Rational basis functions in discrete least squares approximation on the unit circle

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Abstract

We consider a problem that arises in the field of frequency domain system identification. If a discrete-time system has an input-output relation $Y(z) = G(z)U(z)$, with transfer function G, then the problem is to find a rational approximation $\hat{G}_n$ for G. The data given are measurements of input and output spectra in the frequency points ω_k: $\{U(e^{i\omega_k}), Y(e^{i\omega_k})\}_{k=1}^N$. The approximation criterion is to minimize the discrete least squares norm of the vector obtained by evaluating $Y - \hat{Y}_n$ in the measurement points where $\hat{Y}_n(z) = \hat{G}_n(z)U(z)$ is the output of the model.

If the poles of the system are fixed, then the problem reduces to a linear least squares solution of an overdetermined system of equations. However, when the transfer function $\hat{G}_n$ is represented with respect to a non-orthogonal basis, then this least squares problem can be very ill conditioned. Even if a orthogonal basis is used, but with respect to the wrong inner product (e.g., the Lebesgue measure on the unit circle) numerical instability can be fatal in practice.

Key words: approximation, least squares, orthogonal rational functions, system identification

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1 Introduction

It is a well known fact from undergraduate numerical analysis, that the sensitivity of a problem for rounding error perturbations can be (and should be) reduced by solving the problem in an orthogonal basis. This explains why many numerical approximation procedures make use of systems of orthogonal polynomials.

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Rather recently, the system engineering literature started using orthogonal rational basis functions for solving identification problems. A good survey of system identification is found in [10].

Consider a stationary discrete time linear SISO system described in the z-domain by

$$Y(z) = G(z)U(z), \tag{1.1}$$

where G is the tranfer function, $U(z) = \sum_k u_k z^{-k}$ is the z-transform of the input and similarly Y is z-transform of the output. Assume the system is stable, so that $G(z)$ is analytic for $|z| > 1$. Suppose we fix the poles of the rational approximant $\hat{G}_n$ of degree n for G. The problem is then linear and $\hat{G}_n$ belongs to an $(n+1)$-dimensional space. Assume this space is spanned by a set of rational basis functions $\{\phi_k\}_{k=0}^n$ so that we can write

$$\hat{G}_n(z) = \sum_{k=0}^{n} \theta_k \phi_k(z)$$

with the parameter vector $\underline{\theta} = [\theta_0, \ldots, \theta_n]^T$ to be chosen in an optimal way. For numerical (and many other theoretical [14]) reasons, one should choose these basis functions to be orthogonal.

The idea of representing the transfer function G as a linear combination of orthogonal rational functions with prescribed poles is not new, but it has received much attention in the literature lately [1,4–9,12–16,18,22,23].

All this identification literature is restricted to rational functions that are orthogonal with respect to a uniform continuous weight. One of the reasons is probably that an explicit expression for these basis functions exists. It is the so-called Takenaka-Malmquist basis [17,11]. In some applications however this is not the appropriate basis for solving the least squares problem to find the parameters $\underline{\theta} = [\theta_0, \ldots, \theta_n]^T$ from given spectral data, i.e., from values $\{U(z_k), Y(z_k)\}_{k=1}^N$, with all points $\{z_k\}$ on the unit circle. Using this basis can cause severe numerical problems, which are easily avoided using the appropriate basis as we will show.

The simple message of this short note is that the use of an orthogonal basis is not sufficient to overcome ill conditioning and hence numerical problems during the computation. To be really a remedy, the orthogonality should be with respect to the appropriate inner product.

The text is built up as follows: In section 2, we state some results about rational functions, orthogonal with respect to a general measure. Section 3 describes the identification problem and the reason why the rational functions of section 2 are necessary for numerically stable calculations. An example is provided in section 4

and some notes abote the optimal location of the poles is given in section 5.

