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RESEARCH REPORT 9916

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D/1999/2376/16

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Abstract

In this paper we investigate the dependence in Fréchet spaces containing mutually exclusive risks. It is shown that, under some reasonable assumptions, the safest dependence structure, in the sense of the minimal stop-loss premiums for the aggregate claims involved, is obtained with the Fréchet lower bound and precisely corresponds to the mutually exclusive risks of the Fréchet space. In that respect, the present paper complements some previous studies by Heilmann (1986), Dhaene & Goovaerts (1996, 1997), Müller (1997) and Taizhong & Zhiqiang (1998). A couple of actuarial applications enhance the interest of the results derived here.

Keywords : Fréchet spaces, Fréchet bounds, comonotonicity, mutual exclusivity, stop-loss order, supermodular order

1 Introduction

Consider a portfolio consisting of n insurance policies. The aggregate claims S of the portfolio is the sum of all amounts $X_1, X_2, \cdots, X_n$ payable during the reference period (one year, say), i.e.

$$S = \sum_{i=1}^{n} X_i;$$

the random variables X_i, assumed to possess a finite mathematical expectation, are commonly called "risks" in the actuarial literature. The X_i's are non-negative and model the total claims generated by policy i, $i = 1, 2, \cdots, n$. The calculation of the stop-loss premiums related to such a portfolio is one of the main topics of risk theory. Therefore not only the marginal distributions of the X_i's have to be known, but also the knowledge of the dependence structure among the X_i's is required. In practice, the problem is almost always simplified by assuming that the X_i's are mutually independent so that knowledge of the marginal distributions suffices to compute stop-loss premiums. Of course, the independence hypothesis obviously

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relies on computational convenience rather than realism, and dependencies may have disastrous effects on stop-loss premiums (for numerical illustrations, the interested reader is referred, e.g., to Kaas (1993) or Dhaene & Goovaerts (1997)).

The present paper is devoted to the safest dependence structure between the risks $X_1, X_2, \cdots, X_n$ with given marginals, i.e. the one giving rise to the smallest stop-loss premiums for S. Several authors (e.g., Dhaene & Goovaerts (1996,1997) and Müller (1997)) have already determined the worst dependence structure, i.e. the one generating the largest stop-loss premiums for S. They showed that the dependence structure of the riskiest portfolio is described by the so-called Fréchet upper bound. We will see below that the Fréchet lower bound plays a symmetric role for the safest portfolio, although some mathematical conditions are involved.

In order to compare the riskiness of insurance portfolios, we will use the stop-loss order. Therefore, we briefly recall the following result which will be used in the sequel.

Definition 1. *A risk X is said to be smaller in stop-loss order than a risk Y, which is denoted as $X \leq_{s\ell} Y$, if any of the two following conditions hold :*

1. *the stop-loss premiums associated to X and Y are ordered for any level d of the deductible, i.e.*

$$E\left[(X-d)_+\right] \leq E\left[(Y-d)_+\right], \text{ for all } d \geq 0;$$

2. *X is preferred over Y by all the risk-averse profit-seeking decision-makers, i.e.*

$$E\left[u\left(-X\right)\right] \geq E\left[u\left(-Y\right)\right]$$

for every concave non-decreasing utility function u, provided that the expectations exist.

For a proof of the equivalence of the two conditions contained in Definition 1 for stop-loss order, see e.g. Kaas, van Heerwaarden & Goovaerts (1994, Theorem 1.1 on page 21).

The paper is organized as follows. In Section 2, we provide some mathematical background about Fréchet spaces and Fréchet bounds. It is explained why the n-dimensional case ($n \geq 3$) and the bivariate case are so different. In Section 3, we give bounds in the stochastic dominance sense on the largest and smallest claims affecting an insurance portfolio. In Section 4, we investigate the safest dependence structure among the risks $X_1, X_2, \cdots, X_n$ and we extend to general risks a recent result obtained by Taizhong & Zhiqiang (1998) in the case of two-point distributions. Therefore, we introduce the concept of mutually exclusive risks, which is particularly relevant in actuarial sciences. In Section 5, we show that, under some circumstances, the Fréchet lower bound is the minimal element for the supermodular order in a given Fréchet space. This complements a recent result obtained by

Müller (1997). We end the paper by providing two applications of the theory. We first derive bounds on reinsurance premiums when dependent risks are involved. For exponentially distributed risks, elegant explicit formulas are available. Secondly, we examine optimality among some life insurance contracts and we prove a result in the vein of Bowers *et al.* (1996) and Kling & Wolthuis (1992); namely that from the insurer's point of view, it is safer to issue an n-year endowment on a single person than to sell an n-year pure endowment together with an n-year term insurance to two different people.

