# Some notes about signals, orthogonal polynomials and linear algebra

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**Abstract**

These short notes contain an introduction to filtering of deterministic and stochastic signals. The connection with algorithms from classical complex analysis (the Schur algorithm) and with the recurrence relation (Szegő) for polynomials orthogonal with respect to a measure supported on the unit circle are both given. The interpretation of these algorithms in terms of linear algebra lead to fast algorithms for structured matrices. It is explained how these algorithms can be generalized to situations where the signal is not stationary. The corresponding notion of matrices with low displacement rank is introduced.

The notes are organized like lecture notes and contain several exercises that form an essential part of the text. Solutions are provided at the end.

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# Some notes about signals, orthogonal polynomials and linear algebra

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## 1 Discrete signals and digital filters

Let $s(t)$ be a *continuous time signal* that is sampled to obtain a *discrete time signal* or *time series* $s(nT)$. $T$ is the sampling period and $1/T$ is the *sampling frequency*. We shall abbreviate $s(nT)$ as $s_n$.

Denote by $\ell$ the space of complex double sequences

$$\ell = \{s = (s_n) : s_n \in \mathbb{C}, n \in \mathbb{Z}\}.$$

**Note:** In practical applications, a signal is real valued, but considering complex signals does not realy complicate the analysis. On the contrary, we shall make use of complex analysis and considering complex signals seems to be quite natural. On the other hand, it is easier to leave away the complex conjugate bar if you want to specialize to real signals than to introduce it at appropriate places if you want to use results from complex analysis.

Let $\ell_D$ and $\ell_R$ be two subspaces of $\ell$ and $T : \ell_D \to \ell_R$ a linear operator. As a special case we can define the *shift operator* $S$ by

$$(Ss)_n = s_{n-1}, \quad s \in \ell, \quad n \in \mathbb{Z}.$$

A transformation $T$ is called *shift-invariant* if

$$ST = TS.$$

Note that this implies immediately $S^m T = TS^m, m \in \mathbb{Z}$.

A *digital filter* is a *linear* shift invariant transformation.

A signal $s = (s_n) = (\delta_{n0}) = \delta$ is called *unit pulse* (at time $n = 0$) and $h = T\delta$ for this pulse is called the (unit) *impulse response* of the filter $T$.

Consider the normed spaces

$$\begin{aligned} \ell_p &= \{s \in \ell : \|s\|_p = (\sum_{k\in\mathbb{Z}} |s_k|^p)^{1/p} < \infty\}, \quad 0 < p < \infty \\ \ell_\infty &= \{s \in \ell : \|s\|_\infty = \sup_{k\in\mathbb{Z}} |s_k| < \infty\}. \end{aligned}$$

The *convolution* of two signals $u, h \in \ell$ is defined as $u * h = s$, with $s = (s_n)$ given by

$$s_n = \sum_{m\in\mathbb{Z}} u_m h_{n-m}, \quad n \in \mathbb{Z}$$

if these sums converge.

Note that the impuse response $h$ completely characterizes the filter.

$$\begin{aligned} Ts &= T(\sum_m s_m(S^m u)) = \sum s_m(TS^m u) \\ &= \sum_m S^m T u = \sum_m s_m(S^m h) \\ &= \sum_m s_m h_{n-m} = (s * h)_n. \end{aligned}$$

**Exercise 1.1** Let $h \in \ell_1$, then the sums in the convolution $u * h$ converge if $u \in \ell_\infty$ and

$$Tu = u * h = h * u, \quad u \in \ell_\infty, \quad h \in \ell_1$$

defines a digital filter $T : \ell_\infty \to \ell_\infty$. $T$ is a continuous transformation and $h$ is the unit impulse response.

$$T\delta = h.$$

◇

A filter $T$ is called *BIBO* (bounded input, bounded output) *stable* if the sequence $Tu$ is bounded whenever $u$ is bounded (in some $\ell_p$ sense).

Sometimes $\|u\|_2^2$ for $u \in \ell_2$ is called the *energy* of the signal $u$.

A signal $u \in \ell$ is called *causal* if $u_k = 0$ for $k < 0$.

A filter is called *causal* if it maps causal signals into causal signals.

**Exercise 1.2** Prove that $T$ is causal iff

$$u_k = v_k \quad \text{for } k < m \Rightarrow (Tu)_k = (Tv)_k \quad \text{for } k < m$$

◇

**Exercise 1.3** A filter $T : l_\infty \to \ell_\infty$ defined by $Tu = h * u$ and $h \in \ell_1$ with $h$ causal is a causal filter. Prove this. ◇

The *z-transform* $U(z)$ of a signal $u$ is defined as the formal series

$$U(z) = \sum_{n \in \mathbb{Z}} u_n z^n$$

**Note:** In the engineering literature this is usually defined with $z^{-1}$ instead of $z$. We use here the mathematical convention.

The space of signals $\ell$ is often called the *time domain*, while the space of all the $z$-transforms is called the *frequency domain*.

Note that the $z$-transform of a convolution is the product of the $z$-transforms. For example let $h \in \ell_1$, $u \in \ell_\infty$ and $y = Tu = h * u$, then $Y(z) = H(z)U(z)$.

If $h \in \ell_1$, then $H(z)$ converges for $|z| = 1$ to some function. If $h \in \ell_1$ and causal, then $H(z)$ converges for $|z| \leq 1$ to a (complex) function analytic in $|z| < 1$.

Another useful observation. The shift operator $S$ in the time domain $(Su)_n = (u_{n-1})$ corresponds in the frequency domain to a multiplication with $z$.

$$\sum_n u_{n-1} z^n = \sum_n u_n z^{n+1} = z \sum_n u_n z^n = zU(z).$$

We shall abuse the notation $z$ for $S$, also in the time domain. Thus from now on $zu_n = u_{n-1}$. Thus $z$ represents the (backward) shift or *time delay* operator.

**Exercise 1.4** Let $u = (u_k)$ be a given signal with samples of the form

$$u_k = \sum_{i=-I}^{I} \alpha_i e^{j\omega_i k} = \lambda_0 + \sum_{i=1}^{I} \lambda_i \cos(\omega_i k + \varphi_i), \quad k \in \mathbb{Z}$$

for $1 \leq I < \infty$, $\alpha_0 = \lambda_0 \geq 0$, $\omega_0 = 0$ and $\lambda_i > 0$ for $1 \leq i \leq I$, $\omega_{-i} = -\omega_i \in \mathbb{R}$, $\varphi_{-i} = \varphi_i = -\varphi_i \in \mathbb{R}$, $\alpha_{-i} = \bar{\alpha}_i = (1/2)\ \lambda_i e^{j\varphi_i}$.
Let $T : \ell_\infty \to \ell_\infty$ be defined by $Tu = h * u$, $h \in \ell_1$. Then

$$\begin{aligned}(Tu)_k &= \sum_{i=-I}^{I} \alpha_i H(e^{-j\omega_i})\ e^{j\omega_i k} \\ &= \lambda_0 H(1) + \sum_{i=1}^{I} \lambda_i |H(e^{-j\omega_i})| \cos\left(\omega_i k - \varphi_i + \arg H(e^{-j\omega_i})\right)\end{aligned}$$

◇

The *autocorrelation function* $R$ for $u \in \ell_2$ is defined by

$$R_n = \sum_k u_{n+k}\bar{u}_k, \quad R = (R_n), \quad n = 0, \pm 1, \pm 2, \ldots$$

and $R_0 = \sum |u_k|^2$ is called the *total energy* or *average power* of the signal $u$. Note $0 < R_0 < \infty$ if $u \in \ell_2$.

**Exercise 1.5** Prove $R_{-n} = \bar{R}_n$ ◇

Define $P_u(z)$ as the $z$-transform of the autocorrelation function: $P_u(z) = \sum_n R_n z^n$. The function $P_u(e^{j\omega})$ defined on the unit circle is called the *energy spectrum* or *power spectrum* of $u$. If $(R_n) \in \ell_1$, then $P_u(z)$ is a function defined on $|z| = 1$ and its Fourier coefficients are given by

$$R_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-jn\omega} P_u(e^{j\omega}) d\omega.$$

The autocorrelation function $(R_n)$ and the energy spectrum form a *Fourier transform pair.*

Note that

$$\begin{aligned}P_u(z) &= \sum_n (\sum_k u_{n+k}\bar{u}_k) z^n \\ &= \sum_{i,k} u_i \bar{u}_k z^{i-k} \\ &= (\sum_i u_i z^i)\ (\sum_k \bar{u}_k z^{-k}) \\ &= U(z)\ U_*(z)\end{aligned}$$

with $U_* = \overline{U(1/\bar{z})}$. If $u \in \ell_2$,

$$P_u(e^{j\omega}) = |U(e^{j\omega})|^2.$$

Indeed, $U_*(e^{j\omega}) = \overline{U(e^{j\omega})}$.

From Fourier analysis we borrow the following isomorphy relation :

$$u = (u_n) \in \ell_p \Leftrightarrow U(e^{j\omega}) = \sum_n u_n e^{jn\omega} \in L_p, \quad 1 \leq p \leq \infty$$

where $L_p$ is the normed space

$$L_p = \left\{ f : \|f\|_p = \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{j\omega})|^p d\omega \right]^{1/p} < \infty \right\}$$

and Parseval's identity $\|u\|_p = \|U\|_p$ holds.

If $u \in \ell_p$ is a causal signal in $\ell_p$, then $U(z) = \sum_{k=0}^{\infty} u_k z^k$ is analytic in $|z| < 1$ and it belongs to $L_p$, then we say that it is an element from the Hardy space $H_p$

$$H_p = \{f \in L_p \text{ and } f \text{ analytic in } |z| < 1\}.$$

## 2 Stochastic processes and signals

Due to components in the signal such as measurement noise, a signal is often considered as a stochastic entity. A sequence $s = (s_n)$ of identically distributed random variables is called a (discrete) (stochastic) signal or a discrete *stochastic process.* In general, a stochastic variable $x$ is a complex function on a probability space $(\Sigma, m)$, that is a set $\Sigma$ with a probability measure $m$ ($\int_\Sigma dm = 1$). The *expectation operator* E is defined for a random variable $s$ as

$$\mathsf{E}[s] = \int_\Sigma s(\sigma) dm(\sigma) = \text{the } \textit{mean} \text{ of } s.$$

If $s = (s_n)$ is a stochastic process with

$$\mathsf{E}[|s_n|^2] < \infty, \quad n \in \mathbb{Z},$$

we define $\mathcal{L}_2$ to be the closure of the linear hull

$$\{\sum_{-m}^{m} c_n s_n : c_n \in \mathbb{C}, n \in \mathbb{Z}\}.$$

This $\mathcal{L}_2$ is a Hilbert space with inner product

$$< u, v >= \int_\Sigma u(\sigma)\overline{v(\sigma)} dm(\sigma), \quad u, v \in \mathcal{L}_2.$$

The *autocorrelation* function is defined by

$$\begin{aligned} R(k,l) &= < s_k, s_l >= \int_\Sigma s_k(\sigma)\overline{s_l(\sigma)} dm(\sigma), \quad k, l \in \mathbb{Z} \\ &= \mathsf{E}[s_k \bar{s}_l]. \end{aligned}$$

The *autocovariance* is defined as

$$\begin{aligned} C(k,l) &= \mathsf{E}[(s_k - \mathsf{E}[s_k])\overline{(s_l - \mathsf{E}[s_l])}] \\ &= \mathsf{E}[s_k \bar{s}_l] - \mathsf{E}[s_k]\mathsf{E}[\bar{s}_l] - \mathsf{E}[\bar{s}_l]\mathsf{E}[s_k] + \mathsf{E}[s_k]\overline{\mathsf{E}[s_l]} \\ &= \mathsf{E}[s_k \bar{s}_l] - \mathsf{E}[s_k]\overline{\mathsf{E}[s_l]}. \end{aligned}$$

Clearly, if we have a zero-mean stochastic process then $R(k,l) = C(k,l)$ and in that case $R$ is also called the (auto)covariance function.

The process $s = (s_n)$ is called *stationary* if

$$R(k,l) = R(k+m, l+m), \quad k, l, m \in \mathbb{Z}$$

This means that the covariance function of a stochastic process depends only on the difference of the indices $k$ and $l$.
We denote then

$$R_m = R(k+m, k) =< s_m, s_0 >$$

For a stationary process, $\|s_k\|^2 = \|s_0\|^2 = R_0$ is called its *energy* or *average power.*

**Exercise 2.1** Prove that for a stationary process $R_{-m} = \bar{R}_m$ and the Toeplitz forms $\sum_{i,j=0}^{n} c_i \bar{c}_j \, R_{i-j}$ are all positive semi-definite. $\diamond$

The Fourier transform $P$ of the autocorrelation function is called the (power) *spectrum* of the stochastic process:

$$P(e^{j\omega}) = \sum_m R_m e^{jm\omega}; \quad R_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-jm\omega} P(e^{j\omega}) d\omega.$$

Note that the spectrum has the properties

$$\begin{aligned} \text{real} \quad &: \quad P(e^{j\omega}) = \overline{P(e^{j\omega})} \\ \text{positive} \quad &: \quad P(e^{j\omega}) \geq 0. \end{aligned}$$

Hence it can be used as a weight function to define another Hilbert space $L_2$ with inner product

$$< f, g >= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{j\omega}) \overline{g(e^{j\omega})} P(e^{j\omega}) d\omega.$$

It consists of the square integrable complex functions for which

$$\|f\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{j\omega})|^2 P(e^{j\omega}) d\omega < \infty.$$

In this formalism we can for example write

$$R_m =< 1, z^m > .$$

More generally, the mapping $s_n \rightarrow e^{-jn\omega}$ is an isomorphism (the *Kolmogorov isomorphism*) between the two Hilbert spaces $\mathcal{L}_2$ and $L_2$ defined in this section. Moreover, the $L_2$ space of the deterministic case and the $L_2$ space in the stochastic case are mathematically the same. This shows that the deterministic and the stochastic case can be treated in a uniform mathematical framework.

Let $H(z)$ be a causal filter with impulse response $h = (h_n) \in \ell_1$ and $z$-transform (transfer function of the filter) $H(z) = \sum_{n=0}^{\infty} h_n z^n$. Note that we assume $h$ to be a deterministic sequence. Apply as input to the filter a stationary stochastic process $u = (u_n)$, then the output $s = (s_n)$ is also a stationary process and they are related by

$$s_n = \sum_m u_m h_{n-m} \quad \text{or} \quad S(z) = H(z)U(z).$$

It can be shown that their power spectra are related by

$$\begin{aligned} P_s(e^{j\omega}) &= P_u(e^{j\omega}) |H(e^{j\omega})|^2 \\ &= P_u(e^{j\omega}) H(e^{j\omega}) H_*(e^{j\omega}). \end{aligned}$$

A stationary process $w = (w_n)$ is called *white noise* if

$$\mathsf{E}[w_n \bar{w}_m] = \delta_{nm} \|w\|^2, \quad \mathsf{E}[w_n] = 0.$$

Hence its power spectrum is constant : $P(e^{j\omega}) = \|w\|^2$. Often, one considers normalized white noise, i.e., one takes $\|w\| = 1$.
Note that white noise is the stochastic equivalent of a deterministic unit pulse in the sense that they have both the same perfectly flat spectrum.

A signal having a flat spectrum means that it does not contain any significant information. The samples are completely uncorrelated to each other. The process forgets immediately everything about its past. There is no strategy to be detected in the signal observed. Thus it is useless to

try and predict the next output after observing what has happened in the past. In a modern terminology, one would say that it has a completely chaotic behaviour.

As in the deterministic case, we shall also for stochastic signals use $z$ as a notation for the time delay for example

$$A(z)u_m = \sum_{k=0}^{n} a_k u_{m-k} \quad \text{for } A(z) = \sum_{k=0}^{n} a_k z^k.$$

# 3 Linear prediction

A major problem in signal processing is to predict its behaviour from the observations of the signal itself. That is to say, we want to model the filter (= system) which produces the particular signal we are observing. In many application, (e.g. EEG-signals, speech processing,...) we do not have access to the input signal and we can only observe the output. From the observation of the present and the past, we want to *predict* the next observation by a linear relation. Thus we estimate $s_n$ by $\hat{s}_n$, given some observations $s_{n-1}, s_{n-2}, \ldots, s_{n-p}$ where

$$\hat{s}_n = -\sum_{k=1}^{p} a_k s_{n-k}.$$

The coefficients $a_k$ are the *predictor coefficients* and $A(z) = \sum_{k=0}^{n} a_k z^k$, $a_0 = 1$ is called the *predictor.*

The *error* or *residual* for $s_n$ is

$$e_n = s_n - \hat{s}_n = s_n + \sum_{k=1}^{p} a_k s_{n-k} = A(z)s_n, \quad a_0 = 1.$$

We want to minimize the least squares error, i.e. the energy of the error signal

$$E = \|e\|^2$$

which is given by $\sum_n |e_n|^2$ in the deterministic case and by $\mathsf{E}[|e_n|^2]$ in the stochastic case. Note that $\mathsf{E}[|e_n|^2]$ is independent of $n$ in the stationary case.

## 3.1 Deterministic case

We can describe this as a linear least squares problem with infinitely many equations.

Define the column vectors $s = (s_n)$ (the signal), $e = (e_n)$ (the prediction errors), $\hat{s} = (\hat{s}_n)$ (the predicted signal), and $a = (a_i)_{i=1}^{p}$ (predictor coefficients) and also the $\infty \times p$ matrix $S$ with columns $zs, z^2 s, \ldots, z^p s$. Then $e = s - \hat{s} = s + Sa$. Minimizing $\|e\|$ corresponds to finding the least squares solution of the system $Sa = -s$. Its solution is given by the normal equations.

$$(S^H S)a = -S^H s$$

where $S^H S$ is a $p \times p$ matrix and $S^H s$ a $p \times 1$ vector. These are obtained by setting

$$\frac{\partial E}{\partial a_i} = 0, \quad i = 1, \ldots, p; \quad E = \|e\|^2$$

or more explicitly:

$$\sum_{k=1}^{p} \left( \sum_n s_{n-k} \bar{s}_{n-i} \right) a_k = -\sum_n s_n \bar{s}_{n-i}, \quad i = 1, \ldots, p.$$

**Exercise 3.1** Show that the least squared prediction error is given by

$$E_p = \sum_n |s_n|^2 + \sum_{k=1}^{p} a_k (\sum_n \bar{s}_n s_{n-k}).$$

◇

In principle, the summation for $n$ should range over $\mathbf{Z}$. Practically however, we can't work with infinite sequences or an infinite system of equations. That is why we did not specify the range for the summation of $n$. We can then take two approaches: either we consider only a moderate number of the infinite set of equations $Sa = -s$ (and just hope that it will give a meaningful solution as well) or we want to consider the infinite set of equations by all means. As we shall see below, this assumes that it is possible by some device to perform infinite summations (or at least approximate them adequately). We accordingly get two different situations. The latter being the simpler one (mathematically that is). The methods are known in the literature as covariance and autocorrelation method respectively. This terminology is certainly misleading because it has nothing in common with the difference between correlation and covariance.

### 3.1.1 The autocorrelation method

Here the error is minimized over an infinite duration. The normal equations reduce to

$$\sum_{k=1}^{p} R_{i-k} a_k = -R_i, \quad 1 \leq i \leq p$$

where $R_i = \sum_{-\infty}^{\infty} \bar{s}_n s_{n+i}$. Thus $(S^H S)_{ik} = R_{i-k}$ is a $p \times p$ Toeplitz matrix (hermitian and positive (semi) definite). The least squared error is given by

$$E_p = R_0 + \sum_{k=1}^{p} R_{-k} a_k$$

**Note:** In practice, an infinite signal can be made finite by putting a window over it. This means that we consider a window $(w_n)$ which is only different from zero outside some interval, say $0 \leq n \leq N-1$. Thus we consider $s'$ where

$$s'_n = \begin{cases} s_n w_n & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}.$$

The autocorrelation function is then

$$R_i(N) = \sum_{n=0}^{N-1-i} \bar{s}'_n s'_{n+i}, \quad i \geq 0.$$

Of course a more direct method is to minimize only over a finite interval. This is done in the next subsection.