2 Notation

In this section we introduce some notation and state some results which we need throughout the rest of the paper. The notation is very similar to the one used in [20].
By $\mathbb{T}$, we denote the unit circle in the complex plane. The complex numbers satisfying $|z| < 1$ and $|z| > 1$ are denoted by $\mathbb{D}$ and $\mathbb{E}$, respectively.
Given N weights w_k, we can define an inner product on $\mathcal{H}(\mathbb{E})$, the space of analytic functions on $\mathbb{E}$, as follows

$$\langle f, g \rangle_w = \sum_{k=1}^{N} f(e^{i\omega_k}) \overline{g(e^{i\omega_k})} w_k, \quad f, g \in \mathcal{H}(\mathbb{E}). \tag{2.1}$$

We will work with rational functions whose poles belong to a prescribed sequence $\underline{\alpha} = \{\alpha_k\}_{k=1}^{\infty}$. We assume that $\underline{\alpha} \subset \mathbb{D}$ so that the model is stable. The denominator polynomials are denoted by

$$\pi_n(z) = \prod_{k=1}^{n} (z - \alpha_k).$$

With Π_n we denote the space of polynomials of degree less than or equal to n.
We put $\alpha_0 = 0$, for technical reasons and define the Blaschke factors ζ_k as

$$\zeta_k(z) = \frac{1 - \overline{\alpha}_k z}{z - \alpha_k}, \quad k \in \mathbb{N}.$$

The Blaschke products B_k are defined as

$$B_0(z) = 1, \quad B_k(z) = B_{k-1}(z)\zeta_k(z), \quad k = 1, 2, \ldots$$

The space of rational functions of degree n is denoted by

$$\mathcal{L}_n = \operatorname{span}\{B_0(z), B_1(z), \ldots, B_n(z)\} = \left\{ \frac{p_n(z)}{\pi_n(z)} : p_n(z) \in \Pi_n \right\}$$

and we set $\mathcal{L} = \bigcup_{n=0}^{\infty} \mathcal{L}_n$.

The following lemma is found in [2, Corollary 7.1.3].

Lemma 2.1 *The space $\mathcal{L}$ is dense in the Hardy spaces $H^2(\mathbb{E})$ iff the Blaschke con-*

dition is satisfied, i.e.

$$\sum_{k=0}^{\infty}(1-|\alpha_k|)=\infty.$$

In practice the Blaschke condition is always fullfilled, since we look for an approximation of degree $n \ll N$ and thus we can put $\alpha_k = 0$ if $k > n$.

For a function $f \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$, we define the superstar transform by

$$f^*(z) = B_n(z)\overline{f(1/\overline{z})}.$$

By a Gram-Schmidt procedure we can orthonormalize the basis of Blaschke products and these orthogonal rational functions, denoted by ϕ_k, satisfy a recurrence relation [2,19].

Theorem 2.2 *The orthonormal rational functions ϕ_k satisfy the following recurrence relation*

$$\begin{bmatrix}\phi_k(z)\\ \phi_k^*(z)\end{bmatrix} = e_k\frac{z-\alpha_{k-1}}{z-\alpha_k}\begin{bmatrix}1 & \overline{L_k}\\ L_k & 1\end{bmatrix}\begin{bmatrix}\zeta_{k-1}(z) & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\phi_{k-1}(z)\\ \phi_{k-1}^*(z)\end{bmatrix},$$

where

$$L_k = -\frac{\left\langle \phi_l, \frac{1-\overline{\alpha}_{k-1}z}{z-\alpha_k}\phi_{k-1}\right\rangle_w}{\left\langle \phi_l, \frac{z-\alpha_{k-1}}{z-\alpha_k}\phi_{k-1}^*\right\rangle_w},\ \forall l \le k-1,$$

$$e_k = \left(\frac{1-|\alpha_k|^2}{1-|\alpha_{k-1}|^2}\frac{1}{1-|L_k|^2}\right)^{1/2}.$$

The initial conditions are $\phi_0(z) \equiv 1/\sqrt{\langle B_0, B_0\rangle_w} \equiv \phi_0^*(z)$.

