On the Cognitive Potential of Derivative Meaning in Aristotelian Diagrams*

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Abstract. In this paper we investigate the cognitive potential of Derivative Meaning — defined in terms of Abstraction Tracking (Shimojima 2015) — in order to characterise various families of Aristotelian diagrams. In a first part we consider the notion of subdiagrams — i.e. uniform triangles of contrariety relations and implication relations — inside Aristotelian hexagons and octagons. In a second part we look at different strategies for embedding complete Aristotelian diagrams — i.e. classical and degenerate squares — into Aristotelian hexagons and octagons.

Keywords: Cognitive potential · Derivative Meaning · Aristotelian diagram · Abstraction Tracking · informational/computational equivalence · Logical Geometry · classical/degenerate Aristotelian square.

1 Introduction

The overall aim of this paper is to apply the general semantic and cognitive framework for the analysis of diagrams proposed by Shimojima [13] to the analysis of Aristotelian diagrams developed by Demey and Smessaert in the framework of Logical Geometry. In our joint Diagrams 2020 paper [16] we demonstrated the relevance of Shimojima's first cognitive potential — namely that for Free Ride in Inference — in drawing the distinction between consequential constraint tracking by consequence with Logical Space Diagrams and consequential constraint

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tracking by correlation with Aristotelian diagrams. In the present paper we focus on Shimojima's fourth cognitive potential — namely that for Derivative Meaning — in order to characterise various families of Aristotelian diagrams. In this introductory section we lay out the basic ingredients of the two frameworks. On the one hand, we present the phenomenon of Derivative Meaning and its technical analysis in terms of Abstraction Tracking. On the other hand, we introduce the Aristotelian relations, diagrams and subdiagrams.

Derivative meaning. In the realm of visualising quantitative information, tables with numerical data are standardly 'translated' into statistical graphs such as scatter plots, line graphs or bar charts. The fundamental constraints in these graphical representation systems concern 'point-wise' facts: if a dot appears at X-coordinate m and Y-coordinate n in a scatter plot or line graph, then there is an instance in the data with the X-value m and the Y-value n. These semantic constraints are not 'natural' but CONVENTIONAL or arbitrary: they hold because a group of people started to conform to them at some time in history and kept respecting them for the common interest of effective communication [13, p. 103]. We call these historically prior conventions BASIC SEMANTIC CONVENTIONS, to distinguish them from additional conventions that are logically derivable in the way described later in this paper. In this limited point-wise sense, the statistical graphs have the same informational content as their corresponding tables.

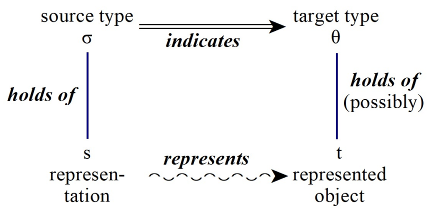
Nevertheless, there is a clear sense in which each of these graphs expresses more information than its corresponding table. With scatter plots, the particular shapes of dot configurations indicate different types of correlations — e.g. linear versus quadratic — between the variables X and Y. With line diagrams, differences in degree or direction of the inclines made by the various lines indicate differences in speed, trend or intensity of the changes in the data. In both cases, the observation of general trends or overall shapes yields additional informational relations which are *not* part of the basic semantic conventions of the representation system. Instead, these patterns indicate more abstract or general facts about the represented data. It is important to stress that these 'new' facts have a different logical and historical status. In particular, it is *not* necessary for the establishment of these more abstract relations that the relevant group of people be conforming to a new basic semantic convention. These constraints, by contrast, seem to be holding *naturally* — or logically — once the basic conventions are established, and in this sense can be considered *derived* constraints. We will therefore refer to these additional informational relations as instances of DERIVATIVE MEANING [13, p. 103]. It is precisely these derivative informational relations which significantly contribute to the *informational utility* of statistical graphs: their existence is often the very reason why a given type of graphs is more effective than others as a method of displaying certain information. Furthermore, the distinction between basic semantic conventions and derivative meaning has led researchers to distinguish 'levels of meaning', and to consider derivative or higher-level meanings as the 'main messages' of statistical graphs.

The phenomenon of derivative meaning is by no means restricted to the representation of numerical data in statistical charts. First of all, it is also relevant when displaying spatial information in the form of topographic contour maps or meteorological maps. With such maps, the basic semantic conventions are concerned with individual points on individual contour lines or isobars, whereas overall patterns formed by several proximate contour lines or isobars also carry important information about the mapped topographic or meteorological reality. Such more abstract constraints also illustrate another crucial property of this type of derivative meaning relations, namely the importance of expertise, reading skills and recognition memory in appreciating perceptual patterns [13, p. 108]. Secondly, derivative meaning also plays a crucial role in graphical representation systems for more abstract, symbolic — i.e. non-numerical — data. With node-edge graphs, for instance — such as route maps for underground or subway systems in large cities — the basic semantic convention is that if a (single) edge connects two (adjacent) nodes, it means that the objects denoted by the two nodes stand in the particular relationship represented by the edge. Derivative meaning in these graphs, by contrast, gives rise to 'overview effects' when we observe the presence of a *path* consisting of multiple, consecutive edges connecting two (non-adjacent) nodes, or when we derive the hub-like nature of a particular node from the number of nodes directly connected to it. To sum up: graphical systems can support derivative meaning relations that go beyond their basic semantic conventions. This enables us to read off a richer set of informational relations directly from graphics, expanding the expressive coverage of a system with a relatively simple set of basic semantic conventions.

General framework for the analysis of diagrams. In order to characterise the semantic content of a diagrammatic representation, the framework adopted in this paper [13, p. 23ff.] has a two-tier semantics. It draws a distinction between a TOKEN level at the bottom of Fig. 1(a) — with a REPRESENTATION relation $\rightsquigarrow$ from a representation s to represented object t — and a TYPE level at the top of Fig. 1(a) — with an INDICATION relation $\Rightarrow$ from a source type σ to a target type θ. In the case of a street map, for instance, the representation s is a particular sheet of paper (token) and the arrangement of lines and symbols is the source type σ or property holding of (or 'being supported' by) that s. The actual streets and buildings constitute the represented object t (token) and their overall arrangement is the target type θ or property holding of that t. A representation s represents an object or situation t as being of target type θ if s represents t and s supports a source type σ that indicates θ. Since the notions of Derivative Meaning and Abstraction Tracking will be defined in terms of source and target types, this paper will focus on the type level and the indication relation established by the semantic conventions for the representational practice. A set Γ of source types COLLECTIVELY INDICATES a set Δ of target types ($\Gamma \Rightarrow \Delta$) if Γ and Δ stand in a one-to-one correspondence under the indication relation $\Rightarrow$.

