Schopenhauer's Partition Diagrams and Logical Geometry

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Abstract. The paper examines Schopenhauer's complex diagrams from the Berlin Lectures of the 1820s, which show certain partitions of classes. Drawing upon ideas and techniques from logical geometry, we show that Schopenhauer's partition diagrams systematically give rise to a special type of Aristotelian diagrams, viz. (strong) α-structures.

Keywords: Arthur Schopenhauer · logic diagrams · partition diagram · Aristotelian diagram · bitstring semantics · logical geometry.

1 Introduction

For almost 200 years the name of the philosopher Arthur Schopenhauer, who was born in Gdansk in 1788 and died in Frankfurt in 1860, was not associated with logic. It was not until the middle of the 2010s that it became known that for his Berlin lectures in the 1820s, Schopenhauer composed a treatise on logic that covered the scope of an entire book. Since the discovery of this 'logica maior', philosophers, linguists and logicians have made numerous discoveries in Schopenhauer's work, of which only a few are mentioned here as examples: Schopenhauer's logic anticipates several important linguistic principles that later became prominent through the Vienna Circle [12], the Lvov-Warsaw School [11], and generative grammar [14]. Long before John Venn, Schopenhauer drew complex logic diagrams for n terms [23] and, at the same time as Joseph Gergonne, he extended Euler diagrams to the so-called Gergonne relations [26]. Furthermore, Schopenhauer already used logical notations which could have paved the way to mathematical logic towards the end of the 19th century, had they been known at that time [18].

However, Schopenhauer's logic is not only interesting from a historical point of view, but also offers numerous systematic points of departure for taking ideas further, rethinking old ones or developing new ones. In recent years, for example, Schopenhauer's approach has been modernized with the help of transition rules of elementary cellular automata [24]. A formalism called 'Schopenhauer diagrams' was developed to analyze processes of abstraction and reification in ontology and conceptual engineering [13,22]. For the present paper it is particularly noteworthy that a number of these Schopenhauer diagrams can provide general insights and theorems about Aristotelian diagrams in the contemporary framework of logical geometry [6].

In the present paper, we will further develop this latter approach, by underpinning it with new material from Schopenhauer's original manuscripts. We will show how Schopenhauer came to the modern idea of investigating not only Boolean algebras consisting of *propositions*, but also of *sets*. In particular, we will present a series of logic diagrams for complex partitions that Schopenhauer drew in his *Berlin Lectures*, and argue that these partition diagrams closely correspond to what are nowadays sometimes called (strong) α-structures [6,27]. Typical partition diagrams can be found below in Figs. 3 and 5.

The paper is organized as follows. We start in Section 2 by briefly discussing some key notions from logical geometry that are useful for studying Schopenhauer's logic. In Section 3 we then introduce Schopenhauer's logica maior, and describe in particular the logical context of those passages that are important for our argument. In Section 4 we show how Schopenhauer uses logic diagrams to visualize set partitions. Finally, in Section 5 we focus on a particularly interesting partition diagram from Schopenhauer's manuscripts, and argue that it gives rise to several (strong) α-structures, including a strong α_7-structure. Roughly speaking, Sections 3 and 4 are primarily historically oriented, whereas Sections 2 and 5 are of more systematic interest.

2 Aristotelian Relations and α-Structures

This section introduces some key notions from logical geometry that have turned out to be very fruitful for studying Schopenhauer's logic [6], and that will also take center stage in Sections 4 and 5 of the present paper. We begin by discussing the *Aristotelian relations*, which can be characterized with various degrees of abstractness and generality [5,8]. For the purposes of this paper, it will be useful to consider a very general definition, in the setting of Boolean algebra [17].

Definition 1. *Let $\mathbb{B} = \langle B, \wedge, \vee, \neg, \top, \bot \rangle$ be an arbitrary Boolean algebra. Two elements $x, y \in B$ are said to be*

$\mathbb{B}$-contradictory	*iff*	$x \wedge y = \bot$	*and*	$x \vee y = \top$,
$\mathbb{B}$-contrary	*iff*	$x \wedge y = \bot$	*and*	$x \vee y \neq \top$,
$\mathbb{B}$-subcontrary	*iff*	$x \wedge y \neq \bot$	*and*	$x \vee y = \top$,
in $\mathbb{B}$-subalternation	*iff*	$\neg x \vee y = \top$	*and*	$x \vee \neg y \neq \top$.

More informal and familiar characterizations of the Aristotelian relations can be obtained from this definition by plugging in concrete Boolean algebras for $\mathbb{B}$. For example, we can take $\mathbb{B}$ to be a Boolean algebra of *propositions*. In this case, two propositions P and Q being contrary means that $P \wedge Q$ is contradictory while $P \vee Q$ is not tautological, i.e. P and Q cannot be true together, but can be false together. Similarly, there is a subalternation from P to Q iff P entails Q but not vice versa. For a second example, we can take $\mathbb{B}$ to be a Boolean algebra of *sets*, e.g. the powerset $\wp(D)$ of some domain of discourse D. In this second case, two sets X and Y being contrary means that $X \cap Y = \emptyset$ while $X \cup Y \neq D$, i.e. X and Y are disjoint but do not exhaust D. Similarly, there is a subalternation from X to Y iff $X \subseteq Y$ but not $X \supseteq Y$. The fact that Definition 1 allows us to deal not only with Aristotelian relations between propositions (as is usually done), but also between sets, will be absolutely crucial when we turn to Schopenhauer's diagrams. After all, the latter also represent relations between sets, viz. spheres/extensions of concepts (cf. Section 3). Concrete examples, which we will also discuss further, are e.g. the contrariety between fish and bird, the subalternation from bird to vertrebrate, the contradiction between fish and ¬fish, and the subcontrariety between ¬fish and ¬bird.