### 3.1.2 The covariance method

Here we minimize the prediction error over a finite interval $0 \leq n \leq N-1$. The normal equations become

$$\sum_{k=1}^{p} \varphi_{ik} a_k = -\varphi_{i0}, \quad 1 \leq i \leq p$$

where now

$$\varphi_{ik} = \sum_{n=0}^{N-1} \bar{s}_{n-i} s_{n-k}.$$

The matrix of the system is no longer Toeplitz, but it is still hermitian and positive (semi) definite. It has still a special structure as the product of two Toeplitz matrices: $S^H S$.
Of course, for $N \to \infty$, the covariance method reduces to the autocorrelation method.

### 3.2 Stochastic case

Here similar results can be obtained as for the deterministic case. Minimizing

$$E = \mathsf{E}[|e_n|^2] = \mathsf{E}[|s_n + \sum_{k=1}^{p} a_k s_{n-k}|^2]$$

gives the normal equations:

$$\sum_{k=1}^{p} a_k \mathsf{E}[s_{n-k}\bar{s}_{n-i}] = -\mathsf{E}[s_n \bar{s}_{n-i}], \quad 1 \le i \le p$$

with least squared error

$$E_p = \mathsf{E}[|s_n|^2] + \sum_{k=1}^{p} a_k \mathsf{E}[\bar{s}_n s_{n-k}].$$

Now we can also distinguish two cases:

#### 3.2.1 Stationary process

If the process is stationary, then the autocorrelation function $R_i$ of the process

$$R_{i-k} = \mathsf{E}[s_{n-k}\bar{s}_{n-i}]$$

is independent of $n$ and we get the same Toeplitz system as we had before, which in the stochastic literature is known as the set of *Yule-Walker equations.*

#### 3.2.2 Nonstationary case

The autocorrelation function $R(n-k, n-i) = \mathsf{E}[s_{n-k}\bar{s}_{n-i}]$ does depend on $n$ now and the system we have to solve is no longer Toeplitz. However, when the process is "close to stationary", the system will have a matrix that is "close to Toeplitz". For more details see later sections.

## 4 The filter model

### 4.1 The autoregressive model

In the previous analysis, we did not take into account what the input was. However, if we want to simulate (model) the signal, we should produce it by some system (filter) which has got to have some input, otherwise nothing would happen.
Observe that if we filter the signal $s$ through the finite impulse response filter (predictor) $A(z) = \sum_0^p a_k z^k$, $a_0 = 1$, then the output is the error signal $(e_n)$:

$$A(z)S(z) = E(z) \quad \text{or} \quad A(z)s_n = e_n, \quad n \in \mathbb{Z}$$

or explicity:

$$e_n = \sum_{k=0}^{p} a_k s_{n-k} = s_n + \sum_{k=1}^{p} a_k s_{n-k} = s_n - \hat{s}_n.$$

If we invert this filter and feed the error $e$ as input, we shall get the signal $s$ reconstructed. Mathematically this is simply writing $S(z) = \frac{1}{A(z)}E(z)$, which means that we have to build a filter with transfer function $1/A(z)$, which has an infinite impulse response.

Unfortunately, for practical reasons, it is impossible to apply the exact error. So the real problem is to find out what input signal to apply which can replace the error $e$. Therefore we make the following reasoning. Our objective was to extract as much information as possible from the signal. Thus what remains (the error) should be completely meaningless, i.e. it should be random noise which is as white as possible.

In the deterministic case, this is a unit impulse. However, when the filter with transfer function $1/A(z)$ is driven by a unit pulse, the impulse response will always start wit $h_0 = 1$ because $A(0) = 1$. This need not be true for the signal we want to model, since $s_0$ need not be 1. So, we need some scaling factor $G$. We propose a *modeling filter* of the form $G/A(z)$, $G > 0$, such that if $u$ (impulse or white noise) is the input (with flat spectrum $|U|^2 = 1$), then the output $s'$ is a model (approximation) for the true signal $s$. The factor $G$ is called the *gain* of this filter.

Thus the system which models the signal looks like

$$S'(z) = \frac{G}{A(z)}U(z) = \frac{1}{A(z)}E'(z), \quad u_n \rightarrow \boxed{\frac{G}{A(z)}} \rightarrow s'_n.$$

We see that an equivalent formulation is that we use $1/A(z)$ as transfer function and apply the input $E'(z) = GU(z)$.

Thus we can also say that we model the signal by replacing the true error $e$ by the approximation

$$e'_n = Gu_n$$

with $u = (w_n)$ (unit) white noise in the stochastic case and $u = (\delta_{n0})$ a unit impulse in the deterministic case.

The filter $A(z)$ itself is called the *whitening filter* because it makes $e_n = A(z)s_n$ as white as possible.

So, our problem is to find a value of the gain $G$. It is defined by requiring that the energy of the model $s'$ is equal to the energy of the true signal $s$. Now, requiring that $\|s'\|^2 = \|s\|^2$, corresponds to requiring that we have equality of the Fourier coefficients $R'_0 = R_0$. Thus we have chosen the model

$$s'_n = -\sum_{k=1}^{p} a_k s'_{n-k} + Gu_n \quad \text{for} \quad s_n = -\sum_{k=1}^{p} a_k s_{n-k} + e_n$$

so that it satisfies $R'_0 = R_0$.

**Exercise 4.1** Prove that the autocorrelation function $R'_i$ of the deterministic signal $(s'_n)$ satisfies

$$\begin{aligned} R'_0 &= -\sum_{k=1}^{p} a_k R'_{-k} + G^2 \\ R'_i &= -\sum_{k=1}^{p} a_k R'_{i-k}, \quad i \geq 1 \end{aligned}$$

(Assume the system is causal, thus that $s'_n = 0$ for $n < 0$). ◇

**Exercise 4.2** Prove that if $R'_0 = R_0$ then (use previous exercise) $R'_i = R_i$ for $i = 1, \ldots, p$ (and hence also $R'_i = R_i, i = -1, \ldots, -p$).

Warning: this exercise is very hard to solve with our current knowledge. You better try again later. ◇

If the previous exercises are solved, then it follows easily that

$$G^2 = \sum_{k=0}^{p} a_k R'_{-k} = \sum_{k=0}^{p} a_k R_{-k} = E_p = \|e\|^2 \quad (a_0 = 1).$$

From the previous exercises, it follows that we can reformulate our modeling problem as: find a filter of the form $G/A(z)$, $A(z) = \sum_{k=0}^{p} a_k z^k$, $a_0 = 1$, such that the autocorrelation function $R'_i$ of the output (impulse response) matches the autocorrelation function of the signal for $-p \leq i \leq p$. Thus

$$P_{s'}(z) = \frac{G^2}{A(z)A_*(z)} = \sum_{-\infty}^{\infty} R'_k z^k \quad \text{and} \quad P_s(z) = \sum_{-\infty}^{\infty} R_k z^k$$

with $R'_k = R_k$, $k = -p, \ldots, p$. This is an example of a $(0/p)$ Laurent-Padé approximant.

### 4.2 Other models

The previous example of a model is just one possibility to model a signal. A more general model could be a filter of the form

$$H(z) = G\frac{B(z)}{A(z)} = G\frac{1 + \sum_{\ell=1}^{q} b_\ell z^\ell}{1 + \sum_{k=1}^{p} a_k z^k}.$$

This is *pole-zero model.* When $a_k = 0$, $1 \leq k \leq p$, we get an *all-zero model* or a *moving average* model (MA). If $b_\ell = 0$, $1 \leq \ell \leq q$, we obtain an *all-pole model* or an *autoregressive* (AR) model. The general case is an ARMA model.

When we know the transfer function of the given filter, e.g., $H(z)$, then we can try Padé approximation to find $B(z)$ and $A(z)$. And as we have seen, we can always assume that we know the transfer function. Either the input $U(z)$ and the output $S(z)$ is known, and then we get $H(z) = S(z)/U(z)$, or we do not know the input, and then we assume that the $S(z)$ is generated by some filter for which it is the transfer function. If we assume that not the transfer function, but its spectrum, i.e., its autocorrelation coefficients are known, then we can try to approximate it with a more general Laurent-Padé approximant. To find the transfer function itself, we need the approximating spectrum in the factored form

$$P_{s'}(z) = \left(G\frac{B(z)}{A(z)}\right)\left(G\frac{B_*(z)}{A_*(z)}\right)$$

which is easily obtained for the denominator, but is more difficult for the numerator. One problem is that the spectrum has to be nonnegative on the unit circle. The approximating spectrum which is constructed by Laurent-Padé approximation can take negative values there.

We shall make another sugestion to solve the ARMA case later on.

## 5 Spectral estimation

The previous solution to the prediction or modeling problem can also be used to solve a spectral estimation problem in the following sense. Note that the prediction error $e$ is related to the signal $s$ by $E(z) = A(z)S(z)$ where $A(z)$ is the predictor polynomial and thus is the spectrum of the signal $s$ given by

$$P(e^{j\omega}) = |S(e^{j\omega})|^2 = \frac{|E(e^{j\omega})|^2}{|A(e^{j\omega})|^2}.$$

The spectrum of the approximating signal $s'$ is given by

$$P'(e^{j\omega}) = |S'(e^{j\omega})|^2 = \frac{G^2}{|A(e^{j\omega})|^2}.$$

Thus

$$\frac{P(e^{j\omega})}{P'(e^{j\omega})} = \frac{|E(e^{j\omega})|^2}{G^2}.$$

So that

$$E_p = G^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |E(e^{j\omega})|^2 d\omega = \frac{G^2}{2\pi}\int_{-\pi}^{\pi} \frac{P(e^{j\omega})}{P'(e^{j\omega})} d\omega.$$

This implies that the model is chosen so as to make

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{P(e^{j\omega})}{P'(e^{j\omega})} d\omega = 1.$$

Thus we approximate the given spectrum $P$ by $P'$ in the previous sense, rather than e.g. minimizing $\|P - P'\|^2$! We can also write it as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left[\frac{1}{P(e^{j\omega})} - \frac{1}{P'(e^{j\omega})}\right] P(e^{j\omega}) d\omega = 0.$$

One can in fact prove (but we shall not do it) that the linear prediction method proposes actually minimizes

$$\left\|\frac{1}{S} - \frac{A}{G}\right\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\frac{1}{S(e^{j\omega})} - \frac{A(e^{j\omega})}{G}\right|^2 P(e^{j\omega}) d\omega.$$

At the same time we have solved a *spectral factorization problem* approximately because

$$P(e^{j\omega}) = |S(e^{j\omega})|^2 \approx \frac{G}{A(e^{j\omega})} \cdot \frac{G}{A_*(e^{j\omega})}$$

with (as we shall see later) one factor $(G/A(z))$ analytic in $|z| < 1$ and the other $(G/A_*(z))$ analytic in $|z| > 1$.

## 6 Toeplitz systems and orthogonal polynomials

An alternative form to write the normal equations (autocorrelation method/stationary case) is

$$\begin{bmatrix} R_0 & R_{-1} & \dots & R_{-p} \\ R_1 & R_0 & \ddots & \\ \vdots & \ddots & \ddots & \\ R_p & & & R_0 \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} E_p \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \equiv \quad T_p A_p = E_p e_0.$$

This shows the importance in this context to solve Toeplitz systems. Let $\tilde{I}$ be the reversal matrix: with 1's on the main antidiagonal, then a Toeplitz matrix $T$ satisfies

$$\tilde{I}\, T\, \tilde{I} = T^T$$

Using this property, we can rephrase the previous system as (recall $T^H = T$ here)

$$\tilde{I} T_p \tilde{I} \tilde{I} A_p = \tilde{I} E_p e_0 = E_p e_p$$

or

$$\begin{bmatrix} R_0 & R_{-1} & \dots & R_{-p} \\ R_1 & R_0 & \ddots & \\ \vdots & \ddots & \ddots & \vdots \\ R_p & & & R_0 \end{bmatrix} \begin{bmatrix} \bar{a}_p \\ \vdots \\ \bar{a}_1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ E_p \end{bmatrix} \quad \equiv \quad T_p A_p^* = E_p e_p.$$

These normal equations are related to the optimization problem $\min \|e\|^2$, which in the frequency domain translates into: find

$$E_p = \min_{a_1,\dots,a_p} \left\| 1 + \sum_{k=1}^{p} a_k z^k \right\|^2 = \|A_p\|^2$$

with optimal solution the predictor polynomial $A_p(z) = \sum_{k=0}^{p} a_k z^k$, $a_0 = 1$, and the minimum being

$$\begin{aligned} E_p = \|A_p(z)\|^2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |\sum_{k=0}^{p} a_k e^{j\omega k}|^2 P(e^{j\omega}) d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |\sum_{k=0}^{p} \bar{a}_{p-k} e^{j\omega k}|^2 P(e^{j\omega}) d\omega \\ &= \|\varphi_p(z)\|^2 \end{aligned}$$

where $\varphi_p(z) = z^p + \bar{a}_1 z^{p-1} + \cdots + \bar{a}_p = z^p \overline{A_p(1/\bar{z})} = z^p A_{p*}(z)$. In the sequel we shall denote this transformation as $\varphi_p(z) = A_p^*(z)$. Note that the column $A_p^*$ indeed contains the coefficients of the polynomial $A_p^*(z)$.
Note also that in this Hilbert space

$$\begin{aligned} < \varphi_p, z^j > &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\sum_{k=0}^{p} \bar{a}_{p-k} z^k) z^{-j} P(z) d\omega, \quad z = e^{j\omega} \\ &= \sum_{k=0}^{p} \bar{a}_{p-k} \frac{1}{2\pi} \int_{-\pi}^{\pi} z^{k-j} P(z) d\omega \\ &= \sum_{k=0}^{p} \bar{a}_{p-k} R_{j-k}. \end{aligned}$$

Thus the previous Toeplitz system expresses that

$$\begin{aligned} < \varphi_p, z^j > &= 0 \qquad \text{for} \quad j = 0, 1, \dots, p-1 \\ &= E_p \qquad \text{for} \quad j = p. \end{aligned}$$

In other words, $\varphi_p$ is an orthogonal (monic) polynomial with minimal norm

$$< \varphi_p, z^p >= \|\varphi_p\|^2 = E_p.$$

Thus finding a $p$-th order predictor is equivalent to finding a (monic) orthogonal polynomial in this weighted Hilbert space.

**Exercise 6.1** Prove that in any Hilbert space, the polynomial monic and of minimal norm is an orthogonal one. Thus

$$\min\{\|p\|^2, p \in \Pi_n \text{ and monic}\}$$

is obtained for $p = \varphi_n$ where $\varphi_n$ is the monic orthogonal polynomial. It is uniquely defined up to a constant factor of modulus 1. $\diamond$

# 7 Computation of the predictor

Special methods exist that require only $O(p^2)$ operations to solve structured $p \times p$ systems such as Toeplitz systems. A basic method is called the *Levinson algorithm* (1947)[7].

Define for $p = 0, 1, \ldots$ the Toeplitz matrices $T_p$ by

$$[T_p]_{ij} = R_{i-j}; \quad i, j = 0, \ldots, p, \quad R_{-i} = \bar{R}_i$$

and define $A_p = [1 \ a_1 \ \ldots \ a_p]^T$ such that

$$T_p A_p = [E_p \ 0 \ \ldots \ 0]^T = E_p e_0.$$

Furthermore set $A_p^* = [\bar{a}_p \ldots \bar{a}_1 \ 1]^T$. Then, as long as $E_{i-1} \neq 0$ it holds that

$$\begin{array}{l} E_0 = R_0 \\ \textbf{for} \quad p = 1, 2, \ldots \\ \left| \begin{array}{rcl} \eta_{p-1} & = & [R_p \ \ldots \ R_1] A_{p-1} \\ \rho_p & = & \eta_{p-1} / E_{p-1} \\ A_p & = & \begin{bmatrix} A_{p-1} \\ 0 \end{bmatrix} - \rho_p \begin{bmatrix} 0 \\ A_{p-1}^* \end{bmatrix} \\ E_p & = & (1 - |\rho_p|^2) E_{p-1} \end{array} \right. \end{array}$$

**Exercise 7.1** Prove that this algorithm gives indeed the solutions of the systems $T_i A_i = E_i e_0$ for $i = 0, 1, \ldots$. ◇

**Exercise 7.2** Compute the complexity of this algorithm. ◇

Note that in this algorithm, to compute the prediction of order $p$, we also compute all the previous ones and as a by-product we also get all the prediction errors $E_i$.
Because $E_p$ has to be decreasing with $p$, and it is non-negative

$$0 \leq E_p \leq E_{p-1} \leq \ldots \leq E_0 = R_0.$$

It thus follows that $|\rho_i| \leq 1$, and if $|\rho_p| = 1$, then $E_p = 0$, which means that the model matches the given signal perfectly.

The coefficients $\rho_i$ are crucial and they are called *reflection coefficients* (derived from scattering theory — see later) or *partial correlation* (PARCOR) coefficients (from their stochastic interpretation).

The predictor $A_i$ is called the *forward* predictor and $A_i^*$ is the *backward* predictor. The latter is explained as follows: Interpreting again $z$ as a delay operator, we have

$$e_n = A_i(z) s_n, \quad A_i(z) = [1 \ z \ldots z^i] A_i, \quad A_i = [1 \ a_1 \ldots a_i]^T$$

where $A_i(z)$ is constructed to minimize the least squared error $\|e\|^2$. Similarly $A_i^*(z)$ appearing in

$$\begin{array}{rcl} f_n & = & A_i^*(z) s_n, \quad A_i^*(z) = [1 \ z \ldots z^i] A_i^*, \quad A_i^* = [\bar{a}_i \ldots \bar{a}_1 \ 1]^T \\ & = & s_{n-i} + \bar{a}_1 s_{n-i+1} + \cdots + \bar{a}_i s_n \end{array}$$

can be seen as a (backward) "prediction" of $s_{n-i}$, given $s_{n-i+1}, \ldots, s_n$, resulting in a backward prediction error $f_n$. It is constructed so as to minimize the least squared error $\|f\|^2$. In a stochastic context, the processes $(f_n)$ and $(e_n)$ are called the backward and forward *innovation* processes.

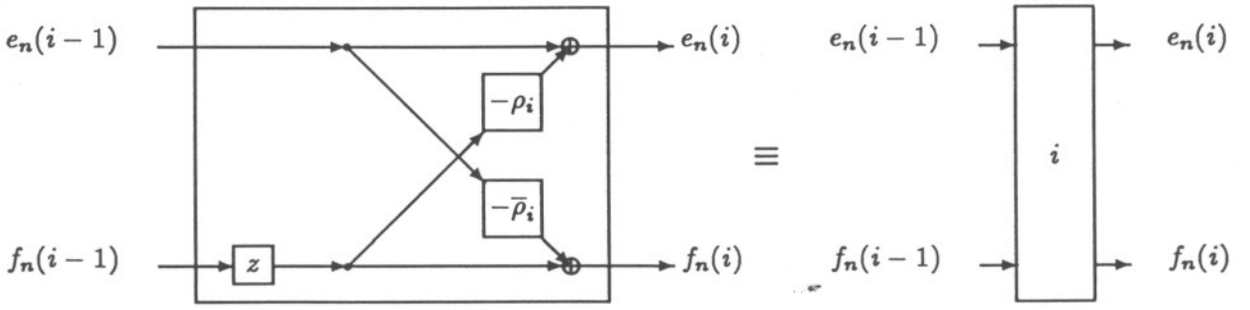



Figure 1: Lattice filter

**Exercise 7.3** Prove that in terms of the polynomials $A_i(z)$ and $A_i^*(z)$, the Levinson algorithm gives the recursion

$$\begin{bmatrix} A_i(z) \\ A_i^*(z) \end{bmatrix} = \begin{bmatrix} 1 & -\rho_i \\ -\bar{\rho}_i & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix} \begin{bmatrix} A_{i-1}(z) \\ A_{i-1}^*(z) \end{bmatrix}.$$

◇

With the interpretations above, we can derive the circuit of figure 1 where $e_n(i)$ means the forward prediction error for $s_n$ when using predictor $A_i(z)$ etc. This is called a *lattice* realization of one section of the filter. Thus for such a realization, we do not need the predictor coefficients $a_i$ explicitly, it is sufficient to know the reflection coefficients $\rho_i$.
The whitening filter (which analyses the signal) can be realized with such blocks as illustrated in figure 2. The modeling filter (which synthesises the signal) has the realization of figure 3 where the

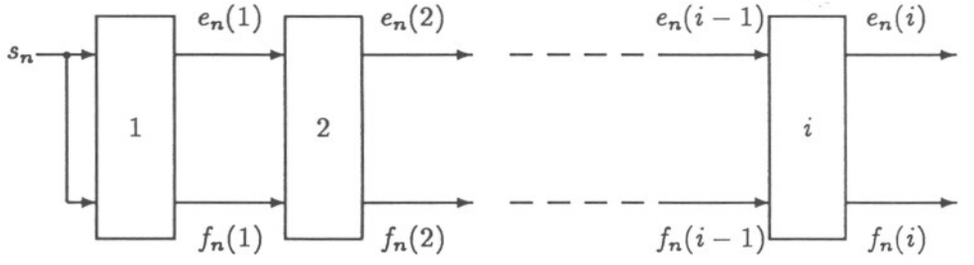



Figure 2: Whitening filter

$i$-th block has now the inverse lattice form as depicted in figure 4.