Abstraction Tracking. In order to characterise the notion of Derivative Meaning in a more technical manner in terms of relations between source types and target types, we take the simple example of a so-called Round-Robin table in Fig. 1(b) as our starting point.[4] Such a table is used to represent the results

[4] The figure is a considerably simplified/modified version of [13, p. 112, Figure 84(a)].



	US	CH	KO	JP
US	-	○	○	○
CH	●	-	●	●
KO	●	○	-	●
JP	●	○	○	-

Fig. 1. (a) General framework [13, Figure 21] (b) Round-Robin Table

Table 1. Sets of source types Γ_n and target types Δ_n.

$\Gamma_1 = \{\circ(\text{JP,CH}), \circ(\text{JP,KO}), \bullet(\text{JP,US})\}$ $\Delta_1 = \{W(JP,CH),\ W(JP,KO),\ L(JP,US)\}$
$\Gamma_2 = \{\circ(\text{JP,CH}), \circ(\text{JP,US}), \bullet(\text{JP,KO})\}$ $\Delta_2 = \{W(JP,CH),\ W(JP,US),\ L(JP,KO)\}$
$\Gamma_3 = \{\circ(\text{JP,KO}), \circ(\text{JP,US}), \bullet(\text{JP,CH})\}$ $\Delta_3 = \{W(JP,KO),\ W(JP,US),\ L(JP,CH)\}$

of a (sports) competition in which each contestant or team — in this case the US, China, Korea and Japan — meets all other contestants or teams in turn. The basic semantic conventions of a Round-Robin table are as follows: if a white (resp. black) circle appears at the intersection of a row headed by name X and a column headed by name Y, then this means that the team with name X has won against (resp. lost to) the team with name Y. The two source types (characterising the graphical representation) will be abbreviated as $\circ$(X,Y) and $\bullet$(X,Y) respectively, whereas the two target types (characterising the represented situation) will be abbreviated as $W(X,Y)$ and $L(X,Y)$, assuming that the italicised X and Y are the teams denoted by the names X and Y. This allows us to reformulate the basic semantic conventions as two general CONSTRAINTS between source and target types, namely (i) if $\circ$(X,Y) holds, then $W(X,Y)$ holds, and (ii) if $\bullet$(X,Y) holds, then $L(X,Y)$ holds [13, p. 115]. These constraints are concerned with individual symbols in individual cells, i.e. with the results of individual games. Derivative meaning relations, by contrast, are concerned with an entire row or column, or multiple rows or columns, specifying the information carried by the particular *distributions or patterns* of white and black circles on them. From the two white circles on the final row in Fig. 1(b), for instance, we can read off that the Japanese team won two of its three games (namely against China and Korea). This kind of derivative meaning relation can also be characterised in terms of a relation between a source type — n white circles appear in a row with name X, abbreviated as $\circ_n$(X-row) — and a target type — team X has won against n teams, abbreviated as $W_n(X)$. The crucial question now will be how we can account for the valid informational relation in Fig. 1(b) going from the source type $\circ_2$(JP-row) to the target type $W_2(JP)$, since this is *not* directly specified by the system's basic semantic conventions [13, p. 115].

First of all, it is important to note that both the source type $\circ_2$(JP-row) and the target type $W_2(JP)$ are *abstract* conditions in the sense that there are several alternative ways in which they are true. Table 1 lists some sets Γ_n of source types, and Δ_n of target types (for $1 \leq n \leq 3$). When we say that Γ_n is

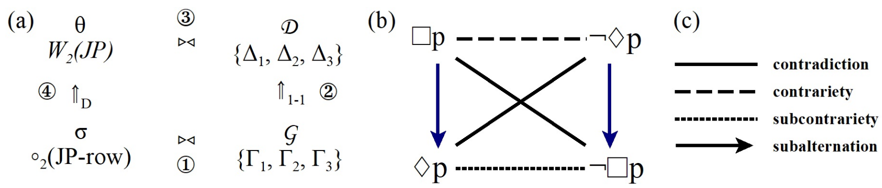


Fig. 2. (a) Abstraction Tracking(b) Aristotelian diagram; (c) coding conventions.

a way in which $\circ_2$(JP-row) holds, we mean that there is a *constraint* holding on possible properties of Round-Robin tables stating that 'if all members of Γ_n hold, then $\circ_2$(JP-row) holds'. Notice that Γ_1 is satisfied by the particular table in Fig. 1(b), whereas Γ_2 and Γ_3 are two alternative ways in which $\circ_2$(JP-row) holds. We now obtain the collection $\mathcal{G}$ of sets of source types — i.e. $\mathcal{G} = \{\Gamma_1, \Gamma_2, \Gamma_3\}$ — which exhausts all alternative ways in which $\circ_2$(JP-row) holds (since ${}_3C_2 = 3$). Intuitively, $\circ_2$(JP-row) captures a piece of information commonly implied by the members of $\mathcal{G}$, *abstracting away* the specific information particular to individual members. We therefore say that $\circ_2$(JP-row) is an ABSTRACTION OVER $\mathcal{G}$ — written as $\circ_2$(JP-row) $\bowtie \mathcal{G}$ — iff (i) for every set Γ_n in $\mathcal{G}$, if all members of Γ_n hold, then $\circ_2$(JP-row) holds, and (ii) if $\circ_2$(JP-row) holds, then there is some set Γ_n in $\mathcal{G}$ whose members all hold. Completely analogously we can obtain the collection $\mathcal{D}$ of sets of target types — i.e. $\mathcal{D} = \{\Delta_1, \Delta_2, \Delta_3\}$ — which exhausts all alternative ways in which target type W_2*(JP)* holds (since ${}_3C_2 = 3$), with Δ_1 being indicated by the table in Fig. 1(b). As a consequence, we can say that W_2*(JP)* is an ABSTRACTION OVER $\mathcal{D}$ — written as W_2*(JP)* $\bowtie \mathcal{D}$ — since (i) for every set Δ_n in $\mathcal{D}$, if all members of Δ_n hold, then W_2*(JP)* holds, and (ii) if W_2*(JP)* holds, then there is some set Δ_n in $\mathcal{D}$ whose members all hold [13, pp. 116–8]. Given (i) the two general constraints capturing the basic semantic conventions for individual cells of the Round-Robin table and (ii) the relation of collective indication between sets of source and target types, Table 1 shows that each Γ_n collectively indicates its Δ_n counterpart. This relation straightforwardly carries over to the entire collections $\mathcal{G}$ and $\mathcal{D}$. In other words, $\mathcal{G}$ and $\mathcal{D}$ are in a one-to-one correspondence under the collective indication relation by the system's basic semantic conventions.