Now that the Aristotelian relations have been defined relative to arbitrary Boolean algebras, we can likewise define the class of Aristotelian diagrams and one of its important subclasses: the so-called α-structures or α-diagrams [27].[3]

Definition 2. *Let $\mathbb{B}$ be as before, and consider a fragment $\mathcal{F} \subseteq B \backslash \{\top, \bot\}$. Suppose that $\mathcal{F}$ is closed under $\mathbb{B}$-complementation, i.e. if $x \in \mathcal{F}$ then $\neg x \in \mathcal{F}$. An* Aristotelian diagram for $\mathcal{F}$ in $\mathbb{B}$ *is a diagram that visualizes an edge-labeled graph $\mathcal{G}$. The vertices of $\mathcal{G}$ are the elements of $\mathcal{F}$, and the edges of $\mathcal{G}$ are labeled by the Aristotelian relations between those elements, i.e. if $x, y \in \mathcal{F}$ stand in some Aristotelian relation, then this is visualized according to the code in Fig. 1(a).*

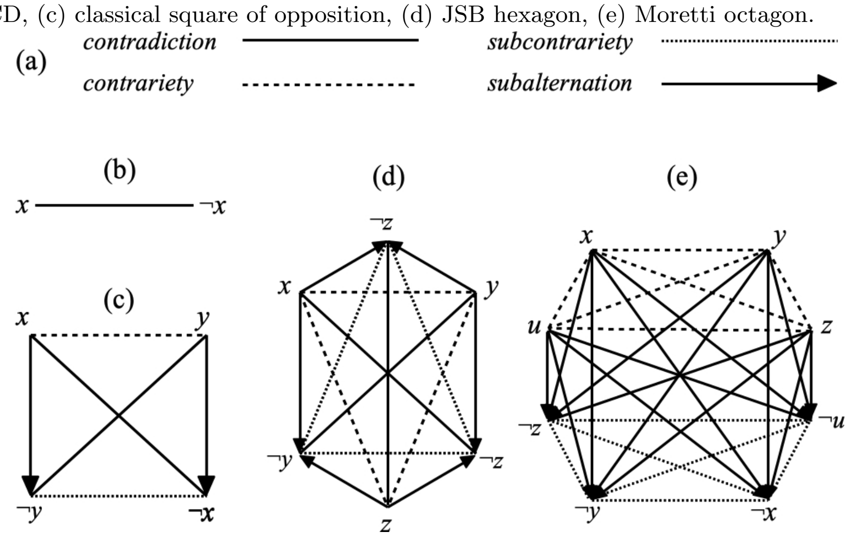
Definition 3. *Let $\mathbb{B}$ be as before, and consider a natural number $n \geq 1$. An* α_n-structure in $\mathbb{B}$ *is an edge-labeled graph $\mathcal{G}$. The vertices of $\mathcal{G}$ form a fragment $\{x_1, \ldots, x_n, \neg x_1, \ldots, \neg x_n\} \subseteq B \backslash \{\top, \bot\}$, where all distinct x_i, x_j are pairwise $\mathbb{B}$-contrary, i.e. x_i and x_j are $\mathbb{B}$-contrary for all $1 \leq i \neq j \leq n$. The edges of $\mathcal{G}$ are labeled by the Aristotelian relations between those elements. An* α_n-diagram in $\mathbb{B}$ *is an Aristotelian diagram that visualizes such an α_n-structure in $\mathbb{B}$.*

Note that by Definition 2, Aristotelian diagrams are closed under complementation and only contain non-trivial elements (i.e. neither $\top$ nor $\bot$). The historical and systematic reasons for these restrictions are discussed in more detail in [33, Subsection 2.1]. Furthermore, the condition in Definition 3 regarding pairwise $\mathbb{B}$-contrariety between all distinct x_i, x_j immediately implies that there are several other Aristotelian relations in an α_n-structure as well. In particular, it follows that $\neg x_i$ and $\neg x_j$ are $\mathbb{B}$-subcontrary and that there are $\mathbb{B}$-subalternations from x_i to $\neg x_j$, for all $1 \leq i \neq j \leq n$. And of course, as in any Aristotelian diagram, it holds that x_i and $\neg x_i$ are $\mathbb{B}$-contradictory, for all $1 \leq i \leq n$. Several of the most well-known Aristotelian diagrams are indeed α-structures:

- The α_1-structure is simply a *pair of contradictory elements* (PCD); cf. Fig. 1(b). PCDs do not frequently appear in the literature, but they have considerable theoretical importance, since they can be thought of as the fundamental 'building blocks' for all other, larger Aristotelian diagrams [7,9].

[3] Strictly speaking, the term 'α-structure' refers to the (abstract) underlying graph, while the term 'α-diagram' refers to the (concrete) diagram. However, this distinction will not matter much in this paper, so we will usually not distinguish between these two terms, and follow Moretti [27] in simply talking about 'α-structures'.

Fig. 1. (a) code for visually representing the Aristotelian relations; examples of (b) PCD, (c) classical square of opposition, (d) JSB hexagon, (e) Moretti octagon.



