A look-ahead method to compute vector Padé-Hermite approximants

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1 Introduction

Given a matrix of power series, we shall describe an algorithm which will compute recursively the vector Padé-Hermite approximants which are located at successive perfect points on a diagonal of the vector Padé-Hermite table. In fact our algorithm does more since it computes a basis for all possible solutions that can be obtained in that point of the table.

A vector Padé-Hermite approximant with order indices $\vec{\sigma}$ for a matrix of power series $F(z) \in \mathbb{F}[[z]]^{s \times m}$ is a polynomial vector $\vec{p}(z) \in \mathbb{F}[z]^m$ such that

$$F(z)\vec{p}(z) = O_+(z^{\vec{\sigma}}).$$

When $s = 1$, $m = 2$ and $F(z) = [-1 \ f(z)]$ with $f(z)$ a scalar power series, we get a Padé approximant [1, 2]. When $s = 1$ and σ, which reduces to a scalar integer number, is the sum of the allowed degrees of $\vec{p}$ plus $(m-1)$, it is the definition of a Padé-Hermite approximant. For more details, we refer the interested reader to the introduction of [20, 8] and the references therein.

The vector Padé-Hermite table considered in this paper, is a generalization of the Padé-Hermite table [17, 13] and the classical Padé table [12]. The algorithms of [10, 18] are based on relationships between neighboring approximants in the Padé-Hermite table. However, the algorithms are only guaranteed to work when the vector of power series is *perfect* (see [13]). To overcome this restriction, we developed in [20] an algorithm computing nonperfect Padé-Hermite approximants. This algorithm is very similar to the one developed by Beckermann in [4] to solve the M-Padé approximation problem. When all the interpolation points coincide, the M-Padé approximation problem reduces to a Padé-Hermite approximation problem. Beckermann also investigated the singular structure of M-Padé table [3]. In [19] the simultaneous Padé approximation problem is solved in a similar way. The previous two types of approximation and more general matrix rational interpolation problems are all special cases of the *vector* M-Padé approximation problem. In [21], we derived basic algorithmic steps to follow any path of approximation. Beckermann and Labahn [5] introduce the concept of *power Hermite-Padé approximants* to solve different types of matrix Padé approximation problems. They even derive a superfast algorithm to accomplish this. The basic steps taken in all these algorithms are the most elementary steps one can take, i.e. adding only *one* new interpolation condition, changing only *one* degree of the interpolant. However, when using floating point arithmetic, this way of working does not (yet) allow to control the numerical stability of the algorithm. It turns out that algorithms which do bigger jumps but which require a special structure for the solution, (and which were originally not designed to overcome numerical stability problems but to overcome the nonperfectness of the given power series) have more flexibility to control the stability. Such a special structure for the solution is found in so-called *perfect* points. Hence, instead of doing elementary steps to go from one perfect point to another using some intermediate nonperfect points, these algorithms "look-ahead" from a previous perfect

point immediately to the next perfect one. Another reason to have updating formulas for bigger jumps, not necessarily with a special structure for the solution, is to construct superfast algorithms. For a successfull adaptation of these ideas to the scalar Padé approximation problem, we refer to [9]. See also [6].

The algorithm developed in this paper is such a "look-ahead" procedure. Hence, it can be seen as a first step towards a numerically (forward) stable algorithm when floating point arithmetic is used. Instead of jumping from one perfect point to the next one, the updating formulas derived here are also valid to go from one perfect point to *a* next one. Hence, we could compute successive "wel-conditioned" perfect points. However, in this paper, we will not concentrate on these stability considerations.

Our algorithm generalizes an algorithm of Cabay, Labahn and Beckermann [8], which treats a similar problem for the scalar case. Instead of taking a row vector of scalar power series, we replace each scalar power series by a column vector of power series. Hence, we get in general an $(s \times m)$ matrix of power series (with $s < m$). Cabay and Labahn also give a generalization in [7], the difference being that each scalar power series is replaced by a square matrix power series. Therefore, the number of columns of the resulting matrix of power series is a multiple of the number of rows. Hence, the problem they consider can be directly transformed into the problem solved here but not vice-versa. However, they use the recurrence relation from one perfect point to *a* next one to construct a superfast algorithm.

Besides generalizing the previous work of Cabay, Labahn and Beckermann, we wanted to develop the updating formulas to go from one perfect point to another as a special case of a very general and simple updating formula (allowing to construct superfast algorithms where the intermediate points don't need to be perfect). In spite of the simple general updating formula, it turns out that when special structure conditions are imposed, this formula becomes more detailed and complex.

In section 7, we also relate the problem to matrix Padé approximation and give the connection with [16]. In section 8, we use the algorithm to compute the solution of block Hankel systems. We make the comparison with [14] where the inverse of a block Hankel matrix is computed.

Only minor modifications are needed to obtain an updating of the basis when moving from one perfect point to another one that need not be on the same diagonal. Also adaptations for a superfast version of the algorithms are standard and can be applied easily to our algorithm.

We start with some preliminary definitions and notations. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of natural numbers and $\mathbb{Z}$ the set of integers. An s-tuple will be denoted with the vector notation like e.g., $\vec{\sigma} = (\sigma_1, \ldots, \sigma_s) \in \mathbb{N}^s$.

By $\mathbb{F}$ we denote an arbitrary (finite or infinite) field. $\mathbb{F}[z]^{s\times m}$ and $\mathbb{F}[[z]]^{s\times m}$ denote the set of $s \times m$ polynomials and formal McLaurin series respectively. All formal McLaurin series in this paper will be in $\mathbb{F}[[z]]^{s\times m}$ with $s < m$.

For $F(z) \in \mathbb{F}[[z]]^{s\times m}$ and $\vec{\sigma} \in \mathbb{N}^s$, we mean by

$$F(z) = O_+(z^{\vec{\sigma}})$$

that the (i,j)-th entry of $F(z)$ is of the form

$$f_{ij}(z) = \sum_{k \geq \sigma_i} f_{ijk} z^k$$

i.e., σ_i indicates the order of row i of $F(z)$. A similar definition can be given for the notation $O_{-}(z^{\vec{\sigma}})$.

For $\vec{\sigma} \in \mathbb{Z}^s$ and $F(z) \in \mathbb{F}[[z]]^{s \times m}$, we write $z^{\vec{\sigma}} F(z)$ for $\mathrm{diag}(z^{\sigma_1}, \ldots, z^{\sigma_s}) F(z)$.

Let $\vec{\tau} \in \mathbb{Z}^m$ and $\vec{p}(z) \in \mathbb{F}[z]^m$, then

$$\vec{\tau}\text{-deg}\,(p_1(z), \ldots, p_m(z)) = \max\{\deg p_i(z) - \tau_i, i = 1, 2, \ldots, m\}.$$

With such a polynomial vector $\vec{p}(z)$, of $\vec{\tau}$-degree δ, we associate its coefficient vector

$$\mathrm{coeff}(\vec{p}) = [p_{0,1}, p_{1,1}, \ldots, p_{\delta+\tau_1,1} \mid p_{0,2}, p_{1,2}, \ldots, p_{\delta+\tau_2,2} \mid \ldots \mid p_{0,m}, p_{1,m}, \ldots, p_{\delta+\tau_m,m}]^T \tag{1}$$

where

$$p_i(z) = \sum_{k=0}^{\delta+\tau_i} p_{k,i} z^k, \quad i = 1, 2, \ldots, m.$$

For a general polynomial matrix

$$P(z) = [\vec{p}_1(z), \vec{p}_2(z), \ldots, \vec{p}_k(z)]$$

we say that $\vec{\tau}$-deg $P(z) = \vec{\alpha} \in \mathbb{Z}^k$ when $\vec{\tau}$-deg $\vec{p}_i(z) = \alpha_i$ for $i = 1, \ldots, k$. If each of the columns is considered to have a $\vec{\tau}$-degree $\leq \delta$, we define the coefficient matrix $\mathrm{coeff}(P)$ of $P(z)$ as

$$\mathrm{coeff}(P) = [\mathrm{coeff}(\vec{p}_1), \ldots, \mathrm{coeff}(\vec{p}_k)]$$

with $\mathrm{coeff}(\vec{p}_j)$, $j = 1, 2, \ldots, k$ given by (1).

For more details about the folowing concepts, we refer to [21].

If $\vec{p}(z) \in \mathbb{F}[z]^m$ and $\vec{\tau}$-deg $\vec{p}(z) = \delta$, then $\vec{\tau}$-hdc $\vec{p}(z)$ is the highest degree coefficient with respect to $\vec{\tau}$, that is the vector in $\mathbb{F}^m$ whose i-th component equals the coefficient of $z^{\delta-\tau_i}$ in $p_i(z)$. A set of polynomial vectors is $\vec{\tau}$-reduced iff their $\vec{\tau}$-hdc vectors are linearly independent.

For an s-tuple $\vec{z} = (z_1, \ldots, z_s) \in \mathbb{Z}^s$, the norm means

$$\|\vec{z}\| = \sum_{i=1}^{s} |z_i|.$$

We define a function $\mathrm{vect} : \mathbb{N} \to \mathbb{N}^s : t \mapsto \mathrm{vect}(t) = \vec{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_s)$ by

$$\sigma_1 = \sigma_2 = \cdots = \sigma_r = q + 1 \quad \text{and} \quad \sigma_{r+1} = \cdots = \sigma_s = q$$

where q and r are defined by the unique decomposition of t as $t = qs + r$ with $0 \leq r < s$, i.e., $r = t \bmod s$. Note that $\|\vec{\sigma}\| = \|\,\mathrm{vect}(t)\| = t$, so that $\vec{\sigma}$ is uniquely defined by its norm.

The rationale behind this definition is that if we have t interpolation conditions, to be distributed over an s dimensional vectorial interpolation problem and $\vec{\sigma} = \text{vect}(t)$, then a natural choice is to distribute these conditions cyclically such that σ_i conditions are imposed on component i.

In this paper, we shall assume that the vectors $\vec{\sigma}$ of order indices (see below) are always in this natural ordering, so that $\vec{\sigma} = \text{vect}(\|\vec{\sigma}\|)$ and hence that $\vec{\sigma}$ is uniquely defined by its norm.

We also reserve the notation $\vec{e}_s = (1, \ldots, 1, 0, \ldots, 0) \in \mathbb{Z}^m$ with s ones and $\vec{e} = (1, \ldots, 1)$. By $\vec{\epsilon}$ we mean the s-dimensional analogue of $\vec{e}$.

2 Vector Padé-Hermite approximation

The approximants, we are interested in, are defined as follows.