Now we can finally characterise the derivative meaning relation between source type $\circ_2$(JP-row) and target type W_2*(JP)* in terms of the notion of ABSTRACTION TRACKING. A source type σ is said to *track* a target type θ *in abstraction* if there are a collection $\mathcal{G}$ of sets of source types and a collection $\mathcal{D}$ of sets of target types such that (i) $\sigma \bowtie \mathcal{G}$, (ii) $\theta \bowtie \mathcal{D}$ and (iii) $\mathcal{G}$ and $\mathcal{D}$ are in a one-to-one correspondence. As is shown in Fig. 2(a), $\circ_2$(JP-row) abstracts over collection $\mathcal{G}$ (bottom $\bowtie$) and this abstraction 'tracks' the abstraction relation between W_2*(JP)* and $\mathcal{D}$ (top $\bowtie$). This tracking of abstraction is mediated by the one-to-one correspondence between $\mathcal{G}$ and $\mathcal{D}$ by the collective indication relation $\Rightarrow_{1-1}$. The source type σ and the target type θ are thus taken to stand in a derivative indication relation $\Rightarrow_D$ whenever σ and θ stand in an

Table 2. Aristotelian relations between two propositions α and β.

a.	contradictory	$CD(\alpha,\beta)$	iff	α and β cannot be true together and α and β cannot be false together
b.	contrary	$CR(\alpha,\beta)$	iff	α and β cannot be true together but α and β can be false together
c.	subcontrary	$SCR(\alpha,\beta)$	iff	α and β can be true together but α and β cannot be false together
d.	in subalternation	$SA(\alpha,\beta)$	iff	α entails β but β doesn't entail α

abstraction tracking relation. Figuratively speaking, the information relation in Fig. 2(a) goes (1) from $\circ_2$(JP-row) to $\mathcal{G}$, then (2) from $\mathcal{G}$ to $\mathcal{D}$ and then (3) from $\mathcal{D}$ to $W_2(JP)$. Therefore, it goes (4) directly from $\circ_2$(JP-row) to $W_2(JP)$. This is precisely the sense in which the latter is a *derivative* relation, based on the system's basic semantic conventions, plus the two instances of abstraction relations holding within the source and the target domains.

Aristotelian relations and diagrams. In the research programme of Logical Geometry [6, 15] a central object of investigation is the so-called 'Aristotelian square' or 'square of opposition', which visualises logical relations of opposition and implication. Table 2 gives an informal definition and abbreviations for the different ARISTOTELIAN RELATIONS in which two propositions α and β can stand. In order to draw an ARISTOTELIAN DIAGRAM (AD for short), we first of all need a (non-empty) fragment F of a language L, i.e. a subset of formulas of that language. The formulas in the fragment F are typically assumed to be contingent and pairwise non-equivalent, and the fragment is standardly closed under negation: if formula φ belongs to F, then its negation $\neg\varphi$ also belongs to F. For the language of the modal logic S5 (with operators $\Box$ for necessity and $\Diamond$ for possibility), for instance, such a fragment F could be $\{\Box p, \neg\Box p, \Diamond p, \neg\Diamond p\}$. An Aristotelian diagram AD for F is then defined as a diagram that visualises an edge-labeled graph G. Fig. 2(b) presents the AD for the modal fragment $\{\Box p, \neg\Box p, \Diamond p, \neg\Diamond p\}$. The vertices of G are the elements of F, whereas the edges of G are labeled by *all* the Aristotelian relations holding between those elements, using the coding conventions in Fig. 2(c) and abbreviations in Table 2: full line for CD, dashed line for CR, dotted line for SCR, and arrow for SA. In this respect, it is worth observing that the systematic study of Aristotelian diagrams is a very recent and emerging field of research, and that an international research community is very much in the process of being established.[5] Hence, it should come as no surprise that the actual basic semantic conventions — in the sense of a community's graphical 'practice' — are themselves also still being established.[6]

[5] See the *World Congress on the Square of Opposition* (Montreux 2007, Corsica 2010, Beirut 2012, Vatican 2014, Easter Island 2016, Crete 2018, Leuven 2022).

[6] The basic colour code (red $\approx$ CD, blue $\approx$ CR, green $\approx$ SCR and black $\approx$ SA) has become the de facto standard in the Square community. As for black and white line style counterparts, full line $\approx$ CD and full line arrow $\approx$ SA are default, but some variation remains as to dotted/dashed/... lines for CR and SCR.

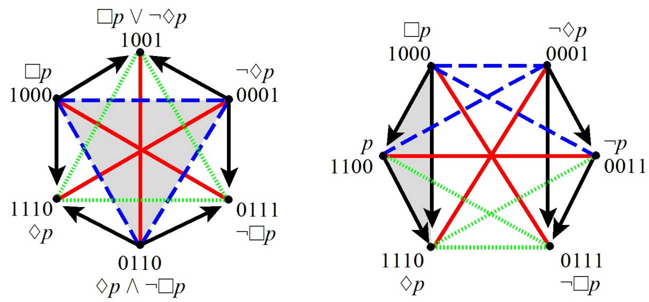


Fig. 3. (a) Jacoby-Sesmat-Blanché hexagon (b) Sherwood-Czeżowski hexagon.