Remark 2.3 Note that the reflection coefficients L_k can be computed by expressing orthogonality of ϕ_k to ϕ_l with $l \le k-1$. For numerical stability it is often better to choose $l = k-1$, since in general, the numbers $\phi_n(e^{i\omega_k})$ and $\phi_l(e^{i\omega_k})$ are more similar if l is closer to n, which causes less rounding errors and so it is better for preserving orthogonality. But even then, it may be necessary to introduce reorthogonalization after a while. This is a well known phenomenon when recursive orthogonalization is used. We are not going in detail here.

3 System approximation by rational functions

In this section we describe the problem we want to solve and give an algorithm that uses the orthogonality of the rational functions ϕ_k, described in section 2. We also

point out that the methods used in [22,23,6,18] can suffer from severe numerical problems due to the fact that the basis functions are not orthogonal with respect to the problem.

If the sequence $\underline{\alpha}$ satisfies the Blaschke condition (Lemma 2.1), then $\mathcal{L}$ is dense in $H^2(\mathbb{E})$ and we can expand the transfer function $G \in H^2(\mathbb{E})$ in a series with respect to the orthogonal rational functions obtained by orthogonalization of the Blaschke products:

$$G(z) = \sum_{k=0}^{\infty} \theta_k \phi_k(z), \quad z \in \mathbb{E}, \tag{3.1}$$

converging in $\mathbb{E}$. Note that if all poles $\alpha_k = 0$ and the measure is the Lebesgue measure, then the expansion (3.1) reduces to a Fourier series expansion when $z \in \mathbb{T}$ and the θ_k are nothing else but the Fourier coefficients of G.

In the case where one considers rational functions orthogonal with respect to a uniform measure on the unit circle, there exists an explicit form that can be found e.g. in [13]. These functions are known as the Takenaka-Malmquist basis functions.

Lemma 3.1 *The rational functions ϕ_k, orthogonal with respect to the Lebesgue measure are*

$$\phi_k(z) = \frac{\sqrt{1-|\alpha_k|^2}z}{z-\alpha_k} B_{k-1}(z),$$

with B_{k-1} the Blaschke product of order $k-1$.

This would be the appropriate basis to be used if we wanted the approximant to minimize

$$\|G - \hat{G}_n\|^2 = \frac{1}{2\pi} \int_0^{2\pi} |G(e^{i\omega}) - \hat{G}_n(e^{i\omega})|^2 \, \mathrm{d}\omega.$$

In that case, $\hat{G}_n(z) = \sum_{k=0}^{n} \theta_k \phi_k(z)$ is just the truncated expansion (3.1).

However, it is more common to minimize the prediction error, that is the difference between the output of the real system and the output of the model. Thus we want to minimize the norm of

$$E_n(z, \underline{\theta}) = Y(z) - \hat{Y}(z, \underline{\theta}) = \Big(G(z) - \hat{G}_n(z, \underline{\theta})\Big) U(z).$$

More precisely, since we know the input and output only in discrete frequency points, we want to minimize $\sum_{k=1}^{N} |E_n(z_k, \underline{\theta})|^2$. If we have an idea about the uncertainty of

the measurements, then we could weight this by σ_k^{-2} if σ_k represents the variance of the measurement error.

Writing again for some $\underline{\theta} = [\theta_1, \ldots, \theta_n]^T$

$$\hat{G}_n(z, \underline{\theta}) = \sum_{k=0}^{n} \theta_k \phi_k(z),$$

then we have to minimize

$$||\mathbf{E}||_{\sigma^{-2}}^2 = ||\mathbf{Y} - \hat{\mathbf{Y}}||_{\sigma^{-2}}^2 = ||G - \hat{G}_n(\cdot, \underline{\theta})||_w^2,$$

where we have set $\mathbf{E} = [E_n(z_1, \underline{\theta}), \ldots, E_n(z_N, \underline{\theta})]^T$, $\hat{\mathbf{Y}} = [\hat{Y}(z_1), \ldots, \hat{Y}(z_N)]^T$, $\hat{Y}(z_k) = \hat{G}_n(z_k, \underline{\theta})U(z_k)$ and w is the discrete measure taking the values $w_k = |U(z_k)/\sigma_k|^2$ at the points $z_k \in \mathbb{T}$. The norm $\|g\|_w = \sqrt{\langle g, g \rangle_w}$ is defined as usual by the inner product (2.1).