Definition 2.1 (vector Padé-Hermite approximation) *Given $F(z) \in \mathbb{F}[[z]]^{s \times m}$ with $s < m$. We say that a polynomial vector $\vec{p}(z) \in \mathbb{F}[z]^{m \times 1}$ is a vector Padé-Hermite approximant with order indices $\vec{\sigma} \in \mathbb{N}^s$ for $F(z)$ if it satisfies the vector interpolation conditions*

$$F(z)\vec{p}(z) = O_+(z^{\vec{\sigma}}). \tag{2}$$

$S^{\vec{\sigma}}$ denotes the set of all vector Padé-Hermite approximants for given $\vec{\sigma}$ and given $F(z)$.

Recall that the vectors $\vec{\sigma}$ are supposed to be in the natural ordering, so that $\vec{\sigma}$ and hence also $S^{\vec{\sigma}}$ is completely fixed by the natural number $t = \|\vec{\sigma}\|$.

We define the vector Padé-Hermite table as follows.

Definition 2.2 (vector Padé-Hermite table) *Given the formal $s \times m$ McLaurin series $F(z) \in \mathbb{F}[[z]]^{s \times m}$ with $s < m$. Point $\vec{\tau} \in \mathbb{N}^m$ of the m-dimensional vector Padé-Hermite table of $F(z)$ contains a matrix $M_{\vec{\tau}}(z) = [\vec{b}_1(z), \vec{b}_2(z), \ldots, \vec{b}_\alpha(z)]$ whose columns form a basis of the $\mathbb{F}$-vector space $S_{\vec{\tau}}$. $S_{\vec{\tau}}$ is the set of all polynomial vectors $\vec{p}(z)$ having $\vec{\tau}$-degree ≤ 0 and being vector Padé-Hermite approximants for $F(z)$ with order indices $\vec{\sigma} = \text{vect}(\|\vec{\tau}\| + s)$.*

Note that the vector of degree indices $\vec{\tau}$ gives an upper bound for the degrees which we allow for the polynomial components of $\vec{p}(z)$, but at the same time fixes the number of interpolation conditions by $\|\vec{\sigma}\| = \|\vec{\tau}\| + s$, and hence also $\vec{\sigma}$ and therefore also the specific interpolation conditions to be satisfied. In other words, $S_{\vec{\tau}} \subset S^{\vec{\sigma}}$ contains all the elements $\vec{p}(z) \in S^{\vec{\sigma}}$ with $\vec{\tau}$-deg $\vec{p}(z) \leq 0$. The converse is not true: $\vec{\sigma}$ will not define $\vec{\tau}$ uniquely.

The computation of vector Padé-Hermite approximants is equivalent to the solution of a structured system of homogeneous linear equations. This can be seen as follows. We denote the columns of $F(z)$ by $\vec{f}_j(z)$ where

$$\vec{f}_j(z) = \sum_{k=0}^{\infty} \vec{f}_{k,j} z^k \quad \text{with} \quad \vec{f}_{k,j} \in \mathbb{F}^{s \times 1} \quad (\vec{f}_{k,j} = \vec{0} \text{ for } k < 0).$$

The elements of $S_{\vec{\tau}}$ can be found by computing the solutions of the following set of linear homogeneous equations which incorporates the degree conditions as well as the corresponding interpolation conditions

$$T_{\vec{\tau}} \text{ coeff}(\vec{p}) = 0. \tag{3}$$

The generalized block Sylvester matrix $T_{\vec{\tau}}$ is defined as follows ($t = \|\vec{\sigma}\| = \|\vec{\tau}\| + s$)

$$T_{\vec{\tau}} = [T_{\vec{\tau},1} \mid T_{\vec{\tau},2} \mid \ldots \mid T_{\vec{\tau},m}] \in \mathbb{F}^{t \times (\|\vec{\tau}\|+m)}$$

where

$$T_{\vec{\tau},j} = \begin{bmatrix} \vec{f}_{0,j} & & & \\ \vec{f}_{1,j} & \vec{f}_{0,j} & & \\ \vdots & \vdots & \ddots & \\ \vec{f}_{q-1,j} & \vec{f}_{q-2,j} & \cdots & \vec{f}_{q-\tau_j-1,j} \\ \vec{f}^{*}_{q,j} & \vec{f}^{*}_{q-1,j} & \cdots & \vec{f}^{*}_{q-\tau_j,j} \end{bmatrix} \in \mathbb{F}^{t \times (\tau_j+1)}$$

and

$$\vec{f}^{*}_{i,j} \in \mathbb{F}^{r \times 1}$$

containing the first r elements of $\vec{f}_{i,j}$ with $r = \|\vec{\tau}\| \bmod s$ (recall $t = \|\vec{\sigma}\| = \|\vec{\tau}\| + s = sq + r$).
The number of unknowns is $\|\vec{\tau} + \vec{e}\| = \|\vec{\tau}\| + m$. The number of equations (interpolation conditions) is $\|\vec{\sigma}\| = \|\vec{\tau}\| + s$. Because $s < m$, the basis $M_{\vec{\tau}}(z)$ will at least contain $(\|\vec{\tau}\| + m) - (\|\vec{\tau}\| + s) = m - s$ elements.

Definition 2.3 (perfect point) *If the basis for $S_{\vec{\tau}}$ contains precisely $m - s$ elements, we say that $\vec{\tau}$ is a perfect point.*

The following theorems are simple observations.

Theorem 2.1 (characterization of a perfect point) *The point $\vec{\tau}$ is perfect iff $T_{\vec{\tau}}$ has full rank, i.e. the $\|\vec{\tau}\| + s$ rows of $T_{\vec{\tau}}$ are linearly independent.*

Note that the nontrivial part of the first block row of $T_{\vec{\tau}}$ is equal to the constant coefficient of $F(z)$, i.e. $F(0)$. This implies the following theorem.

Theorem 2.2 *The generalized block Sylvester matrix $T_{\vec{\tau}}$ has full rank iff $F(0)$ has full rank, i.e. the s rows of $F(0)$ are linearly independent, and the square generalized block Sylvester matrix $T^{*}_{\vec{\tau}}$ is nonsingular, where $T^{*}_{\vec{\tau}}$ is equal to $T_{\vec{\tau}}$ but with the first block row and the first column of the m block columns deleted.*

In the sequel (unless mentioned otherwise) we assume that $F(0)$ has full rank and that the columns of $F(z)$ are ordered such that the first s columns of $F(0)$ are linearly independent.

Corollary 2.1 *Assume that $F(0)$ has full rank. Then the following are equivalent.*

1. *The point $\vec{\tau}$ is perfect.*

2. *$T_{\vec{\tau}}$ has full rank.*

3. *$T^*_{\vec{\tau}}$ is nonsingular.*

4. *There is no nontrivial solution with $\vec{\tau}$-degree ≤ -1 satisfying the first $\|\vec{\tau}\|$ interpolation conditions.*

The following theorem says that for a perfect point, a basis matrix for $S_{\vec{\tau}}$ will always be $\vec{\tau}$-reduced.

Theorem 2.3 ($\vec{\tau}$-reduced basis matrix) *Let $\vec{\tau}$ be a perfect point for $F(z) \in \mathbb{F}[[z]]^{s\times m}$ and let*

$$M_{\vec{\tau}}(z) = [\vec{b}_1(z), \vec{b}_2(z), \ldots, \vec{b}_\alpha(z)]$$

be a basis matix for $S_{\vec{\tau}}$. Then $\alpha = m - s$ and $M_{\vec{\tau}}(z)$ is $\vec{\tau}$-reduced with $\vec{\tau}$-degree 0.

proof. Suppose $M_{\vec{\tau}}(z)$ is not $\vec{\tau}$-reduced, i.e. the $\vec{\tau}$-highest degree coefficient of one of the basis vectors is already zero or can be made equal to zero by recombination of the basis vectors. However, multiplying this basis vector by z would give a nontrivial solution satisfying (3) with constant term equal to zero. Hence, there would exist a nontrivial solution of

$$T^*_{\vec{\tau}}\text{coeff}(\vec{p}) = 0.$$

This means that $T^*_{\vec{\tau}}$ is singular, hence $\vec{\tau}$ is not a perfect point which contradicts our assumption. □

Theorem 2.4 *If $\vec{\tau}$ is a perfect point for $F(z) \in \mathbb{F}[[z]]^{s\times m}$ and $M_{\vec{\tau}}(z)$ is a basis matrix for $S_{\vec{\tau}}$, then the $m-s$ columns of $M_{\vec{\tau}}(0)$ are linearly independent. Moreover, if the columns $i_1, i_2, \ldots, i_s$ of $F(0)$ are linearly independent (form a nonsingular matrix), then the $(m-s)\times(m-s)$ matrix obtained by deleting rows $i_1, i_2, \ldots, i_s$ in $M_{\vec{\tau}}(0)$ is nonsingular.*

proof. If the basis vectors have linearly dependent constant coefficients, they can be combined to give a nontrivial solution of (3) having constant term equal to zero.
As in the previous proof, this means that $\vec{\tau}$ can not be a perfect point.
To prove the second part, observe that we get $F(0)M_{\vec{\tau}}(0) = 0$ as the first block row of (3). □

3 Basis matrix for $S^{\vec{\sigma}}$

In this section, we look for a compact way to denote all elements of $S^{\vec{\sigma}}$, i.e., all the solutions of (2). For more details and proofs, we refer to [21].
Because $S^{\vec{\sigma}}$ forms an $\mathbb{F}[z]$-submodule of $\mathbb{F}[z]^m$, we can represent $\vec{p}(z) \in S^{\vec{\sigma}}$ as a

polynomial linear combination of some basis vectors $\vec{b}_1(z), \ldots, \vec{b}_m(z)$. Arranging these basis vectors as the columns of a basis matrix $B^{\vec{\sigma}}(z) \in \mathbb{F}[z]^{m \times m}$, we see that all solutions $\vec{p}(z)$ of (2) can be written as

$$\vec{p}(z) = B^{\vec{\sigma}}(z)\vec{c}(z)$$

with $\vec{c}(z) \in \mathbb{F}^{m \times 1}[z]$. This $\vec{c}(z)$ contains the uniquely defined coordinates of $\vec{p}(z)$ with respect to this basis matrix.
To construct a basis matrix $B^{\vec{\sigma}}(z)$ we can use the following characterization theorem.

