ANALYSIS AND DESIGN OF STRONGLY STABILIZING PID CONTROLLERS FOR TIME-DELAY SYSTEMS

PIETER APPELTANS*, SILVIU-IULIAN NICULESCU†, AND WIM MICHIELS*

Abstract. This paper presents the analysis of the stability properties of PID controllers for dynamical systems with multiple state delays, focusing on the mathematical characterization of the potential sensitivity of stability with respect to infinitesimal parametric perturbations. These perturbations originate for instance from neglecting feedback delay, a finite difference approximation of the derivative action, or neglecting fast dynamics. It is also shown that adding a low pass filter, which might be necessary to reduce sensitivity to high frequency noise, under certain conditions on the derivative feedback gain may destabilize the closed loop even for arbitrarily large cut-off frequencies. The analysis of these potential sensitivity problems leads us to the introduction of a 'robustified' notion of stability called *strong stability*, inspired by the corresponding notion for neutral functional differential equations. We prove that strong stability can be achieved by adding a low-pass filter with a sufficiently large cut-off frequency to the control loop on the condition that the filter itself does not destabilize the nominal closed loop system. Throughout the paper, the theoretical results are illustrated by examples that can be analyzed analytically, including, among others, a third-order unstable system for which both proportional and derivative control action are necessary for achieving stability, while the regions in the gain parameter-space for stability and strong stability are not identical. Besides the analysis of strong stability, a computational procedure is provided for designing strongly stabilizing PID controllers. Computational case-studies illustrating this design procedure complete the presentation.

Key words. PID control, Stability, Modelling uncertainty, Robustness

AMS subject classifications. 93B35, 93B52, 93C23, 93D15, 93D22

1. Introduction.

In this paper we analyze proportional-integral-derivative (PID) output feedback control of multiple-input-multiple-output (MIMO) linear time invariant (LTI) dynamical systems with discrete state delays. In such controllers, the derivative action allows for improved transient response, while the integral action allows to eliminate steady state errors. PID controllers are used in many control applications and are well-established in industry due to their implementation simplicity, in the sense that they only require the tuning of a limited number of parameters for which good tuning procedures are available [1, 27]. The implementation simplicity is particularly apparent for time-delay systems, for which a full order controller would be infinite-dimensional. Furthermore, for classes of delay systems, in particular systems with feedback delay, analytic characterization of the stabilization and stabilizability is still possible, as witnessed by a plentiful recent literature including, among others, [30, 9, 28, 25, 7, 37, 36, 26, 18, 14] and the references therein. On the other hand, the property that PID controllers are structured and of low-order imposes limitations on stabilizability and performance. These controllers are therefore not generally applicable.

An alternative for PI(D) control is proportional-integral-retarded (PIR) control, in which the derivative signal is replaced by feeding back the output signal with a purposely introduced delay [4, 29]. This approach is motivated by the sensitivity of the derivative signal with respect to high-frequency noise, which can deteriorate the performance of the controller. Note that by using a finite difference approximation of

*NUMA Section, Department of Computer Science, KU Leuven, Leuven, Belgium (Pieter.Appeltans@cs.kuleuven.be, Wim.Michiels@cs.kuleuven.be).

† Laboratory of Signals and Systems (L2S), Inria Team "DISCO", CNRS, CentraleSupélec, Université Paris-Saclay, Gif-sur-Yvette, France (Silviu.Niculescu@l2s.centralesupelec.fr).

the derivative, i.e., $\dot{y}(t) \approx \frac{y(t)-y(t-\tau)}{\tau}$, derivative feedback can be seen as a limit case of delayed feedback in which the size of the delay τ goes to zero.

Central in the presented work is the analysis of a problem of well-posedness, namely the sensitivity of stability of the closed loop system with respect to arbitrarily small modelling errors, which include for instance neglected feedback delay, a finite-difference approximation of the derivative and neglected high-frequency behavior. Similar sensitivity problems are examined in [8], in which the concept of *w-stability* is introduced. w-Stability requires that the closed loop system remains stable for sufficiently small perturbations, in which the considered perturbation class consists of approximate identities (see [34, Section V.C] for a definition of approximate identities). Of interest for this work, is Section 4 which examines non-proper rational controllers, such as PID controllers, for the control of rational (finite-dimensional) plants. More specifically, Theorem 2 in [8] suggests that w-stability can be induced by adding a low-pass filter with a sufficiently large cut-off frequency to the control loop. However, as suggested by [8, Theorem 3] such a low-pass filter can itself destroy the stability of the nominal closed loop system. In this work, we will derive similar results but we will focus on time-delay systems, which are infinite dimensional, and PID control. Furthermore, we will use a different perturbation class and besides perturbations on the input and output channels, the framework presented here also includes a perturbation inside the (derivative part of the) controller. This additional perturbation is motivated by the implementation of the derivative using a finite-difference approximation.

Another important starting point is article [22], which examines the stabilizability of a controllable LTI system using state-derivative feedback control, i.e., using a control law of the form $u(t) = K\dot{x}(t)$ instead of the conventional state feedback $u(t) = Kx(t)$. While the nominal system can be stabilized if and only if the system matrix of the open-loop system has no zero eigenvalue, this stability can be destroyed by infinitesimal perturbations if the open loop system matrix has an odd number of eigenvalues in the open right half plane (the so-called odd number limitation). However, if the number of right half plane eigenvalues is even, then robust stability in the presence of sufficiently small perturbations can always be achieved by adding a low-pass filter to the control law.

Finally, there exists an abundant literature that addresses fragility problems with respect to infinitesimal perturbations, that includes, among others, [16, 5, 10, 23, 15] (and the references therein) discussing the sensitivity of stability of controlled systems with respect to feedback delay, [11, 20, 17] (and the references therein) discussing the sensitivity of stability of neutral functional differential equations with respect to infinitesimal (changes on the) delays, and [31] discussing the fragility of a gradient play dynamics model against a derivative approximation.

In this work we will extend the theoretical framework of [22] to systems with state delays controlled by PID output feedback, and complement it with an algorithm for the design of the controller gain matrices. More specifically, after introducing the considered set-up and illustrating the aforementioned sensitivity problems in Section 2, we derive conditions on the derivative gain matrix under which a stable closed loop system loses stability under arbitrarily small feedback delay (Subsection 3.1) and under an arbitrary close finite-difference approximation of the derivative (Subsection 3.2). In Section 4, we introduce the notion of *strong stability*, in analogy with the corresponding notion for neutral functional differential equations [11], which requires the closed loop system to remain stable under sufficiently small perturbations.