Derivative Meaning and Abstraction Tracking in ADs. In the next two parts of this paper we will explore the phenomenon of Derivative Meaning and its technical analysis in terms of Abstraction Tracking in the realm of Aristotelian diagrams. Whereas the basic semantic conventions in ADs are concerned with *individual* Aristotelian relations between two formulas (i.e. individual edges between two vertices), Derivative Meaning mainly arises when we consider larger *patterns* or constellations of such relations in order to identify or distinguish different families of ADs beyond the basic square. Such Derivative Meaning patterns — the recognition of which clearly involves expert knowledge and training — hence play an important heuristic and methodological role in establishing a systematic and exhaustive typology of Aristotelian families. In section 2 we consider 'uniform' triangular shapes consisting of three contrariety relations or three subalternation relations. These two patterns first of all allow us to distinguish between the so-called Aristotelian family of Jacoby-Sesmat-Blanché hexagons (JSB for short) and that of the Sherwood-Czeżowski (SC for short) hexagons (§ 2.1). Secondly, they allow us to connect the 'arbitrariness' of graphical conventions to the distinction between informational and computational equivalence of diagrams (§ 2.2). In section 3 we go into a second mechanism which plays a central role in drawing up a typology of Aristotelian families, namely the way in which smaller ADs are systematically embedded into larger ADs. On the one hand we look at the embedding of the so-called classical Aristotelian squares in JSB hexagons and SC hexagons (§ 3.1). On the other hand, the embedding of the so-called degenerate square inside two types of Aristotelian octagons — namely the Buridan and the Béziau octagon — nicely illustrates the idea that the observation of a pattern can also be based on the *absence* of certain (individual) relations (§ 3.2).

2 Abstraction Tracking with triangular subdiagrams

2.1 JSB triangles versus SC triangles

The hexagonal diagram in Fig. 3(a) was discovered and described roughly simultaneously in the middle of the twentieth century by the logicians Jacoby [7], Sesmat [12] and Blanché [2], whence its name JSB HEXAGON. If we compare it to the original Aristotelian square for the modal fragment $\{\Box p, \neg\Box p, \Diamond p, \neg\Diamond p\}$ in Fig. 2(b), we observe the addition of an extra PAIR OF CONTRADICTORY FORMULAS (or PCD) — namely $\{\Box p \vee \neg\Diamond p, \Diamond p \wedge \neg\Box p\}$ — which constitutes a third diagonal, crossing the original square vertically. Furthermore, we have added the BITSTRING encoding of the six formulas involved since that will greatly facilitate the systematic description and comparison with diagrams further on. For this particular fragment, these bitstrings consist of four bitpositions β_n, which have the value 1 or 0, and which correspond to four anchor formulas α_n — together constituting a partition Π of logical space.[7] Formulas can now be classified as level one (L1), level two (L2) or level three (L3) according to the number of values 1 in their bitstring. Thus the two PCDs constituting the diagonals of the original square in Fig. 2(b) connect a L1 and a L3 formula, whereas the extra vertical diagonal in the JSB hexagon of Fig. 3(a) connects two L2 formulas. The addition of this third PCD gives rise to the typical downward pointing equilateral triangular shape (marked in grey) which is defined by three interconnected contrariety relations (marked by the dashed lines). Such a triangle thus serves as a diagnostic for identifying the hexagon as a member of the Aristotelian family of JSB hexagons. It is important to stress that this shape — which we refer to as a triangular Aristotelian SUBDIAGRAM (or AsD) — is *not* itself an Aristotelian diagram: since ADs are standardly closed under negation, they consist of an even number of vertices/formulas (i.e. they are built out of PCDs), and therefore triangles are excluded in principle.[8]

The observation of such an overall shape — over and beyond that of the individual Aristotelian relations that it consists of — can now straightforwardly be captured in terms of Derivative Meaning and Abstraction Tracking. On the level of source types we first of all need the set of individual graphical components — i.e. the dashed lines between the pairs of vertices. If we represent the presence of a dashed line between vertex β and β' as $\|(\beta, \beta')$, then the grey triangle in Fig. 3(a) corresponds to the set of source types $\Gamma_1 = \{\ \|(1000, 0001), \|(0001, 0110), \|(0110, 1000)\ \}$. An alternative JSB hexagon could be characterised as $\Gamma_2 = \{\ \|(1000, 0001), \|(0001, 0100), \|(0100, 1000)\ \}$. It can easily be shown that, with bitstrings of length 4, exactly ten of these sets of source types Γ_n can be built.[9] Taking these ten Γ_n sets together, we get the source type collection $\mathcal{G}$. Their common property — namely that they constitute

[7] For the technical details see [6]. In this particular case $\Pi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} = \{\Box p, \neg\Box p \wedge p, \Diamond p \wedge \neg p, \neg\Diamond p\}$ and for every formula φ, its bitstring representation $\beta(\varphi) = \beta_1\beta_2\beta_3\beta_4$ is such that $\beta_n = 1$ iff $\models \alpha_n \to \varphi$. Thus, $\beta(\Box p) = 1000$, since only $\models \alpha_1 \to \Box p$ and $\beta(\Diamond p) = 1110$, since $\models \alpha_1 \to \Diamond p$, $\models \alpha_2 \to \Diamond p$ and $\models \alpha_3 \to \Diamond p$ but $\not\models \alpha_4 \to \Diamond p$. Strictly speaking, bitstrings of length 3 suffice for the JSB hexagon in Fig. 3(a). However, for the sake of uniformity with the SC hexagon in Fig. 3(b) — which does require length 4 — and the octagons later on, we stick to length 4.

[8] The contrariety triangle in the JSB hexagon thus closely resembles the four Aristotelian subdiagrams — left/right triangle, hour glass and bow tie — in [16].