Theorem 3.1 (characterization of a basis matrix) *A polynomial matrix $B(z) \in \mathbb{F}^{m \times m}[z]$ is a basis matrix for $S^{\vec{\sigma}}$ iff*

1. *the m columns $\vec{b}_j(z)$ of $B(z)$ belong to $S^{\vec{\sigma}}$ i.e. satisfy (2);*
2. $\deg \det B(z) = k_S$ *with k_S the number of independent equations of (2).*

Notes:

- k_S is minimal in the following sense. Take m polynomial vectors $\vec{p}_j(z)$, $j = 1, 2, \ldots, m$ satisfying (2) and set

  $$P(z) = [\vec{p}_1(z), \vec{p}_2(z), \ldots, \vec{p}_m(z)].$$

  If $\det P(z)$ is not identically 0, then $\deg \det P(z) \geq k_S$.

- If $F(0)$ has full rank, all interpolation conditions are linearly independent, i.e. $k_S = \|\vec{\sigma}\|$.

All elements of $S^{\vec{\sigma}}$ of limited $\vec{\tau}$-degree can be represented as follows.

Theorem 3.2 ($\vec{\tau}$-reduced basis matrix) *If the basis matrix*

$$B^{\vec{\sigma}}(z) = [\vec{b}_1(z), \vec{b}_2(z), \ldots, \vec{b}_m(z)]$$

for $S^{\vec{\sigma}}$ is $\vec{\tau}$-reduced, then all the elements of $S^{\vec{\sigma}}$ having $\vec{\tau}$-degree $\leq \alpha$ can be represented as

$$\vec{p}(z) = B^{\vec{\sigma}}(z)\vec{c}(z)$$

with $\deg c_i(z) \leq \alpha - \alpha_i$ and $\alpha_i = \vec{\tau}$-deg of $\vec{b}_i(z)$.

The latter result will allow us to construct an entry $M_{\vec{\tau}}(z)$ of the vector Padé-Hermite table corresponding to a perfect point $\vec{\tau}$. By the previous theorem it is obvious that for a perfect point $\vec{\tau}$, we can generate the subset $S_{\vec{\tau}}$ of all elements in $S^{\vec{\sigma}}$ with $\vec{\tau}$-degree ≤ 0 where $\vec{\sigma} = \text{vect}(\|\vec{\tau}\| + s)$ if we have a $\vec{\tau}$-reduced basis matrix $B^{\vec{\sigma}}(z)$ for the submodule $S^{\vec{\sigma}}$. Thus the construction of a basis matrix $M_{\vec{\tau}}(z)$ for the vector space $S_{\vec{\tau}}$ is reduced to the construction of a $\vec{\tau}$-reduced basis matrix $B_{\vec{\tau}}(z)$ for the submodule $S^{\vec{\sigma}}$.
We use the following definition.

Definition 3.1 (residual) *If $\vec{p}(z) \in S^{\vec{\sigma}}$, we define the residual $R \in \mathbb{F}^{s\times 1}$ of $\vec{p}(z)$ with respect to $F(z)$ for the order indices $\vec{\sigma}$ as*

$$F(z)\vec{p}(z) = z^{\vec{\sigma}}R + O_+(z^{\vec{\sigma}+\vec{\epsilon}}) \quad \textit{with} \quad \vec{\epsilon} = (1,1,\ldots,1).$$

The residual series $R(z)$ is defined by

$$R(z) = z^{-\vec{\sigma}}F(z)\vec{p}(z).$$

Theorem 3.3 *Let $\vec{\tau}$ be a perfect point of $F(z) \in \mathbb{F}[[z]]^{s\times m}$ and $M_{\vec{\tau}}(z)$ a basis matrix which is the entry of the Padé-Hermite table at point $\vec{\tau}$. The matrix*

$$B_{\vec{\tau}}(z) = [\underbrace{z^2B^*_{\vec{\tau}}(z)}_{s} \; \underbrace{M_{\vec{\tau}}(z)}_{m-s}]$$

with

$$T^*_{\vec{\tau}}\,\mathrm{coeff}(B^*_{\vec{\tau}}) = [0,0,\ldots,0,X]^T, \qquad X \in \mathbb{F}^{s\times s}, \textit{ nonsingular}$$

and $\vec{\tau}$-deg $B^*_{\vec{\tau}}(z) \le -\vec{e}$ (columnwise) forms a $\vec{\tau}$-reduced basis matrix for $S^{\vec{\sigma}}$ with $\vec{\sigma} = \mathrm{vect}(\|\vec{\tau}\| + s)$.
The $\vec{\tau}$-column degrees of $B_{\vec{\tau}}(z)$ are $\vec{e}_s$ (s ones, $m-s$ zeros) .
Hence, deg det $B_{\vec{\tau}}(z) = \|\vec{\sigma}\|$.
In other words, the columns of $B^*_{\vec{\tau}}(z)$ are vector Padé-Hermite approximants for order indices $\vec{\sigma} - 2\vec{\epsilon}$, having $\vec{\tau}$-degree $= -1$ and having linearly independent residuals.

The basis matrix $B_{\vec{\tau}}(z)$ can be made unique by choosing $X = I_s$ and $M_{\vec{\tau}}(0) = I_{m-s}$.

proof. First, note that $B^*_{\vec{\tau}}(z)$ can always be computed because $T^*_{\vec{\tau}}$ is nonsingular if $\vec{\tau}$ is a perfect point. Second, note that all the columns of $B_{\vec{\tau}}$ are in $S^{\vec{\sigma}}$. The columns of $B_{\vec{\tau}}(z)$ are linearly independent because the columns of $M_{\vec{\tau}}(0)$ are linearly independent and the residuals of the columns of $B^*_{\vec{\tau}}(z)$ or $z^2B^*_{\vec{\tau}}(z)$ are linearly independent. Hence, deg det $B_{\vec{\tau}}(z) \ge k_S = \|\vec{\sigma}\|$.
On the other hand, the $\vec{\tau}$-deg of the columns of $M_{\vec{\tau}}(z)$ are 0 and the $\vec{\tau}$-deg of the columns of $B^*_{\vec{\tau}}$ are ≤ -1.
Therefore,

$$z^{-\vec{\tau}}B_{\vec{\tau}}(z) = O_-(z^{\vec{e}_s})$$

or

$$\deg\det B_{\vec{\tau}}(z) \le \|\vec{\tau}\| + s = \|\vec{\sigma}\| = k_S.$$

We get that deg det $B_{\vec{\tau}}(z) = k_S$ so that by Theorem 3.1, this $B_{\vec{\tau}}$ is a basis matrix for $S^{\vec{\sigma}}$. Moreover, it is $\vec{\tau}$-reduced with $\vec{\tau}$-column degrees $\vec{e}_s$.

Because X is nonsingular and also $M_{\vec{\tau}}(0)$ is nonsingular by theorem 2.4, we may consider the matrix

$$[z^2B^*_{\vec{\tau}}(z)|M_{\vec{\tau}}(z)]\begin{bmatrix} X & 0 \\ 0 & M_{\vec{\tau}}(0) \end{bmatrix}^{-1}.$$

Because the $\vec{\tau}$-degrees of the columns of the first part of $B_{\vec{\tau}}(z)$ are all -1 and those of the second part are all 0, we can recombine them and keep a $\vec{\tau}$-reduced basis. Hence the latter matrix is a $\vec{\tau}$-reduced basis matrix.

It is unique because the first part is unique by construction and the second part is unique because there is only one basis matrix $M_{\vec{\tau}}(z)$ for the $\mathbb{F}$-vector space $S_{\vec{\tau}}$ with $M_{\vec{\tau}}(0) = I_{m-s}$. □

4 Going from one perfect point to another one

Assume that we know a $\vec{\tau}$-reduced basis matrix $B_{\vec{\tau}}(z)$ for $S^{\vec{\sigma}}$ corresponding to the perfect point $\vec{\tau}$ given by Theorem 3.3. Suppose that $\vec{\tau} + k\vec{e}$ is also a perfect point. Can we compute a $(\vec{\tau}+k\vec{e})$-reduced basis matrix $B_{\vec{\tau}+k\vec{e}}(z)$ starting from the basis matrix $B_{\vec{\tau}}(z)$? The following observation is trivial.

Lemma 4.1 ($\vec{\tau}$-reduced) *A polynomial matrix $P(z) \in \mathbb{F}[z]^{m\times m}$ is $\vec{\tau}$-reduced iff $P(z)$ is $(\vec{\tau} + k\vec{e})$-reduced. Moreover*

$$\vec{\tau}\text{-deg}\,(P) = (\vec{\tau} + k\vec{e})\text{-deg}\,(P) + k\vec{e}.$$

When adding interpolation conditions, we can use a general updating theorem. It says that a basis with order indices $\vec{\sigma} + \vec{\nu}$ can be obtained by combining an order $\vec{\sigma}$ basis for $F(z)$ with an order $\vec{\nu}$ basis of its residual.

Theorem 4.2 (general updating theorem) *Given a $\vec{\tau}$-reduced basis matrix $B^{\vec{\sigma}}(z)$ for $S^{\vec{\sigma}}$. The $\vec{\tau}$-degrees of the columns of $B^{\vec{\sigma}}(z)$ are $\vec{\alpha}$. The polynomial matrix $B^{\vec{\sigma}+\vec{\nu}}(z) = B^{\vec{\sigma}}(z)B^{\vec{\nu}}(z)$ is a $\vec{\tau}$-reduced basis matrix for $S^{\vec{\sigma}+\vec{\nu}}$ iff $B^{\vec{\nu}}(z)$ is a $(-\vec{\alpha})$-reduced basis matrix for the submodule $S^{\vec{\nu}}$ with*

$$S^{\vec{\nu}} = \{\vec{p}(z) \mid R(z)\vec{p}(z) = O_+(z^{\vec{\nu}})\}$$

with $R(z)$ the residual series of $B^{\vec{\sigma}}(z)$, i.e. $F(z)B^{\vec{\sigma}}(z) = z^{\vec{\sigma}}R(z)$.

proof. A proof of this theorem can be found in appendix A. □

When we apply this theorem to update from one perfect point to another one, We have to be careful however. Suppose $\|\vec{\sigma} + \vec{\nu}\| = q's + r'$, $0 \le r' < s$. Then we impose $q' + 1$ interpolation conditions for the rows $1, \ldots, r'$ of $F(z)$ and q' interpolation conditions for the rows $r' + 1, \ldots, s$. If $\|\vec{\sigma}\| = qs + r$, $0 \le r < s$, then the elements from $S^{\vec{\sigma}}$ satisfy $q + 1$ interpolation conditions for rows $1, \ldots, r$ and q conditions for rows $r+1, \ldots, s$. So the order $\vec{\nu}$ approximant for the residual should exactly fill up the difference in conditions per row. Thus the interpolation conditions for the residuals are imposed on the $\|\vec{\nu}\|$ rows, repeated cyclically, starting with row $r + 1$. In orther words, the order $\vec{\nu}$ approximant for the residual series $R(z)$ is in fact computed for $C^r R(z)$ where C represents the permutation operator which shuffles the rows cyclically one place up. Applying the previous theorem to $B_{\vec{\tau}}(z)$ and $B_{\vec{\tau}+k\vec{e}}(z)$, this results in the following theorem.