2 Fréchet spaces and Fréchet bounds

Let $F_1, F_2 \cdots, F_n$ be univariate cumulative distribution functions (c.d.f.'s, in short) and consider the Fréchet space $\mathcal{R}_n(F_1, F_2, \cdots, F_n)$ consisting of all n-dimensional c.d.f.'s $F_{\mathbf{X}}$ (or equivalently of all the n-dimensional random vectors $\mathbf{X} = (X_1, X_2, \cdots, X_n)$) possessing $F_1, F_2, \cdots, F_n$ as marginal c.d.f.'s. In this paper, we restrict ourselves to (c.d.f.'s of) non-negative random variables with finite expectations, further called risks. We have that for all $\mathbf{X}$ in $\mathcal{R}_n(F_1, F_2, \cdots, F_n)$ the following inequality holds :

$$M_n(\mathbf{x}) \leq F_{\mathbf{X}}(\mathbf{x}) \leq W_n(\mathbf{x}) \text{ for all } \mathbf{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n,$$

where W_n is usually referred to as the Fréchet upper bound of $\mathcal{R}_n(F_1, F_2, \cdots, F_n)$ and is defined by

$$W_n(\mathbf{x}) = \min\{F_1(x_1), F_2(x_2), \cdots, F_n(x_n)\}, \ \ \mathbf{x} \in \mathbb{R}^n,$$

while M_n is usually referred to as the Fréchet lower bound of $\mathcal{R}_n(F_1, F_2, \cdots, F_n)$ and is defined by

$$M_n(\mathbf{x}) = \max\left\{\sum_{i=1}^{n} F_i(x_i) - n + 1, 0\right\}, \ \ \mathbf{x} \in \mathbb{R}^n.$$

Remark that W_n is reachable in $\mathcal{R}_n(F_1, F_2, \cdots, F_n)$. Indeed, given a random variable U, uniformly distributed on $[0,1]$, it can be shown that W_n is the c.d.f. of

$$(F_1^{-1}(U), F_2^{-1}(U), \cdots, F_n^{-1}(U)) \in \mathcal{R}_n(F_1, F_2, \cdots, F_n),$$

where the generalized inverses of the F_i's are defined as

$$F_i^{-1}(u) = \inf\{x \in \mathbb{R} \text{ such that } F_i(x) \geq u\}, \ \ u \in [0,1], \ \ i = 1, 2, \ldots, n.$$

The elements of the Fréchet space $\mathcal{R}_n(F_1, F_2, \cdots, F_n)$ which have a multivariate c.d.f. given by $W_n(\mathbf{x})$, are said to be comonotonic. Applications of the notion of comonotonicity in the actuarial literature can be found e.g. in Wang (1996), Wang & Dhaene (1997), Dhaene, Wang, Young & Goovaerts (1997) and Wang, Young & Panjer (1998).

On the contrary, when $n \geq 3$, M_n is not always a c.d.f. anymore, as shown by the following counterexample proposed by Tchen (1980) : for $n = 3$, take X_1, X_2 and X_3 uniformly distributed on the unit interval $[0, 1]$; then, the "probability" that $\mathbf{X}$ lies in $[0.5, 1] \times [0.5, 1] \times [0.5, 1]$ is equal to -0.5 when the dependence structure is described by M_3, so that M_3 cannot be a proper c.d.f. (another counterexample is provided by Joe (1997, Example 3.1)). The following necessary and sufficient condition for M_n to be a c.d.f. in $\mathcal{R}_n(F_1, F_2, \cdots, F_n)$ can be found e.g. in Joe (1997, Theorem 3.7).

Theorem 2. *A necessary and sufficient condition for M_n to be a c.d.f. in $\mathcal{R}_n(F_1, F_2, \cdots, F_n)$ is that either*

1. $\sum_{j=1}^{n} F_j(x_j) \leq 1$ *for all* $\mathbf{x} \in \mathbb{R}^n$ *with* $0 < F_j(x_j) < 1$, $j = 1, 2, \cdots, n$; *or*
2. $\sum_{j=1}^{n} F_j(x_j) \geq n - 1$ *for all* $x \in \mathbb{R}^n$ *with* $0 < F_j(x_j) < 1$, $j = 1, 2, \cdots, n$.