**Exercise 7.4** Check the inversion as suggested by figure 4. ◇

Note that the normalized predictors which generate

$$e_n'(i) = \frac{e_n(i)}{G_i} = \frac{A_i(z)}{G_i} s_n = A_i'(z) s_n; \quad G_i^2 = E_i$$

are given by the normalized recursion

$$\begin{bmatrix} A_i'(z) \\ A_i'^*(z) \end{bmatrix} = \frac{1}{\sqrt{1-|\rho_i|^2}} \begin{bmatrix} 1 & -\rho_i \\ -\bar{\rho}_i & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix} \begin{bmatrix} A_{i-1}'(z) \\ A_{i-1}'^*(z) \end{bmatrix}.$$

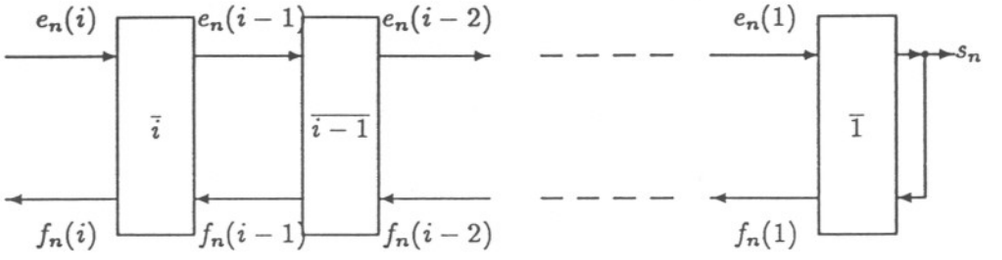



Figure 3: Modeling filter

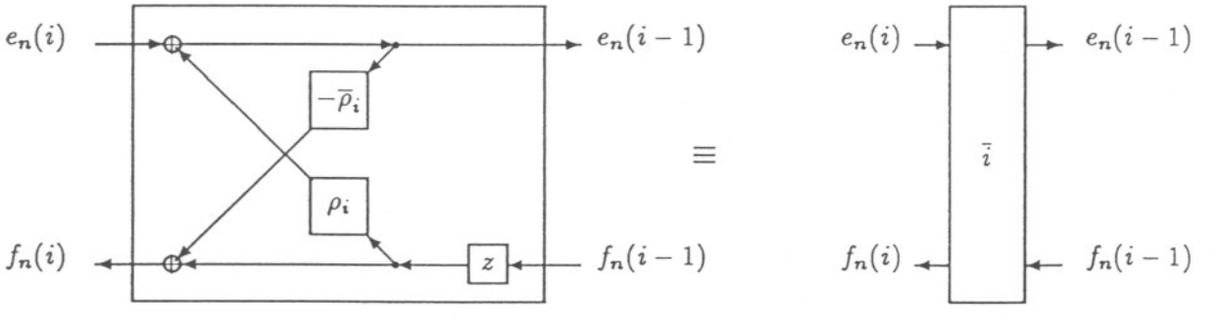



Figure 4: Inverse lattice section

# 8 Example : seismic signal processing

Consider a layered medium (like the earth surface) with layers of equal thickness. (See figure 5.) A wave is propagating through this medium. At every interface between two layers the wave is

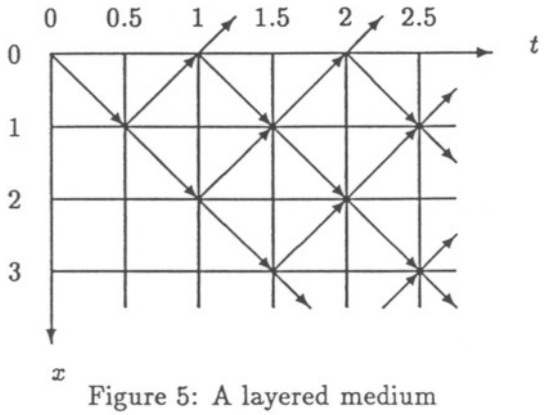



Figure 5: A layered medium

partially reflected and partially transmitted, according to the relative physical properties of the two layers meeting there. The wave equation

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u(x,t)}{\partial t^2} \quad (c = \text{speed})$$

has two particular solutions

$$\begin{aligned} u(x,t) &= \bar{e}(t + \frac{x}{c}) \quad \text{(ascending wave)} \\ u(x,t) &= \bar{f}(t - \frac{x}{c}) \quad \text{(descending wave)} \end{aligned}$$

We also normalized the time axis, so that it takes half a time unit for a wave to travel through a layer. Thus at integer time intervals, a wave reaches the surface. To describe the scattering of this wave in the medium we use the following notations :

$$\begin{aligned} \tilde{f}'_n(i) &= \text{descending wave, emerging from boundary } i \text{ at time } n \\ \tilde{f}_n(i) &= \text{descending wave, reaching boundary } i+1 \text{ at time } n \\ \tilde{e}_n(i) &= \text{ascending wave, emerging from boundary } i+1 \text{ at time } n \\ \tilde{e}'_n(i) &= \text{ascending wave, reaching boundary } i \text{ at time } n \end{aligned}$$

These are illustrated in figure 6. We also suppose that the medium is *passive* and *lossless*, which means that no energy is added to the wave, nor that it looses energy.

Using $z$ as a time delay : $z^{1/2} s_n(\cdot) = s_{n-\frac{1}{2}}(\cdot)$, we have that in the homogeneous layer $i$ :

$$\begin{bmatrix} \tilde{f}_n(i) \\ \tilde{e}_n(i) \end{bmatrix} = \begin{bmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{bmatrix} \begin{bmatrix} \tilde{f}'_n(i) \\ \tilde{e}'_n(i) \end{bmatrix}$$

which says that e.g., signal $\tilde{f}'_n(i)$ which emerges from boundary $i$ at time $n-1/2$ will reach boundary $i+1$ unaltered, half a time unit later, where according to our notation, it is called $\tilde{f}_n(i)$. Thus $\tilde{f}_n(i) = z^{1/2}\tilde{f}'_n(i) = \tilde{f}'_{n-\frac{1}{2}}(i)$.

The *scattering matrix* $\Sigma_{i+1}$ which describes the interaction at boundary $i+1$ should be unitary (orthogonal), because if

$$\begin{bmatrix} \tilde{f}'_n(i+1) \\ \tilde{e}_n(i) \end{bmatrix} = \Sigma_{i+1} \begin{bmatrix} \tilde{f}_n(i) \\ \tilde{e}'_n(i+1) \end{bmatrix}$$

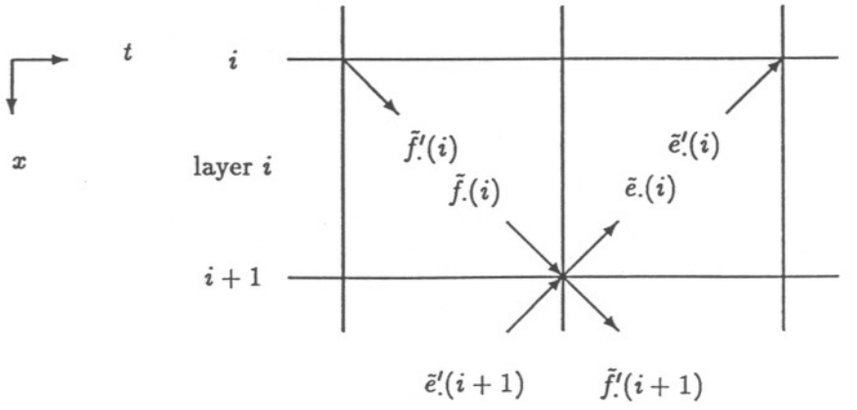



Figure 6: Notation of interacting waves

then, no energy should be lost; which means

$$\sum_n |\tilde{f}_n(i)|^2 + \sum_n |\tilde{e}'_n(i+1)|^2 = \sum_n |\tilde{f}'_n(i+1)|^2 + \sum_n |\tilde{e}_n(i)|^2$$

(the incoming energy equals the outgoing energy). This is guaranteed when $\Sigma_{i+1}^H \Sigma_{i+1} = I$. Thus we may choose $\Sigma_{i+1}$ to be a Givens rotation

$$\Sigma_{i+1} = \begin{bmatrix} \cos\theta_{i+1} & -\sin\theta_{i+1} \\ \sin\theta_{i+1} & \cos\theta_{i+1} \end{bmatrix}$$

or, in the complex case

$$\Sigma_{i+1} = \begin{bmatrix} c & -\bar{s} \\ s & c \end{bmatrix}, \quad c^2 + |s|^2 = 1, \quad c \in \mathbb{R}.$$

**Exercise 8.1** Prove that with this choice of the scattering matrix, the interaction can also be written in the form

$$\begin{bmatrix} \tilde{e}'_n(i+1) \\ \tilde{f}'_n(i+1) \end{bmatrix} = \theta_{i+1} \begin{bmatrix} \tilde{e}_n(i) \\ \tilde{f}_n(i) \end{bmatrix}$$

with

$$\theta_{i+1} = \frac{1}{c} \begin{bmatrix} 1 & -s \\ -\bar{s} & 1 \end{bmatrix} = \frac{1}{\sqrt{1-|\rho_{i+1}|^2}} \begin{bmatrix} 1 & -\rho_{i+1} \\ -\bar{\rho}_{i+1} & 1 \end{bmatrix}, \quad \rho_{i+1} = s.$$

◇

The $\theta_{i+1}$-matrix of the previous exercise is called a *chain scattering matrix*. It represents a hyperbolic rotation. It is unitary with respect to the indefinite metric

$$J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

i.e., it is $J$-unitary, which means that

$$\theta_{i+1}^H J \theta_{i+1} = J.$$

This relation is (in the real case) satisfied by a matrix of the form

$$\theta_{i+1} = \begin{bmatrix} \cosh\theta_{i+1} & -\sinh\theta_{i+1} \\ \sinh\theta_{i+1} & \cosh\theta_{i+1} \end{bmatrix}.$$

Thus we may describe the complete effect of a layer, including its boundary by

$$\begin{bmatrix} \tilde{e}'_n(i+1) \\ \tilde{f}'_n(i+1) \end{bmatrix} = \theta_{i+1} \begin{bmatrix} \tilde{e}_n(i) \\ \tilde{f}_n(i) \end{bmatrix} = \theta_{i+1} \begin{bmatrix} z^{-1/2} & 0 \\ 0 & z^{1/2} \end{bmatrix} \begin{bmatrix} \tilde{e}'_n(i) \\ \tilde{f}'_n(i) \end{bmatrix}.$$

When switching to the time-delayed signals (waves)

$$e'_n(i) = z^{i/2}\tilde{e}'_n(i) \quad \text{and} \quad f'_n(i) = z^{i/2}\tilde{f}'_n(i),$$

this reduces to

$$\begin{bmatrix} e'_n(i+1) \\ f'_n(i+1) \end{bmatrix} = \frac{1}{\sqrt{1-|\rho_{i+1}|^2}} \begin{bmatrix} 1 & -\rho_{i+1} \\ -\bar{\rho}_{i+1} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix} \begin{bmatrix} e'_n(i) \\ f'_n(i) \end{bmatrix}$$

which corresponds to the normalized recursions for the forward and backward prediction errors. This application explains the term reflection coefficient for $\rho_i$. Looking at figure 6 and the defining relation of $\Sigma_{i+1}$, you see when we write the outgoing waves $\tilde{f}_m(i+1)$ and $\tilde{e}_n(i)$ in terms of the incoming waves $\tilde{e}'_n(i+1)$ and $\tilde{e}_n(i)$ as

$$\begin{aligned} \tilde{f}'_n(i+1) &= c\tilde{f}_n(i) - \bar{s}\tilde{e}'_n(i+1) \\ \tilde{e}_n(i+1) &= s\tilde{f}_n(i) + c\tilde{e}'_n(i+1) \end{aligned}$$

then you can check on the figure that the parts of the right hand side with a coefficient $c$ correspond to transmitted parts and the parts with an $s$-coefficient to reflected parts. Hence the term "reflection coefficient" for $\rho_{i+1} = s$.

The previous analysis suggests the following method to get a model for the earth surface, which is of crucial intrest for e.g., oil companies. We apply an impulse as input at the surface (e.g., an explosion) and measure the response of the (pressure) wave at the surface. When the Levinson algorithm computes the reflection coefficients for the signal, it actually computes the relative density properties of the different earth layers below the surface, which may disclose the presence of a fluid (oil) or a gas bubble.

A problem of the above type, where you know the effect of some scattering medium to a signal, and where you want to model the medium is called an *inverse scattering* problem.

The same analysis can be applied to e.g., an acoustic signal, scattered by an organ pipe or the vocal tract for a speech signal. For small time intervals, a speech signal may be considered as stationary. Computing the reflection coefficients results in a model for the speech-sound and they can be used to synthesize the speech again by a modeling filter. Sending all the reflection coefficients (plus some extra information about pitch etc.) uses much less bandwidth than sampling the speech signal and send these samples. It is possible to get a considerable data reduction in this way. Of course, in such applications the analysis and synthesis should be done in real time. The CORDIC (coordinate rotation for digital computing) chip performs a hyperbolic rotation in a few clockcycles.

## 9 Stability

If $H(z) \in L_2$ is a filter which is causal (and has finite energy) then $H(z) = \sum_{k=0}^{\infty} h_k z^k$ represents a function analytic in $|z| < 1$. Thus a rational $H(z) \in L_2$ is stable ($\in H_2$) if all its poles are in $|z| > 1$. A rational filter is called *minimal phase* if all its zeros are in $|z| > 1$.
For an AR model

$$H(z) = \frac{G}{A_n(z)} = \frac{G}{\varphi_n^*(z)}, \quad A_n(z) = \varphi_n^*(z) = z^n\overline{\varphi_n(1/\bar{z})} = z^n\varphi_n^*(z)$$

the filter is certainly minimal phase (all zeros at $z = \infty$) and it will be stable if all the zeros of $A_n(z)$ are in $|z| > 1$ or equivalently, the zeros of $\varphi_n(z)$ are all in $|z| < 1$.

**Exercise 9.1** Prove that if $\varphi_n(\alpha) = 0$ then $\varphi_n^*(1/\bar{\alpha}) = 0$ too. ◇

Since we identified $\varphi_n$ as an orthogonal polynomial, this property of its zeros is a consequence of that theory. We can however give a direct proof.

**Lemma 9.1** *Define* $F_n(z) = A_n^*(z)/A_n(z)$ *with*

$$A_n(z) = \sum_{k=0}^{n} a_k z^k.$$

*Let* $A_n(z) \neq 0$ *for* $|z| \leq 1$. *Then* $F_n(z)$ *(and* $zF_n(z)$*) maps the unit disc onto itself.*

**Proof.** Writing $A_n^*(z) = \bar{a}_0 \prod_{i=1}^{n}(z - \alpha_i)$, with all $|\alpha_i| < 1$, we check easily that

$$\left|\frac{z - \alpha_i}{1 - \bar{\alpha}_i z}\right| < 1, = 1, > 1 \quad \text{when} \quad |z| < 1, = 1, > 1$$

and thus, because

$$F_n(z) = \frac{\bar{a}_0}{a_0} \prod_{i=1}^{n} \frac{z - \alpha_i}{1 - \bar{\alpha}_i z},$$

also

$$|F_n(z)| < 1, = 1, > 1 \quad \text{when} \quad |z| < 1, = 1, > 1.$$

The same holds for $zF_n(z)$.
Hence the functions are injective. They are also surjective because the inverse of

$$z \to w = \frac{z - \alpha}{1 - \bar{\alpha} z} \quad \text{is} \quad w \to z = \frac{w + \alpha}{1 + w\bar{\alpha}}$$

is of the same form and thus also injective. □

**Theorem 9.2** $A_n(z)$ *has all its zeros in* $|z| > 1$ *iff all* $|\rho_i| < 1, i = 1, \ldots, n$.

**Proof.** This goes by induction on $n$. We only prove the induction step. Let $A_k(z) \neq 0$ for $|z| < 1$ and $|\rho_{k+1}| < 1$. By the previous lemma, $zF_k(z) = -\frac{1}{\rho_{k+1}}$ has no solution in $|z| \leq 1$. Thus

$$A_{k+1}(z) = A_k(z) - \rho_{k+1} z A_k^*(z) \neq 0 \quad \text{in} \quad |z| \leq 1.$$

Conversely, if the zeros of $A_k$ are in $|z| > 1$, the product of its zeros is $(-1)^{k+1}\rho_k^{-1}$ ($a_0 = 1$, $a_k = -\rho_k$). Hence $|\rho_k| < 1$. Furthermore

$$A_{k-1}(z) = \frac{1}{1 - |\rho_k|^2}[A_k(z) + \rho_k A_k^*(z)]$$

and this can only be zero if $F_k = -1/\rho_k$, which can only happen for $|z| > 1$. □

We may thus conclude that the Levinson algorithm generates a filter that is guaranteed to be minimal phase and stable, as long as the reflection coefficients are bounded by 1. When $|\rho_n| = 1$, then $E_n = 0$ and the approximation is exact.