– The α_2-structure is a *classical square of opposition*; cf. Fig. 1(c). Without a doubt, this is the oldest and most well-known type of Aristotelian diagram.
– The α_3-structure is a so-called *Jacoby-Sesmat-Blanché (JSB) hexagon*, which is named after Jacoby [19], Sesmat [32] and Blanché [2]; cf. Fig. 1(d). After the classical square, this is the most well-known type of Aristotelian diagram.
– The α_4-structure is a so-called *Moretti octagon*, which is named after Moretti [27] (who drew it as a cube, rather than an octagon); cf. Fig. 1(e).

One of the key insights of logical geometry is that a given family of Aristotelian diagrams can have multiple *Boolean subtypes*, i.e. it is possible for two Aristotelian diagrams to exhibit exactly the same configuration of Aristotelian relations among their respective sets of elements, and yet have different Boolean properties [10]. The first concrete example of this phenomenon was pointed out by Pellissier [29], who showed that there are two Boolean subtypes of JSB hexagons: in a *strong* JSB hexagon, the join of the three pairwise contrary elements equals $\top$, whereas in a *weak* JSB hexagon, this join is not equal to $\top$. Such Boolean differences are nowadays usually characterized in terms of *bitstrings*, which are formally introduced below (for more details, see [10,34]).

Definition 4. *Let the Boolean algebra* $\mathbb{B}$ *and the fragment* $\mathcal{F}$ *be as before. The partition of* $\mathbb{B}$ *induced by* $\mathcal{F}$ *is defined as* $\Pi_{\mathbb{B}}(\mathcal{F}) := \{\pm x_1 \wedge \cdots \wedge \pm x_n \mid x_1, \ldots, x_n \in \mathcal{F}\} \backslash \{\bot\}$, *where* $+x_i = x_i$ *and* $-x_i = \neg x_i$. *For every* $y \in \mathcal{F}$ *we have* $y = \bigvee \{a_i \in \Pi_{\mathbb{B}}(\mathcal{F}) \mid a_i \leq y\}$. *The bitstring representation of* $y \in \mathcal{F}$ *keeps track of which* $a_i \in \Pi_{\mathbb{B}}(\mathcal{F})$ *enter into this join; for example, if* $\Pi_{\mathbb{B}}(\mathcal{F}) = \{a_1, a_2, a_3, a_4\}$ *and* $y = a_1 \vee a_3 \vee a_4$, *then* y *will be represented as the bitstring* 1011.

Using this technique, one can show, for example, that representing a strong JSB hexagon requires bitstrings of length 3 (so the join of its three pairwise contrary elements is $100 \vee 010 \vee 001 = 111$), whereas a weak JSB hexagon requires bitstrings of length 4 (so the join of its three pairwise contrary elements is $1000 \vee 0100 \vee 0010 = 1110 \neq 1111$). (Again, see [10,34] for more details.)

In general, determining the Boolean subtypes of a given type of Aristotelian diagrams is highly non-trivial [4]. However, for the specific subclass of α-structures, the situation is relatively straightforward, as is summarized by Theorem 1.

Theorem 1. – *All α_1-structures (i.e., PCDs) require bitstrings of length 2,*
- *all α_2-structures (i.e., classical squares) require bitstrings of length 3,*
- *for $n \geq 3$, there are two Boolean subtypes of α_n-structures: (i) a* strong *subtype, which requires bitstrings of length n, and (ii) a* weak *subtype, which requires bitstrings of length $n+1$.*

Note that the important cutoff happens at $n = 3$. This is not a coincidence: because of their *binary* nature, the Aristotelian relations cannot capture the full Boolean complexity that may arise in larger sets [4]. Furthermore, note that the case $n = 3$ says that the family of JSB hexagons has two Boolean subtypes, viz. the strong JSB hexagons (requiring bitstrings of length 3) and the weak JSB hexagons (requiring bitstrings of length 4). In other words, Pellissier's original result on JSB hexagons [29] is thus subsumed as a special case of Theorem 1.

In a Boolean algebra $\mathbb{B} = \langle B, \wedge, \vee, \neg, \top, \bot \rangle$, a finite set $\Pi = \{x_1, \ldots, x_n\} \subseteq B \backslash \{\top, \bot\}$ (with $n \geq 2$) is said to be an *n-partition of* $\mathbb{B}$ iff (i) $x_i \wedge x_j = \bot$ for all distinct $x_i, x_j \in \Pi$ and (ii) $\bigvee \Pi = \top$. There is a clear correspondence between partitions and (strong) α-structures.[4] This is made fully precise in Theorem 2 below. Note that there is again a cutoff at $n = 3$, and that α_2-structures (i.e. classical squares of opposition) do *not* correspond to any partitions.

Theorem 2.

1. *Each 2-partition $\{x, \neg x\}$ gives rise to an α_1-structure with elements $\{x, \neg x\}$.*
2. *For $n \geq 3$, each n-partition $\{x_1, \ldots, x_n\}$ gives rise to a strong α_n-structure with elements $\{x_1, \ldots, x_n, \neg x_1, \ldots, \neg x_n\}$.*

Concretely, each 2-partition corresponds to a PCD, each 3-partition to a strong JSB hexagon, and each 4-partition to a strong Moretti octagon.