As in [22], we will show that a strongly stable closed loop can be obtained by adding a low-pass filter to the control loop, under the condition that this low-pass filter itself does not destabilize the system. We will see that this last condition induces an algebraic constraint on the derivative gain matrix. Subsequently, Section 5 presents a computational procedure to design strongly stabilizing PID controllers with low pass filtering. This procedure consists of minimizing the spectral abscissa of the nominal, i.e., without low-pass filter, closed loop system in function of the elements of the feedback matrices under the aforementioned constraint on the derivative gain matrix. The presented method thus fits in the direct optimization framework as, for example, used in [19, 6]. This framework has the advantage that a particular structure of the feedback matrices can be easily incorporated in the design procedure. This comes, however, at the cost of having to solve a small non-convex non-smooth optimization problem. Section 6 presents some remarks on how to generalize the presented results to systems with input delay and to systems with bounded uncertainties on the system matrices and the state delays. Finally, Section 7 concludes the paper.

2. Problem statement. This work considers LTI systems with discrete state delays of the form:

$$\begin{cases} \dot{x}(t) = A_0\, x(t) + \sum_{k=1}^{K} A_k\, x(t-\tau_k) + B\, u(t), \\ y(t) = C\, x(t) \end{cases} \tag{2.1}$$

with $x \in \mathbb{R}^n$ the internal state, $u \in \mathbb{R}^m$ the input, $y \in \mathbb{R}^p$ the output, $0 < \tau_1 < \cdots < \tau_K < +\infty$ discrete delays, A_0, A_1, …, $A_K \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$. Furthermore, without loss of generality, we will assume that B is of full column rank and that C is of full row rank. The goal is to design a PID output feedback control law

$$u(t) = K_p\, y(t) + K_d\, \dot{y}(t) + K_i \int_0^t y(s)\, \mathrm{d}s, \tag{2.2}$$

with K_p, K_d and K_i real-valued $m \times p$ matrices, that exponentially stabilizes the system. We will examine the exponential stability of the closed loop system consisting of (2.1) and (2.2) using the following characteristic function

$$H_0(\lambda) := \det\left(\lambda \begin{bmatrix} I_n - BK_dC & 0 \\ 0 & I_q \end{bmatrix} - \begin{bmatrix} A_0 + BK_pC & BU_i \\ V_iC & 0 \end{bmatrix} - \sum_{k=1}^{K} \begin{bmatrix} A_k & 0 \\ 0 & 0 \end{bmatrix} e^{-\lambda\tau_k}\right) \tag{2.3}$$

with I_n the identity matrix of size n, $q = \operatorname{rank}(K_i)$ and $K_i = U_iV_i$, a rank revealing decomposition with $U_i \in \mathbb{R}^{m\times q}$ of full column rank and $V_i \in \mathbb{R}^{q\times p}$ of full row rank. More specifically, the closed loop system is exponentially stable if and only if all roots of (2.3) lie inside the open left half plane bounded away from the imaginary axis. However, the following example illustrates that a stable closed loop system of this form can loose stability due to arbitrarily small implementation errors.

Example 2.1. Consider the following second-order, single-input single-output system:

$$\begin{cases} \dot{x}(t) &= \begin{bmatrix} 0 & \omega_0 \\ \omega_0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} -1 \\ 0 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{cases}$$

with as corresponding transfer function $-s/(s^2-\omega_0^2)$. We want to control this system using the feedback law $u(t) = k_p y(t)+k_d\dot{y}(t)$, that is nothing else than a PD controller. For this set-up, characteristic function (2.3) reduces to the following second-order polynomial:

$$(1 + k_d)\lambda^2 + k_p\lambda - \omega_0^2.$$

It follows from the Routh-Hurwitz stability criterion for polynomials of degree two that the roots of this polynomial lie inside the open left half plane if (and only if) $k_d < -1$ and $k_p < 0$. The system is thus stabilized by any control law with $k_d < -1$ and $k_p < 0$, meaning that proportional or derivative feedback on their own can not exponentially stabilize the system. Next, we examine the effect of delayed feedback and a finite-difference approximation of the derivative on the stability of the closed loop and we will show that any stable closed loop system loses stability under these implementation errors, even when the size of the error is arbitrarily small.

As first implementation error, we consider delayed feedback, i.e., $u(t) = k_p y(t - r) + k_d\dot{y}(t - r)$ with $r > 0$. In this case, the exponential stability of the closed loop system is characterized by the roots of the following quasi-polynomial

$$(1 + k_d e^{-\lambda r})\lambda^2 + k_p\lambda e^{-\lambda r} - \omega_0^2. \tag{2.4}$$

It follows from [21, Proposition 1.28] that (2.4) has a sequence of roots $\{\lambda_k\}_{k=1}^{\infty}$ which satisfies

$$\lim_{k\to\infty} |\Im(\lambda_k)| = +\infty \text{ and } \lim_{k\to\infty} \Re(\lambda_k) = \frac{1}{r}\ln\left(|k_d|\right).$$

Characteristic function (2.4) thus has (infinitely many) roots in the closed right half plane for any $k_d < -1$ and any $r > 0$. Under delayed feedback, the closed loop system is thus no longer stable, even if the feedback delay is arbitrarily small. As second implementation error, we consider a finite-difference approximation of the derivative:

$$\dot{y}(t) \approx \frac{y(t) - y(t - r)}{r},$$

with $r > 0$. Such an approximation might be necessary if there is no sensor to measure the actual derivative. For the considered example, the exponential stability of the closed loop system is now characterized by the roots of

$$f(\lambda) := \lambda^2 + \left(k_p + k_d\frac{1 - e^{-\lambda r}}{r}\right)\lambda - \omega_0^2. \tag{2.5}$$

This characteristic function can be rewritten as $f(\lambda) = \lambda^2\Big(f_1(\lambda) - f_2(\lambda)\Big)$ with

$$f_1(\lambda) := 1 + \frac{k_p}{\lambda} - \frac{\omega_0^2}{\lambda^2} \quad \text{and} \quad f_2(\lambda) := -k_d\frac{1 - e^{-\lambda r}}{\lambda r}.$$

As $\lambda = 0$ is not a root of $f(\cdot)$, the roots of (2.5) thus correspond to the crossings of $f_1(\cdot)$ and $f_2(\cdot)$ in the complex plane. Figure 1 shows these functions for λ real-valued and positive, $k_d < -1$, $k_p < 0$ and $r > 0$. For the considered parameter region, characteristic function (2.5) thus has a real-valued root in the right half plane. Furthermore, this root moves to $+\infty$ as $r \to 0+$. So also under an arbitrary accurate finite-difference approximation of the derivative, stability is lost. ◇

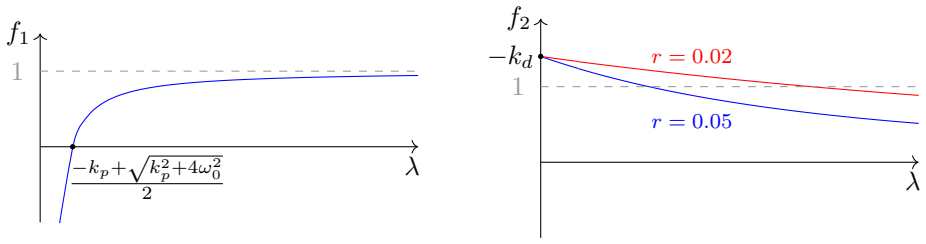


Fig. 1: The functions $f_1(\lambda)$ and $f_2(\lambda)$ for λ real-valued and positive, $k_d < -1$, $k_p < 0$ and $r > 0$.