3 Stochastic bounds on the smallest and largest claims

Despite M_n is not always a proper c.d.f., Tchen (1980, Theorem 4) proved that there exists $\mathbf{X} \in \mathcal{R}_n(F_1, F_2, \cdots, F_n)$ achieving the lower bound M_n when all the x_i's are equal. This is formally stated in the next result.

Theorem 3. *There exist* $\mathbf{X} \in \mathcal{R}_n(F_1, F_2, \cdots, F_n)$ *such that*

$$P[\max\{X_1, X_2, \cdots, X_n\} \leq x] = M_n(x, x, \cdots, x)$$

for any $n \in \mathbb{N}_0$ *and* $x \in \mathbb{R}$.

As a corollary, Tchen (1980, Corollary 4.1) provided the following bounds on the distribution of $\max\{X_1, X_2, \cdots, X_n\}$ and $\min\{X_1, X_2, \cdots, X_n\}$: for any $\mathbf{X} \in \mathcal{R}_n(F_1, F_2, \cdots, F_n)$,

$$1 - \min\{F_1(x), F_2(x), \cdots, F_n(x)\} \leq P[\max\{X_1, X_2, \cdots, X_n\} > x]$$
$$\leq \min\left\{1, \sum_{i=1}^{n}(1 - F_i(x))\right\}, \text{ for all } x \in \mathbb{R},$$

and

$$\max\{F_1(x), F_2(x), \cdots, F_n(x)\} \leq P[\min\{X_1, X_2, \cdots, X_n\} \leq x]$$
$$\leq \min\left\{1, \sum_{i=1}^{n} F_i(x)\right\}, \text{ for all } x \in \mathbb{R}.$$

The latter inequalities provided useful bounds on the distribution of the largest and smallest claims affecting an insurance portfolio consisting of dependent risks.

Therefore, they can be used to get bounds on the premium of a $LCR(1)$ treaty (such a reinsurance agreement covers the largest claim occuring during a given reference period (one year, say)). Of course, when the X_i's are thought of as being time-until-death random variables, these inequalities also yield bounds on life insurances or annuities based on either a joint-life status or a last-survivor status. These bounds have been used by Dhaene, Vanneste & Wolthuis (1997) in order to find extremal joint-life and last-survivor statuses (in terms of stochastic dominance).

4 Extremal dependence structures

Dhaene & Goovaerts (1997) considered the Fréchet spaces $\mathcal{R}_n(F_1, F_2, \cdots, F_n)$ of all n-dimensional multivariate risks $(X_1, X_2, \cdots, X_n)$ with each X_i having a two-point distribution (with probability masses in 0 and $\alpha_i > 0$). They investigated the dangerousness of this Fréchet space by looking for the element with the most dangerous mutual dependence between the risks, i.e. the one leading to the highest stop-loss premiums. They found that the most dangerous dependence structure is described by the Fréchet upper bound W_n; see also Dhaene & Goovaerts (1996) and Müller (1997) for an extension of this result to general risks. The following theorem is borrowed from Dhaene, Wang, Young & Goovaerts (1997).

Theorem 4. *Let U be a random variable uniformly distributed on $[0,1]$. Then,*

$$\sum_{i=1}^{n} X_i \leq_{s\ell} \sum_{i=1}^{n} F_i^{-1}(U)$$

for any multivariate risk $\mathbf{X}$ in $\mathcal{R}_n(F_1, F_2, \cdots, F_n)$.

Other proofs for this theorem can be found in Müller (1997) (in terms of supermodular ordering) and, in a special setting, in Heilmann (1986) (in terms of convex mappings).