3 The Context of Schopenhauer's Partition Diagrams

Schopenhauer's partition diagrams can be found in his logica maior, that is, in the Berlin Lectures that he held in the course of the 1820s. The logic in the Berlin Lectures [31, 234–368] can roughly be divided into four parts. The first part contains a doctrine of concepts [31, 242–260] and enriches classical positions

[4] For another perspective on the correspondence between partitions and Aristotelian diagrams, cf. [35].

of Aristotelian logic especially with diagrams and discussions on philosophy of language. The second part concerns the doctrine of judgement [31, 260–293]. It contains, among other things, treatises on the laws of thought, on truth, conceptual relations and contraposition. The third part concerns the theory of inferences [31, 293–356], in which Schopenhauer uses Euler diagrams to argue for the validity and naturalness of the original Aristotelian syllogistics. Furthermore, Stoic (propositional) logic and Aristotelian modal logic can also be found in this section. The short last part [31, 356–368] contains some additions and remarks about history and philosophy of logic, and about eristic dialectics.[5]

Schopenhauer's partition diagrams are given in the second part, more precisely in the treatise on relations between concepts. This treatise deals with many themes that had become popular through Kantian philosophy, and enriches them with logic diagrams. First, he explains the distinction between analytic and synthetic judgements by means of diagrams [31, 269–272]. The main part of this treatise, however, goes on to deal with Kant's theory of the four properties of judgements: quantity, quality, relation and modality. In contrast to Kant [20, III: 86ff.], Schopenhauer argues that this division should not simply be taken from textbooks of logic, since this leads to numerous problems [25], but that the properties of judgements only become apparent by analyzing the various ways in which two concepts can relate to one another. In order to find these relations, Schopenhauer makes use of geometric figures based on conceptual spheres. He therefore speaks of a 'clue of diagrams' ([31, 272], "Leitfaden sind die Schemata"). As a result, Schopenhauer uses six basic types of relational diagrams (*RD*) which can be described as follows.

Definition 5. *Let a circle or part of a circle in a given diagram represent a conceptual sphere. The diagrams RD1–6 in Figure 2 depict the possible spatial positions of at least two conceptual spheres:*

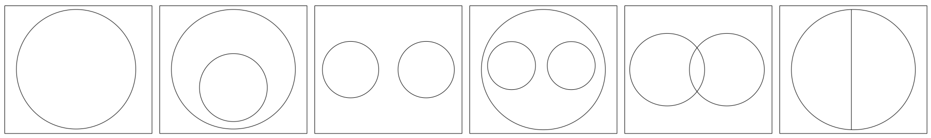
RD1 *Two conceptual spheres exactly overlap, so that only one sphere can be seen.*
RD2 *One conceptual sphere completely contains another conceptual sphere.*
RD3 *Two conceptual spheres are completely disjoint.*
RD4 *A conceptual sphere includes two or more further spheres, such that the included spheres are mutually disjoint but do not exhaust the first sphere.*
RD5 *Two conceptual spheres partly intersect each other.*
RD6 *A conceptual sphere includes two or more further spheres, such that the included spheres do not intersect each other but do exhaust the first sphere.*

From a contemporary perspective, Schopenhauer's relational diagrams *RD1–6* are clearly related to other diagrammatic systems.[6] In particular, *RD2*, *RD3*

[5] Schopenhauer's logica maior is thus structured in a way that had become standard in the history of logic (compare, for example, with the structure of William Ockham's *Summa Logicae* [28] or that of the *Port-Royal Logic* [1]), and that ultimately finds its roots in the division of Aristotle's logical works: (i) *Categories* on concepts/terms, (ii) *On interpretation* on judgements/propositions, and (iii) the remaining four works of the *Organon* on inferences/syllogisms.

[6] A terminological remark: we name a diagrammatic system after an author A to indicate that A drew or described at least some diagrams belonging to this system.

Fig. 2. Schopenhauer's six basic types of relational diagrams.



(a) *RD1* (b) *RD2* (c) *RD3* (d) *RD4* (e) *RD5* (f) *RD6*

and *RD5* are Euler diagrams [15], *RD1*, *RD2*, *RD3* and *RD5* are Gergonne diagrams [16]. Note that *RD4* and *RD6*, which are more or less proper to Schopenhauer, depict strictly more than two conceptual spheres [31, 281].

In the Berlin Lectures, *RD1–6* are used to determine what the basic properties of judgements (PJ) are. For Schopenhauer, a proposition is called a 'judgement' when it expresses the relationship between at least two concepts. The first concept in the judgement is called the 'subject' (S), the second is called the 'predicate' (P). Using his six relational diagrams, Schopenhauer argues that there are six possible relations between two concepts S and P, which are listed below as PJ_S (the S-subscript stands for 'Schopenhauer'):

PJ_S1	universal
PJ_S2	particular
PJ_S3	affirmative

PJ_S4	negative
PJ_S5	disjunctive
PJ_S6	hypothetical

This treatise (and also further ones) shows that Schopenhauer on the one hand clearly orients his doctrine of judgements to Kant, but on the other hand also acts as a strong critic of him. After all, in his *Critique of Pure Reason*, Kant had also dealt with the PJs, which were to structure his entire system in many ways. Kant's so-called 'table of judgements' contains a total number of 12 PJs, grouped into 4 'titles' consisting of 3 PJs each. The titles are ($T1$) Quantity, ($T2$) Quality, ($T3$) Relation and ($T4$) Modality. Kant's 12 properties of judgements are listed below as PJ_K (the K-subscript stands for 'Kant'):