**Exercise 9.2** Prove that if $|\rho_k| < 1, k = 1, 2, \ldots, n-1$ and $|\rho_n| = 1$, then $A_n(z) = \eta A_n^*(z)$ with $|\eta| = 1$ and its zeros are all of modulus 1. ◇

# 10 Orthogonal polynomials

We have seen before that the backward predictors $\varphi_n = A_n^*$ are orthogonal polynomials w.r.t. an inner product whose weight is the power spectrum $P(e^{j\omega})$. Its Fourier coefficients can be expressed as $< 1, z^n >= R_n$, $n \in \mathbf{Z}$. They satisfy the recurrence relation (recall that $\varphi_n$ is monic)

$$\varphi_n(z) = z\varphi_{n-1}(z) - \bar{\rho}_n \varphi_{n-1}^*(z), \quad \varphi_0 = 1$$

or also

$$\begin{bmatrix} \varphi_n(z) \\ \varphi_n^*(z) \end{bmatrix} = \theta_n \begin{bmatrix} \varphi_{n-1}(z) \\ \varphi_{n-1}^*(z) \end{bmatrix}$$

with

$$\theta_n = \begin{bmatrix} 1 & -\bar{\rho}_n \\ -\rho_n & 1 \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix}.$$

**Exercise 10.1** The polynomial $\varphi_n$ is orthogonal to $\text{span}\{1, z, \ldots, z^{n-1}\}$. Prove that $\varphi_n^*$ is orthogonal to $\text{span}\{z, \ldots, z^n\}$. ◇

**Exercise 10.2** Prove that the Levinson algorithm can be reformulated in terms of orthogonal polynomials as follows

$$\begin{array}{l} \varphi_0 = 1 \\ E_0 = \|1\|^2 = \|\varphi_0\|^2 = R_0 \\ \textbf{for } i = 1, 2, \ldots \\ \left| \begin{array}{l} \eta_{i-1} =< \varphi_{i-1}^*, z\varphi_{i-1} > \\ \rho_i = \eta_{i-1}/E_{i-1} \\ \varphi_i(z) = z\varphi_{i-1}(z) - \bar{\rho}_i \varphi_{i-1}^*(z) \\ E_i = E_i(1 - |\rho_i|^2) \end{array} \right. \end{array}$$

◇

**Exercise 10.3** Prove that $\det\ T_n = E_0 E_1 \ldots E_n$ and thus that $E_n = \det\ T_n / \det\ T_{n-1}$ ◇

**Exercise 10.4** Derive from the recurrence relation the Christoffel-Darboux type formula

$$\begin{aligned} (1 - x\bar{y}) \sum_{k=0}^{n} \frac{\varphi_k(x)\overline{\varphi_k(y)}}{E_k} &= \frac{\varphi_n^*(x)\overline{\varphi_n^*(y)} - x\bar{y}\varphi_n(x)\overline{\varphi_n(y)}}{E_n} \\ &= \frac{\varphi_{n+1}^*(x)\overline{\varphi_{n+1}^*(y)} - \varphi_{n+1}(x)\overline{\varphi_{n+1}(y)}}{E_n} \end{aligned}$$

◇

The *polynomials of the second kind* are defined by

$$\psi_n(z) = \begin{cases} \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} \frac{e^{j\omega} + z}{e^{j\omega} - z} [\varphi_n(e^{j\omega}) - \varphi_n(z)] P(e^{j\omega}) d\omega & , \quad n > 1 \\ R_0 & , \quad n = 0 \end{cases}$$

**Exercise 10.5** Prove that $\psi_n$ is a polynomial of degree $n$. ◇

**Exercise 10.6** Define

$$\Omega(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} D(t,z)P(t)d\omega, \qquad D(t,z) = \frac{t+z}{t-z}, \quad t = e^{j\omega}$$

Prove that

$$\varphi_n\Omega + \psi_n = \begin{cases} \Omega + R_0, & n = 0 \\ z^n g(z), & n > 0, \quad g(z) \text{ analytic in } |z| < 1 \end{cases}$$

$$\varphi_n^*\Omega - \psi_n^* = \begin{cases} \Omega - R_0, & n = 0 \\ z^{n+1}h(z), & n > 0, \quad h(z) \text{ analytic in } |z| < 1 \end{cases}$$

This shows that $\varphi_n/\psi_n$ is a two-point Padé approximant for $\Omega$. ◇

**Exercise 10.7** Prove that the $\psi_n$ satisfy the recurrence relation

$$\psi_n(z) = z\psi_{n-1}(z) + \bar{\rho}_n\psi_{n-1}^*(z)$$

◇

**Exercise 10.8** Prove that the recurrences for $\varphi_n$ and $\psi_n$ can be collected in one relation as

$$\begin{bmatrix} \varphi_n(z) & \psi_n(z) \\ \varphi_n^*(z) & -\psi_n^*(z) \end{bmatrix} = \theta_n \begin{bmatrix} \varphi_{n-1}(z) & \psi_{n-1}(z) \\ \varphi_{n-1}^*(z) & -\psi_{n-1}^*(z) \end{bmatrix}, \quad \theta_n = \begin{bmatrix} 1 & -\bar{\rho}_n \\ -\rho_n & 1 \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix}$$

and thus that the following holds

$$\Theta_n = \theta_n \ldots \theta_1 = \frac{1}{2R_0}\begin{bmatrix} R_0\varphi_n + \psi_n & R_0\varphi_n - \psi_n \\ R_0\varphi_n^* - \psi_n^* & R_0\varphi_n^* + \psi_n^* \end{bmatrix}$$

which becomes particularly simple if $R_0$ is normalized to be 1. ◇

**Exercise 10.9** Take determinants in the previous exercise to obtain

$$\varphi_n(z)\psi_n^*(z) + \varphi_n^*\psi_n(z) = 2R_0\prod_1^n(1 - |\rho_i|^2)z^n = 2E_n z^n. \tag{10.1}$$

◇

These polynomials were first studied by Szegő. Therefore, they are called *Szegő polynomials* and in this context, the parameters $\bar{\rho}_n$ are called *Szegő coefficients*. They are the complex conjugates of the reflection coefficients.
The Levinson algorithm (or its reformulation we gave above) can thus be considered as a fast Gram-Schmidt procedure (only $O(n^2)$ flops) to construct orthogonal polynomials.

**Exercise 10.10** Set $x = y = z$ in the Christoffel-Darboux formula and derive in this way that $|\varphi_n^*(z)|^2 > 0$ for $|z| < 1$. Thus all the zeros of $\varphi_n^*$ are in $|z| \geq 1$. The zeros of $\varphi_n^*(z)$ can not be on $|z| = 1$, because if $\varphi_n^*(z_0) = 0$, $|z_0| = 1$, then $\overline{\varphi_n(z_0)} = z_0^n\varphi_n^*(z_0) = 0$, and this is impossible by (10.1).
This is an alternative proof for the stability of the filter in the AR model. ◇

**Note:** The orthonormal polynomials are $\phi_k = \varphi_k/G_k$ with $G_k^2 = E_k = \|\varphi_k\|^2$. These satisfy the normalized recursion

$$\begin{bmatrix} \phi_n \\ \phi_n^* \end{bmatrix} = \theta_n \begin{bmatrix} \phi_{n-1} \\ \phi_{n-1}^* \end{bmatrix}$$

with normalized transition matrix

$$\theta_n = \frac{1}{\sqrt{1-|\rho_n|^2}} \begin{bmatrix} 1 & -\bar{\rho}_n \\ -\rho_n & 1 \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix}.$$

We have proved that the AR filter is stable by construction. However, the system which defines the polynomials may be ill conditioned. (Note that we basically use the unmodified moments $R_n =< 1, z^n >$). The better the approximation of the signal, the smaller the prediction error $E_n$, thus the smaller $\det T_n$ which means that the matrix of the defining Toeplitz system comes closer and closer to a singular matrix. This ill conditioning means that small errors in the computation of the moments $R_n$, perturb the Toeplitz matrix so that it is no longer positive definite and this will imply that the $\varphi_n$ no longer have all their zeros in $|z| < 1$. Moreover rounding errors during the computation in the Levinson algorithm may propagate fast, which gives errorneous predictor coefficients and also this will affect its stability properties. Thus it is advisable to check the stability of the actually computed filter.
This reasoning is still true in a much wider context: inverse problems (such as the inverse scattering problem) are often very ill conditioned.

## 11 Stability test

If we have to check if all the zeros of a polynomial are in $|z| < 1$, we could compute its zeros, but this is of course a costly procedure. Instead, we could use the previous theory in reverse order. We know that a (monic) polynomial $\varphi_n$ has all its zeros in $|z| < 1$ iff all the $|\rho_k| < 1$ iff all the $E_k = \|\varphi_k\|^2 > 0$ iff all the Toeplitz matrices $T_k$ are positive definite iff $\det T_k > 0$ for all $k = 1, 2, \ldots, n$.

All this information can be extracted from $\varphi_n$ by inverting the Szegő recursion. This gives the following algorithm:

$$\begin{array}{l} \bar{\rho}_n = -\varphi_n(0) \\ \text{for } k = n, n-1, \ldots, 2 \\ \left\lfloor \begin{array}{l} \varphi_{k-1}(z) = \dfrac{\varphi_k(z) + \bar{\rho}_k \varphi_k^*(z)}{z(1-|\rho_k|^2)} \\ \bar{\rho}_{k-1} = -\varphi_{k-1}(0) \end{array} \right. \end{array}$$

**Exercise 11.1** Prove the previous algorithm. What is its complexity? ◇

The previous algorithm is known as the *Lehmer-Schur* or *Jury* test.

## 12 Fast (inverse) Cholesky factorization

Suppose that we have computed the least squares (polynomial) predictors and the corresponding least squares errors. In terms of the (monic) orthogonal polynomials $\varphi_k$, we thus have

$$< z^k, \varphi_i >= \delta_{ki} E_k, \quad k \leq i = 0, 1, \ldots, n.$$

We can express this orthogonality relation in terms of matrices as follows. Denote by $U_n$ the unit upper triangular matrix whose $k$-th column consists of the coefficients of $\varphi_k$ (and zeros below the diagonal). The orthogonality can then be expressed as

$$T_n U_n = \begin{bmatrix} R_0 & R_{-1} & \ldots & R_{-n} \\ R_1 & R_0 & \ddots & \\ \vdots & \ddots & \ddots & \vdots \\ R_n & & \ldots & R_0 \end{bmatrix} U_n = \begin{bmatrix} E_0 & & & \\ \times & E_1 & & \\ \vdots & \vdots & \ddots & \\ \times & \times & \ldots & E_n \end{bmatrix} =: F_n$$

Because $T_n$ is hermitian,

$$U_n^H T_n^H = U_n^H T_n = F_n^H$$

is upper triangular and therefore

$$U_n^H T_n U_n = D_n$$

is diagonal: $D_n = \mathrm{diag}(E_0, \ldots, E_n)$. Note also that $U_n^H = D_n F_n^{-1}$.

This relation implies the upper triangular $\times$ diagonal $\times$ lower triangular factorization of $T^{-1}$

$$T_n^{-1} = U_n D_n^{-1} U_n^H$$

This is a kind of Cholesky factorization for $T_n^{-1}$. We would rather expect $T_n^{-1}$ to be factorized as a product of a lower triangular $\times$ diagonal $\times$ upper triangular. But the latter can also be obtained as follows. Let us as before denote by $\tilde{I}$ the reversal operator (matrix) with 1's on the anti-diagonal. Then $\tilde{I} T_n \tilde{I} = (T_n)^T$ and thus $T_n^{-1} = \tilde{I} T_n^{-T} \tilde{I}$. Therefore

$$T_n^{-1} = (\tilde{I} \bar{U}_n \tilde{I})(\tilde{I} D_n^{-1} \tilde{I})(\tilde{I} U_n^T \tilde{I})$$

is indeed of the required form: lower triangular $\times$ diagonal $\times$ upper triangular.

**Exercise 12.1** Check that $\tilde{I} \bar{U}_n \tilde{I}$ is the unit lower triangular matrix which contains in its $k$-th column the coefficients of the polynomial $\varphi_{n-k}^*$ (and zeros above the diagonal). $\diamond$

Thus we have identified the Levinson (or Szegő) algorithm as a fast ($O(n^2)$ instead of $O(n^3)$ flops) Cholesky algorithm for computing the triangular (Cholesky) factors of the (hermitian positive definite) inverted Toeplitz matrix $T_n^{-1}$, starting from the data in the matrix $T_n$ itself. Therefore, it is sometimes called a *fast inverse Cholesky* algorithm.

The natural question now is whether we can also find the Cholesky factors of $T_n$ itself? Of course

$$T_n = U_n^{-H} D_n U_n^{-1} = L_n D_n L_n^H = F_n D_n^{-1} F_n^H$$

with $L_n = F_n D_n^{-1} = U_n^{-H}$, a unit lower triangular matrix. But, the question is, can we find the columns of $L_n$ (or $F_n$) directly, i.e., without first computing $U_n$? The answer is yes. Therefore, consider the vectors

$$\begin{aligned} V_1 &= [0\ \bar{a}_k\ \ldots\ \bar{a}_0\ 0\ \ldots\ 0]^T \in \mathbb{C}^{(n+1)\times 1} \\ V_2 &= [a_0\ \ldots\ a_k\ 0\ 0\ \ldots\ 0]^T \in \mathbb{C}^{(n+1)\times 1} \end{aligned}$$

where $\varphi_k^*(z) = \sum_{i=0}^k a_i z^i = A_k(z)$. Then we can easily check that

$$T_n[V_1 \mid V_2] = \begin{bmatrix} \bar{\eta}_k & E_k \\ 0 & 0 \\ \vdots & \vdots \\ E_k & \eta_k \\ \times & \times \\ \vdots & \vdots \\ \times & \times \end{bmatrix} =: [r_1' \mid r_2'] \qquad (k \text{ zeros})$$

where

$$\eta_k = [R_{k+1}\ \ldots\ R_1][a_0\ \ldots\ a_k]^T$$

and hence

$$\bar{\eta}_k = [R_{-1}\ \ldots\ R_{-k-1}][\bar{a}_k\ \ldots\ \bar{a}_0]^T$$

and

$$\begin{aligned} E_k = \|\varphi_k\|^2 &= [R_k \ \ldots \ R_0][\bar{a}_k \ \ldots \ \bar{a}_0]^T \\ &= [R_0 \ \ldots \ R_{-k}][a_0 \ \ldots \ a_k]^T. \end{aligned}$$

Because $V_1 - \bar{\rho}_{k+1}V_2$ gives the $(k+1)$-st column of $U_n$, also $r_1 - \bar{\rho}_{k+1}r_2$ must give the $(k+1)$-st column of $F_n$ which has the form $[0 \ \ldots \ 0 \ E_{k+1} \ \times \ \ldots \ \times]^T$ ($k+1$ zeros). In other words, $\bar{\eta}_k - \bar{\rho}_{k+1}E_k = 0$ or $\bar{\rho}_{k+1} = \bar{\eta}_k/E_k$, which identifies $\eta_k$ as $< \varphi_k^*, z\varphi_k >$ (compare with the $\eta_k$ fom sections 10 and 7). Now cut the first elements from $r_1'$ and $r_2'$ and call the result

$$[\Delta_{1k} \mid \Delta_{2k}] = \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ E_k & \eta_k \\ \times & \times \\ \vdots & \vdots \\ \times & \times \end{bmatrix} \in \mathbb{C}^{n\times 2} \qquad (k \text{ zeros})$$

Now we note that

$$T_n[V_1 \mid V_2] \begin{bmatrix} 1 & -\rho_{k+1} \\ -\bar{\rho}_{k+1} & 1 \end{bmatrix} = \begin{bmatrix} 0 & E_{k+1} \\ 0 & 0 \\ \vdots & \vdots \\ E_{k+1} & 0 \\ \times & \eta_{k+1} \\ \vdots & \vdots \\ \times & \times \end{bmatrix}$$

where the first column of the right hand side is $\Delta_{1,k+1}$ if we chop off its last element and the second column becomes $\Delta_{2,k+1}$ if we chop off its first element. Therefore, it is true that (delete the first element)

$$[\Delta_{1k} \mid \Delta_{2k}] \begin{bmatrix} 1 & -\rho_{k+1} \\ -\bar{\rho}_{k+1} & 1 \end{bmatrix} = [\Delta'_{1,k+1} \mid \Delta_{2,k+1}]$$

where the first column $\Delta'_{1,k+1}$ can be turned into $\Delta_{1,k+1}$ if we shift it down one place, i.e., if we delete its last element and introduce a zero at the top. If we agree to denote this operation as a multiplication with $z$, then we have the recurrence

$$[\Delta_{1k} \mid \Delta_{2k}] \begin{bmatrix} 1 & -\rho_{k+1} \\ -\bar{\rho}_{k+1} & 1 \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} = [\Delta_{1,k+1} \mid \Delta_{2,k+1}]$$

and $\rho_{k+1}$ is found as the ratio of element $k$ of $\Delta_{2k}$ over element $k$ of $\Delta_{1k}$ for $k = 0, 1, \ldots$. This procedure allows us to compute the factor $F_n$ independently from $V_1$ and $V_2$ (thus independently from the the columns of $U_n$, i.e., from the orthogonal polynomials). The initial conditions for this procedure are easily found to be

$$[\Delta_{10} \mid \Delta_{20}] = \begin{bmatrix} R_0 & R_1 \\ \vdots & \vdots \\ R_{n-1} & R_n \end{bmatrix}$$

An operation count for this procedure will reveal that it requires also $O(n^2)$ flops and it is thus a *fast Cholesky* algorithm for (hermitian positive definite) Toeplitz matrices.

The representation of the down-shift operation on $\Delta'_{1,k+1}$ by a multiplication with $z$ makes sense in the following way. Let $n \to \infty$ and define the functions (or formal series) $\Delta_{ik}(z) = [1\ z\ z^2\ \ldots]\Delta_{ik}$, then the previous recurrence shows that

$$[\Delta_{1k}(z) \mid \Delta_{2k}(z)] \begin{bmatrix} 1 & -\rho_{k+1} \\ -\bar{\rho}_{k+1} & 1 \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} = [\Delta_{1,k+1}(z) \mid \Delta_{2,k+1}(z)]$$

where the operation denoted by $z$ is now a genuine multiplication with $z$. Note that if the $\Delta_{ik}(z)$ converge in a neighborhood of $z = 0$ and indeed represent functions there, then

$$\rho_{k+1} = \lim_{z \to 0} \frac{\Delta_{2k}(z)}{\Delta_{1k}(z)}.$$

# 13 The Schur algorithm

The functions $\Delta_{ik}(z)$ introduced at the end of the previous section have a meaning.

**Exercise 13.1** Define $\Gamma_k(z) = \Delta_{2k}(z)/\Delta_{1k}(z)$. Prove that the recurrence for the $\Delta_{ik}(z)$ of the previous section implies the recurrence

$$\Gamma_{k+1}(z) = \frac{1}{z} \frac{\Gamma_k(z) - \rho_{k+1}}{1 - \bar{\rho}_{k+1}\Gamma_k(z)}, \quad \rho_{k+1} = \Gamma_k(0),$$

for the $\Gamma_k$ functions. ◇

The functions $\Gamma_k$ are all *Schur functions*. This means that they belong to the class

$$\mathcal{S} = \{\Gamma : |\Gamma| \leq 1 \text{ in } |z| \leq 1 \text{ and analytic in } |z| < 1\}.$$

We shall prove this by induction.
We start with the function

$$\Omega_0(z) = R_0 + 2\sum_{k=1}^{\infty} R_k z^k.$$

This is a *Carathéodory function* or *positive real function*, which means that it belongs to the class

$$\mathcal{C} = \{\Omega : \Re\Omega \geq 0 \text{ in } |z| \leq 1 \text{ and analytic in } |z| < 1\}.$$

To prove the latter, note that for $|z| < 1$,

$$\begin{aligned} \Omega_0(z) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(1 + 2\sum_{k=1}^{\infty} e^{-jk\omega} z^k\right) P(e^{j\omega})d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{j\omega} + z}{e^{j\omega} - z} P(e^{j\omega})d\omega \end{aligned}$$

and its real part is

$$\begin{aligned} \Re\Omega_0(z) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Re\frac{e^{j\omega} + z}{e^{j\omega} - z} P(e^{j\omega})d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 - |z|^2}{|e^{j\omega} - z|^2} P(e^{j\omega})d\omega \end{aligned}$$

which is positive for $|z| < 1$ because the kernel (Poisson kernel) and the weight $P(e^{j\omega})$ are positive. On the boundary,

$$\begin{aligned} \Re\Omega_0(e^{j\omega}) &= \frac{1}{2}\left[R_0 + 2\sum_{k=1}^{\infty} R_k e^{jk\omega} + R_0 + 2\sum_{k=1}^{\infty} R_{-k} e^{-jk\omega}\right] \\ &= \sum_{-\infty}^{\infty} R_k e^{jk\omega} = P(e^{j\omega}) \end{aligned}$$

and hence positive. Because $\Omega_0(z)$ has a "causal" expansion in $|z| < 1$, it is analytic. This proves that $\Omega_0 \in \mathcal{C}$.

Next we note that

$$\Gamma_0(z) = \frac{1}{z}\frac{\Omega_0(z) - \Omega_0(0)}{\Omega_0(z) + \Omega_0(0)}, \quad \Omega_0(0) = R_0 > 0.$$

The ratio satisfies

$$\left|\frac{\Omega_0(z) - \Omega_0(0)}{\Omega_0(z) + \Omega_0(0)}\right|^2 = \frac{|\Omega_0(z)|^2 - \Omega_0(0)2\Re(\Omega_0(z)) + \Omega_0(0)^2}{|\Omega_0(z)|^2 + \Omega_0(0)2\Re(\Omega_0(z)) + \Omega_0(0)^2} \leq 1$$

Because the numerator is zero for $z = 0$, we can divide out $z$ (Schwarz' lemma) and get a function $\Gamma_0$ which is thus a Schur function.

All the other $\Gamma_k$ are also Schur functions as is proved in the induction step below.

Note that the mapping

$$z \to \frac{z - \alpha}{1 - \overline{\alpha} z}, \quad |\alpha| < 1$$

is a bijection of $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ onto itself, which maps the unit circle onto itself, the open unit disk onto itself and the open outside of the unit disk onto itself.

**Exercise 13.2** Prove this bijection property for the previous Moebius map. ◇

Thus

$$\frac{\Gamma_k - \rho_{k+1}}{1 - \overline{\rho}_{k+1}\Gamma_k}$$

will again be a Schur function (since $\rho_{k+1} = \Gamma_k(0)$ is bounded by 1). Because this function is zero for $z = 0$, we can divide out $z$ and still get a Schur function, which proves that $\Gamma_{k+1}$ is in the class $\mathcal{S}$.

I. Schur used the recursion for $\Gamma_k$ to check whether a given function $\Gamma_0$ was in the class $\mathcal{S}$. He proved the following theorem (for this reason, the $\rho_k$ are also referred to as *Schur coefficients*).