Remark 2.2. We consider (2.3) as characteristic function instead of

$$\det\left(\lambda\begin{bmatrix} I_n - BK_dC & 0 \\ 0 & I_p \end{bmatrix} - \begin{bmatrix} A_0 + BK_pC & B \\ K_iC & 0 \end{bmatrix} - \sum_{k=1}^{K}\begin{bmatrix} A_k & 0 \\ 0 & 0 \end{bmatrix} e^{-\lambda\tau_k}\right),$$

which one would naively obtain by bringing the characteristic equation associated with the closed loop system to linear form. This is motivated by the fact that this last equation might introduce roots in the origin that have no relation with the actual asymptotic behavior of the closed loop system. Furthermore, note that q $\big(= \operatorname{rank}(K_i)\big)$ corresponds to the minimal number of integrators needed to implement the (integral) control action. ◇

3. Losing stability under arbitrarily small implementation errors. Motivated by Example 2.1, this section will derive conditions on the spectrum of BK_dC under which stability is lost under feedback delay (Subsection 3.1) and a finite-difference approximation of the derivative (Subsection 3.2), even when the size of the perturbation is arbitrarily small.

3.1. Feedback delay. For studying the effect of delayed feedback, on a stabilizing feedback law of form (2.2), consider now the following system:

$$\dot{x}(t) = A_0\,x(t) + \sum_{k=1}^{K} A_k\,x(t-\tau_k) + B\,u(t-r), \tag{3.1}$$

derived from (2.1) by including a delay $r > 0$ in the input.

The following proposition gives a *sufficient condition* on the spectrum of BK_dC under which stability is lost under delayed feedback with an arbitrarily small delay.

PROPOSITION 3.1. *Assume that the gain matrices K_p, K_d and K_i are such that the closed loop system (2.1)-(2.2) is exponentially stable. If the spectral radius of BK_dC is larger than one, then the closed loop of (3.1) and (2.2) is unstable for all $r > 0$.*

Proof. Under delayed feedback, the closed loop system is unstable if

$$H_1(\lambda;r) := \det\left(\lambda\left(\begin{bmatrix} I_n & 0 \\ 0 & I_q \end{bmatrix} - \begin{bmatrix} BK_dC & 0 \\ 0 & 0 \end{bmatrix} e^{-\lambda r}\right) - \begin{bmatrix} A_0 & 0 \\ V_iC & 0 \end{bmatrix} - \begin{bmatrix} BK_pC & BU_i \\ 0 & 0 \end{bmatrix} e^{-\lambda r} - \sum_{k=1}^{K}\begin{bmatrix} A_k & 0 \\ 0 & 0 \end{bmatrix} e^{-\lambda\tau_k}\right) \tag{3.2}$$

has at least one root inside the right half plane. Note that (3.2) corresponds to the characteristic function of a neutral delay eigenvalue problem [21, Section 1.2]. It follows therefore from [21, Proposition 1.28] that for non-zero BK_dC, characteristic function (3.2) has a sequence of characteristic roots $\{\lambda_k\}_{k=1}^{\infty}$ for which

$$\lim_{k\to\infty} |\Im(\lambda_k)| = +\infty \text{ and } \lim_{k\to\infty} \Re(\lambda_k) = \frac{1}{r} \ln\big(\rho(BK_dC)\big)$$

with $\rho(\cdot)$ the spectral radius of its matrix argument. If $\rho(BK_dC) > 1$, then for all $r > 0$, characteristic function $H_1(\cdot; r)$ has thus (infinitely many) characteristic roots in the right half plane. ∎

3.2. Approximation of derivatives. Next, we derive a similar condition for the effect of using a finite-difference approximation instead of the actual derivative. The feedback signal now becomes:

$$u(t) = K_p\, y(t) + K_d\, \frac{y(t) - y(t-r)}{r} + K_i \int_0^t y(s)\, \mathrm{d}s. \tag{3.3}$$

PROPOSITION 3.2. *Assume that the gain matrices K_p, K_d and K_i are such that the closed loop system (2.1)-(2.2) is exponentially stable. If at least one of the eigenvalues of BK_dC lies outside* $\mathrm{clos}(S)$, *with*

$$S := \left\{ \begin{array}{l} \lambda \in \mathbb{C} :\ \Im(\lambda) \in (-\pi,\ \pi) \text{ and} \\ \qquad\qquad \Re(\lambda) < \left\{ \begin{array}{ll} \Im(\lambda)\ \mathrm{cotan}(\Im(\lambda)), & \Im(\lambda) \in (-\pi,\ 0) \cup (0,\ \pi), \\ 1, & \Im(\lambda) = 0, \end{array} \right. \end{array} \right\}$$

then there exists a $\hat{r} > 0$ such that the closed loop of (2.1) and (3.3) is unstable for all $r \in (0, \hat{r})$.

