Conditional Sampling for Option Pricing under the LT Method

Nico Achtsis,
Dept. of Computer Science, K.U.Leuven, Belgium

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Outline

- The Linear Transformation (LT) algorithm.
- Conditional sampling:
  - MC+CS,
  - LT+CS.
- Results.
- Extensions:
  - Rootfinding,
  - Levy market models.

Assumptions

We assume (for now):

- n assets, m time steps;
- Black-Scholes dynamics, i.e.,

$$S_i(t_j) = S_i(0)e^{(r-\sigma_i^2/2)t_j + Z_j^i},$$

  where Z_j^i and Z_ℓ^k have covariance $\rho_{ik}\sigma_i\sigma_k\sqrt{|t_j - t_\ell|}$.
- A European payoff represented as

$$V(S_1(t_1), \ldots, S_n(t_m)) = \max\left[g(S_1(t_1), \ldots, S_n(t_m)), 0\right].$$

Covariance matrix

- Under the LT method g is simulated from

$$\mathbf{Z} = A\mathbf{z} = CQ\mathbf{z}$$

where CC' is the Cholesky decomposition of Σ and
Q is a carefully chosen according to the following optimization problem:

$$\begin{aligned} \underset{Q_{\cdot k} \in \mathbb{R}^{mn}}{\text{maximize}} \quad & \text{variance contribution of } g \text{ due to } k\text{th dimension} \\ \text{subject to} \quad & Q^T Q = 1. \end{aligned}$$

- Note that introducing Q would not change the behaviour of ordinary Monte Carlo, but it makes a lot of difference to quasi-Monte Carlo methods.

LT algorithm

Imai & Tan (2006) proposed to approximate the objective function by linearizing it using a first-order Taylor expansion around a point $\mathbf{z} = \hat{\mathbf{z}} + \Delta\mathbf{z}$,

$$g(\mathbf{z}) \approx g(\hat{\mathbf{z}}) + \sum_{\ell=1}^{mn} \left.\frac{\partial g}{\partial z_\ell}\right|_{\mathbf{z}=\hat{\mathbf{z}}} \Delta z_\ell.$$

Using this expansion, the variance contributed to the kth component is

$$\left(\left.\frac{\partial g}{\partial z_k}\right|_{\mathbf{z}=\hat{\mathbf{z}}} \right)^2.$$

The expansion points are chosen as $\hat{\mathbf{z}}_k = (1, \ldots, 1, 0, \ldots, 0)$, the vector with $k-1$ leading ones. The optimization problem becomes

$$\begin{aligned} \underset{Q_{\cdot k} \in \mathbb{R}^{mn}}{\text{maximize}} \quad & \left(\left.\frac{\partial g}{\partial z_\ell}\right|_{\mathbf{z}=\hat{\mathbf{z}}_k} \right)^2 \\ \text{subject to} \quad & Q^T Q = 1. \end{aligned}$$

Problem outline

- Suppose we are interested in pricing an up-&-out option with payoff

$$g(S_1(t_1), \ldots, S_n(t_m)) = f(S_1(t_1), \ldots, S_n(t_m))\, \mathbb{I}\left\{ \max_j S_1(t_j) < B \right\}.$$

- When the probability of survival becomes low (e.g., if $B - S_1(t_0)$ is small) a lot of the generated paths will result in a knock-out.
- This can make the variance among all paths quite large relative to the price.
- Remedy this problem by using conditional sampling.

MC+CS

- Suppose we are interested in pricing an up-&-out option with payoff

$$g(S_1(t_1), \ldots, S_n(t_m)) = f(S_1(t_1), \ldots, S_n(t_m)) \, \mathbb{I}\left\{ \max_j S_1(t_j) < B \right\}.$$

- For regular Monte Carlo Glasserman & Staum (2001) proposed the following estimator:

$$\mathbb{E}\left[e^{-rT} V\right] = \mathbb{E}\left[e^{-rT} L \max(f, 0)\right],$$

  where L is the "likelihood" of survival.
- Using this estimator there are no more paths that cross the barrier.

LT+CS

- We devised a similar construction for the LT algorithm.
- For the option to stay alive we need

$$\sum_{i=1}^{mn} a_{j,i}\Phi^{-1}(u_i) < \log(B/S_1(0)), \quad j = 1, \ldots, m.$$

- For $u_2, \ldots, u_{mn}$ fixed, the restriction on u_1 can be written as (assuming $a_{j,1} > 0$)

$$u_1 < \Phi\left(\min_j \left[\frac{\log(B/S_1(0)) - a_{j,2}\Phi^{-1}(u_2) - \ldots - a_{j,mn}\Phi^{-1}(u_{mn})}{a_{j,1}}\right]\right).$$

- Our algorithm thus samples u_1 to u_{mn}, then calculates the upper bound on u_1 using $u_2, \ldots, u_{mn}$, and afterwards rescales u_1 to satisfy the barrier condition.