This shows that it is necessary to use a basis orthogonal with respect to the weights w_k, thus with respect to a data-dependent measure.

The linear system of equations to be solved can be written as

$$\mathbf{Y} = \Psi \underline{\theta},$$

with

$$X_l(z_k) = \phi_l(z_k) \frac{U(z_k)}{\sigma_k}, \ \mathbf{X}_l = [X_l(z_1), \ldots, X_l(z_N)]^T, \ \Psi = [\mathbf{X}_0, \ldots, \mathbf{X}_n].$$

If we choose the rational functions ϕ_k to be orthogonal with respect to the discrete inner product with weights $\{w_k\}$, then the matrix Ψ is orthogonal. This implies that

$$\underline{\theta}^{LS} = \Psi^H \mathbf{Y}. \tag{3.2}$$

If we do not use the appropriate basis functions, but e.g. the Takenaka-Malmquist basis of Lemma 3.1, then the least-squares solution (3.2) becomes

$$\underline{\theta}^{LS} = \left(\Psi^H \Psi\right)^{-1} \Psi^H \mathbf{Y},$$

and if the condition number of the system matrix $\Psi^H \Psi$ is high, the accuracy of the calculations can become disastrous as we show in the example in the next section. The algorithm with the Takenaka-Malmquist basis is not only less accurate, it requires also more work.

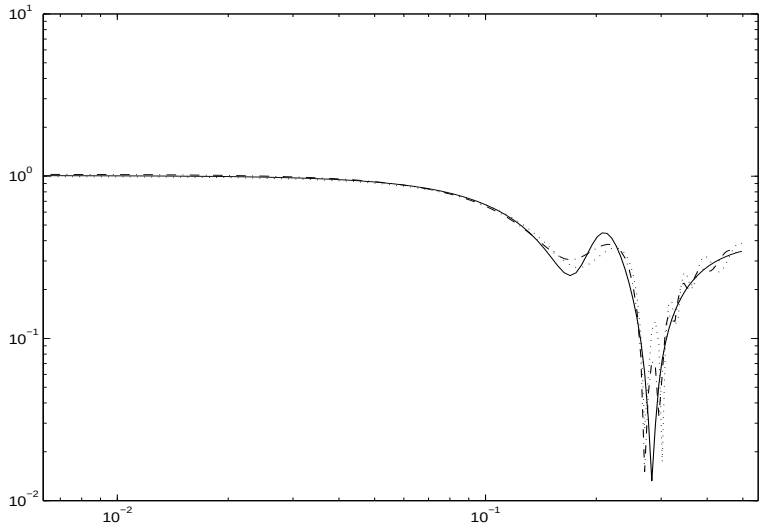


Fig. 1. The transfer function of the example. The solid line is the exact system. The approximant constructed by the Takenaka-Malmquist basis is the dotted line and the dash-dotted line is the approximant

4 Example

Here we give a little example to show that the explicit form of the orthogonal rational functions does not guarantee the best solution, which we expect theoretically.

We consider the following transfer function (see [18])

$$G(z) = \frac{.25299 - .9724z^{-1} + 1.4282z^{-2} - .9493z^{-3} + .241z^{-4}}{1 - 4.15z^{-1} + 6.8831z^{-2} - 5.6871z^{-3} + 2.3333z^{-4} - .3787z^{-5}},$$

which has poles $.95 \pm .2i$, $.85 \pm .09i$ and $.55$.

As input, we consider a zero-mean unit-variance white noise signal of length $N = 1200$ and frequency data equidistant in $[-\pi/6, \pi/6]$. For the model of degree 20 with poles $.9 \pm .3i$ and $.7 \pm .2i$ repeated 5 times as in [18], the condition number of Ψ is $O(10^{17})$ for the Takenaka-Malmquist basis and $1 + O(10^{-12})$ for the basis of appropriate orthogonal rational functions. The relative error $\|Y - \hat{Y}\|/\|Y\|$ for the orthogonal rational functions turns out to be 63% better!