In view of the above, an interesting problem is to look for the safest element in the Fréchet class $\mathcal{R}_n(F_1, F_2, \cdots, F_n)$, with "safest" meaning that the corresponding dependence structure leads to the lowest stop-loss premiums for S. By symmetry, we would like to say that the Fréchet lower bound M_n provides the least dangerous mutual dependence between the risks. Nevertheless, if this were true, the problem will not have a solution in general, because the Fréchet lower bound is not always a proper c.d.f. (see Theorem 2). Therefore, we will restrict our study to a Fréchet class $\mathcal{R}_n(F_1, F_2, \cdots, F_n)$ for which the condition

$$\sum_{i=1}^{n} q_i \leq 1 \text{ where } q_i = 1 - F_i(0), \qquad i = 1, 2, \cdots, n, \tag{1}$$

is fulfilled, i.e. the probability mass of the marginal distributions outside 0 is at most one. According to Theorem 2(2), (1) is a sufficient condition for the lower

Fréchet bound M_n to be a proper c.d.f. in $\mathcal{R}_n(F_1, F_2, \cdots, F_n)$. As we will see the study of Fréchet spaces satisfying (1) has some actuarial relevance.

In a recent paper by Taizhong & Zhiqiang (1998), the aforementionned problem has already been investigated when the marginals $F_1, F_2, \cdots, F_n$ are two-point distributions. They found the following result.

Theorem 5. *Consider a Fréchet space $\mathcal{R}_n(F_1, F_2, \cdots, F_n)$ satisfying (1), such that for $i = 1, 2, \cdots, n$, the F_i are two-point distributions with probability masses in 0 and $\alpha_i > 0$. Consider the risk $\mathbf{X} \in \mathcal{R}_n(F_1, F_2, \cdots, F_n)$ with dependence structure given by*

$$P[X_i = \alpha_i, X_j = \alpha_j] = 0, \text{ for all } i \neq j.$$

Then,

$$\sum_{i=1}^{n} X_i \leq_{s\ell} \sum_{i=1}^{n} Y_i.$$

holds for any $\mathbf{Y} \in \mathcal{R}_n(F_1, F_2, \cdots, F_n)$.

We mention that condition (1) for two-point distributions is equivalent to the conditions of Theorem 2, so that (1) is in this particular case necessary and sufficient for M_n to be a proper c.d.f..

We are now going to generalize Theorem 5 to the case of general risks. First, we introduce the notion of mutually exclusive risks. Roughly speaking, the risks $X_1, X_2, \cdots, X_n$ are said to be mutually exclusive when at most one of them can be different from zero. This can be considered as a sort of dual notion of comonotonicity. Indeed, the knowledge that one risk assumes a positive value directly implies that all the other ones vanish. Mutually exclusive risks are therefore "anti-monotonic".

Definition 6. *The risks $X_1, X_2, \cdots, X_n$ are said to be mutually exclusive (or, equivalently, the multivariate risk $\mathbf{X}$ is said to possess this property) when*

$$P[X_i > 0, X_j > 0] = 0 \text{ for all } i \neq j.$$

Let us point out that mutually exclusivity of $\mathbf{X}$ means that its multivariate c.d.f. $F_{\mathbf{X}}$ is concentrated on the axes.

Examples of mutually exclusive risks abound in actuarial sciences : think for instance of

1. the present value of the benefit associated with a whole life insurance A_x written on a status (x) which can be decomposed as

$$A_x = A^1_{x:\overline{k|}} + {}_{k|}A_x.$$

The benefit funtions associated with $A^1_{x:\overline{k|}}$ and ${}_{k|}A_x$ (and written on the same status (x)) are mutually exclusive;

2. a life annuity a_x written on a status (x) can be decomposed as

$$a_x = \sum_{k=1}^{\omega-x} {}_kE_x,$$

where ω is the ultimate age of the lifetable. The benefit functions associated with the ${}_kE_x$, all written on the same status (x), are comonotonic. On the other hand, a life annuity can also be decomposed as

$$a_x = \sum_{k=0}^{\omega-x} {}_{k|}q_x a_{\overline{k|}}.$$

the benefit functions associated with the ${}_{k|}q_x a_{\overline{k|}}$'s (all written on the same status (x)) are mutually exclusive;

3. a term insurance with doubled capital in case of accidental death;

4. an n-year endowment insurance (with payment in case of death and survival) – see also further;

5. a franchise deductible splitting the risk X up in two parts $X = X_1 + X_2$, with the retained part given by

$$X_1 = \begin{cases} X \text{ if } X < d, \\ 0 \text{ if } X \geq d, \end{cases}$$

and the insured part

$$X_2 = \begin{cases} 0 \text{ if } X < d, \\ X \text{ if } X \geq d. \end{cases}$$

The risk-sharing scheme (X_1, X_2) is mutually exclusive.