[9] First of all, there are six so-called *strong* JSBs that form their contrariety triangle like Γ_1, i.e. by first choosing two L1 formulas (${}_4C_2 = 6$) and then adding the one

(upside down) dashed line triangles — can be captured by means of the source type $\sigma = \|\bigtriangledown$. This σ counts as an *abstraction over* $\mathcal{G}$ — i.e. $\|\bigtriangledown \bowtie \mathcal{G}$ — as defined in section 1. On the level of target types, the individual contrariety relations between two formulas φ and φ' can be represented as $CR(\varphi, \varphi')$. The grey triangle in Fig. 3(a) then corresponds to the set of target types $\Delta_1 = \{$ *CR(1000,0001)*, *CR(0001,0110)*, *CR(0110,1000)* $\}$. The target level counterpart of the alternative JSB hexagon with the Γ_2 source type set could be characterised as $\Delta_2 = \{$ *CR(1000,0001)*, *CR(0001,0100)*, *CR(0100,1000)* $\}$. Obviously, all ten Γ_n source type sets defined above have Δ_n counterparts, which, taken together, yield the target type collection $\mathcal{D}$. Its defining property — namely that all its members yield a JSB contrariety constellation — can be captured by means of the target type $\theta =$ *CR(JSB)* which counts as an *abstraction over* $\mathcal{D}$ — i.e. *CR(JSB)* $\bowtie \mathcal{D}$. The overall result is a constellation of *abstraction tracking* similar to the one depicted in Fig. 2(a): by virtue of the one-to-one correspondence between $\mathcal{G}$ and $\mathcal{D}$ under the basic semantic conventions in Fig. 2(c) and Table 2, the source level abstraction $\|\bigtriangledown \bowtie \mathcal{G}$ 'tracks' the target level abstraction *CR(JSB)* $\bowtie \mathcal{D}$. As a consequence, the source type $\|\bigtriangledown$ is said to stand in a derivative meaning relation with the target type *CR(JSB)*.

As illustrated in the top hexagon of Fig. 4(a), whenever we observe an upside down (dark grey) CR triangle ($\|\bigtriangledown$), the logic of the basic semantic conventions — and in particular the central symmetry of the contradiction diagonals — predicts the existence of additional meaningful objects, i.e. an overlapping or intertwined (light grey) SCR triangle ($\dagger\bigtriangleup$), and alternating SA arrows constituting the outer edges of the hexagon ($\downarrow\bigcirc$). We can think of these three basic shapes as 'first-order' source type abstractions which can be combined into the more complex shape $\Gamma_n = \{\|\bigtriangledown, \dagger\bigtriangleup, \downarrow\bigcirc\}$. The resulting source type collection $\mathcal{G}$ is characterised by the 'higher-order' source type abstraction $\sigma = \bigtriangledown\bigtriangleup \bigcirc \bowtie \mathcal{G}$, 'fusing' the two triangles and the arrow edges. Analogously, we combine the three first-order target type abstractions into $\Delta_n = \{$*CR(JSB)*, *SCR(JSB)*, *SA(JSB)*$\}$. The resulting target type collection $\mathcal{D}$ is then characterised by the higher-order target type abstraction denoting the complete JSB hexagon, i.e. $\theta =$ *HEX(JSB)* $\bowtie \mathcal{D}$. Fig. 4(a) thus illustrates the mechanism of HIGHER-ORDER ABSTRACTION TRACKING: the source type $\bigtriangledown\bigtriangleup \bigcirc$ stands in a HIGHER-ORDER DERIVATIVE MEANING RELATION with the target type *HEX(JSB)*.[10]

We can now turn to the second family of Aristotelian hexagons, namely the so-called Sherwood-Czeżowski (SC) hexagon illustrated in Fig. 3(b). It was described in the mid twentieth century by Czeżowski [3], but Khomskii [8] convincingly demonstrated that the figure already occurs in the work of the mediaeval logician William of Sherwood [10]. If we again compare this hexagon to

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'complementary' L2 formula. In addition, there are four so-called *weak* JSBs that form their contrariety triangle like Γ_2, i.e. by means of 3 L1 formulas (${}_4C_3 = 4$).

[10] A very similar process occurs with spatial information in topographic contour maps, where the basic, first-order abstraction shape of concentric contour lines indicates a single mountain top, whereas a series of adjacent basic shapes indicates the higher-order abstraction shape of a mountain range.

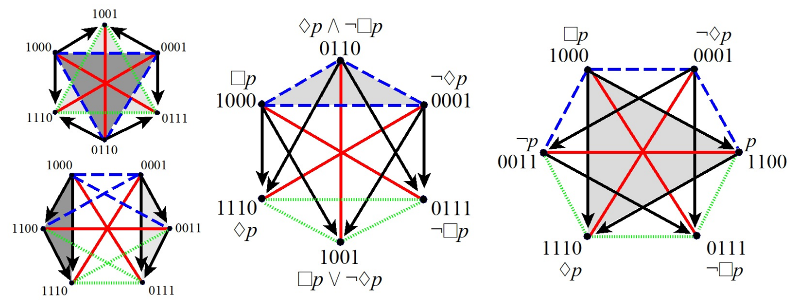

Fig. 4. (a) Double JSB/SC triangles (b) alternative JSB (c) alternative SC.

the original Aristotelian square for the modal fragment $\{\Box p,\ \neg\Box p,\ \Diamond p,\ \neg\Diamond p\}$ in Fig. 2(b), we observe that the extra PCD — namely the third diagonal $\{p, \neg p\}$ — now crosses the original square horizontally, as opposed to vertically in the JSB of Fig. 3(a). The addition of this third PCD yields an obtuse triangular shape (marked in grey) which is defined by three interconnected subalternation relations (marked by the arrows). Such a triangle thus serves as a diagnostic for identifying the hexagon as a member of the Aristotelian family of SC hexagons.[11] Within the framework of Logical Geometry, identifying which of the five logically possible Aristotelian families a particular hexagon belongs to, provides crucial (typo)logical information, in particular concerning the bitstring length required to encode the fragment involved.[12] The crucial difference between the symmetric relation of contrariety — $CR(\alpha, \beta) \Leftrightarrow CR(\beta, \alpha)$ — and the asymmetric relation of subalternation — $SA(\alpha, \beta) \Leftrightarrow \neg SA(\beta, \alpha)$ — is graphically reflected in the 'undirectedness' of the JSB triangle in Fig. 3(a) as opposed to the top down 'directedness' of the subalternation triangle in Fig. 3(b). With all its arrows pointing downwards, the latter represents the 'transitive closure' of the subalternation relation: if a first SA arrow gets you from α to β and a second SA arrow gets you from β to γ, then a third one gets you directly from α to γ.