Theorem 4.3 (updating from one perfect point to another) *Let $\vec{\tau}$ and $\vec{\tau}+k\vec{e}$ be two perfect points on a diagonal of the Padé-Hermite table for $F(z)$. Suppose $B_{\vec{\tau}}(z)$ is a $\vec{\tau}$-reduced basis for $S^{\vec{\sigma}}$, with $\vec{\sigma} = \text{vect}(\|\vec{\tau}\| + s)$ (e.g., as given by Theorem 3.3). Let $R_{\vec{\tau}}(z)$ be the residual series of $B_{\vec{\tau}}(z)$, i.e., $F(z)B_{\vec{\tau}}(z) = z^{\vec{\sigma}}R_{\vec{\tau}}(z)$. Finally for $0 < k \in \mathbb{N}$ define $\vec{\nu} \in \mathbb{N}^s$ as the vector of order indices $\vec{\nu} = \text{vect}(km)$.*

Then the polynomial matrix $B_{\vec{\tau}+k\vec{e}}(z) = B_{\vec{\tau}}(z)B_{k\vec{e}-\vec{e}_s}(z)$ is a $(\vec{\tau}+k\vec{e})$-reduced basis matrix for $S^{\vec{\sigma}+\vec{\nu}}$ iff $B_{k\vec{e}-\vec{e}_s}(z)$ is a $(k\vec{e}-\vec{e}_s)$-reduced basis matrix for $S^{\vec{\nu}}$ for the perfect point $(k\vec{e}-\vec{e}_s)$ in the vector Padé-Hermite table for $C^TR_{\vec{\tau}}(z)$ with $\|\vec{\sigma}\| = qs + r$, $0 \le r < s$.

proof. Let us first prove that if $\vec{\tau}$ is a perfect point for $F(z)$, then $\vec{\tau}+k\vec{e}$ is also a perfect point for $F(z)$ iff $(k\vec{e}-\vec{e}_s)$ is a perfect point for $R^c_{\vec{\tau}}(z) = C^TR_{\vec{\tau}}(z)$.
First assume that $(k\vec{e}-\vec{e}_s)$ is not a perfect point for $R^c_{\vec{\tau}}(z)$. Hence, there exists a nontrivial polynomial vector $\vec{p}(z)$ having $(k\vec{e}-\vec{e}_s)$-degree ≤ -1 satisfying the first $\|k\vec{e}-\vec{e}_s\| = km - s$ interpolation conditions.
Note that $\vec{q}(z) = B_{\vec{\tau}}(z)\vec{p}(z)$ has $(\vec{\tau}+k\vec{e})$-degree ≤ -1 and $\vec{q}(z)$ is a nontrivial solution satisfying the first

$$(\|\vec{\tau}\| + s) + \|k\vec{e}-\vec{e}_s\| = \|\vec{\tau}+k\vec{e}\|$$

interpolation conditions for $F(z)$. Hence, $\vec{\tau}+k\vec{e}$ is not a perfect point.
For the "if"-part we assume that $(k\vec{e}-\vec{e}_s)$ is a perfect point for $R^c_{\vec{\tau}}(z)$ and as above, it follows easily that $\vec{\tau}+k\vec{e}$ is a perfect point for $F(z)$.

The rest of the theorem follows immediately from Lemma 4.1 and Theorem 4.2. □

Note that if $B_{\vec{\tau}}(z)$ and $B_{k\vec{e}-\vec{e}_s}(z)$ are constructed using Theorem 3.3 also $B_{\vec{\tau}}(z)B_{k\vec{e}-\vec{e}_s}(z)$ will have the form described in this theorem.

5 The border of the vector Padé-Hermite table

In the previous section, we have assumed that $\vec{\tau} \ge \vec{0}$ (componentwise). Theorem 4.3 shows how to jump from the perfect point $\vec{\tau}$ to the perfect point $\vec{\tau}+k\vec{e}$.
Suppose now that we are interested in a $\vec{\tau}$-reduced basis matrix $B_{\vec{\tau}}(z)$ for the submodule $S^{\vec{\sigma}}$ where some of the components of $\vec{\tau}$ may be negative.
The first nontrivial point on the diagonal $\vec{\tau}+k\vec{e}$, $k \in \mathbb{Z}$ is

$$\vec{\tau} - \tau_{\max}\vec{e} \quad \text{with } \tau_{\max} = \max\{\tau_j \mid j = 1, 2, \ldots, m\}.$$

However, some of the components of $\vec{\tau}-k\vec{e}$, $k = \tau_{\max}, \tau_{\max}-1, \ldots, 0$ can be negative. Hence, we have to extend the definition of $S^{\vec{\sigma}}$, $B_{\vec{\tau}}(z)$, $S_{\vec{\tau}}$, $M_{\vec{\tau}}(z)$, ... when $\exists \tau_j$, $j = 1, 2, \ldots, m : \tau_j < 0$.

Definition 5.1 (projection of the diagonal $\vec{\tau} - k\vec{e}$) *We define the projection of $\vec{z} = (z_1, \ldots, z_m) \in \mathbb{Z}^m$ on the positive cone by* $\mathrm{proj}(\vec{z}) = \vec{\pi}$ *with*

$$\pi_j = \max\{0, z_j\}, \quad j = 1, \ldots, m.$$

We call the piecewise linear path $\vec{\pi}_k =$ $\mathrm{proj}(\vec{\tau}_k)$ the projection of the diagonal path $\vec{\tau}_k = \vec{\tau} - k\vec{e}$, $k = \tau_{\max}, \tau_{\max} - 1, \ldots, 0$ when each point is replaced by its projection on the positive cone. The entities $B_{\vec{\tau}}(z)$, $S_{\vec{\tau}}$, $M_{\vec{\tau}}(z)$, ... when $\exists \tau_j$: $\tau_j < 0$, are defined as the corresponding entities for the projection $\vec{\pi} = \mathrm{proj}(\vec{\tau})$. Thus for example $B_{\vec{\tau}}(z) = B_{\vec{\pi}}(z)$.

We show now that the results of the previous section are also valid when $\exists \tau_j$: $\tau_j < 0$.

From now on, we shall assume, without loss of generality, that the columns of $F(z) \in \mathbb{F}[[z]]^{s \times m}$ are ordered such that for a perfect point $\vec{\tau}$, $\sum_{j=1}^{s} \tau_j$ is maximal with $F_L(0)$ nonsingular. We use $F_L(z)$ to denote the $s \times s$ matrix containing the first s columns of $F(z)$.

Theorem 5.1 *Given $\vec{\tau} \in \mathbb{Z}^m$ and $F(z) \in \mathbb{F}[[z]]^{s \times m}$. The columns of $F(z)$ are ordered as explained above.*
Let the point $\vec{\pi} = \mathrm{proj}(\vec{\tau})$ be perfect. Let $B_{\vec{\tau}}(z) = B_{\vec{\pi}}(z)$ be the unique $\vec{\pi}$-reduced basis matrix given by theorem 3.3. Then, $B_{\vec{\tau}}(z)$ is also a $\vec{\tau}$-reduced basis matrix for $S^{\vec{\sigma}}$, $\vec{\sigma} = \mathrm{vect}(\|\vec{\pi}\| + s)$. Its $\vec{\pi}$-column degrees are $\vec{e}_s$ and its $\vec{\tau}$-column degrees are $\vec{e}_s + (\vec{\pi} - \vec{\tau})$. Moreover, the first s components of $(\vec{\pi} - \vec{\tau})$ are always zero.

proof. The proof of this theorem is given in appendix B. □

We extend now Theorem 4.3 to the more general case when one of the components of $\vec{\tau}$ could be negative.

Theorem 5.2 (updating from one perfect point to another) *Let $B_{\vec{\tau}}(z) = B_{\vec{\pi}}(z)$ with $\vec{\pi} = \mathrm{proj}(\vec{\tau})$ be the unique $\vec{\tau}$-reduced basis matrix for $S^{\vec{\sigma}}$, $\vec{\sigma} = \mathrm{vect}(\|\vec{\pi}\| + s)$, given by theorem 5.1.*
For $0 < k \in \mathbb{N}$, the polynomial matrix $B_{\vec{\tau}}(z)B_{\vec{\delta}}(z)$ is the unique $(\vec{\tau} + k\vec{e})$-reduced basis matrix $B_{\vec{\tau}+k\vec{e}}(z) = B_{\vec{\pi}'}(z)$ for $S^{\vec{\sigma}+\vec{\nu}}$, with $\vec{\pi}' = \mathrm{proj}(\vec{\tau} + k\vec{e})$ for the perfect point $\vec{\tau} + k\vec{e}$ iff $B^{\vec{\nu}}(z) = B_{\vec{\delta}}(z)$ is the unique $\vec{\delta} = k\vec{e} - \vec{e}_s - (\vec{\pi} - \vec{\tau})$-reduced basis matrix for the perfect point $\vec{\delta}$ in the vector Padé-Hermite table for $R^c_{\vec{\tau}}(z) = C^r R_{\vec{\tau}}(z)$ with $R_{\vec{\tau}}(z)$ the residual series for $B_{\vec{\tau}}(z)$, i.e. $F(z)B_{\vec{\tau}}(z) = z^{\vec{\sigma}} R_{\vec{\tau}}(z)$, C the cyclical up-shift matrix and $r = \|\vec{\sigma}\| \bmod s$.
The $\vec{\delta}$-column degrees of $B_{\vec{\delta}}(z)$ are $\vec{e}_s + (\vec{\pi}' - \vec{\tau} - k\vec{e})$.

proof. The proof of this theorem is given in appendix C. □

6 Algorithm

The previous theorem allows us to construct an algorithm jumping from one perfect point to the next one on the diagonal $\vec{\tau} - l\vec{e}$, $l = \tau_{\max}, \tau_{\max} - 1, \ldots$ in the vector Padé-Hermite table.