Proof. The stability of the closed loop of (2.1) and (3.3) is characterized by the roots of

$$H_2(\lambda; r) := \det\left(\lambda \left(\begin{bmatrix} I_n & 0 \\ 0 & I_q \end{bmatrix} - \begin{bmatrix} BK_dC & 0 \\ 0 & 0 \end{bmatrix} \tfrac{1-e^{-\lambda r}}{\lambda r} \right) - \begin{bmatrix} A_0 + BK_pC & BU_i \\ V_iC & 0 \end{bmatrix} - \sum_{k=1}^{K} \begin{bmatrix} A_k & 0 \\ 0 & 0 \end{bmatrix} e^{-\lambda \tau_k} \right). \tag{3.4}$$

Multiplying $H_2(\cdot; r)$ with r^{n+q} and introducing the change of variables $\widehat{\lambda} = \lambda r$, we obtain

$$G(\widehat{\lambda}; r) := \det\left(\widehat{\lambda} \begin{bmatrix} I_n & 0 \\ 0 & I_q \end{bmatrix} - \begin{bmatrix} BK_dC & 0 \\ 0 & 0 \end{bmatrix} \left(1 - e^{-\widehat{\lambda}}\right) - r \begin{bmatrix} A_0 + BK_pC & BU_i \\ V_iC & 0 \end{bmatrix} - \sum_{k=1}^{K} r \begin{bmatrix} A_k & 0 \\ 0 & 0 \end{bmatrix} e^{-\widehat{\lambda}(\tau_k/r)} \right).$$

As $r \searrow 0$, $G(\cdot; r)$ uniformly converges to

$$\widetilde{G}(\widehat{\lambda}) := \det\left(\widehat{\lambda} \begin{bmatrix} I_n & 0 \\ 0 & I_q \end{bmatrix} - \begin{bmatrix} BK_dC & 0 \\ 0 & 0 \end{bmatrix} \left(1 - e^{-\widehat{\lambda}}\right) \right)$$

on compact regions in the open right half plane. This function $\widetilde{G}(\cdot)$ has a root of multiplicity (at least) q at the origin; the remaining roots of $\widetilde{G}(\cdot)$ are the solutions of one of the following n equations

$$\widehat{\lambda} - \lambda_i \left(1 - e^{-\widehat{\lambda}}\right) = 0 \quad \text{for } i = 1, \ldots, n,$$

in which $\{\lambda_i\}_{i=1}^n$ are the eigenvalues of BK_dC. It follows from [22, Lemma A.1] that $\widetilde{G}(\cdot)$ has a root in the open right half plane if (and only if) BK_dC has an eigenvalue outside $\text{clos}(S)$. As $G(\cdot; r)$ uniformly converges to $\widetilde{G}(\cdot)$ on compact regions in the open right half plane, it follows from a similar argument as in [22, Proposition 3.1 - Case 1] that there exist $c > 0$ and $\hat{r} > 0$ such that the function $G(\cdot; r)$ has at least one root in the right half plane $\{\lambda \in \mathbb{C} : \Re(\lambda) > c\}$ for all $r \in (0, \hat{r})$. Because the roots of $H_2(\cdot; r)$ correspond to the roots of $G(\cdot; r)$ divided by r, $H_2(\cdot; r)$ has at least one root in the right half plane $\{\lambda \in \mathbb{C} : \Re(\lambda) > c/r\}$. ∎

4. Strong stability. The previous section demonstrated that, under certain conditions on the spectrum of BK_dC, there exist arbitrarily small implementation errors that render a stable closed loop system unstable. Or in other words, under certain conditions on the derivative gain matrix, the stability of the closed loop system is *sensitive* with respect to infinitesimal perturbations. As observed in the proofs of the previous section, these perturbations introduced characteristic roots in the right half plane that move to $+\infty$ as the perturbation size goes to zero. An intuitive way to avoid such right half plane roots is to apply a low-pass filter to the derivative signal, leading to the following control law:

$$u(t) = K_p y(t) + K_d \zeta(t) + K_i \int_0^t y(s)\, \mathrm{d}s \tag{4.1}$$

with

$$T\dot{\zeta}(t) + \zeta(t) = \dot{y}(t),$$

and $1/T$ the cut-off frequency. However, as observed in [8, Theorem 3] and [22, Secion 4.1] this low-pass filter might itself be destabilizing.

The following subsection shows that this low-pass filter does not destroy stability for sufficiently large cut-off frequencies if a certain algebraic constraint on K_d is fulfilled. Subsequently, Subsection 4.2 introduces the notion of strong stability, which requires the closed loop to remain stable under sufficiently small perturbations. In that subsection we will also show that if the aforementioned constraint on K_d is fulfilled, then the closed loop with low-pass filtering is strongly stable. Finally, these theoretical results and those of the previous section are illustrated using a third-order system in Subsection 4.3.

4.1. Losing stability under low-pass filtering. The following proposition shows that, under a particular condition on the spectrum of BK_dC, the stability of the closed loop system (2.1)-(2.2) is sensitive with respect to low pass filtering of the derivative signal. However, this proposition also shows that if the real parts of the eigenvalues of BK_dC are smaller than 1, then stability of the closed loop system is preserved when replacing control law (2.2) by control law (4.1) with T sufficiently small.

PROPOSITION 4.1. *Assume that the gain matrices K_p, K_d and K_i are such that the closed loop system (2.1)-(2.2) is exponentially stable. If BK_dC has an eigenvalue*

with real part larger than one, then there exists a $\widehat{T} > 0$ such that the closed loop of (2.1) and (4.1) is unstable for all $T \in (0, \widehat{T})$. On the other hand, if $BK_dC - I_n$ is Hurwitz then there exists a $\widetilde{T} > 0$ such that the closed loop of (2.1) and (4.1) is exponentially stable for all $T \in (0, \widetilde{T})$.

Proof. The closed loop stability of system (2.1)-(4.1) is characterized by the roots of

(4.2)
$$H_3(\lambda; T) := \det\left(\lambda\left(\begin{bmatrix} I_n & 0 \\ 0 & I_q \end{bmatrix} - \tfrac{1}{\lambda T+1}\begin{bmatrix} BK_dC & 0 \\ 0 & 0 \end{bmatrix}\right) - \begin{bmatrix} A_0 + BK_pC & BU_i \\ V_iC & 0 \end{bmatrix} - \sum_{k=1}^{K}\begin{bmatrix} A_k & 0 \\ 0 & 0 \end{bmatrix} e^{-\lambda\tau_k}\right).$$

If BK_dC has an eigenvalue with real part larger than one, then one can use a similar derivation as in the proof of Proposition 3.2 to show that there exist $c > 0$ and $\widehat{T} > 0$ such that $H_3(\cdot; T)$ has a root in the right half plane $\{\lambda \in \mathbb{C} : \Re(\lambda) > c/T\}$ for all $T \in (0, \widehat{T})$.