The observation of such an overall triangular shape — in addition to that of the individual arrows that it consists of — can again straightforwardly be captured in terms of Derivative Meaning and Abstraction Tracking. On the level of source types, we first of all represent the presence of an arrow from vertex β to β' as $\downarrow(\beta, \beta')$. The grey triangle in Fig. 3(b) thus corresponds to the set of source types $\Gamma_1 = \{\ \downarrow(1000, 1100), \downarrow(1100, 1110), \downarrow(1000, 1110)\ \}$. It can easily be shown that, with bitstrings of length 4, exactly 24 of these sets of source types Γ_n can

[11] As with the contrariety triangle in Fig. 3(a), the subalternation triangle in Fig. 3(b) is not itself an AD, but an AsD, since it is not closed under negation.

[12] For an analysis of the Boolean differences between JSB and SC hexagons — the two most common and well-studied Aristotelian families of hexagons — see [14].

be built.[13] Taking these 24 Γ_n sets together, we get the source type collection $\mathcal{G}$. Their common property — namely that they constitute obtuse arrow triangles — can be captured by means of the source type $\sigma = \downarrow\triangleleft$. This σ counts as an *abstraction over* $\mathcal{G}$ — i.e. $\downarrow\triangleleft \bowtie \mathcal{G}$. On the level of target types, the individual subalternation relations from formulas φ to φ', can be represented as $SA(\varphi, \varphi')$. The grey triangle in Fig. 3(b) then corresponds to the set of target types $\Delta_1 = \{$ *SA(1000,1100)*, *SA(1100,1110)*, *SA(1000,1110)* $\}$. Obviously, all 24 Γ_n source type sets defined above have Δ_n counterparts, which, taken together, yield the target type collection $\mathcal{D}$. Its defining property — namely that all its members yield a SC subalternation constellation — can be captured by means of the target type $\theta =$ *SA(SC)* which counts as an *abstraction over* $\mathcal{D}$ — i.e. *SA(SC)* $\bowtie \mathcal{D}$. The overall result is again a constellation of *abstraction tracking* (see Fig. 2(a)): by virtue of the one-to-one correspondence between $\mathcal{G}$ and $\mathcal{D}$ under the basic semantic conventions in Fig. 2(c) and Table 2, the source level abstraction $\downarrow\triangleleft \bowtie \mathcal{G}$ 'tracks' the target level abstraction *SA(SC)* $\bowtie \mathcal{D}$. Hence, source type $\downarrow\triangleleft$ is said to stand in a derivative meaning relation with target type *SA(SC)*.

As illustrated in the bottom hexagon of Fig. 4(a), whenever we observe an obtuse (dark grey) SA triangle on the left side of a SC hexagon ($\downarrow\triangleleft$), the basic semantic conventions and the centrally symmetric CD diagonals predict the existence of additional meaningful objects, in particular a non-overlapping (light grey) SA triangle on the right side of the hexagon ($\downarrow\triangleright$). Furthermore, the relations of (sub)contrariety also yield easily recognizable 'hour glass' patterns, where the $\barwedge$ shaped constellation for CR at the top is the mirror image of the $\veebar$ shape for SCR at the bottom. As argued above for the JSB hexagon, this idea of combining basic shapes into more complex shapes can be analysed in terms of HIGHER-ORDER ABSTRACTION TRACKING and derivative meaning.[14]

2.2 Alternative JSB versus SC triangles

In this subsection, we briefly go into the aspect of 'arbitrariness' in the way in which basic semantic conventions — and graphical practice in general — come about within a given community and at a given point in history. As we argued above, both the JSB hexagon and the SC hexagon start out from the same original Aristotelian square and then add a third PCD diagonal: with the JSB hexagon the latter is inserted vertically, with the SC hexagon it is inserted horizontally. Notice that — exceptionally — an alternative visualisation shows up in which a minimal change takes place at the moment of inserting the respective third PCD, i.e. not so much by changing its fundamental orientation (vertical vs horizontal) but simply by switching around its two formulas [5].

[13] For each of the four L1 starting points α, three L3 end points γ can be chosen (the contradictory L3 being excluded), and for each of these twelve L1-L3 pairs two L2 intermediate steps β can be chosen: ${}_4C_1 \times {}_3C_1 \times {}_2C_1 = 4 \times 3 \times 2 = 24$.

[14] In particular, the higher-order source type set $\Gamma_n = \{\downarrow\triangleleft, \downarrow\triangleright, \|\barwedge, \dagger\veebar\}$ corresponds to the higher-order target type set $\Delta_n = \{SA_1\textit{(SC)}, SA_2\textit{(SC)}, \textit{CR(SC)}, \textit{SCR(SC)}\}$ and the higher-order source type abstraction $\sigma = \triangleleft\triangleright\barwedge\veebar \bowtie \mathcal{G}$ tracks the higher-order target type abstraction of the complete SC hexagon $\theta =$ *HEX(SC)* $\bowtie \mathcal{D}$.

In the case of the alternative JSB in Fig. 4(b), this means that the 1001-0110 vertices for $\{\Box p \vee \neg\Diamond p, \Diamond p \wedge \neg\Box p\}$ are switched from top to bottom, as compared to the original JSB hexagon in Fig. 3(a). In the case of the alternative SC in Fig. 4(c) this means that the 1100-0011 vertices for $\{p, \neg p\}$ are switched from left to right, as compared to the original SC hexagon in Fig. 3(b).