As an example in finance, consider a stock with price X at time t. Consider two European options (a put and a call) on this stock with expiration date t and exercise price d. The writer of the options bears the risk

$$X_1 = \max(0, X - d)$$

for the call and

$$X_2 = \max(0, d - X)$$

for the put. X_1 and X_2 are mutually exclusive.

Let us now emphasize the central role of condition (1) in the theory of mutually exclusive risks.

Theorem 7. *A Fréchet space $\mathcal{R}_n(F_1, F_2, \cdots, F_n)$ contains mutually exclusive risks if, and only if, it satisfies (1).*

Proof. First, assume that $\mathbf{X}$ is mutually exclusive and belongs to $\mathcal{R}_n(F_1, F_2, \cdots, F_n)$. Define the indicator variables $I_1, I_2, \cdots, I_n$ as

$$I_i = \begin{cases} 1 \text{ if } X_i > 0 \\ \\ 0 \text{ if } X_i = 0 \end{cases},$$

so that $P[I_i = 1] = q_i$, $i = 1, 2, \cdots, n$. Note that

$$\begin{aligned} P[I_1 = I_2 = \cdots = I_n = 0] &= 1 - P[\exists\ i \text{ such that } I_i = 1] \\ &= 1 - \sum_{i=1}^{n} q_i, \end{aligned}$$

so that (1) has to be fulfilled. Conversely, assume that $\mathcal{R}_n(F_1, F_2, \cdots, F_n)$ satisfies (1). From Theorem 2, we know that M_n is a c.d.f. in $\mathcal{R}_n(F_1, F_2, \cdots, F_n)$. Consider $\mathbf{X} \in \mathcal{R}_n(F_1, F_2, \cdots, F_n)$ with c.d.f. M_n. Then, we find

$$P[X_i = 0, X_j = 0] = 1 - q_i - q_j \text{ for all } i \neq j,$$

whence it follows that

$$P[X_i > 0, X_j > 0] = 0 \text{ for all } i \neq j,$$

which, in turn, means that $\mathbf{X}$ is mutually exclusive. $\diamond$

Let us prove the following characterization of mutual exclusivity, which relates this notion to the Fréchet lower bound (as comonotonicity corresponds to the Fréchet upper bound). More precisely, we prove that when (1) is fulfilled, the multivariate c.d.f. $F_{\mathbf{X}}$ of the mutually exclusive risk $\mathbf{X}$ in the Fréchet space $\mathcal{R}_n(F_1, F_2, \cdots, F_n)$ is given by the Fréchet lower bound $M_n(\mathbf{x})$.

Theorem 8. *Consider a Fréchet space $\mathcal{R}_n(F_1, F_2, \cdots, F_n)$ satisfying (1). The risk $\mathbf{X} \in \mathcal{R}_n(F_1, F_2, \cdots, F_n)$ is said to be mutually exclusive if, and only if,*

$$F_{\mathbf{X}}(\mathbf{x}) = M_n(\mathbf{x}), \ \ \mathbf{x} \in \mathbb{R}^n.$$

Proof. Assume that $\mathbf{X}$ is mutually exclusive. Then, we have that

$$\begin{aligned}
F_{\mathbf{X}}(\mathbf{x}) &= \sum_{i=1}^{n} P[X_1 \leq x_1, X_2 \leq x_2, \cdots, X_n \leq x_n | I_i = 1] P[I_i = 1] \\
&\qquad + P[I_1 = I_2 = \cdots = I_n = 0] \\
&= \sum_{i=1}^{n} P[X_i \leq x_i | I_i = 1] q_i + 1 - \sum_{i=1}^{n} q_i \\
&= \sum_{i=1}^{n} (F_i(x_i) - F_i(0)) + 1 - \sum_{i=1}^{n} q_i \\
&= \sum_{i=1}^{n} F_i(x_i) + 1 - n = M_n(\mathbf{x}),
\end{aligned}$$

which achieves the proof of the necessity part. The opposite direction immediately follows from the second part of the proof of Theorem 7. ◇

Combining Theorems 7 and 8, we find that a Fréchet space $\mathcal{R}_n(F_1, F_2, \cdots, F_n)$ satisfies (1) if, and only if, the Fréchet lower bound is the unique c.d.f. of $\mathcal{R}_n(F_1, F_2, \cdots, F_n)$ which is concentrated on the axes.

Let us now derive the following result, which states that the expected utility is additive for a sum of mutually exclusive risks.