$T1$: Quantity	
PJ_K1	universal
PJ_K2	particular
PJ_K3	singular

$T2$: Quality	
PJ_K4	affirmative
PJ_K5	negative
PJ_K6	infinite

$T3$: Relation	
PJ_K7	categorical
PJ_K8	hypothetical
PJ_K9	disjunctive

$T4$: Modality	
PJ_K10	problematic
PJ_K11	assertoric
PJ_K12	apodictic

Every PJ_S corresponds to some PJ_K, and just like Kant, Schopenhauer also classifies PJ_S1 and PJ_S2 under the title Quantity, PJ_S3 and PJ_S4 under Quality, and PJ_S5 and PJ_S6 under Relation. But unlike Kant, Schopenhauer rejects PJ_K10, PJ_K11, PJ_K12 from the title of Modality, PJ_K7 from Relation, PJ_K6 from Quality and PJ_K3 from Quantity, for various reasons. For example, Schopenhauer does not believe that modality is a property of the judgement, but rather of the one who judges, since modality only indicates the degree of certainty of the judge.

Schopenhauer argues with Aristotle that categoricity, i.e. PJ_K7, is not an independent property of judgements, but rather results from the cross-combination of $PJ_{K/S}1$ and $PJ_{K/S}2$ with PJ_K4/PJ_S3 and PJ_K5/PJ_S4. He thus holds that all judgements which represent a relationship between S and P are categorical. Drawing an analogy to Wittgenstein (TLP, 4.442), Schopenhauer's argument can be reformulated as follows: Kant's notion of 'categorical judgement' is logically quite meaningless; it simply indicates that the uttered proposition concerns a relationship between S and P. Schopenhauer reads from the set of Gergonne diagrams $\{RD1, RD2, RD3, RD5\}$ various instantiations of PJ_S1–4. These four properties of judgements traditionally originate from Aristotelian assertoric syllogistics; cross-combining them yields the following four categorical judgements:

	PJ_S3: affirmative	PJ_S4: negative
PJ_S1: universal	All S is P.	No S is P.
PJ_S2: particular	Some S is P.	Some S is not P.

Schopenhauer excludes the categorical judgements (PJ_K7) from Kant's title of relation ($T3$), but retains the disjunctive (PJ_K8) and hypothetical (PJ_K9) judgements, i.e. those judgements which traditionally do not originate from Aristotelian syllogistics but rather from Stoic logic, and thus correspond in certain aspects to contemporary propositional logic [3]. Schopenhauer thus argues that PJ_K8 and PJ_K9 are not properties of *judgements* in the sense described above; after all, these properties do not concern any relationship between concepts S and P, but rather a relationship between two or more propositions (expressed by means of connectives such as "or" or "if ... then ...").

Nevertheless, Schopenhauer does integrate PJ_K8 and PJ_K9 into his own list (as PJ_S6 and PJ_S5, respectively), because the spatial combinations in the diagrams $RD2$, $RD4$ and $RD6$ provide him with an astonishing insight: connectives can not only be applied to combine *propositions* in order to obtain new, more complex propositions, but also to *concepts* in order to obtain new, more complex concepts. By applying logical connectives to concepts, Schopenhauer is thus able in his Berlin Lectures to develop complex partition diagrams.

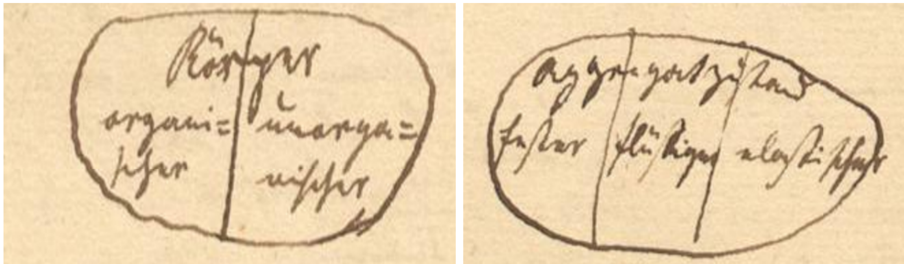
4 Schopenhauer's Partition Diagrams

As we have just seen, Schopenhauer set up different properties of judgements (PJs) through a guideline of schemes, called $RD1-6$. At first he noticed that many PJ_Ks do not have a diagrammatic equivalent, but rather seem to be arranged arbitrarily. Moreover, Schopenhauer realized that Kant's title of relation ($T3$) was problematic in several respects: categorical judgements (PJ_K7) do not have a unique function, but result from the cross-combination of the meaningful PJ under the title of quality ($T1$) and quantity ($T2$), i.e. PJ_K1, PJ_K2, PJ_K4, PJ_K5. Schopenhauer also noticed that hypothetical and disjunctive judgements, i.e. PJ_K8 and PJ_K9, are actually not regarded by Kant as relations between concepts. In the Kantian sense they are relations between judgements, which means that Kant inserts different criteria in his table of judgements.

Schopenhauer, however, argues that this problem can be solved with the help of the 'schematism of spheres'. Assuming that each concept in a judgement is represented by a circle in Euclidean space, in the case of two concepts there are several possible combinations of the two circles, which correspond by Definition 5 to the Gergonne relations. In contrast to $RD1$, in $RD2$ one circle contains a second one, in such a way that a third circle (disjoint from the second one) can be inserted, thus resulting in either $RD4$ or $RD6$, depending on whether the second and third circle jointly exhaust the first one (as in $RD6$) or not (as in $RD4$). As the insertion of further circles in a diagram such as $RD1$ can be repeated indefinitely, it is possible to create increasingly complex shapes in which a circle contains n other, mutually disjoint circles which are either jointly exhaustive of the first one (thus generalizing $RD6$) or they are not (thus generalizing $RD4$).

