcal{A}_1$ to $\mathcal{A}_2$ to $\mathcal{A}_3$. Within the framework of Logical Geometry, the respective three-, five- and seven-partitions of these logical spaces correspond to the bitstring mappings β_1, β_2 and β_3, yielding bitstrings of increasing complexity, i.e. length 3, 5 and 7 respectively. The final extension, by contrast, is no longer a matter of extending the logical space, but rather of providing a more fine-grained analysis of the same logical space $\mathcal{A}_3$ in terms of a nine-partition, corresponding to the bitstring mapping β'_3 and yielding bitstrings of length 9. These various bitstring analyses allow a straightforward account of the Aristotelian relations between angular predicates, which are graphically represented by means of standard hexagonal or decagonal Aristotelian diagrams, among others.

Notice that the rhetoric of the paper was very much in terms of expansion from small to large, with respect to both logical space and bitstring complexity. On occasion, however, also the reverse perspective is adopted in Logical Geometry, when a distinction is drawn between 'collapsing' and 'eliminating' bit positions [6, p. 349]. Moving 'back' from β'_3 to β_3, for instance, and thus replacing a more fine-grained analysis with a more coarse-grained analysis of the same logical space $\mathcal{A}_3$, would be a matter of *collapsing* bit positions 6, 7 and 8 of β'_3 into the single bit position 6 of β_3. Moving 'back' from β_3 to β_2, or from β_2 to β_1, by contrast, reduces the size of logical space from $\mathcal{A}_3$ to $\mathcal{A}_2$, and from $\mathcal{A}_2$ to $\mathcal{A}_1$, and would be a matter of *eliminating* bit positions 6 and 7 from β_3 and bit positions 1 and 5 from β_2 respectively.

In Section 3, we moved from one-place predicates for angular properties to two-place predicates for angular relations. First, the two standard relations from the literature were analysed, namely *complementarity* — for angles adding up to 90° — and *supplementarity* — for angles adding up to 180°. Secondly, a new relation is proposed, namely *counter-supplementarity* for angles adding up to 360°. Furthermore, equivalence relations were observed between angular relations and the flipping operation ϕ which reverses the bitstrings from left to right. In particular, we established equivalences between *supplementarity* and flipping bitstrings of length 3 and 5, and between *counter-supplementarity* and flipping bitstrings of length 9.

Notice that the flipping operation ϕ is the only operation that has been applied to bitstrings in the present paper. It is perfectly possible, however, to also define a 'switching' operation σ which systematically reverses the values of each bit position, e.g. from 11010 to 00101. As a matter of fact, this is precisely the effect of the predicate negation that showed up time and again between a predicate X and its negative counterpart *non_X*. In future research we aim to investigate the possible interaction between the flipping and switching operations ϕ and σ. What is the relationship, for instance, between $\sigma(\phi(\beta(X)))$ and $\phi(\sigma(\beta(X)))$? We also intend to extend the present analysis to the realm of proportional quantification in natural language — involving expressions such as *two thirds of the students* or *80% of the books*, the underlying scalar structure of which closely resembles that of the logical space of geometric angles. One crucial question in this respect will revolve around the possible connection with duality notions such as internal and external negation [5].

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