On the other hand, if $BK_dC - I_n$ is Hurwitz, then for each $c > 0$ there exists a $\widehat{T} > 0$ such that $I_n - (BK_dC)/(\lambda T + 1)$ is invertible for all λ that lie inside the right half plane $V := \{\lambda \in \mathbb{C} : \Re(\lambda) > -c\}$ and all $T \in (0, \widehat{T})$. This means that for λ inside the right half plane V and $T \in (0, \widehat{T})$, $H_3(\lambda; T) = 0$ is equivalent with

$$\det\left(\lambda\begin{bmatrix} I_n & 0 \\ 0 & I_q \end{bmatrix} - \begin{bmatrix} \left(I_n - \frac{BK_dC}{\lambda T+1}\right)^{-1} & 0 \\ 0 & I_q \end{bmatrix} \times \left(\begin{bmatrix} A_0 + BK_pC & BU_i \\ V_iC & 0 \end{bmatrix} + \sum_{k=1}^{K}\begin{bmatrix} A_k & 0 \\ 0 & 0 \end{bmatrix} e^{-\lambda\tau_k}\right)\right) = 0.$$

A characteristic root, λ_0, of $H_3(\cdot; T)$ for $T \in (0, \widehat{T})$ inside V is thus bounded in modulus by

$$|\lambda_0| \leq \sup_{\substack{\lambda \in V \\ T \in (0, \widehat{T})}} \left\| \begin{bmatrix} \left(I_n - \frac{BK_dC}{\lambda T+1}\right)^{-1} & 0 \\ 0 & I_q \end{bmatrix} \right\| \times \left(\left\| \begin{bmatrix} A_0 + BK_pC & BU_i \\ V_iC & 0 \end{bmatrix} \right\| + \textstyle\sum_{k=1}^{K} \|A_k\| e^{-\Re(\lambda)\tau_k} \right) := \Xi < \infty$$

Because $H_3(\cdot; T)$ uniformly converges to $H_0(\cdot)$ on compact regions in the complex plane as $T \searrow 0$, there exists a $\widetilde{T} \leq \widehat{T}$ such that for all $T \in (0, \widetilde{T})$

$$\max_{\lambda \in \partial\Omega} |H_3(\lambda; T) - H_0(\lambda)| < \min_{\lambda \in \partial\Omega} |H_0(\lambda)|$$

with Ω the intersection of V and $\{\lambda \in \mathbb{C} : |\lambda| \leq \Xi\}$ if $H_0(\cdot)$ has no root on $\partial\Omega$. It then follows from Rouché's theorem [32, pages 116-117] that $H_3(\cdot; T)$ and $H_0(\cdot)$ have the same number of zeros in Ω for all $T \in (0, \widetilde{T})$. The proposition now follows by choosing $-c$ larger than the real part of the right-most root of $H_0(\cdot)$. ∎

4.2. Strongly stable closed loop. In this subsection, we will show that if the low-pass filter itself is not destabilizing, then closed loop system (2.1)-(4.1) does not suffer from the same sensitivity problems as encountered in Section 3. Furthermore,

instead of restricting ourselves to delayed feedback and a finite-difference approximation of the derivative, we will consider all perturbations that fit the framework given in Figure 2, in which perturbation functions $R_1(\cdot;\cdot) : \mathbb{C} \times [0, +\infty) \mapsto \mathbb{C}^{m\times m}$, $R_2(\cdot;\cdot) : \mathbb{C} \times [0, +\infty) \mapsto \mathbb{C}^{p\times p}$ and $R_3(\cdot;\cdot) : \mathbb{C} \times [0, +\infty) \mapsto \mathbb{C}^{p\times p}$ fulfill the following assumptions:

ASSUMPTION 4.2.

1. *for every $r \geq 0$, the functions $\{\lambda \mapsto R_i(\lambda; r)\}_{i=1}^3$ are meromorphic and for every $\lambda \in \mathbb{C}$, the functions $\{r \mapsto R_i(\lambda; r)\}_{i=1}^3$ are continuous;*
2. *the matrices $R_1(\lambda; r)$ and $R_3(\lambda; r)$ are of full rank for all $\lambda \in \mathbb{C}$ and all $r \geq 0$;*
3. *$\lambda \mapsto R_i(\lambda; 0) \equiv I$ for $i = 1, 2, 3$;*
4. *for every compact set $\Omega \subset \mathbb{C}$, we have*

$$\lim_{r\to 0+} \max_{\lambda\in\Omega} \|R_i(\lambda; r) - I\| = 0 \text{ for } i = 1, 2, 3$$

   i.e., the functions $\{\lambda \mapsto R_i(\lambda; r)\}_{i=1}^3$ uniformly converge to the identity matrix on compact regions in the complex plane as r goes to zero;
5. *there exist constants $M, N, \hat{r} > 0$ such that for all $\lambda \in \mathbb{C}$ with $\Re(\lambda) \geq -N$ and for all $r \in (0, \hat{r})$*

$$\|R_i(\lambda; r)\| \leq M \text{ for } i = 1, 2, 3.$$

The implementation errors studied in Section 3 fit this framework:

- $R_1(\lambda; r) = e^{-\lambda r} I_m$ and $R_2(\lambda; r) = R_3(\lambda; r) = I_p$ for (3.1);
- $R_1(\lambda; r) = I_m$, $R_2(\lambda, r) = \begin{cases} \frac{1-e^{-\lambda r}}{\lambda r} I_p & \lambda r \neq 0 \\ I_p & \lambda r = 0 \end{cases}$ and $R_3(\lambda, r) = I_p$ for (3.4);

We note that also the low-pass filter in (4.1) can be interpreted in terms of this perturbation framework: $R_1(\lambda; r) = I_m$, $R_2(\lambda, r) = I_p/(\lambda r + 1)$ and $R_3(\lambda, r) = I_p$.

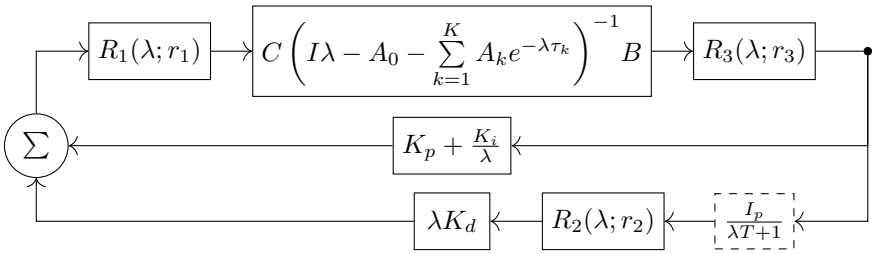


Fig. 2: Closed loop description of the considered perturbation framework.

Next, in analogy with neutral functional differential equations, in which stability can be sensitive to arbitrarily small perturbations on the delays, we define strong stability as follows.

DEFINITION 4.3. *A closed loop system, as depicted in Figure 2, is strongly stable if for every set $\{R_i(\lambda; r_i)\}_{i=1}^3$ that satisfies the five assumptions given in Assumption 4.2, there exists a $\hat{r} > 0$ such that the corresponding perturbed closed loop system is exponentially stable for all r_1, r_2 and r_3 in the open interval $(0, \hat{r})$.*

Remark 4.4. The notion of strong stability used here should not be confused with the one used in [33, Section 5.3], in which strong stability refers to stabilization with controllers that themselves are stable.