**Theorem 13.1 (Schur 1917)** *Let $\Gamma_0$ be a given function. Define recursively*

$$\Gamma_{k+1}(z) = \frac{1}{z}\frac{\Gamma_k(z) - \rho_{k+1}}{1 - \overline{\rho}_{k+1}\Gamma_k(z)}, \quad \rho_{k+1} = \Gamma_k(0).$$

*Then $\Gamma_0 \in \mathcal{S}$ iff $|\rho_k| < 1$ for $k = 1, 2, \ldots$ or $|\rho_k| < 1$ for $k = 1, 2, \ldots, n-1$ and $|\rho_n| = 1$ and $\rho_{n+k} = 0$ for $k = 1, 2, \ldots$. In the latter case, $\Gamma_0$ is a Blaschke product of degree $n$:*

$$\begin{aligned} \Gamma_0(z) &= \frac{P^*(z)}{P(z)}, \quad P(z) \in \Pi_n \\ &= c\prod_{i=1}^{n} \frac{z - \alpha_i}{1 - \overline{\alpha}_i z}, \quad |\alpha_i| < 1, \quad i = 1, \ldots, n, \quad |c| = 1. \end{aligned}$$

# 14 The general pole-zero model

If we allow a rational model of ARMA type, the problem becomes a nonlinear one (AR modelling was essentially reduced to a linear least squares problem). Hence we shall have to apply iterative methods to minimize the error. However, if we have some idea about the location of the zeros, we can propose a filter of the form

$$H(z) = G\frac{(1-\bar{\alpha}_1 z)\ldots(1-\bar{\alpha}_n z)}{A_n(z)} = \frac{G}{D_n(z)}$$

where $1/\bar{\alpha}_i$ are the estimated zeros, with $|\alpha_i| < 1$ to make it minimal phase,

$$D_n(z) = \frac{A_n(z)}{\pi_n(z)}, \quad \pi_n(z) = \prod_{k=1}^{n}(1-\bar{\alpha}_k z), \quad A_n \in \Pi_n.$$

Define

$$\varphi_n(z) = \frac{A_n^*(z)}{\pi_n(z)}$$

then both $\varphi_n$ and $D_n$ are in the space

$$\mathcal{L}_n = \left\{\frac{p_n}{\pi_n}, \quad p_n \in \Pi_n, \quad \pi_n(z) = \prod_{k=1}^{n}(1-\bar{\alpha}_k z)\right\}$$

of rational functions with prescribed poles (outside the closed unit disk).

**Exercise 14.1** Prove that the space $\mathcal{L}_n$ is also spanned by the basis functions $\{\zeta_k, k = 1, 2, \ldots, n\}$ with

$$\zeta_0 = 1, \quad \zeta_k(z) = \frac{z-\alpha_k}{1-\bar{\alpha}_k z}, \quad k = 1, 2, \ldots, n$$

or by $\{B_k, k = 1, \ldots, n\}$ where

$$B_0 = \zeta_0, \quad B_k = B_{k-1}\zeta_k, \quad k = 1, \ldots, n.$$

◇

**Exercise 14.2** Prove that

$$\varphi_n(z) = D_n^*(z) := B_n(z)D_{n*}(z) = B_n(z)\overline{D_n(1/\bar{z})} \in \mathcal{L}_n$$

◇

Note that when all $\alpha_k = 0$, the space $\mathcal{L}_n$ reduces to the space $\Pi_n$ of polynomials and we are back in the AR case. If we want for a stochastic process $(x_n)$ to predict $x_0$ from its past, i.e., from $\text{span}\{x_k, k-1, -2, \ldots\}$ in best $L_2$ sense, we should find $(h_n)$ such that

$$\left\| x_0 + \sum_{k=1}^{\infty} h_k x_{-k} \right\|^2 \quad \text{is minimal.}$$

Under the Kolmogorov isomorphism, this is equivalent to finding the minimum of $\|H\|^2$, where

$$H(z) = 1 + \sum_{k=1}^{\infty} h_k z^k$$

and where the norm is taken in the Hilbert space with inner product

$$< f, g >= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{j\omega}) \overline{g(e^{j\omega})} P(e^{j\omega}) d\omega$$

Thus we look for

$$\min\{\|H\|^2 : H \in H_2, H(0) = 1\}$$

In the AR model, this (infinite dimensional) problem was approximated by minimizing over polynomial subspaces $\Pi_n \subset H_2$. However, it is perfectly possible to minimize over any other finite dimensional subspace of $H_2$, for example the subspaces $\mathcal{L}_n$. We should then solve the problems

$$\min\{\|f\|^2, f \in \mathcal{L}_n \subset H_2, f(0) = 1\}$$

To solve this problem (theoretically and practically) we need the concept of *reproducing kernel.*

Let $\mathcal{H}$ be a Hilbert space, then we call $k(z, w)$ a reproducing kernel for $\mathcal{H}$, if $\forall w$

$$< f(z), k(z, w) > = f(w), \quad \forall f \in \mathcal{H}$$

**Exercise 14.3** Prove that if $\varphi_k$, $k = 0, 1, \ldots$ is an orthogonal basis for $\mathcal{H}$, then

$$k(z, w) = \sum_k \frac{\varphi_k(z)\overline{\varphi_k(w)}}{\|\varphi_k\|^2}$$

is a reproducing kernel in $\mathcal{H}$. ◇

We have now the following

**Theorem 14.1** *The optimization problem*

$$\min\{\|f\|^2, f \in \mathcal{H}, f(w) = 1\}$$

*in the Hilbert space $\mathcal{H}$ with reproducing kernel $k$ is solved for*

$$f(z) = g_w(z) = \frac{k(z, w)}{k(w, w)}$$

*and the minimum is $1/k(w, w)$.*

**Proof.** It is clear that $g_w(w) = 1$ and $\|g_w\|^2 = 1/k(w, w)$. Let $f = \sum_k a_k \varphi_k(z)$ be an arbitrary function in $\mathcal{H}$ such that $f(w) = 1$ (suppose $\varphi_k$ forms an orthonormal basis). Then we have to minimize

$$\|f\|^2 = \sum_k |a_k|^2$$

with the side condition $\sum_k a_k \varphi_k(w) = 1$.

By the Cauchy-Schwarz inequality

$$1 = |\sum_k a_k \varphi_k(w)|^2 \leq (\sum_k |a_k|^2)(\sum_k |\varphi_k(w)|^2)$$

which implies

$$\frac{1}{k(w, w)} = \frac{1}{\sum_k |\varphi_k(w)|^2} \leq \sum_k |a_k|^2$$

The lower bound is indeed reached by choosing $a_k = \overline{\varphi_k(w)}/k(w, w)$ and this proves the theorem. □

A possible solution method for the prediction problem is now clear: we first construct the orthonormal functions $\varphi_0, \varphi_1, \ldots, \varphi_n$, e.g., by the orthogonalization of the basis $B_0, B_1, \ldots, B_n$ and we so get the reproducing kernel

$$k_n(z,w) = \sum_{k=0}^{n} \frac{\varphi_k(z)\overline{\varphi_k(w)}}{\|\varphi_k\|^2}.$$

It is even possible to find a recurrence relation for the reproducing kernels directly. The optimal predictor from $\mathcal{L}_n$ is then given by

$$H(z) = \frac{Gk_n(0,0)}{k_n^*(z,0)}$$

with $k_n^*(z,0) = B_n(z)k_{n*}(z,0)$.

Fortunately, (but this goes beyond the time limit of these lectures), all the previous algorithms generalize nicely to this rational case. Of all the equivalent formulations we have given before in the case of the AR model, the Schur algorithm is the easiest to reformulate. It becomes the rational interpolation algorithm of Pick-Nevanlinna which computes recursively

$$\Gamma_{k+1}(z) = \frac{1}{\zeta_{k+1}(z)} \frac{\Gamma_k(z) - \rho_{k+1}}{1 - \bar{\rho}_{k+1}\Gamma_k(z)}, \quad \rho_{k+1} = \Gamma_k(\alpha_{k+1}).$$

By essentially the same argument as given in the previous section, one can show that all the $\Gamma_k$ belong to the Schur class and that all $|\rho_k| < 1$.

## 15 A more general prediction framework

We shall reformulate the prediction problem again in a more general setting.

Consider a Hilbert space of vectors $y_i$ with inner product

$$< y_i, y_j >= R_{ij}.$$

The Gram matrix of the Hilbert space is $R = [R_{ij}]$.

For the moment you can think of $y_0$ as a signal $s = (s_n)$ and $y_1, y_2, \ldots$ are shifted (time delayed) versions of $y_0$. Then clearly $R$ corresponds to the covariance of the signal: for a stochastic signal, we shall define $< y_i, y_j >= \mathsf{E}[y_i y_j^H]$ and for a deterministic signal, $< y_i, y_j >= y_i y_j^H$.

Now consider a subspace spanned by

$$U = \{y_{t-m}, y_{t-m+1}, \cdots, y_{t-1}\}$$

and denote the projection of $y_t$ onto span $U$ by $\hat{y}_{t|U}$. It is the best (least squares) prediction of $y_t$ from span $U$. The corresponding *forward prediction error* is

$$e_{m,t} = y_t - \hat{y}_{t|U}.$$

### 15.1 Order update

We want to find $e_{m+1,t}$ from $e_{m,t}$, which corresponds to an increase in the prediction order from $m$ to $m+1$. Note that

$$e_{m+1,t} = y_t - \hat{y}_{t|\{U, y_{t-m-1}\}}.$$

The component of $y_{t-m-1}$ orthogonal to span $U$ is

$$f_{m,t-1} = y_{t-m-1} - \hat{y}_{t-m-1|U}.$$

It corresponds to a *backward prediction error*. If we denote by $\bar{y}$ the normalized vector $\bar{y} = y/\|y\|$, then using orthogonality considerations, one gets

$$\hat{y}_{t|\{U,y_{t-m-1}\}} = \hat{y}_{t|U} + \hat{y}_{t|f_{m,t-1}} = \hat{y}_{t|U} + < y_t, \bar{f}_{m,t-1} > \bar{f}_{m,t-1}.$$

Consequently

$$\begin{aligned} e_{m+1,t} &= e_{m,t} - < y_t, \bar{f}_{m,t-1} > \bar{f}_{m,t-1} \\ &= \|e_{m,t}\|(\bar{e}_{m,t} - < \bar{e}_{m,t}, \bar{f}_{m,t-1} > \bar{f}_{m,t-1}) \\ &= \|e_{m,t}\|(\bar{e}_{m,t} - \rho_{m+1,t}\bar{f}_{m,t-1}), \quad \rho_{m+1,t} = < \bar{e}_{m,t}, \bar{f}_{m,t-1} > \end{aligned}$$

We used $f_{m,t-1} \perp U$.

The $\rho_{m+1,t}$ will turn out to generalize the reflection coefficients. They are in general not scalar!

The least squared error is

$$\|e_{m+1,t}\|^2 = \|e_{m,t}\|(I - \rho_{m+1,t}\rho_{m+1,t}^H)\|e_{m,t}\|^H$$

or

$$\|e_{m+1,t}\| = \|e_{m,t}\|(I - \rho_{m+1,t}\rho_{m+1,t}^H)^{1/2}.$$

In normalized form, the order update for the forward predictors becomes thus

$$\bar{e}_{m+1,t} = (I - \rho_{m+1,t}\rho_{m+1,t}^H)^{-1/2}.(\bar{e}_{m,t} - \rho_{m+1,t}\bar{f}_{m,t-1}).$$

Similar computations can be done for the backward predictor:

$$\begin{aligned} f_{m+1,t} &= y_{t-m-1} - \hat{y}_{t-m-1|\{U,y_t\}} \\ &= f_{m,t-1} - < y_{t-m-1}, \bar{e}_{m,t} > \bar{e}_{m,t} \\ &= \|f_{m,t-1}\|(\bar{f}_{m,t-1} - < \bar{f}_{m,t-1}, \bar{e}_{m,t} > \bar{e}_{m,t}) \\ &= \|f_{m,t-1}\|(\bar{f}_{m,t-1} - \rho_{m+1,t}^H\bar{e}_{m,t}) \end{aligned}$$

and

$$\|f_{m+1,t}\| = \|f_{m,t-1}\|(I - \rho_{m+1,t}^H\rho_{m+1,t})^{1/2}.$$

Thus the normalized recursions are

$$\bar{f}_{m+1,t} = (I - \rho_{m+1,t}^H\rho_{m+1,t})^{-1/2}.(\bar{f}_{m,t-1} - \rho_{m+1,t}\bar{e}_{m,t}).$$

Note that

$$\bar{e}_{0,t} = \bar{y}_t = \bar{f}_{0,t}$$

and by Cauchy-Schwarz

$$\|\rho_{m+1,t}\| \leq \|\bar{e}_{m,t}\|\|\bar{f}_{m,t-1}\| = 1.$$

For any two vectors $a$ and $b$, we define its correlation as

$$\rho(a,b) = < \bar{a}, \bar{b} > = \|a\|^{-1} < a, b > \|b\|^{-1}$$

and, given a set of vectors $V$, we denote by $a_V$ the projecting vector of $a$ onto span $V$:

$$a_V = a - < a, \bar{V} > \bar{V}.$$

($\bar{V}$ is an orthonormal basis for span $V$).

The *partial correlation* (PARCOR) *coefficient* for the couple $a, b$ is then defined as

$$\rho_V(a,b) = \rho(\bar{a}_V, \bar{b}_V) = \|a\|^{-1} < a_V, b_V > \|b\|^{-1}.$$

The reflection coefficient can thus be written as a PARCOR coefficient:

$$\rho_{m+1,t} = \rho_U(y_t, y_{t-m-1}), \quad U = \{y_{t-m}, \ldots, y_{t-1}\}$$

which explains the term PARCOR for the reflection coefficient.

**Exercise 15.1** Check the following formulas

$$\begin{aligned} a_V &= \|a\|(\bar{a} - \rho(a,V)\bar{V}) \\ \|a_V\| &= \|a\|(I - \rho(a,V)\rho^H(a,V))^{1/2} \\ \bar{a}_V &= (I - \rho(a,V)\rho^H(a,V))^{-1/2}(\bar{a} - \rho(a,V)\bar{V}) \end{aligned}$$

(and similarly for $b_V$). We get

$$\rho_V(a,b) = F(A,B,C)$$

with

$$F(A,B,C) := (I - BB^H)^{-1/2}(A - BC^H)(I - CC^H)^{-H/2}$$

and

$$A = \rho(a,b), \quad B = \rho(a,V), \quad C = \rho(b,V).$$

◇

**Exercise 15.2** Prove Yule's PARCOR identity:

$$\rho_{\{U,V\}}(a,b) = F(\rho_U(a,b), \rho_U(a,V), \rho_U(b,V)).$$

Hint: to find $a_{\{U,V\}}$, first estimate $a$ from $U$ with error $a_U$ and then estimate $a_U$ from $V_U$ with error $(a_U)_{V_U} = a_{\{U,V\}}$. ◇

### 15.2 Time update

Besides the order update ($m \to m+1$), there is also a time update ($t \to t+1$) which corresponds to taking a new input into account as the time progresses. Of course, for stationary processes, $\rho_{m,t}$ is independent of $t$, but this is not the real life situation. It will take some time before the process arrives in a stationary situation and after a short while it will start migrating to another stationary one. In the deterministic case, time independence appears when considering infinite signals and also that is not a workable condition.

Due to time restrictions, we shall not go into the time updating problem in these short notes.

## 16 Low displacement rank matrices

In the autocorrelation method, the covariance matrix was Toeplitz. In the covariance method, it is not Toeplitz, but it is the product of two Toeplitz matrices and as such it is in a sense "close to Toeplitz". We shall see that it has a low *displacement rank*. We develop here the notion of displacement rank for Toeplitz type matrices, but this is only a special case of displacement rank that can also be developed for other matrices with a special structure like Hankel, Vandermonde, Bezout, etc...

Define $Z$ to be the down shift matrix.

$$Z = \begin{bmatrix} 0 & \dots & 0 & 0 \\ 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \end{bmatrix}$$

and the effect of the operator $\Delta$ (*displacement operator*) on a matrix $R$ as $\Delta R = R - ZRZ^T$. For example, if $R$ is Toeplitz, then

$$\Delta R = \begin{bmatrix} R_0 & R_{-1} & \dots & R_{-n} \\ R_1 & & & \\ \vdots & & 0 & \\ R_n & & & \end{bmatrix}$$

which is of rank $\alpha \leq 2$, whereas for a general matrix $R$, this would have rank $\alpha \leq n+1$. We shall call $\alpha$ the *displacement rank* of R.

Let $R$ be positive definite and hermitian, then $\Delta R$ will be hermitian (but not positive definite). Thus its eigenvalues are real. Suppose $p$ of them are positive and $q$ of them are negative, then the displacement rank is $\alpha = p + q$. It is possible to write then a dyadic decomposition of the form

$$\Delta R = G\Sigma G^H, \quad \Sigma = I_p \oplus -I_q, \quad G \in \mathbb{C}^{(n+1)\times\alpha}$$

$\Sigma$ is called a *signature* matrix (a diagonal with only $+1$'s and $-1$'s). $\{\Sigma, G\}$ is called a *generator*. The choice of $\Sigma$ is unique[1], but $G$ is not. Indeed for any $\Sigma$-unitary matrix $U$, $(U^H\Sigma U = \Sigma)$, we may write

$$\Delta R = (GU)\Sigma(GU)^H = G\Sigma G^H$$

thus $\{\Sigma, GU\}$ will also be a generator.

The couple of integers $\mathrm{In}(\Delta R) = \{p, q\}$ is called the *displacement inertia* of $R$.

**Exercise 16.1** Prove that for a Toeplitz matrix $R$ we can write

$$\Delta R = G\Sigma G^H, \quad \Sigma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad G^H = \frac{1}{\sqrt{R_0}} \begin{bmatrix} R_0 & R_{-1} & \dots & R_{-n} \\ 0 & R_{-1} & \dots & R_{-n} \end{bmatrix}.$$

◇

**Theorem 16.1** *The equation*

$$\Delta R = \sum_{k=1}^{p} x_i x_i^H - \sum_{k=p+1}^{p+q} x_k x_k^H$$

*has a unique solution*

$$R = \sum_{k=1}^{p} L(x_k)L(x_k)^H - \sum_{k=p+1}^{p+q} L(x_k)L(x_k)^H$$

*where $L(x)$ is a lower triangular Toeplitz matrix whose first column is $x$.*

**Proof.** Suppose $p = 1$, $q = 0$, then $(Z^n = 0)$

$$\begin{aligned} \Delta R &= xx^H \text{ has a solution} \\ R &= \sum_{k=0}^{\infty} Z^i x x^H (Z^i)^H = L(x)L(x)^H. \end{aligned}$$

The solution is unique because, if there were two solutions $R_1, R_2$, then $\Delta R_1 = \Delta R_2$ implies

$$R_1 - R_2 = Z(R_1 - R_2)Z^T = Z^2(R_1 - R_2)(Z^2)^T = \ldots = (Z^n)(R_1 - R_2)(Z^n)^T = 0$$

Since $\Delta$ is a linear operator, the result follows for general $p$ and $q$. □

**Theorem 16.2** *Let $\tilde{I}$ denote the reversal matrix (1's on antidiagonal) then* $\mathrm{In}(\Delta R) = \mathrm{In}(\Delta \tilde{I} R^{-1} \tilde{I})$ *if $R$ is hermitian. Hence the displacement rank of $R$ and $\tilde{I}R^{-1}\tilde{I}$ is the same.*

[1] when its dimension is minimal ($\alpha = \#$ of nonzero eigenvalues of $\Delta R$)

**Proof.** Note that

$$\begin{bmatrix} R & Z \\ Z^T & R^{-1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ Z^T R^{-1} & I \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & R^{-1} - Z^T R^{-1} Z \end{bmatrix} \begin{bmatrix} I & R^{-1} Z \\ 0 & I \end{bmatrix}$$
$$= \begin{bmatrix} I & ZR \\ 0 & I \end{bmatrix} \begin{bmatrix} R - ZRZ^T & 0 \\ 0 & R^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ RZ^T & I \end{bmatrix}$$

By Sylvester's law, the inertia are preserved under congruence transformations. Hence

$$\text{In}(R) + \text{In}(R^{-1} - Z^T R^{-1} Z) = \text{In}(R^{-1}) + \text{In}(R - ZRZ^T)$$

But $R = RR^{-1}R$, so that $\text{In}(R) = \text{In}(R^{-1})$. Thus

$$\text{In}(\Delta R) = \text{In}(R^{-1} - Z^T R^{-1} Z) = \text{In}(\Delta(\tilde{I} R^{-1} \tilde{I}))$$

Because the inertia determine the displacement rank, the theorem is proved. □

This theorem proves that the inverse of a Toeplitz matrix, which is not Toeplitz, has the same displacement rank as the Toeplitz matrix itself.