The two JSB hexagons in Fig. 3(a) and Fig. 4(b) represent exactly the same Aristotelian relations between exactly the same formulas, and the same holds for the two SC hexagons in Fig. 3(b) and Fig. 4(c). In the terminology of Larkin & Simon [11], the two variants for both types of hexagons stand in a relation of INFORMATIONAL EQUIVALENCE, which need not coincide with that of COMPUTATIONAL EQUIVALENCE. In other words, two variants representing exactly the same information may still differ in terms of cognitive processing requirements. From the point of view of Derivative Meaning in the previous subsection, the alternative JSB hexagon in Fig. 4(b) resembles the original SC hexagon in Fig. 3(b) in that the CR and SCR triangles have become obtuse (instead of equilateral) and are no longer intertwined. At the same time, all subalternation arrows between the top CR triangle and the bottom SCR triangle are now pointing downwards, thus resembling to some extent an upside down Hasse diagram. One could argue that this property is more in line with the Congruence Principle [17] — the structure of the visualisation should match the represented logical structure as closely as possible — at least if one prefers to focus on implication relations instead of opposition relations [15]. Conversely, the alternative SC hexagon in Fig. 4(c) resembles the original JSB hexagon in Fig. 3(a) in that the two SA triangles — the one in grey pointing to the right, the other one pointing to the left — have now become equilateral (instead of obtuse) as well as intertwined. At the same time the CR and SCR relations have now become two 'semicircles' — the top and the bottom half respectively — along the outer edges of the hexagon. From the perspective of the Congruence Principle, one disadvantage of this alternative SC hexagon in Fig. 4(c) could be that the iconicity for the transitivity closure of the SA arrows is lost. The L1-L3 arrow from 1000 to 1110 has the same length as the two 'intermediate' arrows from 1000 to 1100 (L1-L2) and from 1100 to 1110 (L2-L3), whereas from a logical point of view it has the combined effect of the two intermediate SA relations. Obviously, much more research is needed in order to determine how a representational variant of a given Aristotelian family has obtained 'canonical' status at a given point in history and to what extent cognitive principles played a role in that process (see [5] for some initial observations). But apart from that, the analysis of Derivative Meaning in terms of Abstraction Tracking — which was spelt out in full detail in § 2.1 — straightforwardly carries over to the alternative representations of the JSB and SC hexagons in Fig. 4(b-c).

3 Abstraction Tracking with embedded squares

In addition to the identification of triangular AsD shapes illustrated in § 2, a second important heuristic and diagnostic technique for distinguishing families

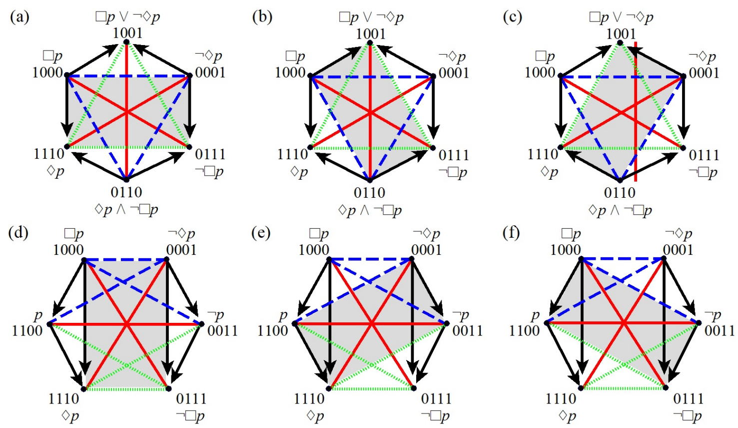

Fig. 5. (a-b-c) classical squares in JSB hexagon (d-e-f) classical squares in SC hexagon.

of ADs concerns the identification of the number and position of smaller ADs EMBEDDED in larger ADs. Aristotelian squares — the smallest non-trivial ADs to be embedded — come in two families, namely the CLASSICAL SQUARE and the DEGENERATE SQUARE. The classical square — represented in Fig. 2(b) — has six Aristotelian relations between four vertices, whereas the degenerate square only has two Aristotelian relations left, namely the two diagonals for contradiction. We consider the embedding of classical squares in two families of hexagons (§ 3.1), and that of degenerate squares in two families of octagons (§ 3.2).

3.1 Classical squares inside JSB versus SC hexagons

In addition to the 'basic' embedding of the classical square in Fig. 5(a), two more classical squares are embedded in a JSB hexagon, with 120° (counter)clockwise rotations in Fig. 5(b-c). Completely analogously, Fig. 5(d) represents the 'basic' embedding of a classical square in a SC hexagon, whereas Fig. 5(e-f) have embedded squares with 30° (counter)clockwise rotations.[15]

The technical analysis in terms of abstraction tracking illustrated in § 2 for the triangular AsD shapes straightforwardly generalises to embedded squares. The grey square in Fig. 5(a) yields the source type set $\Gamma_1 = \{ \,\|((1000, 0001), \dagger(1110, 0111), \downarrow(1000, 1110), \downarrow(0001, 0111) \,\}$. With bitstrings of length 4, the source type collection $\mathcal{G}$ contains 18 of these Γ_n sets.[16] The source type $\sigma = \downarrow \boxtimes \downarrow$ counts as an *abstraction over* $\mathcal{G}$ — i.e. $\downarrow \boxtimes \downarrow \bowtie \mathcal{G}$. Γ_1 collectively indicates target type set $\Delta_1 = \{$ *CR(1000,0001)*, *SCR(1110,0111)*, *SA(1000,1110)*,

[15] As to the Apprehension Principle [17] — the structure/content of the visualisation should be readily/accurately perceived/comprehended — the three SC squares in Fig. 5(d-f) are basically 'upright', whereas with the 120° rotations in Fig. 5(b-c) the two JSB squares are almost upside down, and thus less easily perceivable.

[16] Six of them have a L1-L1 CR relation (${}_4C_2 = 6$) and twelve of them a L1-L2 CR relation (${}_4C_1 \times {}_3C_2 = 4 \times 3 = 12$).

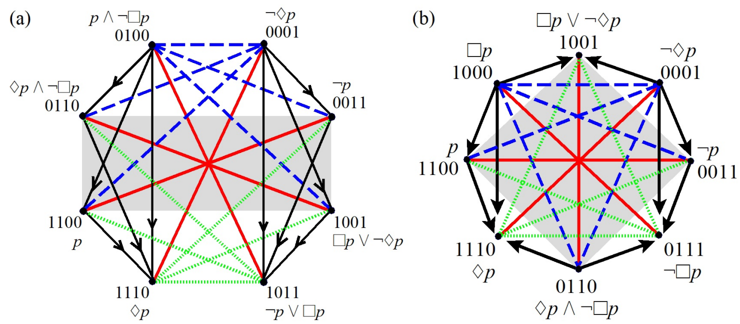

Fig. 6. Degenerate square inside (a) Buridan octagon (b) Béziau octagon.