In the following proposition, it is shown that a stable closed loop system of the form (2.1)-(2.2) can be made strongly stable by including a low-pass filter with a sufficiently large cut-off frequency if the low-pass filter itself is not destabilizing.

PROPOSITION 4.5. *Assume that the gain matrices K_p, K_d and K_i are such that the closed loop system (2.1)-(2.2) is exponentially stable. If $BK_dC - I_n$ is Hurwitz, then the closed loop of (2.1) and (4.1) is strongly stable for sufficiently small T.*

Proof. By incorporating the perturbations $\{R_i(\lambda; r_i)\}_{i=1}^3$ in (4.2), the characteristic function becomes

$$H_4(\lambda; T, r_1, r_2, r_3) := \det\left(\lambda \begin{bmatrix} I_n & 0 \\ 0 & I_q \end{bmatrix} - \begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix} - \sum_{k=1}^{K} \begin{bmatrix} A_k & 0 \\ 0 & 0 \end{bmatrix} e^{-\lambda\tau_k} \right.$$
$$\left. - \begin{bmatrix} BR_1(\lambda; r_1) & 0 \\ 0 & I_q \end{bmatrix} \begin{bmatrix} K_p + \frac{\lambda}{\lambda T+1} K_d R_2(\lambda; r_2) & U_i \\ V_i & 0 \end{bmatrix} \begin{bmatrix} R_3(\lambda; r_3)C & 0 \\ 0 & I_q \end{bmatrix} \right)$$

We choose numbers N and $\hat{r}$ according to item 5 of Assumption 4.2. Furthermore, if $BK_dC - I_p$ is Hurwitz and the closed loop (2.1)-(2.2) is stable, then it follows from Proposition 4.1 that we can choose a sufficiently small T such that there exists a $c \in (0, N]$ for which $H_3(\lambda; T)$ has no root in $V := \{\lambda \in \mathbb{C} : \Re(\lambda) > -c\}$. Subsequently, for all r_1, r_2 and r_3 in the interval $(0, \hat{r})$, the characteristic roots of $H_4(\cdot; T, r_1, r_2, r_3)$ in V are bounded in modulus by

$$|\lambda_0| \leq \sup_{\substack{\lambda \in V \\ r_1, r_2, r_3 \in (0, \tilde{r})}} \|A_0\| + \textstyle\sum_{k=1}^{K} \|A_k\| e^{-\Re(\lambda)\tau_k} +$$
$$\left\| \begin{bmatrix} BR_1(\lambda; r_1) & 0 \\ 0 & I_q \end{bmatrix} \begin{bmatrix} K_p + \frac{\lambda}{\lambda T+1} K_d R_2(\lambda; r_2) & U_i \\ V_i & 0 \end{bmatrix} \begin{bmatrix} R_3(\lambda; r_3)C & 0 \\ 0 & I_q \end{bmatrix} \right\| := \Xi < \infty.$$

By using Rouché's theorem as in Proposition 4.1 and the uniform convergence of $H_4(\cdot; T, r_1, r_2, r_3)$ to $H_3(\cdot; T)$ on compact regions for r_1, r_2 and r_3 going to zero, it follows that for sufficiently small r_1, r_2 and r_3 the functions $H_4(\cdot; T, r_1, r_2, r_3)$ and $H_3(\cdot; T)$ have the same number of roots in the right half plane V, namely zero. ∎

Note that if $CB = 0_{p\times m}$, i.e., the relative degree[1] of each input-output channel of system (2.1) is larger than one, then stability implies strong stability even without low pass filtering, as shown in the following proposition.

PROPOSITION 4.6. *Assume that the gain matrices K_p, K_d and K_i are such that the closed loop system (2.1)-(2.2) is exponentially stable. If $CB = 0_{m\times p}$, then the closed loop of (2.1) and (2.2) is also strongly stable.*

Proof. By incorporating the perturbations $\{R_i(\lambda; r_i)\}_{i=1}^3$ in (2.3), the characteristic function becomes

$$H_5(\lambda; r_1, r_2, r_3) := \det\left(\lambda \begin{bmatrix} I_n - BR_1(\lambda; r_1)K_d R_2(\lambda; r_2)R_3(\lambda; r_3)C & 0 \\ 0 & I_q \end{bmatrix} \right.$$
$$\left. - \begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix} - \sum_{k=1}^{K} \begin{bmatrix} A_k & 0 \\ 0 & 0 \end{bmatrix} e^{-\lambda\tau_k} - \begin{bmatrix} BR_1(\lambda; r_1) & 0 \\ 0 & I_q \end{bmatrix} \begin{bmatrix} K_p & U_i \\ V_i & 0 \end{bmatrix} \begin{bmatrix} R_3(\lambda; r_3)C & 0 \\ 0 & I_q \end{bmatrix} \right).$$

[1] By interpreting the internal dynamics of system (2.1) as an infinite dimensional ordinary differential equation on the head-tail state space $\mathcal{X} := \mathbb{R}^n \times \mathcal{C}([-\tau_K, 0], \mathbb{R}^n)$ equipped with the inner product $\langle (x_1, \phi_1), (x_2, \phi_2) \rangle_{\mathcal{X}} = x_1^T x_2 + \int_{-\tau_K}^{0} \phi_1(s)^T \phi_2(s)\, \mathrm{d}s$, the relative degree of each input-output channel is defined as in [24, Definition 1.3].

Because $CB = 0_{p\times m}$, the Weinstein–Aronszajn identity[2] implies that

$$Q(\lambda; r_1, r_2, r_3) := BR_1(\lambda; r_1)K_d R_2(\lambda; r_2)R_3(\lambda; r_3)C$$

is nilpotent for all $r_1, r_2, r_3 > 0$, all $\lambda \in \mathbb{C}$ and all permissible perturbations. This has as a consequence that $I - Q(\lambda; r_1, r_2, r_3)$ is invertible. Further, by the last item of Assumption 4.2 there exist $\hat{r} > 0$ and $N > 0$ such that the norm of $\big(I-Q(\lambda; r_1, r_2, r_3)\big)^{-1}$ is finite in the right half plane $V := \{\lambda \in \mathbb{C} : \Re(\lambda) > -N\}$ for all r_1, r_2 and r_3 in the interval $(0, \hat{r})$. Using a similar approach as in the proof of Proposition 4.1, one can show that the modulus of a characteristic root of $H_5(\cdot; r_1, r_2, r_3)$ in the right half plane V is finite for r_1, r_2 and r_3 in $(0, \hat{r})$. Furthermore, because $H_5(\cdot; r_1, r_2, r_3)$ uniformly converges to $H_0(\cdot)$ on compact regions in the complex plane for r_1, r_2 and r_3 going to zero, the proposition now follows from a similar argument as in the proof of Proposition 4.5. ∎