Theorem 9. *Consider a Fréchet space $\mathcal{R}_n(F_1, F_2, \cdots, F_n)$ satisfying (1). Let $\mathbf{X}$ be a mutually exclusive risk in $\mathcal{R}_n(F_1, F_2, \cdots, F_n)$. Then,*

$$E\left[u\left(-\sum_{i=1}^{n} X_i\right)\right] = \sum_{i=1}^{n} E\left[u(-X_i)\right]$$

holds for any utility function u.

Proof. As $\mathbf{X}$ is mutually exclusive, the distribution of the aggregate claims S is given by

$$\begin{aligned}
P\left[\sum_{j=1}^{n} X_j \leq x\right] &= \sum_{i=1}^{n} P\left[\sum_{j=1}^{n} X_j \leq x \,| I_i = 1\right] P[I_i = 1] \\
&\qquad + P[I_1 = I_2 = \cdots = I_n = 0] \\
&= \sum_{i=1}^{n} P[X_i \leq x | I_i = 1] q_i + 1 - \sum_{i=1}^{n} q_i \\
&= \sum_{i=1}^{n} F_{X_i}(x) + 1 - n
\end{aligned}$$

whence the desired result directly follows. ◇

As a special case of Theorem 9, we find that

$$E\left[\left(\sum_{i=1}^{n} X_i - d\right)_+\right] = \sum_{i=1}^{n} E\left(X_i - d\right)_+, \tag{2}$$

holds when $\mathbf{X}$ is mutually exclusive, for any deductible $d \geq 0$.

We are now in a position to extend Theorem 5 to general risks.

Theorem 10. *Consider a Fréchet space $\mathcal{R}_n(F_1, F_2, \cdots, F_n)$ satisfying (1). Let $\mathbf{X}$ be a mutually exclusive risk in $\mathcal{R}_n(F_1, F_2, \cdots, F_n)$. Then,*

$$\sum_{i=1}^{n} X_i \leq_{s\ell} \sum_{i=1}^{n} Y_i.$$

holds for any $\mathbf{Y} \in \mathcal{R}_n(F_1, F_2, \cdots, F_n)$.

Proof. Since $\mathbf{X}$ is mutually exclusive, we have from (2) that

$$E\left[\left(\sum_{i=1}^{n} X_i - d\right)_+\right] = \sum_{i=1}^{n} E\left(X_i - d\right)_+,$$

Now, as $\mathbf{X}$ and $\mathbf{Y}$ both belong to $\mathcal{R}_n(F_1, F_2, \cdots, F_n)$, the latter relation yields

$$E\left[\left(\sum_{i=1}^{n} X_i - d\right)_+\right] = \sum_{i=1}^{n} E\left(Y_i - d\right)_+ \leq E\left[\left(\sum_{i=1}^{n} Y_i - d\right)_+\right],$$

where the latter inequality is true in general. This ends the proof. $\diamond$

We have proven that, in the class $\mathcal{R}_n(F_1, F_2, \cdots, F_n)$ of all the risks with given marginals F_i and such that (1) is fulfilled, the mutually exlusive risks lead to the safest portfolio, in the sense that this kind of mutual dependency leads to the smallest stop-loss premiums. This means that all the risk-averse decision-makers (with non-decreasing utility functions) who have to bear a risk S which is the aggregate claims of the components of a freely choosen element in a Fréchet space $\mathcal{R}_n(F_1, F_2, \cdots, F_n)$ containing mutually exclusive risks will prefer the latter elements.

Remark that a slightly generalization of Theorem 10 is possible.

Corollary 11. *Assume that $X_i \leq_{s\ell} Y_i$ for $i = 1, 2, \cdots, n$, and $\mathbf{X}$ is mutually exclusive. Then,*

$$\sum_{i=1}^{n} X_i \leq_{s\ell} \sum_{i=1}^{n} Y_i$$

holds true.

Corollary 11 means that we can leave the original Fréchet space to another one for which the components are larger in stop-loss order.

5 Inequalities of supermodular-type

In this section, we extend the stochastic inequality of Theorem 10 with the aid of the supermodular order. To be specific, we show that in a Fréchet space $\mathcal{R}_n(F_1, F_2, \cdots, F_n)$ satisfying (1), the minimal element with respect to the supermodular order is precisely M_n, the Fréchet lower bound.