Schopenhauer is aware that diagrams such as $RD4$ open up many possibilities of interpretation, but he uses such diagrams mainly to prove the PJs under the titles of Quality and Quantity. For example, if the small contained circles in a diagram such as $RD4$ are designated with S_i (e.g. in the case of two small circles: S_1 and S_2) and the large containing circle with P, the following judgements can be read from $RD4$-type diagrams: (J1) All S_i is P; (J2) All that is not P is not S_i; (J3) Some P is S_i; (J4) Some P is not S_i. Next (and continuing the case of two small circles), he realizes that S_i in (J1–4) can actually be interpreted as S_1 *or* S_2. But to express the complex concept S_1 *or* S_2, the diagram must be drawn in such a way that the two small contained circles together completely exhaust the large containing circle (i.e. the complex concept S_1 *or* S_2 coincides exactly with the concept P, so that J4 no longer holds). Schopenhauer visualizes this by means of a large circle that is bisected by a line; cf. $RD6$. The large containing circle represents the concept P, while the two semicircles represent the disjoint and exhaustive subconcepts S_1 and S_2. In general, if a concept has n mutually disjoint and jointly exhaustive subconcepts, the large circle must be divided by $n - 1$ (non-intersecting) lines. We will call such diagrams *partition diagrams*, because the subconcepts are mutually disjoint and jointly exhaustive, and thus constitute a *partition* of the large concept.

Schopenhauer gives several examples, of which we briefly discuss two. First of all, consider the concept 'body', which can be divided into two disjoint and exhaustive subconcepts, viz. 'organic' and 'inorganic'. Schopenhauer thus divides the circle that represents the concept 'body' with a line into two halves that represent the subconcepts 'organic' and 'inorganic'; cf. the partition diagram in



(a) Körper = body, organischer = organic, unorganischer = inorganic.

(b) Aggregatzustand = state of matter, fester = solid, flüßiger = liquid, elastischer = elastic.

Fig. 3. Schopenhauer's partition diagrams for (a) body and (b) state of matter.

Fig. 3a. Secondly, consider the concept 'state of matter', which can be divided into three mutually disjoint and jointly exhaustive subconcepts, viz. 'solid', 'liquid' and 'elastic'.[7] Schopenhauer thus divides the circle that represents the concept 'state of matter' with two lines into three equal parts that represent these three subconcepts; cf. the partition diagram in Fig. 3b.

Schopenhauer equates exclusive disjunction with contradiction, as is often done in contemporary logic as well [3,30,33]. This is based on the law of excluded middle. Using Fig. 3a as an example, Schopenhauer explains this as follows:

> "Here two judgements are connected in such a way that the affirmation of the one is the negation of the other; both can neither be negated nor affirmed at the same time: according to the law of thought of the excluded third." [31, 280]

For Schopenhauer, the partition diagram in Fig. 3a illustrates the contradiction of the two sub-concepts of 'body', whereas the following proposition describes this diagram by means of an exclusive disjunction:

$$\text{All bodies are either organic or inorganic.} \tag{1}$$

But Schopenhauer goes further and shows that the partition diagram not only facilitates knowledge *representation*, but also visual *reasoning*. If one adopts (1) as a premise, and adds an instance such as 'sea sponge' that belongs to the generic concept 'body', one can draw a conclusion including the subordinate concepts 'organic' and 'inorganic'. Schopenhauer takes the following example:

[7] Nowadays we would probably make a different classification, for example we would certainly add 'plasma' to the states of matter. Schopenhauer represents these classifications according to the state of knowledge of the early 19$^{\text{th}}$ century. However, since he knows from the history of science that (structures of) concepts can change, he advocates an ontological relativism even for analytic judgements [21, Chap. 2.2.5f.].

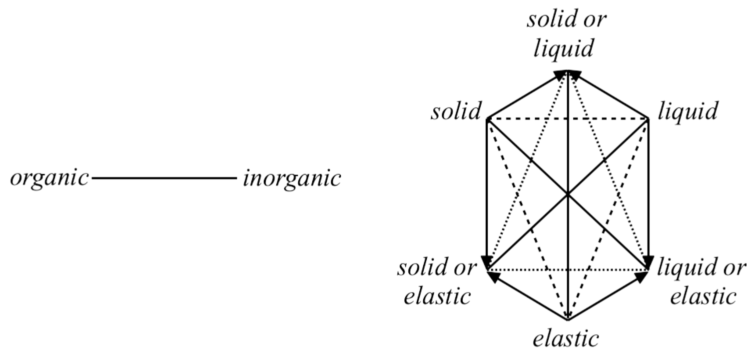


Fig. 4. (a) PCD and (b) strong JSB hexagon corresponding to Schopenhauer's 2- and 3-partitions for body and state of matter, respectively.

A sea sponge is a body. (2)

Thus, a sea sponge is either a organic or an inorganic body. (3)

If we now have further information about the instance of the generic concept, e.g. to which subconcept the sea sponge must be assigned, we can infer (in the sense of Stoic logic) by means of proposition (3) and *modus tollendo ponens* to which subconcept it is not assigned, i.e. we can reason from (4) to (5). Vice versa, we can also use (3) and *modus ponendo tollens* to reason from (5) to (4).