Remark 4.7. For single-input single-output systems without state delay and with relative degree $\nu > 1$, which are described by

$$\begin{cases} \dot{x}(t) = & A\,x(t) + B\,u(t) \\ y(t) = & C\,x(t) \end{cases}$$

with $B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$, $CB = 0$, $CAB = 0$, $\ldots$, $CA^{\nu-2}B = 0$ and $CA^{\nu-1}B \neq 0$, the presented results can be generalized to extended PID control, in the sense of [12, 35], which also includes higher order derivatives in the feedback loop. More specifically, for a system with relative degree ν the feedback signal now becomes,

$$u(t) = K_p\,y(t) + K_i \int_0^t y(s)\,\mathrm{d}s + \sum_{i=1}^{\nu} K_{d,i} y^{(i)}(t).$$

The strong stability analysis of this control law can be addressed by recasting the output and the control law as follows:

$$\begin{aligned} z(t) = & \begin{bmatrix} C^T & (CA)^T & \ldots & \left(CA^{\nu-1}\right)^T \end{bmatrix}^T x(t) \\ u(t) = & \begin{bmatrix} K_p & K_{d,1} & \ldots & K_{d,\nu-1} \end{bmatrix} z(t) + \begin{bmatrix} 0 & \ldots & 0 & K_{d,\nu} \end{bmatrix} \dot{z}(t) \\ & + \begin{bmatrix} K_i & 0 & \ldots & 0 \end{bmatrix} \int_0^t z(s)\,\mathrm{d}s, \end{aligned}$$

meaning that the previously obtained results hold when BK_dC is replaced with $BK_{d,\nu}CA^{\nu-1}$. Note however that such a direct extensions is not possible for the general system class introduced in (2.1), as it relies on the concept of normal form, in which output (derivatives) take the role of state variables, which is not fully understood for time-delay systems.

Next, we present a condition on the system matrices and the gain matrices K_p and K_i under which there does not exist a feedback matrix K_d such that the closed loop system (2.1)-(4.1) is strongly stable.

[2]For two matrices $U \in \mathbb{R}^{n\times m}$ and $V \in \mathbb{R}^{m\times n}$, the equality $\det(I_n + UV) = \det(I_m + VU)$ holds. This has as a consequence that $\det(\lambda I_n - UV) = \lambda^{n-m}\det(\lambda I_m - VU)$. The nilpotence of $Q(\lambda; r_1, r_2, r_3)$ now follows by plugging $U = BR_1(\lambda; r_1)K_dR_2(\lambda; r_2)R_3(\lambda; r_3)$ and $V = C$ into the above equation.

PROPOSITION 4.8. *If for given gain matrices K_p and K_i, and $K_d = 0_{m\times p}$, the characteristic function $H_0(\cdot)$ has an odd number of roots in the closed right half plane then there does not exist a matrix K_d such that the closed loop system (2.1)-(2.2) is stable and $BK_dC - I$ is Hurwitz; and as a consequence there does not exist a matrix K_d such that the closed loop system (2.1)-(4.1) is strongly stable for all sufficiently small T.*

Proof. We use a similar continuation argument as in [22, Theorems 5.1]. More specifically, assume that $K_d^\star$ is such that the closed loop (2.1)-(2.2) is stable. Consider now the feedback law: $u(t) = K_p y(t) + kK_d^\star \dot{y}(t) + K_i \int_0^t y(s)\,\mathrm{d}s$ for $k \in [0,1]$. For $k = 0$ the corresponding characteristic function has an odd number of roots in right half plane, while for $k = 1$ there are no right half plane roots. By increasing k from 0 to 1, all roots are thus moved to the left half plane. Furthermore, if we vary k these characteristic roots move in complex conjugate pairs as all considered matrices are real-valued. Therefore, to bring an odd number of characteristic roots to the left half plane, at least one root has to pass through either the origin or infinity. The former is however not possible as a characteristic root in the origin is invariant with respect to changes in k and the closed loop system is stable for $k = 1$. On the other hand, a root crossing through infinity means that there exists a $\hat{k} \in (0,1)$ such that $\det(I - \hat{k}BK_d^\star C) = 0$, which implies that $BK_d^\star C$ has a real-valued root $1/\hat{k} > 1$. The second part of the proposition now follows from Proposition 4.1. ∎

We conclude the subsection with a comment on parametric uncertainty.

Remark 4.9. If the control law (4.1) achieves strong stability, then the asymptotic stability is robust not only against sufficiently small perturbations satisfying Assumption 4.2 and visualized in Figure 2 but also against (sufficiently) small perturbations on system matrices $A_0, \ldots, A_K$, B, C and delays $\tau_1, \ldots, \tau_K$. This result directly follows from the strong stability criterion in Proposition 4.5.

4.3. Illustration of theory using a third-order SISO system. In this subsection, we illustrate the results of Subsection 4.2 and section 3 using the following third-order, single-input single-output system

$$\left\{\begin{array}{rcl} \dot{x}(t) &=& \begin{bmatrix} -1 & 1/3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} u(t) \\ \\ y(t) &=& \begin{bmatrix} 0.5 & 0 & 0.5 \end{bmatrix} x(t) \end{array}\right. \tag{4.3}$$

with corresponding transfer function

$$\frac{s^2+1}{s^3+s^2-(1/3)s-1}.$$

It is verified in Appendix A that this system can not be stabilized with either P or PI output feedback control. However, as shown next, this system can be stabilized with a PD output feedback controller of the following form

$$u(t) = k_p y(t) + k_d \dot{y}(t), \tag{4.4}$$

with $k_p \in \mathbb{R}$ and $k_d \in \mathbb{R}$. The characteristic function $H_0(\cdot)$ reduces in this case to the following polynomial of degree three:

$$(1-k_d)\lambda^3 + (1-k_p)\lambda^2 + (-k_d - (1/3))\,\lambda - k_p - 1. \tag{4.5}$$

Figure 3 shows the number of right half plane roots of this polynomial in function of k_p and k_d and can be understood as follows. By the Routh-Hurwitz stability criterion for polynomials of degree three, (4.5) has no roots in the closed right half plane for (k_p, k_d) in the set

$$\{(k_p, k_d) : k_d < 1, k_p < -1, k_d < \frac{1}{3} + \frac{2}{3}k_p\} \cup \{(k_p, k_d) : k_d > 1, k_p > 1, k_d < \frac{1}{3} + \frac{2}{3}k_p\}.$$