5 The nonlinear problem

The fact that rational basis functions are used will in the case of slowly decaying impulse responses and when the chosen poles are reasonable approximations of the true poles, result in a good approximant of a rather low degree. The coefficients in the series expansion of G will decay rapidly. Of course this is only because we use a rational model, and this is independent from the fact that we use a basis that satisfies the appropriate orthogonality or not. It will mainly depend on how accurate the estimation of the poles has been. Such considerations can be found

for example in [5, Chapter 6] and the references cited there. All the theoretical conclusions and properties when using rational basis functions can be drawn whether we take the Takenaka-Malmquist basis or not. The contribution of this paper is of a pure numerical order.

Of course the criterion on minimizing the 2-norm of the prediction error is but one of the many cost functions that have been considered in the literature. One could have preferred others like H_∞ or maximum likelihood etc. Moreover, one does not know the position of the poles of the system and it is part of the problem to determine their optimal location. The problem then becomes highly nonlinear, the cost function may be not convex, and iterative methods used can not be guaranteed to converge to the optimal solution, and if they converge it might be to a local minimum that is far from the global one. So these iterative procedures need very good starting points.

However, since the algorithm we describe above is fast and accurate it is possible to successfully use it in a larger algorithm to search for the poles of the underlying system.

One way to solve the non-linear problem is by using a Sanathanan-Koerner type of iteration [3], before a true maximum likelihood iteration is started. Or one can start with a high order approximation, which is then order reduced, and the poles of that approximant are used as new estimates for the poles, or one of the many other techniques discussed in the literature can be used. See e.g. [5, Chapter 6]. It is clear however that in our setting, the poles are the only nonlinear parameters. Indeed, if we estimate the poles $\underline{\alpha}$, then the problem becomes linear and it can be solved efficiently and accurately for the linear parameters. Thus the vector of linear parameters $\underline{\theta}$ can be considered as a function of the nonlinear parameters which are for example the poles $\underline{\alpha}$. Thus, the ϕ_k, which depend on $\underline{\theta}$, and hence on $\underline{\alpha}$, give rise to a nonlinear least squares problem where we have to minimize the norm of the vector

$$\mathbf{E}(\underline{\alpha}) = \Psi(\underline{\alpha}) - \mathbf{Y} = (\Psi(\underline{\alpha})\Psi(\underline{\alpha})^H - I)\mathbf{Y}.$$

A problem is that the poles $\underline{\alpha}$ will not be guaranteed to stay inside $\mathbb{D}$ during the iterations. Thus we have to introduce a constraint $|\alpha_k| < 1$. This means that we write $\alpha_k = \rho_k e^{i\omega_k}$ and put a constraint on ρ_k, viz. $-1 < \rho_k < 1$. For real systems, it is usually required that the poles are either real or appear in complex conjugate pairs. In that case, we can bring a conjugate pair of complex poles together in a quadratic factor $(z-\alpha)(z-\overline{\alpha}) = z^2 - uz + v$ and we can work with two real parameters u and v per conjugate pair. These should satisfy the constraints $-2 < u < 2$ and $-1 < v < 1$. The real poles appear as real parameters with constraint $-1 < \alpha < 1$. So the whole problem is parameterized with real parameters. Such optimization procedures were implemented and gave very promising results, which will be reported elsewhere.

Finally we mention that for control applications it is often desirable to work with a

state space description of the model. Also this can be obtained for our general orthogonal rational functions. We do not include it here for reasons of space limitations. We refer to [21] for this approach.

6 Conclusions

We have given an identification algorithm for the frequency domain. The main advantage of this approach, where we incorporate a weight which depends on the input signal, is that the least squares problem becomes well conditioned and is straightforward. This results in faster algorithms, which are numerically stable. These algorithms can be used in each step of an iterative procedure to find the exact poles of the underlying system.

We only improved numerical results. The theoretical error bounds for the best least squares approximation in a given set of rational functions is of course independent of basis that we choose to represent the approximant, since we are searching for the best solution in the same subspace.

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