Algorithm 6.1 (diagonal path)

* *given* $F(z) \in \mathbb{F}[[z]]^{s \times m}$ *and* $\vec{\tau} \in \mathbb{Z}^m$ *such that* $F_L(0)$ *is nonsingular with* $\sum_{j=1}^{s} \tau_j$ *maximal.*

* $k \leftarrow 1$; $\vec{\mu} \leftarrow \vec{\tau} - (\tau_{\max} + 1)\vec{e}$; $\vec{\pi} \leftarrow -\vec{e}_s$; $B_{\vec{\mu}}(z) \leftarrow I_m$; $R^c_{\vec{\mu}}(z) \leftarrow F(z)$;

* **for** $l = \tau_{\max}, \tau_{\max} - 1, \ldots, 0$
  $\vec{\delta} \leftarrow k\vec{e} - \vec{e}_s - (\vec{\pi} - \vec{\mu})$
  $d \leftarrow \det(T^*_{\vec{\delta}})$ *with* $T^*_{\vec{\delta}}$ *based on* $R^c_{\vec{\mu}}(z)$
  if $d \neq 0$ **then**
    construct the unique $B_{\vec{\delta}}(z)$ *from theorem 3.3*
    $B_{\vec{\mu}+k\vec{e}}(z) \leftarrow B_{\vec{\mu}}(z) B_{\vec{\delta}}(z)$
    $R'_{\vec{\mu}+k\vec{e}}(z) \leftarrow z^{-\vec{\nu}} R^c_{\vec{\mu}}(z) B_{\vec{\delta}}(z)$ *with* $\vec{\nu} = \mathrm{vect}(\|\vec{\delta}\| + s)$
    $\vec{\mu} \leftarrow \vec{\mu} + k\vec{e}$
    $\vec{\pi} \leftarrow \mathrm{proj}(\vec{\mu})$
    $R^c_{\vec{\mu}}(z) = C^r R'_{\vec{\mu}}(z)$ *with* $r = \|\vec{\delta}\| \bmod s$
    $k \leftarrow 1$
  else
    $k \leftarrow k + 1$
  endif
endfor

if $\vec{\mu} = \vec{\tau}$ **then**

  * $\vec{\tau}$ *is a perfect point*
  * $B_{\vec{\mu}}(z)$ *is a* $\vec{\mu}$*-reduced basis matrix connected to this perfect point* $\vec{\mu}$

else

  * $\vec{\tau}$ *is not a perfect point*
  * $\vec{\mu}$ *is the last perfect point on the diagonal before* $\vec{\tau}$ *and* $B_{\vec{\mu}}(z)$ *is the corresponding basis matrix.*
  * *if we are interested in a* $\vec{\tau}$*-reduced basis matrix* $B_{\vec{\tau}}(z)$ *for the non-perfect point* $\vec{\tau}$*, we can compute a* $\vec{\delta}$*-reduced basis matrix* $B_{\vec{\delta}}(z)$ *for the non-perfect point* $\vec{\delta}$ *for* $R^c_{\vec{\mu}}(z)$*. The polynomial matrix* $B_{\vec{\mu}}(z) B_{\vec{\delta}}(z)$ *is a* $\vec{\tau}$*-reduced basis matrix for the non-perfect point* $\vec{\tau}$*.*

endif

Notes

- When $\vec{\mu}$ is initialized as $\vec{\mu} \leftarrow \vec{\tau} - (\tau_{\max}+1)\vec{e}$ and $\vec{\pi}$ as $\vec{\pi} \leftarrow -\vec{e}_s$, we start by looking for a first perfect point of the form

$$(\vec{\delta} \leftarrow) k\vec{e} - \vec{e}_s - (\vec{\pi} - \vec{\mu}) = \vec{\tau} - (\tau_{\max}+1)\vec{e} + k\vec{e}$$

with $k > 0$.

- In an analogous way as in [8], we can show that this algorithm requires only $O(\| \operatorname{proj}(\vec{\tau})\|^2)$ FLOPS if the variable k only reaches small values compared to $\| \operatorname{proj}(\vec{\tau})\|$.

- Note that the special ordering with conditions $F_L(0)$ nonsingular and $\sum_{i=1}^{s} \tau_i$ maximal is only necessary to establish the results when $\vec{\tau} \neq \operatorname{proj}(\vec{\tau})$, i.e. on the border of the vector Padé-Hermite table.

- Suppose $s = 1$ and $m = 2$ with $F(z) = [-1 \ f(z)]$ (Padé-approximation) with $f(z) = O_+(z^1)$. We are interested in a $\vec{\tau}$-reduced basis matrix $B_{\vec{\tau}}(z)$ with $\vec{\tau} = (\tau_1, \tau_2)$ and $0 \leq \tau_1 < \tau_2$. Hence, $\tau_2 = \tau_{\max} = \max\{\tau_1, \tau_2\}$. If we follow the diagonal path $\vec{\tau} - l\vec{e}$, $l = \tau_2, \tau_2 - 1, \ldots, \tau_1, \ldots, 0$, the first point which is possibly a perfect point, is reached for $l = \tau_1$. Indeed, we have shown that the first s components of $\vec{\pi} = \operatorname{proj}(\vec{\tau})$ are equal to the corresponding components of $\vec{\tau}$ for a perfect point $\vec{\tau}$. However,

$$\tau_1 - l \neq \pi_1 = 0, \quad l = \tau_2, \tau_2 - 1, \ldots \tau_1 + 1.$$

  Also $l = \tau_1$ gives no perfect point. This follows also from the fact that the first row of $T^*_{\vec{\pi}}$ is a zero row with $\vec{\pi} = \operatorname{proj}(\vec{\tau} - l\vec{e})$, $l = \tau_2, \tau_2 - 1, \ldots, \tau_1$. Suppose $l = \tau_1 - 1$ gives the first perfect point $\vec{\tau}$. Computing a $\vec{\tau}$-reduced basis matrix $B_{\vec{\tau}}(z)$ for this perfect point, requires $O((\tau_2 - \tau_1)^3)$ FLOPS. Hence, if $\tau_2 \gg \tau_1$, this means that the variable k of Algorithm 6.1 reaches the value $\tau_2 - \tau_1 + 1$ which is not moderate anymore compared to the total number of interpolation conditions.
  Therefore, to reduce the amount of computational work, one could follow other paths in the vector Padé-Hermite table. These paths are chosen such that the distance between two consecutive perfect points is small compared to the total length of the path. Of course, when a non-diagonal step is made, the updating formulas have to be adapted.

- Algorithm 6.1 jumps from one perfect point to the next one on a diagonal path in the vector Padé-Hermite table. The updating formula of Theorem 5.2, however, is valid for stepping from one perfect point to any of the following ones. Hence, instead of choosing the next perfect point, we can choose one of the following "well-conditioned" perfect points.
  I.e., instead of determining the next point based on the determinant of $T^*_{\vec{\delta}}$, we can look for a $\vec{\delta}$ such that the condition number of $T^*_{\vec{\delta}}$ is small enough, implying that this determines also a perfect point.

- Instead of Algorithm 6.1, a divide and conquer strategy could be used as in Cabay and Labahn [7], reducing the amount of computational work to $O(k \log_2^2 k)$ with $k = \| \text{proj}(\vec{\tau}) \|$.

7 Matrix Padé approximation

In this section and the next one, we give two applications of the previous algorithm to related problems like matrix Padé approximation and the solution of block Hankel systems. Both problems can be reduced to a vector Padé-Hermite problem.

We get the following connection between the vector Padé-Hermite approximation problem and matrix Padé approximation.
We split $F(z) \in \mathbb{F}[[z]]^{s \times m}$ as

$$F(z) = [F_L(z) \;\; F_R(z)]$$

with

$$F_L(z) \in \mathbb{F}[[z]]^{s \times s} \quad \text{and} \quad F_R(z) \in \mathbb{F}[[z]]^{s \times (m-s)}$$

and assume that $F_L(0)$ is nonsingular.
Let $\vec{\tau}$ be a perfect point and the basis matrix for the $\mathbb{F}$-vector space $S_{\vec{\tau}}$ be

$$M_{\vec{\tau}}(z) = [\vec{b}_1(z), \vec{b}_2(z), \ldots, \vec{b}_{m-s}(z)].$$

This basis matrix can be split up as

$$M_{\vec{\tau}}(z) = \begin{bmatrix} N(z) \\ D(z) \end{bmatrix} \quad \text{with } N(z) \in \mathbb{F}[z]^{s \times (m-s)} \text{ and } D(z) \in \mathbb{F}[z]^{(m-s) \times (m-s)}.$$

Because $F_L(0)$ is nonsingular, we get that $D(0)$ is nonsingular (see theorem 2.4).
Hence, by rewriting the interpolation condition (2), we derive

$$F_L(z) N(z) D(z)^{-1} = -F_R(z) + O_+(z^{\vec{\sigma}}).$$

As usual $\vec{\sigma} = \text{vect}(\|\vec{\pi}\| + s)$. If all the entries of $\vec{\sigma}$ are the same, we get

$$N(z) D(z)^{-1} = -F_L(z)^{-1} F_R(z) + O_+(z^{\vec{\sigma}}).$$

If $F_L(z) = I_{s \times s}$, then, even if the entries of $\vec{\sigma}$ are different from each other, we can always write

$$N(z) D(z)^{-1} = -F_R(z) + O_+(z^{\vec{\sigma}}).$$

Hence, the $M_{\vec{\tau}}(z)$ for a perfect point $\vec{\tau}$ gives us a matrix Padé approximant having $\vec{\tau}$-McMillan degree 0. For the definition of $\vec{\tau}$-McMillan degree, we refer to [21].

If $m = 2s$ and $\text{proj}(\vec{\tau}) = \vec{\tau} = (\delta_N, \ldots, \delta_N, \delta_D, \ldots, \delta_D)$ (s times δ_N and δ_D), we get that $\vec{\sigma} = (\delta_N + \delta_D + 1)\vec{\epsilon}$. Hence, $N(z)D(z)^{-1}$ is a right matrix Padé fraction as defined in [16] with $\deg N(z) \leq \delta_N$ and $\deg D(z) \leq \delta_D$. When we use our algorithm for this specific data and jump immediately to the diagonal part of the path, we get a similar algorithm as in [16].