# 17 Examples

The prediction problems, introduced before give rise to low displacement rank problems.

## 17.1 The autocorrelation method

The Hilbert space considered here is the space where

$$\begin{aligned} y_0 &= [\ldots\ s_{-1}\ s_0\ s_1\ \ldots] \\ y_t &= [\ldots\ s_{t-1}\ s_t\ s_{t+1}\ \ldots] \end{aligned}$$

with inner product

$$< y_s, y_t >= y_s y_t^H = R_{s,t} = R_{s-t}$$

The Gram matrix is Toeplitz $R_n =< Y_n, Y_n >= Y_n Y_n^H$, with $Y_n = [y_0^T\ y_1^T\ \ldots\ y_n^T]^T$.

## 17.2 The prewindowing case

Here we consider the same Hilbert space, but the signal is prewindowed, i.e., we suppose $s_k = 0$, $k < 0$. Now

$$R_n = Y_n Y_n^H, \quad Y_n = \begin{bmatrix} 0 & \ldots & 0 & s_0 & \ldots \\ \vdots & & s_0 & s_1 & \ldots \\ 0 & & & \vdots & \\ s_0 & \ldots & s_{n-1} & s_n & \ldots \end{bmatrix}$$

which is still Toeplitz. Suppose now that we minimize the prediction error only over a finite time interval, then

$$Y_n = \begin{bmatrix} 0 & \ldots & 0 & s_0 & \ldots & s_{N-n} \\ \vdots & & s_0 & s_1 & \ldots & s_{N-n+1} \\ 0 & & & & & \vdots \\ s_0 & \ldots & & & \ldots & s_N \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_N \end{bmatrix}$$

However the normal equations

$$[a_n \ \dots \ a_1 \ 1]R_n = [0 \ \dots 0 \ E_n], \quad R_n = Y_n Y_n^H$$

have not a Toeplitz matrix (unless $s_{N-n+1} = \cdots = s_n = 0$). But let us compute the displacement rank: (we suppose $n < N$)

$$\begin{aligned} \Delta R_n &= [y_t y_s^H - y_{t-1} y_{r-1}^H]_{t,r=0,\dots,n}, \quad y_{-i} = 0, i \geq 1 \\ &= [s_t \bar{s}_r - s_{t-N} \bar{s}_{r-N}]_{t,r}, \quad s_{-i} = 0, i \geq 0 \\ &= [s_t \bar{s}_r]_{t,r} \\ &= x x^H, \qquad x = [s_{N-n} \dots s_n]^T \end{aligned}$$

This means that $\{1, x\} = \{\Sigma, G\}$ is a generator for $R_n$. The displacement rank is $\alpha = 1$.

## 17.3 The covariance method

Now

$$Y_n = \begin{bmatrix} s_{-n} & \dots & s_{N-n} \\ \vdots & & \vdots \\ s_{-1} & & s_{N-1} \\ s_0 & \dots & s_N \end{bmatrix}$$

and again $R_n = Y_n Y_n^H$ is no longer Toeplitz, but $T^H = \tilde{I} Y_n$ is Toeplitz. Therefore write

$$\begin{aligned} X^H &= \left[\begin{array}{cccc|c|cccc} 0 & s_{-n} & \dots & s_{-1} & & 0 & \dots & 0 & 0 \\ 0 & 0 & & & & s_N & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \tilde{I}Y_n & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & s_{-n} & & & & 0 & 0 \\ 0 & 0 & \dots & 0 & & s_{N-n+1} & \dots & s_N & 0 \end{array}\right] \\ &= [\ \hat{L}_2^H \ | \ T^H \ | \ L_4 \ ], \qquad (T^H := \tilde{I} Y_n) \end{aligned}$$

then

$$X^H X = \hat{T} := \hat{L}_2^H \hat{L}_2 + T^H T + L_4 L_4^H \quad \text{is Toeplitz.}$$

But

$$[\hat{L}_2^H | L_2] \begin{bmatrix} \hat{L}_2 \\ L_2^H \end{bmatrix} = \tilde{T} \quad \text{with} \quad L_2 = \begin{bmatrix} 0 & \dots & 0 & 0 \\ s_{-1} & & & 0 \\ \vdots & \ddots & & \vdots \\ s_{-n} & \dots & s_{-1} & 0 \end{bmatrix}$$

is Toeplitz too. Thus

$$\begin{aligned} \hat{L}_2^H \hat{L}_2 &= \tilde{T} - L_2 L_2^H \quad \text{and} \\ T^H T &= \hat{T} - \hat{L}_2^H \hat{L}_2 - L_4 L_4^H = [\hat{T} - \tilde{T}] + L_2 L_2^H - L_4 L_4^H. \end{aligned}$$

The matrix $\hat{T} - \tilde{T}$ is Toeplitz and its first column is

$$[T^H T - L_2 L_2^H - L_4 L_4^H] e_1 = T^H t_1, \quad t_1 = T e_1.$$

Therefore a generator for $T^H T$ is obtained as $G\Sigma G^H$ with $G = [x_1\ x_2\ x_3\ x_4]$

$$\begin{aligned}
x_1 &= T^H t_1/\|t_1\| \\
x_2 &= \text{first column of } L_2 \\
&= \text{first row of } \bar{T}, \text{ except its first element } x_2(0) = 0 \\
x_3 &= \text{the same as } x_1 \text{ except its first element } x_3(0) = 0 \\
x_4 &= \text{first column of } L_4 \\
&= \text{shifted last row of } T^H \\
\Sigma &= \mathrm{diag}(1, 1, -1, -1)
\end{aligned}$$

Thus the displacement rank is $\alpha \leq 4$.
In our concrete example: $R_n = Y_n Y_n^H$ with $T^H = \tilde{I} Y_n$, we get

$$\begin{aligned}
\Delta R_n &= \begin{bmatrix} R_{00} & R_{01} & \dots & R_{0n} \\ R_{10} & & & \\ \vdots & & R_1 & \\ R_{n0} & & & \end{bmatrix}, R_1 = \begin{bmatrix} s_{N-n+1} \\ \vdots \\ s_N \end{bmatrix} [\bar{s}_{N+n+1} \dots \bar{s}_N] - \begin{bmatrix} s_{-n} \\ \vdots \\ s_{-1} \end{bmatrix} [\bar{s}_{-n} \dots \bar{s}_{-1}] \\
&= [x_1\ x_2\ x_3\ x_4]\Sigma[x_1\ x_2\ x_3\ x_4]^H, \qquad \Sigma = \mathrm{diag}(1, 1, -1, -1) \\
x_1 &= [R_{00}\ R_{10}\ \dots\ R_{n0}]^T/\sqrt{R_{00}}, \qquad x_2 = [0\ s_{N-n+1}\ \dots\ s_N]^T \\
x_3 &= [0\ R_{10}\ \dots\ R_{n0}]^T/\sqrt{R_{00}}, \qquad x_4 = [0\ s_{-n}\ \dots\ s_{-1}]^T
\end{aligned}$$

## 18 Generating functions

Let $R$ be a hermitian matrix of size $n \to \infty$. We associate with it the *generating function* (bilinear form).

$$R(z, w) = \sum_{i,j} z^i \bar{w}^j R_{ij} = [1\ z \dots] R [1\ w \dots]^H$$

The generating function of a positive definite matrix $R$ is called a (two dimensional) *spectral function.*

The effect of the displacement operator $\Delta$ is

$$(\Delta R)(z, w) = R(z, w) - z\bar{w} R(z, w) = (1 - z\bar{w}) R(z, w).$$

If $\Delta R = G\Sigma G^H$, and $G$ has rows $g_0, g_1, \dots, g_i, \dots \in \mathbb{C}^{1\times\alpha}$, then

$$(1 - z\bar{w}) R(z, w) = G(z)\Sigma G(w)^H, \quad G(z) = [1\ z \dots] G = \sum_{k=0}^{\infty} g_k z^k.$$

For the Toeplitz case we get

$$\begin{aligned}
G(z) &= [R_0 + \sum_{k=1}^{\infty} R_k z^k | \sum_{k=1}^{\infty} R_k z^k]/\sqrt{R_0} \\
&= \frac{1}{2}[\Omega_0 + R_0 | \Omega_0 - R_0]/\sqrt{R_0}, \quad \Omega_0 = R_0 + 2\sum_{k=1}^{\infty} R_k z^k \in \mathcal{C}.
\end{aligned}$$

If $R$ is a matrix subdivided as

$$R = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad A \text{ regular},$$

then we call $\Delta = D - CA^{-1}B$ the Schur complement of $A$ (in $R$). Writing

$$R = \begin{bmatrix} R_{00} \\ R_{10} \\ \vdots \end{bmatrix} R_{00}^{-1}[R_{00}\ R_{01}\ \ldots] + \begin{bmatrix} 0 & \ldots & 0 \\ \vdots & R_1 & \\ 0 & & \end{bmatrix}$$

we see that $R_1$ is the Schur complement of $R_{00}$. A lemma due to Schur says that

**Lemma 18.1** *$R$ is positive definite iff $R_{00} > 0$ and its Schur complement is positive definite. Thus $R(z,w)$ is a spectral function iff $R(0,0) = R_{00} > 0$ and the generating function of its Schur complement $R_1$, is also a spectral function.*

**Exercise 18.1** Prove that if $R(z,w)$ is the generating function, then the generating function of its Schur complement $R_1$ is

$$R_1(z,w) = \frac{R(z,w) - R(z,0)R(0,0)^{-1}R(0,w)}{z\bar{w}}$$

◇

# 19 Generalized Schur algorithm

Suppose $R$ has generator $\{\Sigma, G\}$. By a $\Sigma$-unitary matrix, we can always bring $G$ into the canonical form so that $G(0) = g_0 = \sigma_0[1\ 0 \ldots 0]$, with $\sigma_0^2 = R_{00}$. So we write

$$G = \left[\begin{array}{c|c} \sigma_0 & 0 \ldots 0 \\ \gamma_1 & \Gamma_1 \\ \vdots & \vdots \\ \gamma_n & \Gamma_n \end{array}\right], \quad \Sigma = \begin{bmatrix} 1 & 0 \\ 0 & -J \end{bmatrix}, \quad \sigma_0 = \sqrt{R_{00}} > 0.$$

**Exercise 19.1** Define the matrix ($J$ is a $(\alpha-1)\times(\alpha-1)$ signature matrix and $K \in \mathbb{C}^{1\times(\alpha-1)}$)

$$\theta(K,J) = \begin{bmatrix} (1-KJK^H)^{-1/2} & 0 \\ 0 & (J-K^HK)^{-1/2} \end{bmatrix} \begin{bmatrix} 1 & -KJ \\ -K^H & I \end{bmatrix}.$$

Prove that $\theta(K,J)$ is $\Sigma$-unitary: $\theta\Sigma\theta^H = \Sigma$, where $\Sigma = 1 \oplus (-J)$. ◇

**Note:** Any $\Sigma$-unitary matrix has the form

$$\begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} \theta(K,J) \quad \text{with } U_1 \text{ unitary and } U_2 \ J\text{-unitary.}$$

The general Schur algorithm goes as follows.
STEP 1A: shift the $\Gamma_i$ rows one place up and delete the last row:

$$\tilde{G}_1 = \begin{bmatrix} \sigma_0 & \Gamma_1 \\ \gamma_1 & \Gamma_2 \\ \vdots & \vdots \\ \gamma_{n-1} & \Gamma_n \end{bmatrix}.$$

Note that this corresponds to the following transformation on $G(z)$

$$\tilde{G}_1(z) = G(z) \begin{bmatrix} 1 & 0 \\ 0 & zI \end{bmatrix}^{-1}.$$

STEP 1B: find $\theta(K_1, J)$ so that $\tilde{G}_1\theta^{-1}(K_1, J)$ is normalized:

$$G_1 = \tilde{G}_1\theta^{-1}(K_1, J) = \left[\begin{array}{c|c} \sigma_1 & 0\ldots 0 \\ \gamma_1^{(1)} & \Gamma_1^{(1)} \\ \vdots & \vdots \\ \gamma_{n-1}^{(1)} & \Gamma_{n-1}^{(1)} \end{array}\right].$$

This is achieved by choosing

$$K_1 = -\sigma_0^{-1}\Gamma_1 J.$$

**Proof.** The $K_1$ has to be chosen such that

$$\begin{aligned} \text{First row of } \tilde{G}_1 &= [\sigma_1\ 0\ \ldots\ 0]\theta(K_1, J) \\ &= \sigma_1 \text{ first row of } \theta(K_1, J) \\ &= \sigma_1(1 - K_1 J K_1)^{-1/2}[1\ \ -K_1 J]. \end{aligned}$$

Thus

$$\sigma_0 = \sigma_1(1 - K_1 J K_1^H)^{-1/2} \text{ and } \Gamma_1 = -\sigma_0 K_1 J.$$

□

We now repeat these steps on the result $G_1$ etc. until $\tilde{G}_1$ has all zeros (except perhaps the first column). All remaining $K_j$ will be zero.

**Exercise 19.2** Prove that the complexity is $O(\alpha n^2)$. ◇

**Theorem 19.1** *Suppose* $\Delta R = G\Sigma G^H$, $G$ *normalized* ($G(0) = \sigma_0[1\ 0\ldots 0]$). *Suppose* $G = [x_1\ \ldots\ x_\alpha]$. *Then*

$$\Delta_1 \ = \ R - x_1 x_1^H = \left[\begin{array}{ccc} 0 & \ldots & 0 \\ \vdots & R_1 & \\ 0 & & \end{array}\right]$$

*has generator* $\{\Sigma, G_1'\}$

$$G_1' = [Zx_1, x_2, \ldots, x_\alpha].$$

**Proof.**

$$\begin{aligned} \Delta_1 - Z\Delta_1 Z^T &= [R - x_1 x_1^H] - Z[R - x_1 x_1^H]Z^T \\ &= R - ZRZ^T - x_1 x_1^H + Zx_1 x_1^H Z^T \\ &= G\Sigma G^H - x_1 x_1^H + Zx_1 x_1^H Z^T \\ &= G_1'\Sigma G_1'^H. \end{aligned}$$

The first row and column of $\Delta_1$, are zero because

$$\Delta_1 e_0 = (R - x_1 x_1^H)e_0 = Re_0 - x_1 x_1^H e_0 = x_1 x_1^H e_0 - x_1 x_1^H e_0 = 0$$

since $G = [x_1\ \ldots\ x_\alpha]$ is normalized. □

**Corollary 19.2** $\tilde{G}_1$ *as constructed by the general Schur algorithm, together with* $\Sigma$, *forms a generator* $\{\Sigma, \tilde{G}_1\}$ *for* $R_1$, *the Schur complement of* $R_{00}$ *in* $R$. *Since* $\theta$ *is* $\Sigma$ *unitary, also* $G_1 = \tilde{G}_1\theta$ *will give a possible generator* $\{\Sigma, G_1\}$ *for* $R_1$.

Together with what we have shown in the previous section we may now conclude that the generating function for $R_1$ is

$$R_1(z,w) = \frac{G_1(z)\Sigma G_1(w)^H}{1 - z\bar{w}}$$

with

$$G_1(z) = G(z) \begin{bmatrix} 1 & 0 \\ 0 & zI \end{bmatrix}^{-1} \theta^{-1}(K,J)$$

and

$$R(z,w) = \frac{G(z)\Sigma G(w)^H}{1 - z\bar{w}}.$$

**Exercise 19.3** Show that $R_1(0,0)$ is given by (recall $G_0 = [\sigma_0\ 0 \ldots 0]$)

$$\sigma_0^2 - \Gamma_1 J \Gamma_1^H = \sigma_0^2(1 - K_1 J K_1^H).$$

◇

By Schur's lemma we may conclude that $R(z,w)$ is a spectral function ($R$ is pos. def.) iff

$$1 - K_i J K_i^H > 0$$

for $i = 1, 2, \ldots$
This generalizes the property of the reflection coefficients $|\rho_i| < 1$.

## 20 Triangular factorization of $R$

We have now a recursive procedure to compute the generating functions of the Schur complements $R_n$ of $R_0$

$$R_0 = \begin{bmatrix} R_{00} \\ \vdots \\ R_{n0} \end{bmatrix} R_{00}^{-1} [R_{00} \ldots R_{0n}] + \begin{bmatrix} 0 & \ldots & 0 \\ \vdots & R_1 & \\ 0 & & \end{bmatrix}$$

with in general

$$z\bar{w} R_{i+1}(z,w) = R_i(z,w) - R_i(z,0) R_i^{-1}(0,0) R_i(0,w).$$

Setting

$$d_i = R_i(0,0) > 0, \quad l_i(z) = z^i R_i(z,0) d_i^{-1}, \quad l_i(z) = [1\ z \ldots] l_i,$$

we see that

$$R_0(z,w) = \sum_{i=0}^{\infty} d_i l_i(z) l_i(w)^H \iff R_0 = \sum_{i=0}^{\infty} d_i l_i l_i^H = L D L^H$$

or in the finite dimensional case ($R_0 \in \mathbb{C}^{(n+1)\times(n+1)}$)

$$R_0 = L_n D_n L_n^H,$$

with $L_n = [l_0\ l_1\ \ldots\ l_n]$ unit lower triangular.

Thus the Schur algorithm gives a fast ($O(\alpha n^2)$) algorithm to compute the Cholesky factorization of a matrix $R$ with displacement rank $\alpha$.

The question now is: "Can we also find the *inverse* Cholesky factors?" because these will give the prediction coefficients.

## 21 Triangular factorization of $R^{-1}$

If we define

$$\theta_i(z) = \theta(K_i, J) \begin{bmatrix} 1 & 0 \\ 0 & zI \end{bmatrix}, \quad \Theta_n = \theta_n \theta_{n-1} \dots \theta_1$$

then we know that

$$G_n(z)\Theta_n(z) = G(z).$$

If we replace $G_n(z)$ by $G_n(0) = \sigma_n[1\ 0 \dots 0]$, we get a generator $H_n(z) = G_n(0)\Theta_n(z)$.

Since all the information for the first $n$ reflection coefficients is still in $H_n(z)$, we can perform the Schur algorithm backward and this results in a covariance matrix whose reflection coefficients are $K_1, \dots, K_n, 0, 0, \dots$. Note also that $H_n(z)$ is a polynomial of degree $n$. The generator $H_n(z)$ is called a *canonical generator.*

**Exercise 21.1** Show that the covariance matrix of a canonical generator has the form

$$\left[\begin{array}{c|c} X & 0 \\ \hline 0 & \sigma_n^2 \end{array}\right] \text{ with } X \in \mathbb{C}^{n \times n}.$$

◇

The following theorem can be proved:

**Theorem 21.1** *Let $(H, \Sigma)$ and $(G, \Sigma)$ be two generators, then the first reflection coefficients $K_1, \dots, K_n$ are the same iff there exist (unique) coefficients $(a_{0n}, \dots, a_{nn})$ such that*

$$H_n = L(a_n)G_n$$

*with $H_n, G_n$ the first $n+1$ rows of $H$ and $G$ and $L(a_n)$ is a lower triangular Toeplitz matrix with entries $a_{0n}, \dots, a_{nn}$.*

In the transform domain this theorem reads

$$H(z) = a_n(z)G(z) + O(z^{n+1}), \quad a_n(z) = \sum_{k=0}^{n} a_{kn} z^k.$$

Thus

$$\frac{H_n(z)}{a_n(z)} \text{ is an order } n \text{ approximant for } G(z).$$

Now we can identify $a_n$ as being (proportional to) the prediction polynomial.

This predictor should satisfy

$$[a_{nn} \dots a_{1n}\ 1]R_n = [0 \dots 0\ E_n]$$

If $L_n D_n L_n^T = R_n$ is a Cholesky factorization, then $[a_{nn}\ \dots\ a_{1n}\ 1]$ is the last row of $L_n^{-1}$ (unit lower triangular).

But since $L(a_n)L_n$ is lower triangular, with last row equal to $e_n^T$, we have $e_n^T L(a_n) L_n = e_n^T$ or $e_n^T L(a_n) = e_n^T L_n^{-1}$. This means that $[a_{nn}\ \dots\ a_{1n}\ 1]$, which is the last row of $L_n^{-1}$, is equal to the last row of $L(a_n)$.