Results

Consider an Asian barrier option on four assets:

$$g = \max\left(\frac{1}{4\times 130}\sum_{i=1}^{4}\sum_{j=1}^{130} S_i(t_j) - K, 0\right)\mathbb{I}\left\{\max_{j=1,\ldots,130} S_1(t_j) < B\right\}.$$

We will consider the valuation of this option under several model parameters. The fixed parameters are $S_i(0) = 100$ for $i = 1, \ldots, 4$, $\sigma_2 = \sigma_3 = 25\%$, $\sigma_4 = 35\%$, $r = 5\%$ and $T = 6$ months. The correlations are given as $\rho_{12} = -50\%$, $\rho_{13} = 60\%$, $\rho_{14} = 20\%$, $\rho_{23} = -20\%$, $\rho_{24} = -10\%$, $\rho_{34} = 25\%$.

(σ_1, B, K)	LT+CS	LT
$(0.25, 125, 100)$	1153%	517%
$(0.25, 110, 100)$	592%	325%
$(0.25, 105, 100)$	367%	275%
$(0.25, 110, 90)$	776%	343%
$(0.25, 105, 90)$	581%	300%
$(0.25, 125, 110)$	664%	375%
$(0.55, 125, 100)$	538%	333%
$(0.55, 125, 110)$	310%	234%

Table: Single barrier Asian basket. The reported numbers are the standard deviations of the MC+CS method divided by those of the QMC+LT and QMC+LT+CS methods. The MC+CS method uses 163840 samples, while the QMC+LT and QMC+LT+CS methods use 4096 samples and 40 independent shifts.

Root-finding

- We can extend the conditional sampling algorithm by incorporating root-finding as well.
- Numerically obtain the bounds such that $f > 0$.
- Sample $\mathbf{u} \in \mathbb{R}^{mn-1}$ and integrate over u_1 analytically.

(σ_1, B, K)	LT+CS+RF	LT+CS	LT
$(0.25, 125, 100)$	1952%	1153%	517%
$(0.25, 110, 100)$	1159%	592%	325%
$(0.25, 105, 100)$	757%	367%	275%
$(0.25, 110, 90)$	911%	776%	343%
$(0.25, 105, 90)$	625%	581%	300%
$(0.25, 125, 110)$	4734%	664%	375%
$(0.55, 125, 100)$	1089%	538%	333%
$(0.55, 125, 110)$	1378%	310%	234%

Table: Single barrier Asian basket. The reported numbers are the standard deviations of the MC+CS method divided by those of the QMC+LT, QMC+LT+CS and QMC+LT+CS+RF methods. The MC+CS method uses 163840 samples, while the QMC+LT and QMC+LT+CS methods use 4096 samples and 40 independent shifts.

Levy market model

- We assume

$$S(t_i) = S(0)e^{\sum_{j=1}^{i} X_j},$$

  where X_j has an infinitely divisible distribution.
- For the LT method we use that $X \sim F_X^{-1}(\mathbf{U}) \sim F_X^{-1}(\Phi(A\Phi^{-1}(\mathbf{U})))$.
- We use cubic splines to approximate F_X^{-1} (Hörmann & Leydold 2003).
- Conditional sampling gets more involved and we need to numerically approximate the bounds on u_1.
- Assuming we have an efficient way of doing this, our conditional sampling scheme does not change conceptually.

LT+CS under Levy models

- We consider the simple case of an up-&-out call option

$$g = \max\left(S(t_m) - K, 0\right) \mathbb{I}\left\{\max_{j=1,\ldots,m} S(t_j) < B\right\}.$$

We assume X follows a Meixner distribution with parameters $a = 0.3977$, $b = -1.494$ and $d = 0.3462$. Furthermore $S(0) = 100$, $r = 1.9\%$ and $q = 1.2\%$.

(m, K, B)	LT+CS+RF	LT+CS	LT
$(2, 90, 100)$	15993%	3167%	336%
$(2, 100, 105)$	18632%	1453%	455%
$(2, 110, 125)$	8744%	1432%	529%
$(4, 90, 100)$	877%	479%	173%
$(4, 100, 105)$	1962%	588%	180%
$(4, 110, 125)$	2229%	1264%	173%
$(12, 90, 100)$	245%	189%	152%
$(12, 100, 105)$	440%	194%	83%
$(12, 110, 125)$	382%	211%	127%

Table: Single barrier Asian basket. The reported numbers are the standard deviations of the MC+CS method divided by those of the QMC+LT, QMC+LT+CS and QMC+LT+CS+RF methods. The MC+CS method uses 163840 samples, while the QMC+LT and QMC+LT+CS methods use 4096 samples and 40 independent shifts.

Conclusion

- We constructed conditional (unbiased) sampling under the LT algorithm.
- Significant variance reduction is achieved.
- More research is needed for Levy market models.
- We will investigate stochastic volatility models in the near future.
- Our paper can be found on `http://arxiv.org/abs/1111.4808`