8 Solution of block Hankel systems of equations

Let us consider the $(\alpha \times \beta)$ block Hankel matrix

$$H = \begin{bmatrix} H_0 & H_1 & \dots & H_\beta \\ H_1 & H_2 & \dots & H_{\beta+1} \\ \vdots & \vdots & & \vdots \\ H_\alpha & H_{\alpha+1} & \dots & H_{\alpha+\beta} \end{bmatrix} \quad \text{with } H_i \in \mathbb{F}^{s \times q}.$$

Consider the set of linear equations

$$H\vec{x} = \vec{b} \quad \text{with} \quad \vec{b} = [\vec{b}_0^T, \vec{b}_1^T, \dots, \vec{b}_\alpha^T]^T \quad \text{and} \quad \vec{b}_i \in \mathbb{F}^{s \times 1}$$

where we look for

$$\vec{x} = [\vec{x}_\beta^T, \vec{x}_{\beta-1}^T, \dots, \vec{x}_0^T]^T \quad \text{with} \quad \vec{x}_i \in \mathbb{F}^{q \times 1}.$$

Before we can apply the previous techniques, we have to translate this linear algebra problem into a vector Padé-Hermite approximation problem. Let us rewrite the original linear algebra problem in the following equivalent one:

$$\left[\begin{array}{ccccc|ccc|c} & & & & H_0 & & & I_s & \\ & & & \iddots & \vdots & & \iddots & & \\ & & H_0 & & & & I_s & & \\ & H_0 & H_1 & \dots & H_{\beta-1} & I_s & & & \\ H_0 & H_1 & H_2 & \dots & H_\beta & & & -\vec{b}_0 & \\ H_1 & H_2 & H_3 & \dots & H_{\beta+1} & & & -\vec{b}_1 & \\ \vdots & \vdots & \vdots & & \vdots & & & \vdots & \\ H_\alpha & H_{\alpha+1} & H_{\alpha+2} & \dots & H_{\alpha+\beta} & & & -\vec{b}_\alpha & \end{array}\right] \begin{bmatrix} \vec{x} \\ \vec{y} \\ 1 \end{bmatrix} = \vec{0} \qquad (4)$$

with unknowns $\vec{x}$ and $\vec{y} = [\vec{y}_{\beta-1}^T, \vec{y}_{\beta-2}^T, \dots, \vec{y}_0^T]^T$ with $\vec{y}_i \in \mathbb{F}^{s \times 1}$. Note that this set of linear homogeneous equations has $(\alpha+\beta+1)s$ equations and $(\beta+1)q + \beta s + 1$ unknowns (counting the 1 as an unknown).
If we define

$$\begin{aligned} H(z) &= H_0 z^0 + H_1 z^1 + \cdots + H_{\alpha+\beta} z^{\alpha+\beta} + O_+(z^{\alpha+\beta+1}), \\ \vec{b}(z) &= \vec{b}_0 z^0 + \vec{b}_1 z^1 + \cdots + \vec{b}_\alpha z^\alpha + O_+(z^{\alpha+1}), \\ \vec{x}(z) &= \vec{x}_0 z^0 + \vec{x}_1 z^1 + \cdots + \vec{x}_\beta z^\beta \quad \text{and} \\ \vec{y}(z) &= \vec{y}_0 z^0 + \vec{y}_1 z^1 + \cdots + \vec{y}_{\beta-1} z^{\beta-1}, \end{aligned}$$

equation (4) is equivalent to the following vector Padé-Hermite approximation problem:

$$[H(z) \;\; I_s \;\; -z^\beta \vec{b}(z)] \begin{bmatrix} \vec{x}(z) \\ \vec{y}(z) \\ 1 \end{bmatrix} = O_+(z^{\alpha+\beta+1})$$

with $\vec{0}$-deg $\vec{x} \leq \beta$ and $\vec{0}$-deg $\vec{y} \leq \beta - 1$.
This can be rewritten as

$$F(z)\vec{p}(z) = O_+(z^{\vec{\sigma}})$$

with

$$\begin{aligned} m &= q + s + 1 \\ F(z) &= [H(z) \;\; I_s \;\; -z^{\beta}\vec{b}(z)] \in \mathbb{F}[[z]]^{s \times m} \\ \vec{p}(z) &= [\vec{x}^T(z) \;\; \vec{y}^T(z) \;\; 1]^T \in \mathbb{F}[z]^{m \times 1} \\ \vec{\tau}\text{-deg}\,\vec{p} &\leq 0, \\ \vec{\tau} &= (\underbrace{\beta, \beta, \ldots, \beta}_{q}, \underbrace{\beta - 1, \beta - 1, \ldots, \beta - 1}_{s}, 0) \quad \text{and} \\ \vec{\sigma} &= (\alpha + \beta + 1, \alpha + \beta + 1, \ldots, \alpha + \beta + 1). \end{aligned}$$

Because $F(0)$ has full rank, we can use Algorithm 6.1 to compute $\vec{\tau}$-reduced basis matrices $B_{\vec{\mu}}(z)$ for each perfect point on the diagonal $\vec{\tau} - l\vec{e}$, $l = \beta, \beta - 1, \beta - 2, \ldots, l^*$. Note that maybe the columns of $F(z)$ should be reordered such that $F_L(0)$ is nonsingular with $\sum_{j=1}^{s} \tau_j$ maximal. The last value of l, i.e. l^* is chosen such that the number of interpolation conditions to determine $B_{\vec{\tau}}(z)$ is smaller than or equal to the total number of given interpolation conditions or

$$l^* = \min\{l \in \mathbb{Z} \mid \|\vec{\pi}\| + s \leq (\alpha + \beta + 1)s \text{ with } \vec{\pi} = \text{proj}(\vec{\tau} - l\vec{e})\}.$$

Equivalently, l^* is determined such that

$$l^* = \min\{\{l \in \mathbb{Z} \mid (s + q)l \geq q\beta - (\alpha + 1)s\} \cup \{\beta\}\} \tag{5}$$

Let us assume from now on that the block Hankel matrix H is square, i.e. $s(\alpha + 1) = q(\beta + 1)$. Moreover, let H be nonsingular. Hence, the solution space of equation (4) has dimension 1 and there is only one element from this solution space having last component equal to 1.
Equation (5) simplifies to $l^* = 0$. For $l = l^*$, we have that $\vec{\mu} = \vec{\tau}$ and the corresponding number of interpolation conditions is $\|\vec{\tau}\| + s = (s + q)\beta$. The remaining number of interpolation conditions is

$$(\alpha + \beta + 1)s - (\|\vec{\tau}\| + s) = q > 0.$$

Hence, to obtain the final solution, we look for the unique solution of

$$C^r R_{\vec{\tau}}(z)\vec{p}(z) = O_+(z^{\vec{\nu}}), \quad r = \|\vec{\tau}\| \bmod s \tag{6}$$

with

$$\begin{aligned} F(z)B_{\vec{\tau}}(z) &= z^{\vec{\sigma}'}R_{\vec{\tau}}(z), \\ \vec{\sigma}' &= \text{vect}(\|\vec{\tau}\| + s), \quad \text{and } \vec{\nu} = \text{vect}(q) \\ (-\vec{e}_s)\text{-deg}\,\vec{p}(z) &\leq 0 \end{aligned}$$

and the last component of $\vec{p}(z)$ is equal to 1.
Equation (6) translates into a set of q linear homogeneous equations with $q+1$ unknowns. The solution space has dimension one and there exists a unique solution $\vec{p}(z)$ having last component equal to one. The final solution $\vec{x}(z)$ of the block Hankel system of equations is given as

$$\vec{x}(z) = [I_q \;\; 0 \;\; 0] B_{\vec{\tau}}(z) \vec{p}(z). \tag{7}$$

Suppose now that the last perfect point for $l = \beta, \beta-1, \ldots, 0$ is not $\vec{\tau} - l^*\vec{e} = \vec{\tau}$ but a previous point $\vec{\tau} - l\vec{e}$ with $l > 0$. Now, $\vec{p}(z)$ is the unique solution of

$$C^r R_{\vec{\tau}-l\vec{e}}(z)\, \vec{p}(z) = O_+(z^{\vec{\nu}}), \quad r = \| \operatorname{proj}(\vec{\tau} - l\vec{e})\| \bmod s \tag{8}$$

with

$$\begin{aligned} F(z) B_{\vec{\tau}-l\vec{e}}(z) &= z^{\vec{\sigma}'} R_{\vec{\tau}-l\vec{e}}(z), \\ \vec{\sigma}' &= \operatorname{vect}(\| \operatorname{proj}(\vec{\tau} - l\vec{e})\| + s), \\ \vec{\nu} &= \operatorname{vect}(s(\alpha+\beta+1) - \|\vec{\sigma}'\|), \end{aligned}$$

i.e. $\|\vec{\nu}\| = q(l+1) + sl$ and $(l\vec{e} - \vec{e}_s - l\vec{u}_{s+q+1})$-deg $\vec{p}(z) \leq 0$ (for $l < \beta$) and the last component of $\vec{p}(z)$ is equal to one ($u_i = (0, 0, \ldots, 1, \ldots, 0)$ with the one at position i). Translating equation (8) into a linear algebra problem results in a set of $q(l+1)+sl$ linear homogeneous equations with $q(l+1)+sl+1$ unknowns. Scaling the last component to be one, gives us the unique $\vec{p}(z)$ which together with equation (7) leads to the final solution of the block Hankel problem. In case the block Hankel matrix H is singular, the solution space of (6) and (8) will have dimension greater than one. From the basis vectors with last component scaled to be one, all solutions of the block Hankel problem can be expressed using equation (7).

We could also solve the block Hankel system by using the inversion formula of Gohberg and Shalom [11]. The entities needed in this inversion formula can be immediately taken out of the $\vec{\tau}$-reduced basis matrix for $S^{\vec{\sigma}}$ given $F(z)$ with

$$\begin{aligned} F(z) &= [H_{-1} + zH(z) \;\; -I_s], \quad \text{with } H_{-1} \text{ and } H_{\alpha+\beta+1} \text{ arbitrary} \\ \vec{\tau} &= (\beta+1)\vec{e} \\ \vec{\sigma} &= (\alpha+\beta+3)\vec{\epsilon}. \end{aligned}$$

Hence, we can use our algorithm to compute this basis matrix going from one perfect point to the next one. Note that the last point will be perfect if the block Hankel matrix is nonsingular. In [14], Labahn also gives an algorithm jumping from one perfect point to another one. However, he only takes into account the perfect points $\vec{\tau}$ such that $\vec{\sigma} = \operatorname{vect}(\|\vec{\tau}\| + s)$ has equal components. This means that the number of interpolation conditions which is added in each step is at least equal to the common multiple of s and q multiplied by $(1 + s/q)$, while our algorithm adds at least $s + q$ interpolation conditions. Take, e.g.,

$s = 5$ and $q = 7$, then the method of Labahn uses a jump size of at least $35(1 + 3/5) = 56$ while our algorithm uses a jump size of at least 12. Hence, if there are intermediate perfect points between the perfect points found by the algorithm of Labahn our algorithm will be more efficient. When the blocks are square, this difference disappears. We refer the interested reader to [15] where the inversion components of block Hankel-like matrices are computed.