Now let $\lambda$ be some $\alpha \times 1$ vector such that $G(z)\lambda$ is not zero for $z = 0$. Then

$$H_n(z) = a_n(z)G(z) + O(z^{n+1})$$

implies

$$H_n(z)\lambda = a_n(z)G(z)\lambda + O(z^{n+1})$$

or

$$a_n(z) = \sigma_n[1\ 0 \ldots 0]\Theta_n(z)\lambda g(z) + O(z^{n+1}), \quad g(z) = 1/(G(z)\lambda).$$

It can be proved that there always exists a $\lambda$ such that $G(z)\lambda = 1$, so that for this particular choice:

$$a_n(z) = \sigma_n[1\ 0 \ldots 0]\Theta_n(z)\lambda.$$

Thus up to a constant multiple, the predictors $a_n$ are given by

$$\begin{bmatrix} a_n(z) \\ C_n(z) \end{bmatrix} = \theta_n(z) \begin{bmatrix} a_{n-1}(z) \\ C_{n-1}(z) \end{bmatrix}, \quad \begin{bmatrix} a_0 \\ C_0 \end{bmatrix} = \lambda.$$

These $a_n$ are orthogonal polynomials w.r.t. the Gram matrix $R_n$. The $C_n$ are in the Toeplitz case equal to $a_n^*(z) = \overline{a_n(1/\bar{z})}z^n$. In general they are polynomials $C_n \in \mathbb{C}^{(\alpha-1)\times 1}[z]$.

They appear in the *Christoffel-Darboux relations*

$$\sum_{i=0}^{n} a_i^*(z)\overline{a_i^*(w)} = \frac{a_{n+1}^*\overline{a_{n+1}^*(w)} - C_{n+1}^*(z)JC_{n+1}^*(w)^H - \lambda\Sigma\lambda^H}{1 - z\bar{w}}.$$

There is also a nonstationary version of the Levinson algorithm which computes the reflection coefficients and the $a_n, C_n$ directly. It is however considerably more complicated to describe than the general Schur algorithm. We shall not do that here.

## 22 The pole-zero model once more

Also the Pick-Nevanlina procedure can be generalized to matrices with low displacement rank.

One should then replace the formula

$$R_1(z,w) = \frac{R(z,w) - R(z,0)R(0,0)^{-1}R(0,w)}{z\bar{w}}$$

by

$$R_1(z,w) = \frac{R(z,w) - R(z,a)R(a,a)^{-1}R(a,w)}{\left(\frac{z-a}{1-\bar{a}z}\right)\left(\overline{\frac{w-a}{1-\bar{a}w}}\right)}$$

which results in a recursion for the generators

$$G_1(z) = G(z)\begin{bmatrix} 1 & 0 \\ 0 & \frac{z-a}{1-\bar{a}z} \end{bmatrix}^{-1} \theta^{-1}(K_1, J).$$

# Appendix: Solutions of the exercises

**Exercise 1.1:** For $h \in \ell_1$ and $u \in \ell_\infty$,

$$\begin{aligned}(u*h)_n &= \sum_{m\in\mathbb{Z}} u_m h_{n-m} \\ |(u*h)_n| &\le \sum_{m\in\mathbb{Z}} |u_m||h_{n-m}| \\ &\le \left(\sum_{m\in\mathbb{Z}} |h_{n-m}|\right) \|u\|_\infty \\ &= \|h\|_1\|u\|_\infty < \infty\end{aligned}$$

Hence

$$\|u*h\|_\infty = \sup_{n\in\mathbb{Z}} |(u*h)_n| \le \|h\|_1\|u\|_\infty < \infty$$

The mapping $T$ is continuous because

$$\|u\|_\infty \to 0 \quad \Rightarrow \quad \|u*h\|_\infty \le \|u\|_\infty\|h\|_1 \to 0$$

T defines a filter because

$$\begin{aligned}(TSu)_n &= \sum_{m\in\mathbb{Z}} u_{m-1}h_{n-m} = \sum_{m\in\mathbb{Z}} u_m h_{n-m-1} \\ &= (Tu)_{n-1} = (S(Tu))_n = (STu)_n\end{aligned}$$

which proves that $TS = ST$. ◇

**Exercise 1.2:** Clearly $u_k = v_k$, $k < m \Leftrightarrow S^m w$ is causal, where $w = u - v$. Note that any causal signal $s$ can be represented in this form.
Similarly, $(Tu)_k = (Tv)_k$, $k < m \Leftrightarrow S^m(Tu - Tv) = S^m T(u-v) = S^m Tw = TS^m w$ is causal.
Hence, $s' = S^m w$ is causal $\Rightarrow TS^m w = Ts'$ is causal, which means that the filter $T$ is causal. ◇

**Exercise 1.3:** If $h_k = 0$, $k < 0$, then

$$(h*u)_n = (Tu)_n = \sum_{m\in\mathbb{Z}} h_{n-m}u_m = \sum_{n-m\ge 0} h_{n-m}u_m$$

This will transform a causal signal $u$ ($u_m = 0$ for $m < 0$) into a causal signal because $(Tu)_n$, $n < 0$ is

$$\sum_{n-m\ge 0} h_{n-m}u_m,$$

where the first factor $h_{n-m} = 0$ if $n < m$ and the second $u_m = 0$ if $m < 0$. Thus the sum is zero for all negative $n$. ◇

**Exercise 1.4:** Compute

$$\begin{aligned}(Tu)_k &= \sum_n u_n h_{k-n} \\ &= \sum_n \left(\sum_i \alpha_i e^{j\omega_i n}\right) h_{k-n} \\ &= \sum_i \alpha_i \left(\sum_n e^{j\omega_i n} h_{k-n}\right)\end{aligned}$$

$$
\begin{aligned}
&= \sum_i \alpha_i \left( \sum_n e^{j\omega_i(k-n)} h_n \right) \\
&= \sum_i \alpha_i \left( \sum_n e^{-j\omega_i n} h_n \right) e^{j\omega_i k} \\
&= \sum_i \alpha_i H(e^{-j\omega_i}) e^{j\omega_i k}
\end{aligned}
$$

For $i = 0$, the contribution to the sum is

$$\alpha_0 H(1) = \lambda_0 H(1).$$

For $i \neq 0$, the contribution is

$$\sum_{i \neq 0} \alpha_i H(e^{-j\omega_i}) e^{j\omega_i k} = \Re \sum_{i=1}^{I} \lambda_i e^{-j\varphi_i} H(e^{-j\omega_i}) e^{j\omega_i k}$$

The real part of such a term

$$\lambda_i e^{-j\varphi_i} H(e^{-j\omega_i}) = \lambda_i |H(e^{-j\omega_i})| \exp\{j[\omega_i k - \varphi_i + \arg H(e^{-j\omega_i})]\}$$

is

$$\lambda_i |H(e^{-j\omega_i})| \cos[\omega_i k - \varphi_i + \arg H(e^{-j\omega_i})]$$

This gives the required result. ◇

**Exercise 1.5:** This is a simple matter to verify since

$$R_{-n} = \sum_k u_{k-n} \bar{u}_k = \sum_k u_k \bar{u}_{k+n} = \bar{R}_n$$

gives the result. ◇

**Exercise 2.1:** Note that

$$
\begin{aligned}
R_{-n} &= \mathsf{E}[s_{-n} \bar{s}_0] \\
&= \int_\Sigma s_{-n}(\sigma) \overline{s_0(\sigma)} dm(\sigma) \\
&= \int_\Sigma s_{-n+l}(\sigma) \overline{s_k(\sigma)} dm(\sigma), \qquad \forall k
\end{aligned}
$$

which for $k = m$ becomes

$$
\begin{aligned}
R_{-n} &= \frac{\int_\Sigma s_0(\sigma) \overline{s_n(\sigma)} dm(\sigma)}{\int_\Sigma s_n(\sigma) \overline{s_0(\sigma)} dm(\sigma)} \\
&= \bar{R}_n
\end{aligned}
$$

The quadratic forms are positive semi-definite because

$$
\begin{aligned}
0 \leq \mathsf{E}[|\sum_{i=0}^{n} c_i s_i|^2] &= \sum_{i,j=0}^{n} c_i \bar{c}_j \mathsf{E}[s_i \bar{s}_j] \\
&= \sum_{i,j} c_i \bar{c}_j R_{i-j}
\end{aligned}
$$

for any choice of the $c_i$, $i = 0, \ldots, n$. ◇

**Exercise 3.1:** If $a$ is the solution of the normal equations, then $Sa$ is the vector in the column space of $S$ which is closest to $-s$. It is the projection of $-s$ onto the column space of $S$. The projecting vector has length $\|s + Sa\|$ which satisfies because of the orthogonality of the projection

$$\begin{aligned} \|s + Sa\|^2 &= \|s\|^2 - \|Sa\|^2 \\ &= s^H s - a^H S^H S a \\ &= s^H s + s^H S a \\ &= \sum_n |s_n|^2 + \sum_{k=1}^{p} a_k \left( \sum_n \bar{s}_n s_{n-k} \right) \end{aligned}$$

as was claimed. ◇

**Exercise 4.1:** The impulse response is the signal

$$s'_n = \begin{cases} 0, & n < 0, \\ G, & n = 0, \\ -\sum_{k=1}^{p} a_k s'_{n-k}, & n > 0. \end{cases}$$

The relation for $n > 0$ can be written as

$$S'[a_1 \ \ldots \ a_p]^T = -s'$$

where

$$S' = \begin{bmatrix} s'_0 & & \\ s'_1 & s'_0 & \\ s'_2 & s'_1 & \ddots \\ \vdots & \vdots & \end{bmatrix} \quad \text{and} \quad s' = \begin{bmatrix} s'_1 \\ s'_2 \\ s'_3 \\ \vdots \end{bmatrix}$$

Multiplying with $S'^H$ gives

$$S'^H S'[a_1 \ \ldots \ a_p]^T = -S'^H s' = -[R'_1 \ R'_2 \ R'_3 \ldots]^T$$

while

$$S'^H S' = [R'_{i-k}] \, .$$

This proves the exercise for $i \geq 1$.
For $i = 0$, note that

$$\begin{aligned} R'_0 &= |s'_0|^2 + \sum_{n=1}^{\infty} |s'_n|^2 = G^2 + s'^H s' \\ &= G^2 - s'^H S'[a_1 \ \ldots \ a_p]^T \\ &= G^2 - \sum_{k=1}^{p} a_k R'_{-k} \end{aligned}$$

◇

**Exercise 4.2:** Denote by $T_p$ and $T'_p$ the Toeplitz matrices

$$T_p = [R_{i-j}]_{i,j=0,\ldots,p} \quad \text{and} \quad T'_p A = [R'_{i-j}]_{i,j=0,\ldots,p}$$

then we know that $A = [1\ a_1\ \ldots\ a_p]^T$ satisfies

$$T_p A_p = [G^2\ 0\ \ldots\ 0]^T$$

and thus is $G^{-2}A_p$ the first column of $(T_p)^{-1}$. Similarly, $G^{-2}A_p$ is also the first column of $(T_p')^{-1}$. Thus we have to prove that two hermitian Toeplitz matrices with the same diagonal element and whose first column of the inverse is the same will coincide.

This is difficult to prove with the knowledge accumulated so far in these notes. An easier proof follows from results to be given later. For example it follows from the Jury test algorithm (Section 11) that $\varphi_p$ and thus also $A_p$ defines all the previous predictors $A_k$ and all the previous reflection coefficients $\rho_k$ for $k = 1, \ldots, p$ uniquely. Thus if $\rho_1 = R_1/R_0 = \rho_1' = R_1'/R_0'$ and $R_0 = R_0'$, it follows that $R_1 = R_1'$. From the equality of $\rho_2 = \rho_2'$ follows that $R_2 = R_2'$ etc. ◇

**Exercise 6.1:** Suppose $\varphi_0, \varphi_1, \ldots, \varphi_n$ forms an orthogonal basis of monic polynomials. We may thus write

$$p(z) = \varphi_n(z) + a_{n-1}\varphi_{n-1}(z) + \cdots + a_0$$

We look for

$$\min\{\|p\|^2 : a_0, \ldots, a_{n-1}\}.$$

Because

$$\|p\|^2 = \|\varphi\|^2 + \sum_{k=0}^{n-1} a_k^2 \|\varphi_k\|^2,$$

we minimize this by taking $a_0 = a_1 = \cdots = a_{n-1} = 0$. ◇

**Exercise 7.1:** The algorithm is correct for $p = 0$ since

$$T_0 A_0 = [R_0][1] = R_0 = E_0.$$

Suppose

$$T_i A_i = E_i e_0 \quad i = 0, 1, \ldots, k-1,$$

we have to prove that

$$T_k A_k = E_k e_0.$$

Note that $T_{k-1}A_{k-1} = E_{k-1}e_0$ implies

$$\begin{aligned} T_{k-1}A_{k-1} &= E_{k-1}e_0 \\ &= \tilde{I}T_{k-1}\tilde{I}\tilde{I}A_{k-1} = [0\ \ldots\ 0\ E_{k-1}]^T \\ &= T_{k-1}^T \tilde{I} A_{k-1} = [0\ \ldots\ 0\ E_{k-1}]^T \end{aligned}$$

Taking complex conjugates ($E_{k-1}$ is real and $T_{k-1} = T_{k-1}^H$)

$$[0\ \ldots\ 0\ E_{k-1}]^T = T_{k-1}^H A_{k-1}^* = T_{k-1}A_{k-1}^*.$$

Thus we get

$$\begin{aligned} T_k A_k &= T_k\left(\begin{bmatrix} A_{k-1} \\ 0 \end{bmatrix} - \rho_k \begin{bmatrix} 0 \\ A_{k-1}^* \end{bmatrix}\right) = T_k \begin{bmatrix} A_{k-1} \\ 0 \end{bmatrix} - \rho_k T_k \begin{bmatrix} 0 \\ A_{k-1}^* \end{bmatrix} \\ &= [E_{k-1}\ 0\ \ldots\ 0\ \eta_{k-1}]^T - \rho_k[\bar{\eta}_{k-1}\ 0\ \ldots\ 0\ E_{k-1}]^T \\ &= [E_k\ 0\ \ldots\ 0]^T, \end{aligned}$$

which proves the induction step. ◇

**Exercise 7.2:** By $M$ we represent the number of multiplications/divisions and by $A$ the number of additions/subtractions.

| | M | A |
|---|---|---|
| compute $\rho_i$ | $i+1$ | $i-1$ |
| compute $A_i$ | $i$ | $i-1$ |
| compute $E_i$ | 2 | 1 |
| total step $i$ | $2i+3$ | $2i-1$ |
| total for $p$ steps | $(p+4)p$ | $p^2$ |

◇

**Exercise 7.3:** Multiply

$$A_i = [A_{i-1} \;\; 0] - \rho_i [0 \;\; A^*_{i-1}]$$

with $Z = [1 \; z \; \ldots \; z^i]^T$, noting that

$$A_i(z) = A_i Z \quad \text{and} \quad A^*_i(z) = A^*_i Z$$

gives

$$A_i(z) = A_{i-1}(z) - \rho_i z A^*_{i-1}(z)$$

Replace $z$ by $1/\bar{z}$, take complex conjugates and multiply with $z^i$ and you get

$$A^*_i(z) = z A^*_{i-1}(z) - \bar{\rho}_i A_{i-1}(z)$$

which gives the result. ◇

**Exercise 7.4:** We have

$$\begin{aligned} e_n(i) &= e_n(i-1) - \rho_i z f_n(i-1) \\ f_n(i) &= -\bar{\rho}_i e_n(i-1) + z f_n(i-1) \end{aligned}$$

Writing down for the adders of the circuit that what goes out of the adders is equal to the sum of what comes in, we find the previous relations again. ◇

**Exercise 8.1:** We have

$$\begin{aligned} \tilde{f}'_n(i+1) &= c\tilde{f}_n(i) - \bar{s}\tilde{e}'_n(i+1) \\ \tilde{e}_n(i) &= s\tilde{f}_n(i) + c\tilde{e}'_n(i+1) \end{aligned}$$

Solving the second equation for $\tilde{e}'_n(i+1)$ gives

$$\tilde{e}'_n(i+1) = \frac{1}{c}[1 \quad -s] \begin{bmatrix} \tilde{e}_n(i) \\ \tilde{f}_n(i) \end{bmatrix}.$$

Filling this into the first one gives

$$\begin{aligned} \tilde{f}'_n(i+1) &= \frac{|s|^2}{c}\tilde{f}_n(i) - \frac{\bar{s}}{c}\tilde{e}_n(i) + c\tilde{f}_n(i) \\ &= \frac{1}{c}(1 - c^2 + c^2)\tilde{f}_n(i) - \frac{\bar{s}}{c}\tilde{e}_n(i) \\ &= \frac{1}{c}[-\bar{s} \;\; 1] \begin{bmatrix} \tilde{e}_n(i) \\ \tilde{f}_n(i) \end{bmatrix}. \end{aligned}$$

and this gives the result. ◇

**Exercise 9.1:** Let

$$\varphi_n(z) = \bar{a}_0 \prod_{k=1}^{n} (z - \alpha_k).$$

Then

$$\begin{aligned} \varphi_n^*(1/\bar{z}) &= a_0 \prod_{k=1}^{n} (1/z - \bar{\alpha}_k) z^n \\ &= a_0 \prod_{k=1}^{n} (1 - \bar{\alpha}_k z) \end{aligned}$$

which proves the result. ◇

**Exercise 9.2:** From

$$(1 - |\rho_n|^2) A_{n-1}(z) = A_n(z) + \rho_n A_n^*(z)$$

we get for $|\rho_n| = 1$ that

$$A_n(z) = -\rho_n A_n^*(z) \quad \text{or} \quad |A_n(z)| = |A_n^*(z)|$$

If $\alpha$ is a zero of $A_n(z)$, then $A_n^*(\alpha)$ should be zero too. None of these zeros can be in the open unit disk because $|\alpha| < 1$ and $0 = A_n^*(\alpha) = \alpha A_{n-1}^*(\alpha) - \rho_n A_{n-1}(\alpha)$ implies $A_{n-1}(\alpha)/A_{n-1}^*(\alpha) = \rho_n/\alpha$. Since $|\rho_n| = 1$, we have found a point $|\alpha| < 1$ where $|A_{n-1}/A_{n-1}^*| > 1$ and this is impossible because all the $|\rho_i| < 1$ for $i = 1, \ldots, n-1$ and thus is $|A_{n-1}/A_{n-1}^*|$ bounded by 1 inside the unit disk as we have seen before (see theorem 9.2 and lemma 9.1). ◇

**Exercise 10.1:** For $k = 0, 1, \ldots, n-1$ we have

$$\begin{aligned} 0 = < \varphi_n, z^k > &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi_n(z) z^{-k} P(z) d\omega, \quad z = e^{j\omega} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{\varphi_n^*(z) z^{k-n}} P(z) d\omega, \quad z = e^{j\omega} \\ &= \overline{< \varphi_n^*, z^{n-k} >} \end{aligned}$$

which shows the orthogonality. ◇

**Exercise 10.2:** This follows from

$$\varphi_i(z) = [1 \; z \; \ldots \; z^i] A_i^* \quad \text{and} \quad \varphi_i^*(z) = [1 \; z \; \ldots \; z^i] A_i$$

and the fact that

$$\eta_{i-1} = [R_i \; \ldots \; R_1] A_{i-1} = < \varphi_{i-1}^*, z^i >$$

Because $\varphi_{i-1}^* \perp \text{span}\{z, z^2, \ldots, z^{i-1}\}$, is

$$\begin{aligned} < \varphi_{i-1}^*, z\varphi_{i-1} > &= < \varphi_{i-1}^*, z^i > + \sum_{k=1}^{i-1} c_k < \varphi_{i-1}^*, z^k > \\ &= < \varphi_{i-1}^*, z^i > \end{aligned}$$

◇

**Exercise 10.3:** Denote

$$\varphi_n(z) = \sum_{k=0}^{n} f_{kn} z^k, \quad f_{nn} = 1$$

Then

$$\begin{aligned} < \varphi_k, \varphi_l > &= [f_{0k} \ \dots \ f_{kk} \ 0 \dots 0] \begin{bmatrix} & < z^i, z^j > & \end{bmatrix} [f_{0l} \ \dots \ f_{ll} \ 0 \dots 0]^H \\ &= \delta_{kl} E_k = \delta_{kl} \|\varphi_k\|^2 \end{aligned}$$