The supermodular order has been introduced in the actuarial literature by Müller (1997) (see also Bäuerle & Müller (1998) and the references therein) who characterized the riskiest portfolio among all those with identical marginals. For a reference in statistics, see, e.g., Shaked and Shanthikumar (1997). The supermodular order is based on the comparison of expectations of supermodular functions. A real-valued function ϕ defined on the positive orthant $\mathbb{R}_+^n$ is called supermodular if

$$\phi(\max(x_1, y_1), \cdots, \max(x_n, y_n)) + \phi(\min(x_1, y_1), \cdots, \min(x_n, y_n)) \geq \phi(\mathbf{x}) + \phi(\mathbf{y}), \tag{3}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}_+^n$. If ϕ has second partial derivatives then it is supermodular if, and only if,

$$\frac{\partial^2 \phi}{\partial x_i \partial x_j} \geq 0 \text{ on } \mathbb{R}_+^n \text{ for all } i \neq j.$$

Then, given two multivariate risks $\mathbf{X}$ and $\mathbf{Y}$ in $\mathcal{R}_n(F_1, F_2, \cdots, F_n)$, $\mathbf{X}$ is said to precede $\mathbf{Y}$ in the supermodular order, denoted as $\mathbf{X} \preceq_{sm} \mathbf{Y}$, if $E\phi(\mathbf{X}) \leq E\phi(\mathbf{Y})$ for all supermodular function $\phi : \mathbb{R}_+^n \to \mathbb{R}$ for which the expectations exist.

Let us now prove the following result.

Theorem 12. *Consider a Fréchet space $\mathcal{R}_n(F_1, F_2, \cdots, F_n)$ satisfying (1). Let $\mathbf{X}$ be a mutually exclusive risk in $\mathcal{R}_n(F_1, F_2, \cdots, F_n)$. Then,*

$$\mathbf{X} \preceq_{sm} \mathbf{Y}$$

holds for any $\mathbf{Y} \in \mathcal{R}_n(F_1, F_2, \cdots, F_n)$.

Proof. Without loss of generality, the supermodular functions generating $\preceq_{sm}$ may be assumed to vanish on the axes. It suffices indeed to substitute for $\phi : \mathbb{R}_+^n \to \mathbb{R}$ the auxiliary function $\phi^* : \mathbb{R}_+^n \to \mathbb{R}$ defined as

$$\phi^*(\mathbf{x}) = \phi(\mathbf{x}) - \sum_{i=1}^{n} \phi(0, \cdots, 0, x_i, 0, \cdots, 0) + (n-1)\phi(0, 0, \cdots, 0),$$

and to notice then that the inequality $E\phi(\mathbf{X}) \leq E\phi(\mathbf{Y})$ holds if, and only if, $E\phi^*(\mathbf{X}) \leq E\phi^*(\mathbf{Y})$ holds, since $\mathbf{X}$ and $\mathbf{Y}$ both belong to $\mathcal{R}_n(F_1, F_2, \cdots, F_n)$. Now, a supermodular function $\phi : \mathbb{R}_+^n \to \mathbb{R}$ that is zero on the axes is necessarily valued in $\mathbb{R}^+$; this is easily seen from repeated use of (3) which gives

$$\begin{aligned} \phi(x_1, x_2, \cdots, x_n) &\geq \phi(x_1, x_2, \ldots, x_{n-1}, 0) \\ &\geq \phi(x_1, x_2, \ldots, x_{n-2}, 0, 0) \end{aligned}$$

$$\begin{aligned} &\geq \cdots \\ &\geq \phi(0,0,\cdots,0) = 0, \end{aligned}$$

for all $\mathbf{x} \in \mathbb{R}_+^n$. To conclude, it suffices to quote that $E\phi(\mathbf{Y}) \geq 0$ for any $\mathbf{Y} \in \mathcal{R}_n(F_1, F_2, \cdots, F_n)$, while $E\phi(\mathbf{X}) = 0$ since the c.d.f. of $\mathbf{X}$ is concentrated on the axes. $\diamond$

It is well-known that the stochastic inequality contained in Theorem 12 implies the one proposed in Theorem 10. Moreover, it provides a host of useful inequalities in actuarial sciences.

Müller (1997) proved that the most dangerous portfolio (in the supermodular sense) among all those with the same marginals was distributed according to the Fréchet upper bound W_n. The result above states that the least dangerous one follows the Fréchet lower bound M_n, provided that the marginals $F_1, F_2, \cdots, F_n$ satisfy condition (1). In other words, the Fréchet bounds are the extremal distributions with respect to the supermodular order, provided that (1) is fulfilled for the minimum.