9 Conclusion

In this paper, we have constructed an algorithm going from one perfect point to another on a diagonal path in a vector Padé-Hermite table. We have indicated that a slightly modified version of the algorithm can also be used to compute basis matrices for "well-conditioned" perfect points or to compute the solution of a block Hankel system of equations.

References

[1] G.A. Baker, Jr. and P.R. Graves-Morris. *Padé Approximants. Part I: Basic Theory*, volume 13 of *Encyclopedia Math. Appl.* Addison-Wesley, Reading, 1981.

[2] G.A. Baker, Jr. and P.R. Graves-Morris. *Padé Approximants. Part II: Extensions and Applications*, volume 14 of *Encyclopedia Math. Appl.* Addison-Wesley, Reading, MA, 1981.

[3] B. Beckermann. The structure of the singular solution table of the M-Padé approximation problem. *J. Comput. Appl. Math.*, 32(1 & 2):3–15, 1990.

[4] B. Beckermann. A reliable method for computing M-Padé approximants on arbitrary staircases. *J. Comput. Appl. Math.*, 40:19–42, 1992.

[5] B. Beckermann and G. Labahn. A uniform approach for Hermite-Padé and simultane ous Padé approximants and their matrix-type generalizations. *Numer. Alg.*, 3(1-4):45–54, 1992.

[6] S. Cabay, M.H. Gutknecht, and R. Meleshko. Stable rational interpolation? Technical Report 93-12, IPS-ETH, Zürich, 1993. Presented at MTNS'93.

[7] S. Cabay and G. Labahn. A superfast algorithm for multi-dimensional Padé systems. *Numerical Algorithms*, 2:201–224, 1992.

[8] S. Cabay, G. Labahn, and B. Beckermann. On the theory and computation of nonperfect Padé-Hermite approximants. *J. Comput. Appl. Math.*, 39:295–313, 1992.

[9] S. Cabay and R. Meleshko. A weakly stable algorithm for Padé approximants and the inversion of Hankel matrices. *SIAM J. Matrix Anal. Appl.*, 14:735–765, 1993.

[10] J. Della Dora and C. di Crescenzo. Approximants de Padé-Hermite. *Numer. Math.*, 43:23–57, 1984.

[11] I.C. Gohberg and T. Shalom. On Bezoutians of nonsquare matrix polynomials and inversion of matrices with nonsquare blocks. *Lin. Alg. Appl.*, pages 249–323, 1990.

[12] W.B. Gragg. The Padé table and its relation to certain algorithms of numerical analysis. *SIAM Rev.*, 14:1–62, 1972.

[13] H. Jager. A multidimensional generalization of the Padé table. *Proc. Kon. Nederl. Akad. Wet.*, 67:193–249, 1964. (Indagationes Mathematicae, 26).

[14] G. Labahn. Inversion algorithm for rectangular-block Hankel matrices. Technical Report CS-90-52, Univ. of Waterloo, Dept. Computer Sci., 1990.

[15] G. Labahn. Inversion components of block Hankel-like matrices. *Lin. Alg. Appl.*, 177:7–48, 1992.

[16] G. Labahn and S. Cabay. Matrix Padé fractions and their computation. *SIAM J. of Computing*, 18(4):639–657, 1989.

[17] K. Mahler. Perfect systems. *Compositio Math.*, 19:95–166, 1968.

[18] S. Paszkowski. Recurrence relations in Padé-Hermite approximation. *J. Comput. Appl. Math.*, 19(1):99–107, 1987.

[19] M. Van Barel and A. Bultheel. A new approach to the rational interpolation problem: the vector case. *J. Comput. Appl. Math.*, 33(3):331–346, 1990.

[20] M. Van Barel and A. Bultheel. The computation of non-perfect Padé-Hermite approximants. *Numer. Alg.*, 1:285–304, 1991.

[21] M. Van Barel and A. Bultheel. A general module theoretic framework for vector M-Padé and matrix rational interpolation. *Numer. Alg.*, 3:451–461, 1992.

A Proof of theorem 4.2

When $F(0)$ has full rank, we have the following result for the constant coefficient of a basis matrix $B^{\vec{\sigma}}(z)$.

Theorem A.1 *Given $F(z) \in \mathbb{F}[[z]]^{s\times m}$ with $s < m$. Take any basis matrix $B^{\vec{\sigma}}(z)$ for the submodule $S^{\vec{\sigma}}$ with*

$$S^{\vec{\sigma}} = \{\vec{p}(z) \in \mathbb{F}[z]^{m\times 1} \,|\, F(z)\vec{p}(z) = z^{\vec{\sigma}} R(z) \text{ with } R(z) \in \mathbb{F}[[z]]^{s\times m}\}.$$

If the columns $i_1, i_2, \ldots, i_s$ of $F(0)$ are linearly independent, then the $(m-s)\times m$ matrix $DB^{\vec{\sigma}}(0)$ obtained by deleting rows $i_1, i_2, \ldots, i_s$ in $B^{\vec{\sigma}}(0)$ has full rank. The matrix D is obtained by deleting rows $i_1, i_2, \ldots, i_s$ in the $m \times m$ identity matrix. Moreover,

$$\begin{bmatrix} R(0) \\ DB^{\vec{\sigma}}(0) \end{bmatrix}$$

is nonsingular. Hence, $R(0)$ and $DB^{\vec{\sigma}}(0)$ have full rank. Also, $\deg \det B^{\vec{\sigma}}(z) = \|\vec{\sigma}\|$, *i.e. all interpolation conditions are linearly independent.*

proof. Consider the following equality

$$\begin{bmatrix} F(z) \\ D \end{bmatrix} B^{\vec{\sigma}}(z) = \begin{bmatrix} z^{\vec{\sigma}} R(z) \\ DB^{\vec{\sigma}}(z) \end{bmatrix}$$

or

$$B^{\vec{\sigma}}(z) = \begin{bmatrix} F(z) \\ D \end{bmatrix}^{-1} \begin{bmatrix} z^{\vec{\sigma}} R(z) \\ DB^{\vec{\sigma}}(z) \end{bmatrix}.$$

Because

$$\begin{aligned} \deg \det B^{\vec{\sigma}}(z) &= O_-(z^{\|\vec{\sigma}\|}), \\ \deg \det \begin{bmatrix} F(z) \\ D \end{bmatrix}^{-1} &= O_+(z^0), \\ \deg \det \begin{bmatrix} z^{\vec{\sigma}} R(z) \\ DB^{\vec{\sigma}}(z) \end{bmatrix} &= O_+(z^{\|\vec{\sigma}\|}), \end{aligned}$$

we get that $\deg \det B^{\vec{\sigma}}(z) = cz^{\|\vec{\sigma}\|}$ with $0 \neq c \in \mathbb{F}$ and

$$\begin{bmatrix} R(0) \\ DB(0) \end{bmatrix}$$

is nonsingular. Hence, $DB(0)$ and $R(0)$ have full rank. I.e., the rank is respectively $m - s$ and s. □

The proof of theorem 4.2 goes as follows.
proof. The polynomial matrix $B^{\vec{\sigma}}(z) \in \mathbb{F}[z]^{m\times m}$ is a $\vec{\tau}$-reduced basis matrix for $S^{\vec{\sigma}}$ given $F(z)$ iff

- $F(z)B^{\vec{\sigma}}(z) = z^{\vec{\sigma}} R(z)$, with $R(z) \in \mathbb{F}[[z]]^{s\times m}$,
- $\deg \det B^{\vec{\sigma}}(z) = \|\vec{\sigma}\|$,
- $B^{\vec{\sigma}}(z) = z^{\vec{\tau}} B_h^{\vec{\sigma}}(z) z^{\vec{\alpha}}$, with $B_h^{\vec{\sigma}}(z) = O_-(z^0)$ and $\deg \det B_h^{\vec{\sigma}}(z) = 0$.

$\Leftarrow$ The polynomial matrix $B^{\vec{\nu}}(z)$ is a $(-\vec{\alpha})$-reduced basis matrix for $S^{\vec{\nu}}$ given $R(z)$ iff

- $R(z)B^{\vec{\nu}}(z) = z^{\vec{\nu}}R'(z)$, with $R'(z) \in \mathbb{F}[[z]]^{s\times m}$,
- $\deg \det B^{\vec{\nu}}(z) = \|\vec{\nu}\|$,
- $B^{\vec{\nu}}(z) = z^{-\vec{\alpha}}B_h^{\vec{\nu}}(z)z^{\vec{\beta}}$, with $B_h^{\vec{\nu}}(z) = O_-(z^0)$ and $\deg \det B_h^{\vec{\nu}}(z) = 0$.

Then the polynomial matrix $B^{\vec{\sigma}+\vec{\nu}}(z) = B^{\vec{\sigma}}(z)B^{\vec{\nu}}(z)$ is a $\vec{\tau}$-reduced basis matrix for $S^{\vec{\sigma}}$ given $F(z)$ because

- $F(z)B^{\vec{\sigma}+\vec{\nu}}(z) = F(z)(B^{\vec{\sigma}}(z)B^{\vec{\nu}}(z)) = z^{\vec{\sigma}}R(z)B^{\vec{\nu}}(z) = z^{\vec{\sigma}+\vec{\nu}}R'(z)$,
- $\deg \det B^{\vec{\sigma}}(z) = \|\vec{\sigma}\| + \|\vec{\nu}\| = \|\vec{\sigma}+\vec{\nu}\|$,
- $B^{\vec{\sigma}+\vec{\nu}}(z) = (z^{\vec{\tau}}B_h^{\vec{\sigma}}(z)z^{\vec{\alpha}})(z^{-\vec{\alpha}}B_h^{\vec{\nu}}(z)z^{\vec{\beta}}) = z^{\vec{\tau}}B_h^{\vec{\sigma}+\vec{\nu}}(z)z^{\vec{\beta}}$, with $B_h^{\vec{\sigma}+\vec{\nu}}(z) = B_h^{\vec{\sigma}}(z)B_h^{\vec{\nu}}(z) = O_-(z^0)$ and $\deg \det B_h^{\vec{\sigma}+\vec{\nu}}(z) = 0$.