A sea sponge is a organic body. (4)

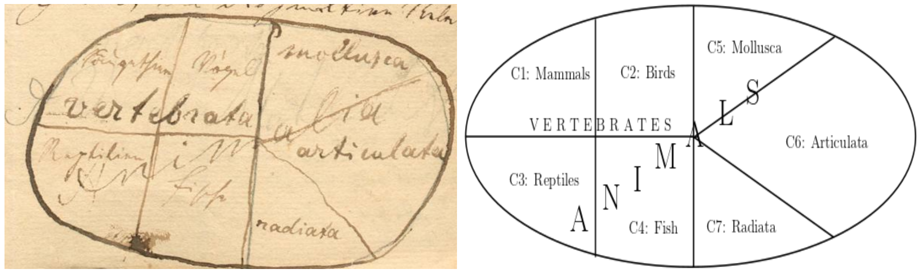
A sea sponge is not an inorganic body. (5)

These ways of reasoning do not seem particularly spectacular, although it can be assumed that the partition diagram provides observational advantages, as it is easier and quicker to read than propositions (1–5). This becomes particularly evident when more complex diagrams are used. Schopenhauer initially drew the diagrams only to represent knowledge. Nevertheless, in several places in the text the added note "Illustrate!" can be found. We can assume that Schopenhauer used the frequently mentioned gesture of indication ('hindeuten') in his lectures to refer to specific regions of a given diagram.[8]

5 From Partition Diagrams to α-Structures

In this section we bring everything together, and show how to apply the insights from logical geometry (cf. Section 2) to Schopenhauer's partition diagrams

[8] Here Schopenhauer still folllows Kant very closely, who uses a similar square diagram in § 29 of the Jäsche logic [20, IX: 108] in order to depict disjunctive judgements. In contrast to Schopenhauer, Kant describes in the text that one should use an x to mark the corresponding region of a disjunctive judgement. Unlike Kant, Schopenhauer's diagrams not only illustrate judgments, but also classes.



(a) Original Diagram (b) Modern Translation

Fig. 5. Schopenhauer's most complex partition diagram

(cf. Sections 3–4). In particular, Theorem 2 tells us that these partition diagrams correspond directly to certain Aristotelian diagrams, viz. α-structures. For example, the 2-partition in Fig. 3a gives rise to an α_1-structure, viz. the PCD in Fig. 4a, while the 3-partition in Fig. 3b gives rise to a strong α_3-structure, viz. the strong JSB hexagon in Fig. 4b.

The most complex partition diagram in Schopenhauer's logica maior is found in [31, 280], and is shown here as Fig. 5a from the original manuscripts. As can also be seen in the translation in Fig. 5b, it shows a large circle for the concept 'animals', which is then subdivided into seven subconcepts (C):

C	Original	Translation	C	Original	Translation
C1	Säugethier	mammals	C5	mollusca	mollusca
C2	Vögel	birds	C6	articulata	articulata
C3	Reptilien	reptiles	C7	radiata	radiata
C4	Fische	fish			

These subconcepts are mutually exclusive and jointly exhaustive, and thus constitute a 7-partition of the concept 'animals'. Appealing once again to Theorem 2, we find that this 7-partition gives rise to a strong α_7-structure, as shown in Fig. 6. This structure consists of the subconcepts C1–C7, which are pairwise contrary to each other, together with their complements (relative to 'animals'), which are pairwise subcontrary to each other. In order to make this more precise, note that C1–C7 can be viewed as the atoms of a Boolean algebra $\mathbb{B}_7$, which can be represented with bitstrings of length 7 [10], i.e. $\mathbb{B}_7$ is isomorphic to $\{0,1\}^7$:

Subconcept	Bitstring	Complementary subconcept	Bitstring
C1: mammals	1000000	$\neg$mammals = animals $\setminus$ mammals	0111111
C2: birds	0100000	$\neg$birds = animals $\setminus$ birds	1011111
C3: reptiles	0010000	$\neg$reptiles = animals $\setminus$ reptiles	1101111
C4: fish	0001000	$\neg$fish = animals $\setminus$ fish	1110111
C5: mollusca	0000100	$\neg$mollusca = animals $\setminus$ mollusca	1111011
C6: articulata	0000010	$\neg$articulata = animals $\setminus$ articulata	1111101
C7: radiata	0000001	$\neg$radiata = animals $\setminus$ radiata	1111110

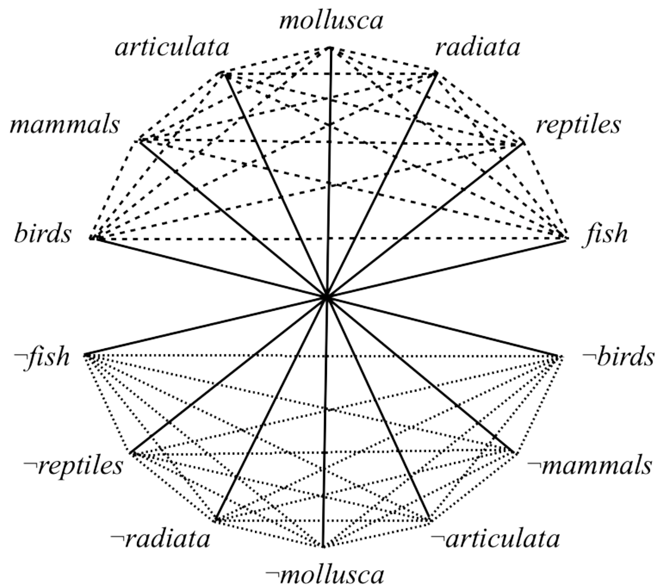


Fig. 6. Strong α_7-structure corresponding to Schopenhauer's 7-partition of 'animals'. For reasons of visual simplicity, the subalternation arrows are not drawn; they go from each 'positive' concept in the upper part of the diagram to each 'negative' concept in the lower part of the diagram (except for its contradictory, of course).