6 Actuarial Examples

6.1 Reinsurance premiums for dependent portfolios

Consider a travel insurance contract, including the following coverages :

1. medical costs (including repatrial costs);
2. a sum in case of death;
3. a sum in case of disablement (proportional to the degree of disability).

The risks resulting from some of these coverages are clearly more or less positively correlated (medical costs and disablement payments, for instance) while others are rather negatively correlated, or even mutually exclusive (death payments and disability payments). Many insurance contracts so provide various coverages generating dependent risks.

Now, consider such a portfolio consisting of ν independent risks of the form

$$X_k = X_{k1} + X_{k2} + X_{k3}, \quad k = 1, 2, \cdots, \nu,$$

where the X_{ki}'s are dependent with distribution functions F_i, $i = 1, 2, 3$, satisfying (1). Note that (1) is reasonable in the present context since the no-claim probabilities are in most cases rather high. The portfolio is protected by an excess of loss treaty, for which the reinsurance benefit is given by

$$\sum_{k=1}^{\nu} (X_k - d)_+.$$

Bounds on the reinsurance premium are provided by the following inequalities :

$$\sum_{i=1}^{3} E\left(F_i^{-1}(U) - d\right)_+ \leq E\left(X_k - d\right)_+ \leq E\left(\sum_{i=1}^{3} F_i^{-1}(U) - d\right)_+, \qquad (4)$$

where U is a random variable uniformly distributed over $[0,1]$. When

$$F_i(x) = \pi + (1-\pi)(1-\exp(-\lambda_i x)), \quad x \geq 0, \quad i = 1,2,3,$$

with $0 \leq \pi \leq 1$ and $3\pi \geq 2$, the bounds in (4) admit a nice closed expression. Indeed, after some algebra, it can be shown that

$$\pi \sum_{i=1}^{3} \frac{\exp(-\lambda_i d)}{\lambda_i} \leq E\left(X_k - d\right)_+ \leq d(1-\pi) \sum_{i=1}^{3} \frac{1}{\lambda_i}.$$

It is important to note that these bounds are the best that can be found in the Fréchet class $\mathcal{R}_3(F_1, F_2, F_3)$ and that hold for all retention levels d.

6.2 Optimality in life insurance

Let Z be the benefit function of an n-year endowment insurance which pays 1 at the moment of the insured's death, or 1 at the end of the n-year term, whichever occurs first. If T denotes the insured's future lifetime, Z is given by

$$Z = \begin{cases} v^T \text{ if } T \leq n, \\ v^n \text{ either,} \end{cases}$$

where v is the discount factor corresponding to the constant yearly interest rate stipulated in the contract.

Now, let T_1 and T_2 be the remaining lifetimes of two persons, such that T, T_1 and T_2 are identically distributed. We do not assume independence among these random variables. If we define

$$Z_1 = \begin{cases} v^{T_1} \text{ if } T_1 \leq n, \\ 0 \text{ either,} \end{cases} \quad \text{and } Z_2 = \begin{cases} 0 \text{ if } T_2 \leq n, \\ v^n \text{ either,} \end{cases}$$

we have from Theorem 10 that

$$Z \leq_{s\ell} Z_1 + Z_2,$$

which means that any risk-averse insurer will prefer to sell a single n-year term insurance, than to issue simultaneously an n-year pure endowment and an n-year term insurance, whatever the dependence between the remaining lifetimes is. Note that the latter inequality in the stop-loss sense holds whatever the dependency between T_1 and T_2 is.

From Kaas *et al.* (1994), it follows that

$$EZ^\alpha \leq E(Z_1 + Z_2)^\alpha \text{ for all } \alpha \geq 1.$$

As the expectations of both random variables are equal, we also get that

$$\mathrm{Var}[Z] \leq \mathrm{Var}[Z_1 + Z_2].$$

The latter inequality can be found in Bowers *et al.* (1996, Section 4.2.2) for independent T_1 and T_2; see also Kling & Wolthuis (1992).

Acknowledgements

The authors wish to thank an anonymous referee for his valuable comments and suggestions which significantly improved the paper. In particular, he pointed out the result contained in Theorem 12. The authors also acknowledge Alfred Müller for precious remarks.

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