Writing this down for $k, l = 0, 1, \dots, n$ and noting that

$$< z^i, z^j > = < 1, z^{j-i} > = < z^{i-j}, 1 > = R_{j-i}$$

we get

$$\begin{bmatrix} f_{00} & & & \\ f_{01} & f_{11} & & \\ \vdots & \vdots & \ddots & \\ f_{0n} & f_{1n} & \dots & f_{nn} \end{bmatrix} \begin{bmatrix} R_0 & R_1 & \dots & R_n \\ \bar{R}_1 & R_0 & \ddots & \\ \vdots & \ddots & \ddots & \vdots \\ \bar{R}_n & & \dots & R_0 \end{bmatrix} \begin{bmatrix} \bar{f}_{00} & \bar{f}_{01} & \dots & \bar{f}_{0n} \\ & \bar{f}_{11} & \dots & \bar{f}_{1n} \\ & & \ddots & \vdots \\ & & & \bar{f}_{nn} \end{bmatrix} = \operatorname{diag}(E_0, \dots, E_n)$$

Because $f_{kk} = 1$, $k = 0, \dots, n$, the determinants of the triangular matrices are 1. Hence, denoting the central Toeplitz matrix by $T_n^T$, we get

$$\det T_n^T = \det T_n = E_0 E_1 \dots E_n$$

which implies $E_n = \det T_n / \det T_{n-1}$. ◇

**Exercise 10.4:** The proof for $n = 0$ is trivial. We suppose $n > 0$.

We shall prove the relation for the normalized polynomials $\Phi_n = \varphi_n / \|\varphi_n\|$. Because $\|\varphi_n\|^2 = \|\varphi_n^*\|^2 = E_n$, we have to prove first

$$\sum_{k=0}^{n} \Phi_k(x) \overline{\Phi_k(y)} = \frac{\Phi_n^*(x) \overline{\Phi_n^*(y)} - x\bar{y} \Phi_n(x) \overline{\Phi_n(y)}}{1 - x\bar{y}}$$

Denote the right hand side of this relation by $l_n$, then we have to prove that

$$l_n(x, y) - l_{n-1}(x, y) = \Phi_n(x) \overline{\Phi_n(y)}$$

The difference in the left hand side is of the form $N/(1 - x\bar{y})$ with

$$\begin{aligned} N(x, y) &= (\Phi_n^*(x) \overline{\Phi_n^*(y)} - x\bar{y} \Phi_n(x) \overline{\Phi_n(y)}) \\ &\quad - (\Phi_{n-1}^*(x) \overline{\Phi_{n-1}^*(y)} - x\bar{y} \Phi_{n-1}(x) \overline{\Phi_{n-1}(y)}) \end{aligned}$$

which should equal $(1 - x\bar{y}) \Phi_n(x) \overline{\Phi_n(y)}$. Use now

$$\begin{bmatrix} \Phi_n(x) \\ \Phi_n^*(x) \end{bmatrix} = \sqrt{\frac{E_{n-1}}{E_n}} \begin{bmatrix} 1 & -\bar{\rho}_n \\ -\rho_n & 1 \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Phi_{n-1}(x) \\ \Phi_{n-1}^*(x) \end{bmatrix}$$

to get

$$\begin{bmatrix} \overline{\Phi_n(y)} & \overline{\Phi_n^*(y)} \end{bmatrix} \begin{bmatrix} \Phi_n(x) \\ \Phi_n^*(x) \end{bmatrix} = \frac{E_{n-1}}{E_n} (1 - |\rho_n|^2) \begin{bmatrix} \overline{\Phi_{n-1}(y)} & \overline{\Phi_{n-1}^*(y)} \end{bmatrix} \begin{bmatrix} x\bar{y} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Phi_{n-1}(x) \\ \Phi_{n-1}^*(x) \end{bmatrix}$$

or

$$\Phi_n^*(x)\overline{\Phi_n^*(y)} - \Phi_n(x)\overline{\Phi_n(y)} = \Phi_{n-1}^*(x)\overline{\Phi_{n-1}^*(y)} - x\bar{y}\Phi_{n-1}(x)\overline{\Phi_{n-1}(y)}$$

so that indeed $N = (1 - x\bar{y})\Phi_n(x)\overline{\Phi_n(y)}$.

The second relation can be proved similarly. ◇

**Exercise 10.5:** This is trivial for $n = 0$. For $n > 1$, we note the following. Because $\varphi_n(t) - \varphi_n(z)$, $(t = e^{j\omega})$ is zero for $z = t$, this is divisible by $t - z$. Therefore

$$\frac{t+z}{t-z}[\varphi_n(t) - \varphi_n(z)]$$

is a polynomial in $z$ of degree $n$. ◇

**Exercise 10.6:** For $n = 0$, this is trivial since $\varphi_0 = 1$ and $\psi_0 = R_0$. For $n > 0$, we note that for $0 \leq k < n$, $t = e^{j\omega}$ and $D(t,z) = (t+z)/(t-z)$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D(t,z)\varphi_n(z)\left[\frac{1}{z^k} - \frac{1}{t^k}\right]P(t)d\omega = 0$$

because it is of the form

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{p(t)}\varphi_n(t)P(t)d\omega$$

where $p(t)$ is a polynomial of degree at most $k$ and since $\varphi_n$ is orthogonal to all polynomials of degree $< n$, this is zero. Hence

$$\begin{aligned}\frac{\psi_n(z)}{z^k} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{D(t,z)}{z^k}[\varphi_n(t) - \varphi_n(z)]P(t)d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} D(t,z)\left[\frac{\varphi_n(t)}{t^k} - \frac{\varphi_n(z)}{z^k}\right]P(t)d\omega\end{aligned}$$

For $k \leq n-1$, this relation becomes

$$\frac{\psi_n(z)}{z^k} = \frac{1}{2\pi}\int_{-\pi}^{\pi} D(t,z)\left[\frac{\varphi_n(t)}{t^k} - \frac{\varphi_n(z)}{z^k}\right]P(t)d\omega$$

Thus is for $k = n-1$

$$\frac{\varphi_n\Omega + \psi_n}{z^{n-1}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} D(t,z)\frac{\varphi_n(t)}{t^{n-1}}P(t)d\omega.$$

This is analytic in $|z| < 1$ as a Cauchy integral and for $z = 0$, it equals

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi_n(t)\overline{t^{n-1}}P(t)d\omega$$

which is 0 because $\varphi_n \perp t^{n-1}$. ◇

The second relation is proved similarly. One should now use for $n > 0$ and $0 \leq k < n$

$$\frac{\psi_n^*(z)}{z^{n-k}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} D(z,t)\left[\frac{\varphi_n^*(t)}{t^{n-k}} - \frac{\varphi_n^*(z)}{z^{n-k}}\right]P(t)d\omega$$

(note $D(z,t) = -D(t,z)$).

**Exercise 10.7:** We prove this first for $n = 1$. Because $\varphi_1(z) = z - \bar{\rho}_1$ we get

$$\begin{aligned}\psi_1(z) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} D(t,z)[\varphi_1(t) - \varphi_1(z)]P(t)d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(t+z)P(t)d\omega\end{aligned}$$

Furthermore, because

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi_1(t)P(t)d\omega = 0 \quad \text{and} \quad \frac{1}{2\pi}\int_{-\pi}^{\pi}P(t)d\omega = R_0$$

it follows that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}tP(t)d\omega = -\bar{\rho}_1 R_0$$

which gives

$$\psi_1(z) = (z - \bar{\rho}_1)R_0.$$

For $n > 1$, use

$$\begin{bmatrix} \psi_{n-1}(z) \\ -\psi^*_{n-1}(z) \end{bmatrix} = -\Omega(z)\begin{bmatrix} \varphi_{n-1}(z) \\ \varphi^*_{n-1}(z) \end{bmatrix} + \frac{1}{2\pi}\int_{-\pi}^{\pi} D(t,z)\begin{bmatrix} \varphi_{n-1}(t) \\ \frac{z}{t}\varphi^*_{n-1}(t) \end{bmatrix} P(t)d\omega.$$

Multiply from the left with $[z \quad -\bar{\rho}_n]$ to find that the right hand side is

$$\begin{aligned} & -\Omega(z)\varphi_n(z) + z\frac{1}{2\pi}\int_{-\pi}^{\pi} D(t,z)\left[\varphi_{n-1}(t) - \frac{\bar{\rho}_n}{t}\varphi^*_{n-1}(t)\right]P(t)d\omega \\ = \quad & -\Omega(z)\varphi_n(z) + z\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{D(t,z)}{t}\varphi_n(t)P(t)d\omega \\ = \quad & -\frac{1}{2\pi}\int_{-\pi}^{\pi} D(t,z)\left[\varphi_n(z) - \frac{z}{t}\varphi_n(t)\right]P(t)d\omega \\ = \quad & \psi_n(z) \end{aligned}$$

The last step follows from the previous exercise. ◇

**Exercise 10.8:** This just couples the relations

$$\begin{bmatrix} \varphi_n(z) \\ \varphi^*_n(z) \end{bmatrix} = \theta_n \begin{bmatrix} \varphi_{n-1}(z) \\ \varphi^*_{n-1}(z) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \psi_n(z) \\ -\psi^*_n(z) \end{bmatrix} = \theta_n \begin{bmatrix} \psi_{n-1}(z) \\ -\psi^*_{n-1}(z) \end{bmatrix}$$

◇

**Exercise 10.9:** Taking determinants of the previous relation gives the result. ◇

**Exercise 10.10:** From the Christoffel-Darboux relations we get

$$\frac{|\varphi^*_n(z)|^2 - |z|^2|\varphi_n(z)|^2}{E_n} = (1 - |z|^2)\sum_{k=0}^{n}\frac{|\varphi_k(z)|^2}{E_k}$$

For $|z| < 1$ is $1 - |z|^2 > 0$ and because $E_n$ and the right hand side are positive, we get for $|z| < 1$

$$|\varphi^*_n|^2 > |z|^2|\varphi_n(z)|^2 \geq 0$$

which proves the result. ◇

**Exercise 11.1:** Because

$$\varphi_n(z) = z\varphi_{n-1}(z) - \bar{\rho}_n\varphi^*_{n-1}(z)$$

the $\varphi_n(z)$ are monic and $\varphi_n(0) = -\bar{\rho}_n$. Because also

$$\varphi^*_n(z) = \varphi^*_{n-1}(z) - \rho_n z\varphi_{n-1}(z) \quad \Rightarrow \quad \varphi^*_{n-1}(z) = \varphi^*_n(z) + \rho_n z\varphi_{n-1}(z)$$

we see that

$$\begin{aligned}\varphi_{n-1}(z) &= \frac{1}{z}[\varphi_n(z) + \bar{\rho}_n \varphi_{n-1}^*(z)] \\ &= \frac{1}{z}[\varphi_n(z) + \bar{\rho}_n \varphi_n^*(z)] + |\rho_n|^2 \varphi_{n-1}(z)\end{aligned}$$

Hence

$$(1 - |\rho_n|^2)\varphi_{n-1}(z) = \frac{1}{z}[\varphi_n(z) + \bar{\rho}_n \varphi_n^*(z)]$$

which proves the algorithm. ◇

**Exercise 12.1:** The $(n-k)$th column of $U_n$ consists of the coefficients of $\varphi_{n-k}$. The multiplication from the left with $\tilde{I}$ and the complex conjugation turns this into the coefficients of $\varphi_{n-k}^*$ and the multiplication from the right with $\tilde{I}$ moves this to the $k$th column. ◇

**Exercise 13.1:** You can easily verify that

$$\frac{\Delta_{2,k+1}}{\Delta_{1,k+1}} = \frac{-\Delta_{1k}\rho_{k+1} + \Delta_{2k}}{z(\Delta_{1k} - \bar{\rho}_{k+1}\Delta_{2k})} = \frac{1}{z}\frac{\frac{\Delta_{2k}}{\Delta_{1k}} - \rho_{k+1}}{1 - \bar{\rho}_{k+1}\frac{\Delta_{2k}}{\Delta_{1k}}}$$

◇

**Exercise 13.2:** Let $\bowtie$ be one of the relational symbols $<$, $=$ or $>$. Then if $|z| \bowtie 1$

$$\begin{aligned}|z - \alpha|^2 &= |z|^2 + |\alpha|^2 - 2\Re\bar{\alpha}z \\ &\bowtie 1 + |\alpha|^2 - 2\Re\bar{\alpha}z \\ &= |1 - \bar{\alpha}z|^2\end{aligned}$$

Thus

$$\left|\frac{1-\alpha}{1-\bar{\alpha}z}\right| \bowtie 1$$

It is clearly injective in $\hat{\mathbb{C}}$ and because the inverse transformation is of the same type (just replace $\alpha$ by $-\alpha$ to get the inverse transformation), it is a bijection. ◇

**Exercise 14.1:** This follows from partial fraction decomposition. ◇

**Exercise 14.2:** Note that

$$B_n(z) = \frac{z^n\overline{\pi_n(1/\bar{z})}}{\pi_n(z)} = B_n(z) = \frac{\pi_n^*(z)}{\pi_n(z)}$$

so that

$$D_n^*(z) = B_n(z)\frac{A_n^*(z)}{\pi_n^*(z)} = \frac{A_n^*(z)}{\pi_n(z)} = \varphi_n(z)$$

as claimed. ◇

**Exercise 14.3:** A function $f \in \mathcal{H}$ can be represented as

$$f(z) = \sum_k a_k \varphi_k(z), \quad a_k = \frac{< f(z), \varphi_k(z) >}{\|\varphi_k\|^2}$$

Hence

$$\begin{aligned} < f(z), \sum_k \frac{\varphi_k(z)\overline{\varphi_k(w)}}{\|\varphi_k\|^2} > &= \sum_k \frac{< f(z), \varphi_k(z) >}{\|\varphi_k\|^2} \varphi_k(w) \\ &= \sum_k a_k \varphi_k(z) = f(w). \end{aligned}$$

This shows the reproducing property. ◇

**Exercise 15.1:** If $\bar{V}_1, \ldots, \bar{V}_n$ is an orthonormal basis for span $V$, then the projection of $a$ onto span $V$ is given by

$$a_V = a - \sum_{k=1}^{n} < a, \bar{V}_k > \bar{V}_k$$

which is written in short hand as

$$a_V = a - < a, \bar{V} > \bar{V} \quad \text{where} \quad \bar{V} = \begin{bmatrix} \bar{V}_1 \\ \vdots \\ \bar{V}_n \end{bmatrix}.$$

This implies

$$\begin{aligned} a_V &= \|a\|(\bar{a} - < \bar{a}, \bar{V} > \bar{V}), \quad \bar{a} = \|a\|^{-1} a \\ &= \|a\|(\bar{a} - \rho(a, V)\bar{V}) \end{aligned}$$

Hence

$$\begin{aligned} \|a_V\|^2 &= \|a\| < (\bar{a} - \rho(a, V)\bar{V}), (\bar{a} - \rho(a, V)\bar{V})\|a\|^H \\ &= \|a\|[\|\bar{a}\|^2 + \rho(a, V) < \bar{V}, \bar{V} > \rho(a, V)^H \\ &\qquad - < \bar{a}, \bar{V} > \rho(a, V)^H - \rho(a, V) < \bar{V}, \bar{a} >]\|a\|^H \\ &= \|a\|[1 - \rho(a, V)\rho(a, V)^H]\|a\|^H \end{aligned}$$

which gives immediately the formula for $\bar{a}_V$. ◇

**Exercise 15.2:** Because of the orthogonal decomposition

$$\text{span}\{U, V\} = \text{span}\, U \oplus \text{span}\, V_U$$

we get $a_{\{U,V\}} = (a_U)_{V_U}$. Hence

$$\begin{aligned} \rho_{\{U,V\}}(a, b) &= \rho(a_{\{U,V\}}, b_{\{U,V\}}) \\ &= \rho((a_U)_{V_U}, (b_U)_{V_U}) \\ &= \rho_{V_U}(a_U, b_U) \\ &= F(\rho(a_U, b_U), \rho(a_U, V_U), \rho(b_U, V_U)) \\ &= F(\rho_U(a, b), \rho_U(a, V), \rho_U(b, V)) \end{aligned}$$

◇

**Exercise 16.1:** We just note

$$\frac{1}{R_0} \begin{bmatrix} R_0 & 0 \\ R_1 & R_1 \\ \vdots & \vdots \\ R_n & R_n \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} R_0 & R_{-1} & \ldots & R_{-n} \\ 0 & R_{-1} & \ldots & R_{-n} \end{bmatrix}$$

$$= \frac{1}{R_0} \begin{bmatrix} R_0 & 0 \\ R_1 & -R_1 \\ \vdots & \vdots \\ R_n & -R_n \end{bmatrix} \begin{bmatrix} R_0 & R_{-1} & \dots & R_{-n} \\ 0 & R_{-1} & \dots & R_{-n} \end{bmatrix}$$

$$= \begin{bmatrix} R_0 & R_{-1} & \dots & R_{-n} \\ R_1 & & & \\ \vdots & & 0 & \\ R_n & & & \end{bmatrix} = \Delta R$$

◇

**Exercise 18.1:** Since

$$R = \begin{bmatrix} R_{00} \\ R_{10} \\ \vdots \end{bmatrix} R_{00}^{-1} [R_{00} \quad R_{01} \quad \dots] + \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \Delta_1 & \\ 0 & & \end{bmatrix}$$

we can multiply from the left with $[1 \; z \; z^2 \; \dots]$ and from the right with $[1 \; w \; w^2 \; \dots]^H$ and we get

$$R(z,w) = R(z,0)R(0,0)^{-1}R(0,w) + z\bar{w}R_1(z,w)$$

which gives the result. ◇

**Exercise 19.1:** Note that

$$\begin{bmatrix} 1 & -KJ \\ -K^H & I \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -J \end{bmatrix} \begin{bmatrix} 1 & -K \\ -JK^H & I \end{bmatrix} = \begin{bmatrix} 1 - KJK^H & 0 \\ 0 & K^HK - J \end{bmatrix}$$

The other factors arrange for a normalization. ◇

**Exercise 19.2:** The inverse of a $\Sigma$-unitary matrix is easily computed by $\theta^{-1} = \Sigma\theta^H\Sigma$ and needs no flops. The normalization by the square roots is not necessary and can be disregarded. Thus the only flops needed in step $i$ are in the computation of $\tilde{G}_i\theta^{-1}(K_i, J)$. This requires $O(\alpha(n-i))$ flops. Since we should repeat this for $i = 1, \dots, n$, we get a total of $O(\alpha n^2)$ flops. ◇

**Exercise 19.3:** We have

$$R_1(0,0) = G_1(0)\Sigma G_1(0)^H = [\sigma_1 \; 0 \; \dots \; 0][\sigma_1 \; 0 \; \dots \; 0]^H = \sigma_1^2$$

Because

$$\sigma_0^2 = \sigma_1^2(1 - K_1JK_1^H)^{-1} \quad \Rightarrow \quad \sigma_1^2 = \sigma_0^2(1 - K_1JK_1^H)$$

Now is $K_1 = -\sigma_0^{-1}\Gamma_1 J$, so that

$$K_1JK_1^H = \sigma_0^{-2}\Gamma_1 J\Gamma_1^H$$

which gives indeed $\sigma_1^2 = \sigma_0^2 - \Gamma_1 J\Gamma_1^H$. ◇

**Exercise 21.1:** Consider the Cholesky decomposition

$$R_0 = L_nD_nL_n^H \quad \text{where} \quad L_n = [l_0 \; \dots \; l_n]$$

is a lower unit triangular matrix and $D_n$ is diagonal with elements $d_i = R_i(0,0)$. Since $d_n = R_n(0,0) = \sigma_n^2$, we only have to prove that the $l_i(z)$, defined by

$$l_i(z) = z^i R_i(z,0)d_i^{-1}$$

are polynomials of degree $n-1$, which means that $L_n$ has the form

$$\begin{bmatrix} 1 & & & & \\ \times & 1 & & & \\ \vdots & \vdots & \ddots & & \\ \times & \times & & 1 & \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

Because

$$l_i(z) = z^i H_i(z) \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = z^i [\sigma_n \; 0 \; \dots \; 0] \theta_n \theta_{n-1} \dots \theta_{i+1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

these are indeed polynomials of degree $n-1$. ◇

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