Interestingly, this 7-partition is itself hierarchically organized. Schopenhauer indicates that the subconcepts of 'mammals' (C1), 'birds' (C2), 'reptiles' (C3) and 'fish' (C4) together constitute the intermediate concept 'vertebrates', which is itself a subconcept of 'animals'. The bitstring representation of 'vertebrates' can easily be calculated in terms of the bitstrings of its four subconcepts:

$$\begin{array}{ccccccccc} \text{vertebrates} & = & \text{mammals} & \text{or} & \text{birds} & \text{or} & \text{reptiles} & \text{or} & \text{fish} \\ 1111000 & = & 1000000 & \vee & 0100000 & \vee & 0010000 & \vee & 0001000 \end{array}$$

For certain reasoning purposes it might not be required to subdivide the vertebrates into mammals, birds, reptiles and fish. In those circumstances, such a further subdivision would only yield unnecessary complexity, and should thus be dispensed with. Formally, this means that the original 7-partition of 'animals' reduces to a 4-partition, consisting of the concepts of 'vertebrates', 'mollusca', 'articulata' and 'radiata'. In terms of bitstring representations, this amounts to focusing exclusively on those bitstrings that have identical values in their first four positions, such as $\underline{1111}000$ and $\underline{0000}100$. Equivalently, one could say that we have moved from bitstrings of length 7 (corresponding to the original, fine-grained 7-partition) to bitstrings of length 4 (corresponding to the new, coarser 4-partition), by systematically collapsing the first four bits into a single bit, e.g. $\underline{1111}000$ and $\underline{0000}100$ reduce to $\underline{1}000$ and $\underline{0}100$, respectively. Appealing one

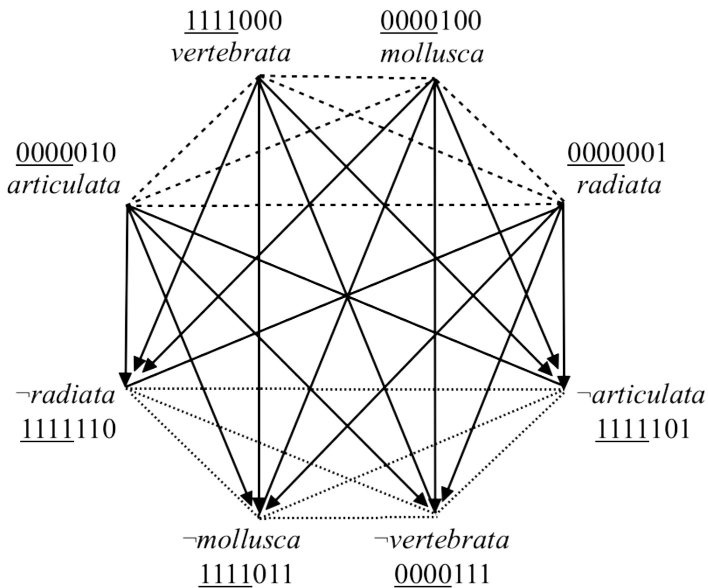


Fig. 7. Strong Moretti octagon corresponding to Schopenhauer's coarsened 4-partition of 'animals', incl. bitstring representations (with respect to the original 7-partition).

final time to Theorem 2, we find that the coarsened 4-partition gives rise to a strong α_4-structure, i.e. a strong Moretti octagon, as shown in Fig. 7.

It bears emphasizing that 'vertebrates' is a primitive/atomic concept with respect to the coarsened 4-partition (where its bitstring representation 1000 contains just a single 1-bit), but it is a complex concept with respect to the original, more fine-grained 7-partition (where its bitstring representation 1111000 is itself the join of four other bitstrings). By contrast, 'mollusca' is an atomic concept with respect to the coarse 4-partition (where its bitstring representation 0100 contains just a single 1-bit) as well as with respect to the original, more fine-grained 7-partition (where its bitstring representation 0000100 again contains just a single 1-bit). This asymmetry between 'vertebrates' on the one hand and 'mollusca' (and 'articulata' and 'radiata') on the other hand captures the hierarchical nature of Schopenhauer's 7-partition of 'animals'.

6 Conclusion

In this paper we have shown how Schopenhauer's criticism of Kantian philosophy led him to the idea of representing Aristotelian relations between sets/concepts. To this end, he developed partition diagrams that went far beyond the diagrammatic techniques known at the time. Drawing upon ideas and techniques from logical geometry, we have shown that Schopenhauer's partition diagrams systematically give rise to a special type of Aristotelian diagrams, viz. (strong) α-structures. These incude a PCD (α_2), a strong JSB hexagon (α_3), a strong Moretti octagon (α_4) and a strong α_7-structure.

As systematic research on Schopenhauer's logic has only just begun, there are still many questions that require further investigation. For example, can one find similar ideas in published logic textbooks from the 19th century? Do partition diagrams play an important role in the interpretation of Schopenhauer's system, which has a structure based on the method of partition or *divisio* in Bacon's *De dignitate et augmentis scientiarum*? To what extent did Schopenhauer use these diagrams to make new discoveries for what was then called Stoic logic, which is considered the precursor of modern propositional logic?

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