SA(0001,0111) }, resulting in a one-to-one correspondence between source and target type collections $\mathcal{G}$ and $\mathcal{D}$. The target type θ = *CL(JSB)* refers to the embedding of a classical square in a JSB hexagon, and abstracts over $\mathcal{D}$ — i.e. *CL(JSB)* $\bowtie \mathcal{D}$. The derivative meaning relation between σ and θ holds, since the source level abstraction 'tracks' the target level abstraction. The embedding of a classical square in the SC hexagon in Fig. 5(d) receives a perfectly analogous treatment. The generalisation — visualised in Fig. 5 — that every JSB and SC hexagon contains exactly three classical squares, could then be analysed in terms of the so-called 'higher-order' abstraction tracking and derivative meaning introduced in § 2. The details of such an analysis would take us too far here, mainly because it involves the simultaneous consideration of three different diagrams, or at least of three different perspectives on the same diagram.[17]

3.2 Degenerate squares in Buridan and Béziau octagons

In the last step in this paper we will consider a very special type of Derivative Meaning. In the examples studied so far — the contrariety and subalternation triangles and the embedded classical squares — Derivative Meaning arose when we shifted the focus from individual Aristotelian relations between pairs of formulas to the observation of shapes or constellations of interconnected relations between three or four formulas inside an Aristotelian diagram. We have characterised the DEGENERATE ARISTOTELIAN SQUARE above as a constellation of four formulas which only consists of the two diagonal relations of contradiction. In other words, both the two horizontal relations of CR and SCR and the two vertical arrows of SA are missing. So when we look for degenerate squares embedded in bigger ADs, we are actually identifying a pattern, not by the *presence* of a number of relations, but by their *absence*. Observing such embedded degenerate squares as constellations of 'missing links' thus — in a figurative sense — boils down to 'seeing invisible squares'.

[17] See [13, Ch. 6] for the closely related notion of ASPECT SHIFTING, an important method of mathematical discovery: diagrammatic proofs of mathematical theorems often involve (constraints between) two decomposition types of the same figure.

One very well studied family of ADs which contain an embedded degenerate square is that of the so-called BURIDAN OCTAGON [4, 9], named after the mediaeval logician John Buridan and illustrated in Fig. 6(a). The latter can be seen as the 'superimposition' of two SC hexagons, in the sense that — in between the 0100 L1 vertex at the top left and the 1110 L3 vertex at the bottom left — not just one L2 vertex is inserted, but two, namely 0110 and 1100, thus interlocking two of the triangular SA arrow shapes that were discussed in full detail in § 2.1.[18] The crucial thing to observe now is that these two L2 formulas themselves do not stand in any Aristotelian relation whatsoever. In [15] these formulas are said to be UNCONNECTED (Buridan himself calls them 'disparatae'). In Fig. 6(a) all four pairs of L2 formulas turn out to be unconnected, as indicated by the grey shaded area identifying the degenerate square embedded in the Buridan octagon.

Transferring the abstraction tracking analysis of the embedded squares in § 3.1, we represent the absence of any line or arrow between vertex β and β' as $\varnothing(\beta, \beta')$. The grey square in Fig. 6(a) corresponds to the source type set $\mathit{\Gamma}_1 = \{\ \varnothing(0110, 0011), \varnothing(0011, 1001), \varnothing(1001, 1100), \varnothing(1100, 0110)\ \}$. With bitstrings of length 4, the source type collection $\mathcal{G}$ contains 3 of these sets $\mathit{\Gamma}_n$.[19] The source type $\sigma = \varnothing\boxtimes$ captures the idea that the latter constitute an 'invisible square' and counts as an *abstraction over* $\mathcal{G}$ — i.e. $\varnothing\boxtimes \bowtie \mathcal{G}$. On the target level, the absence of any Aristotelian relation between 'unconnected' formulas φ and φ' is represented as $UN(\varphi, \varphi')$. $\mathit{\Gamma}_1$ collectively indicates target type set $\mathit{\Delta}_1 = \{$ *UN(0110,0011)*, *UN(0011,1001)*, *UN(1001,1100)*, *UN(1100,0110)* $\}$, resulting in a one-to-one correspondence between source and target type collections $\mathcal{G}$ and $\mathcal{D}$. The target type $\theta =$ *UN(BUR)* refers to the embedding of a degenerate square in a Buridan octagon, and abstracts over $\mathcal{D}$ — i.e. *UN(BUR)* $\bowtie \mathcal{D}$. The derivative meaning relation between σ and θ holds, since the source level abstraction $\varnothing\boxtimes \bowtie \mathcal{G}$ 'tracks' the target level abstraction *UN(BUR)* $\bowtie \mathcal{D}$.

A second Aristotelian family of octagons in which a degenerate square is embedded is the so-called BÉZIAU OCTAGON [1], illustrated in Fig. 6(b). This octagon can be characterised as the 'superimposition' of the JSB hexagon in Fig. 3(a) and the SC hexagon in Fig. 3(b), in the sense that both the horizontal L2-L2 diagonal 1100-0011 and the vertical L2-L2 diagonal 1001-0110 are inserted into the basic square simultaneously. As a consequence, the Béziau octagon contains both the two interlocking equilateral triangles for the CR and SCR relations and the two non-interlocking obtuse triangles for the SA relations. The co-occurrence of two L2-L2 diagonals by definition results in the embedding of a degenerate square in a Béziau octagon, as indicated by the grey shaded area in Fig. 6(b). Notice that — in contrast to the Buridan octagon in Fig. 6(a) — the Béziau octagon does not have any adjacent L2 vertices. In other words, the four L2 vertices are laid out alternatingly around the outer edges of the octagon, resulting in the 'tilted' degenerate square standing on one of its corners. Obviously, the technical analysis in terms of Abstraction Tracking pro-

[18] The analysis of some Buridan octagons requires bitstrings of length 5 and 6 [4].
[19] Each degenerate square consists of two out of the three L2-L2 PCDs (${}_3C_2 = 3$).

vided above for the degenerate square embedded in the Buridan octagon carries over straightforwardly to the analogous constellation in the Béziau octagon.

4 Conclusion

In this paper we have used the mechanism of Abstraction Tracking [13] in order to describe Derivative Meaning arising in various Aristotelian diagrams (ADs). This collaborative enterprise has not only turned out to be fruitful and relevant for a deeper understanding of ADs in the framework of Logical Geometry but has also yielded a deeper understanding of the Cognitive Potential of Derivative Meaning itself, in particular w.r.t. the patterns of so-called higher-order Abstraction Tracking and Derivative Meaning.

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