$\Rightarrow$ In a similar way as for the "if" part, one can show that $B^{\vec{\nu}}(z) = (B^{\vec{\sigma}}(z))^{-1}B^{\vec{\sigma}+\vec{\nu}}(z)$ satisfies all the conditions to be a $(-\vec{\alpha})$-reduced basis matrix for $S^{\vec{\nu}}$ given $R(z)$ except that it is a polynomial matrix. With the notation of the previous theorem, we have the following equality

$$\begin{bmatrix} R(z) \\ DB^{\vec{\sigma}}(z) \end{bmatrix} B^{\vec{\nu}}(z) = \begin{bmatrix} z^{\vec{\nu}}R'(z) \\ DB^{\vec{\sigma}+\vec{\nu}}(z) \end{bmatrix}$$

or

$$\begin{aligned} B^{\vec{\nu}}(z) &= \begin{bmatrix} R(z) \\ DB^{\vec{\sigma}}(z) \end{bmatrix}^{-1} \begin{bmatrix} z^{\vec{\nu}}R'(z) \\ DB^{\vec{\sigma}+\vec{\nu}}(z) \end{bmatrix} \\ &= O_+(z^0)O_+(z^0) \\ &= O_+(z^0). \end{aligned}$$

Hence, $B^{\vec{\nu}}(z)$ is polynomial.

□

B Proof of theorem 5.1

proof. Let us first show the last point, i.e., that $\pi_j = \tau_j$, $j = 1, 2, \ldots, s$. Suppose there is some k, $1 \le k \le s$ such that $\pi_k \neq \tau_k$. Note that this is only possible when $0 = \pi_k > \tau_k$. Hence, $B_{\vec{\tau}}(0)$ has the following form

$$B_{\vec{\tau}}(0) = \left[\begin{array}{c|c} 0\ldots0 & * \quad * \quad *\ldots* \\ \hline 0_{(m-s)\times s} & I_{m-s} \end{array}\right] \longleftarrow k \qquad (9)$$

We prove now that all $*$-entries are zero. Suppose the (k,j)-entry $(0 < k \leq s < j \leq m)$ of $B_{\vec{\tau}}(0)$ is different from zero.
Because $F(0)B_{\vec{\tau}}(0) = 0$, we could switch columns k and j of $F(z)$, keeping $F_L(0)$ nonsingular.
Hence, $\tau_j \leq \tau_k < 0$ ($\sum_{i=1}^{s} \tau_i$ is maximal) or the j-th row of $B_{\vec{\tau}}(z)$ contains only terms of degree 0.
Hence, by expanding the determinant of $B_{\vec{\tau}}(z)$ along row j, we get that it is equal to the determinant of a square matrix of size $m-1$, having the same form as (9).
Hence, by induction on the dimension of these matrices, one can prove that, if $0 = \pi_k > \tau_k$ with $1 \leq k \leq s$, then $\det B_{\vec{\tau}}(z) = 0$ which contradicts the fact that $B_{\vec{\tau}}(z)$ is a basis matrix. Hence $\pi_j = \tau_j$, for all $j = 1, 2, \ldots, s$.
To prove the remainder of the theorem, we assume without loss of generality that $0 = \pi_j > \tau_j$, $j = s+1, \ldots, s+k$ and $\pi_j = \tau_j$ for other j values. Hence, the $\vec{\pi}$-reduced basis matrix $B_{\vec{\tau}}(z)$ can be written as

$$B_{\vec{\tau}}(z) = z^{\vec{\pi}} \left[\begin{array}{c|c|c} * & * & * \\ \hline 0 & I_k & 0 \\ \hline * & * & * \end{array}\right] z^{\vec{e}_s} = z^{\vec{\pi}} B_{\vec{\tau},h}(z) z^{\vec{e}_s}$$

with $\deg \det B_{\vec{\tau},h}(z) = 0$ and $B_{\vec{\tau},h}(z) = O_-(z^0)$ (componentwise). Therefore, $B_{\vec{\tau}}(z)$ can also be written as a $\vec{\tau}$-reduced basis matrix,

$$\begin{aligned} B_{\vec{\tau}}(z) &= z^{\vec{\tau}}[z^{\vec{\pi}-\vec{\tau}} B_{\vec{\tau},h}(z) z^{\vec{\tau}-\vec{\pi}}] z^{\vec{\pi}-\vec{\tau}+\vec{e}_s} \\ &= z^{\vec{\tau}} B'_{\vec{\tau},h}(z) z^{\vec{e}_s+\vec{\pi}-\vec{\tau}} \end{aligned}$$

where from the structure of $B_{\vec{\tau},h}(z)$, it follows that also $\deg \det B'_{\vec{\tau},h}(z) = 0$ and $B'_{\vec{\tau},h}(z) = O_-(z^0)$. This proves the theorem. □

C Proof of theorem 5.2

proof. Similar to the first part of the proof of Theorem 4.3, we can show that $\vec{\tau} + k\vec{e}$ is a perfect point for $F(z)$ iff $\vec{\delta} = (k\vec{e} - \vec{e}_s - (\vec{\pi} - \vec{\tau}))$ is a perfect point for the permuted residual series $R^c_{\vec{\tau}}(z)$.
Let us assume that $B_{\vec{\delta}}(z)$ is the unique $\vec{\delta}$-reduced basis matrix for the perfect point $\vec{\delta}$ in the vector Padé-Hermite table for $R^c_{\vec{\tau}}(z)$. The $\vec{\delta}$-column degrees of the columns of $B_{\vec{\delta}}(z)$ for a perfect point $\vec{\delta}$ are given as $\vec{e}_s + \text{proj}(\vec{\delta}) - \vec{\delta}$ which is equal to $\vec{e}_s + (\vec{\pi}' - \vec{\tau} - k\vec{e})$ as we shall show now.
There are two possibilities for each component of $\vec{\pi}$ and $\vec{\tau}$.

- $\tau_i = \pi_i \geq 0$. Hence

$$\begin{aligned} (\text{proj}(\vec{\delta}) - \vec{\delta})_i &= \text{proj}(k\vec{e} - \vec{e}_s)_i - (k\vec{e} - \vec{e}_s)_i \\ &= 0 \\ &= \text{proj}(\vec{\tau} + k\vec{e})_i - (\vec{\tau} + k\vec{e})_i \\ &= \vec{\pi}'_i - (\vec{\tau} + k\vec{e})_i; \end{aligned}$$

- $\tau_i < \pi_i = 0$ $(s+1 \leq i \leq m)$. Therefore,

$$\begin{aligned}(\mathrm{proj}(\vec{\delta}) - \vec{\delta})_i &= \mathrm{proj}(\vec{\tau} + k\vec{e})_i - (\vec{\tau} + k\vec{e})_i \\ &= \vec{\pi}'_i - (\vec{\tau} + k\vec{e})_i.\end{aligned}$$

Hence, we have shown that

$$\mathrm{proj}(\vec{\delta}) - \vec{\delta} = \vec{\pi}' - (\vec{\tau} + k\vec{e}). \tag{10}$$

In the sequel, we prove that if $B_{\vec{\delta}}(z)$ is connected to the perfect point $\vec{\delta}$ for $R^c_{\vec{\tau}}(z)$, the polynomial matrix $B_{\vec{\tau}}(z)B_{\vec{\delta}}$ is a $(\vec{\tau} + k\vec{e})$-reduced basis matrix connected to the perfect point $(\vec{\tau} + k\vec{e})$ for $F(z)$.
First, we show that the correct number of interpolation conditions is satisfied, i.e. we have to show that

$$\|\vec{\pi}'\| + s = (\|\vec{\pi}\| + s) + (\|\,\mathrm{proj}(\vec{\delta})\| + s).$$

We get

$$\begin{aligned}(\|\vec{\pi}\| + s) + (\|\,\mathrm{proj}(\vec{\delta})\| + s) &= (\|\vec{\pi}\| + s) + (\|\vec{\delta}\| + \|\vec{\pi}'\| - \|\vec{\tau} + k\vec{e}\| + s) \text{ (see (10))} \\ &= (\|\vec{\pi}\| + s) + ((\|-\vec{e}_s\| - \|\vec{\pi}\| + \|\vec{\tau} + k\vec{e}\|) \\ &\qquad + \|\vec{\pi}'\| - \|\vec{\tau} + k\vec{e}\| + s) \text{ (def. of } \vec{\delta}) \\ &= \|\vec{\pi}'\| + s.\end{aligned}$$

In the remaining part of the proof, we show that $B_{\vec{\tau}}(z)B_{\vec{\delta}}(z)$ is $(\vec{\tau}+k\vec{e})$-reduced with $(\vec{\tau} + k\vec{e})$-column degrees $\vec{e}_s + \vec{\pi}' - (\vec{\tau} + k\vec{e})$.
Because $B_{\vec{\tau}}(z)$ is $\vec{\tau}$-reduced with $\vec{\tau}$-column degrees $\vec{e}_s+\vec{\pi}-\vec{\tau}$ and $B_{\vec{\delta}}$ is $\vec{\delta}$-reduced with $\vec{\delta}$-column degrees $\vec{e}_s + \vec{\pi}' - (\vec{\tau} + k\vec{e})$, we can write

$$\begin{aligned}B_{\vec{\tau}}(z)B_{\vec{\delta}}(z) &= (z^{\vec{\tau}}B_{\vec{\tau},h}(z)z^{\vec{e}_s+\vec{\pi}-\vec{\tau}})(z^{k\vec{e}-\vec{e}_s-\vec{\pi}+\vec{\tau}}B_{\vec{\delta},h}(z)z^{\vec{e}_s+\vec{\pi}'-\vec{\tau}-k\vec{e}}) \\ &= z^{\vec{\tau}+k\vec{e}}B_{\vec{\tau},h}(z)B_{\vec{\delta},h}(z)z^{\vec{e}_s+\vec{\pi}'-\vec{\tau}-k\vec{e}}\end{aligned}$$

with $\deg \det(B_{\vec{\tau},h}(z)B_{\vec{\delta},h}(z)) = 0$ and $B_{\vec{\tau},h}(z)B_{\vec{\delta},h}(z) = O_-(z^0)$.
Hence, $B_{\vec{\tau}}(z)B_{\vec{\delta}}$ is $(\vec{\tau}+k\vec{e})$-reduced with $(\vec{\tau}+k\vec{e})$-column degrees $\vec{e}_s+\vec{\pi}'-(\vec{\tau}+k\vec{e})$.
It is easy to see that if $B_{\vec{\tau}}(z)$ and $B_{\vec{\delta}}(z)$ are both the unique basis matrices in the sense of theorem 5.1, also their product is the unique basis matrix. This proves the "if" part of the theorem.
The "and only if" part is proved similarly. □