Further, the number of roots in the closed right half plane can only change in two ways. Firstly, eigenvalues can cross via the imaginary axis. We therefore examine the pairs (k_p, k_d) for which (4.5) has a root on the imaginary axis:

$$-\jmath\omega^3(1 - k_d) - \omega^2(1 - k_p) - \jmath\omega(k_d + \frac{1}{3}) - k_p - 1 = 0.$$

By splitting the real and imaginary part we get:

$$\begin{aligned} -\omega^2(1 - k_p) - k_p - 1 &= 0 \\ -\omega^3(1 - k_d) - \omega(k_d + (1/3)) &= 0. \end{aligned}$$

For $k_p = -1$ and k_d arbitrary, (4.5) thus has an root at the origin ($\omega^\star = 0$) and for $k_p \in (-\infty, -1) \cup (1, \infty)$ and $k_d = \frac{1}{3} + \frac{2}{3}k_p$, (4.5) has a pair of complex conjugate roots on the imaginary axis at $\omega^\star = \pm\sqrt{\frac{-k_p-1}{1-k_p}}$. Furthermore, the crossing direction at these critical parameter values follows from:

$$\left.\frac{\partial \Re(\lambda)}{\partial k_p}\right|_{\lambda=\jmath\omega^\star} = \Re\left(\frac{-\omega^{\star 2} + 1}{-3\omega^{\star 2}(1 - k_d) + 2(1 - k_p)\jmath\omega^\star - (k_d + 1/3)}\right).$$

Secondly, the number of right half plane roots can change by roots moving over infinity at $k_d = 1$. More precisely, for $k_d = 1 - \epsilon$ and ϵ sufficiently small, (4.5) has a root at approximately $\frac{-1+k_p}{\epsilon}$. For $k_p > 1$, there thus is a root that moves from the right half plane to the left half plane as k_d increases from $1-$ to $1+$. For $k_p < 1$, this root moves in the opposite direction.

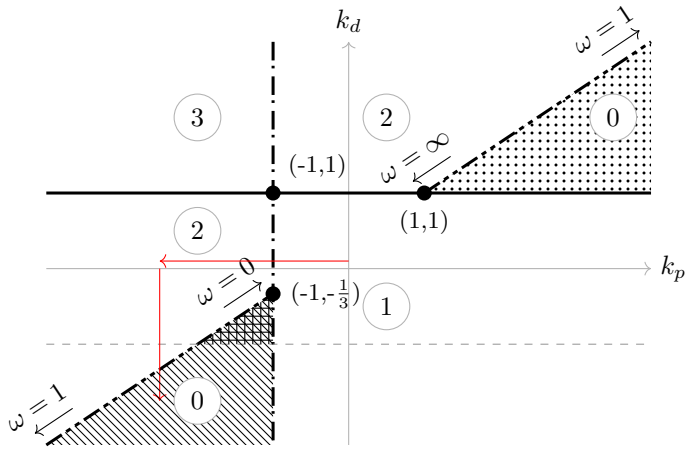
The closed loop system (4.3)-(4.4) is thus stable for (k_p, k_d) inside the set

$$\{(k_p, k_d) : k_d < 1, k_p < -1, k_d < \frac{1}{3} + \frac{2}{3}k_p\} \cup \{(k_p, k_d) : k_d > 1, k_p > 1, k_d < \frac{1}{3} + \frac{2}{3}k_p\}.$$

However, from the theory developed in Section 3 and subsection 4.1, it follows that for $|k_d| > 1$ stability is lost under arbitrarily small feedback delay and that for $k_d > 1$ stability is lost under both a finite-difference approximation of the derivative and the inclusion of a low-pass filter. Furthermore, it can be shown that when considering these three implementation errors and assuming that these errors are sufficiently small, the stability of the closed loop system without low-pass filtering is only preserved for (k_p, k_d) in the set

$$\left\{(k_p, k_d) : k_d > -1, k_p < -1, k_d < \frac{1}{3} + \frac{2}{3}k_p\right\}.$$

Next, if we consider the perturbation class introduced in Subsection 4.2, it follows from Proposition 4.5 that the closed loop system augmented with low pass filtering of



PD controller is stabilizing but not robust against any perturbation type of Section 3 and Subsection 4.1

PD controller is stabilizing and robust against the perturbations of Section 3 and Subsection 4.1

PD controller with low pass filtering is strongly stabilizing

—— Crossing via "infinity" —·— Crossing at origin

—·-·— Crossing via imaginary axis at $\omega = \pm\sqrt{\frac{-k_p-1}{-k_p+1}}$

Fig. 3: The number of roots of (4.5) in the closed right half plane in function of k_p and k_d (grey circles). Transitions between the different regions correspond to eigenvalues crossing via "infinity" (full line), the origin (dot dashed) or the imaginary axis (dot dot dashed). The parameter pairs for which the PD controller is stable but not robust against any of the three types of perturbations discussed in Section 3 and subsection 4.1, for which the PD controller is stable and robust against these perturbations, and for which the PD controller with low-pass filtering is strongly stable are indicated using hatching.

the derivative with a sufficiently large cut-off frequency is strongly stable if (k_p, k_d) lies in the set,

$$\left\{(k_p, k_d) : k_p < -1, k_d < \frac{1}{3} + \frac{2}{3}k_p\right\}.$$

These different stability regions are indicated on Figure 3 using hatching.

Figure 3 can also be used to illustrate Proposition 4.8.

- For $k_p > 1$ and $k_d = 0$, the characteristic function has one root in the right half plane. For fixed k_p, this root can be brought to the left half plane by increasing k_d until $1 < k_d < \frac{1}{3} + \frac{2}{3}k_p$. However, as expected from the proof of Proposition 4.8, the root moves from the right to the left half plane via infinity. As a consequence, for the stabilizing values of k_d, the matrix Bk_dC has a real eigenvalue larger than one. The corresponding closed loop system is hence not strongly stable. Furthermore, adding a low-pass filter to the control loop will not fix this robustness problem as the low pass filter itself is destabilizing.

- On the other hand, for $k_p < -1$ and $k_d = 0$, the characteristic function has two roots in the right half plane. For fixed k_p, these two right half plane roots can be brought to the left half plane via the imaginary axis by decreasing k_d until $k_d < \frac{1}{3} + \frac{2}{3}k_p$. It now follows from Proposition 4.5 that the closed loop system with loop pass filtering is strongly stable.

Finally, combining these results, for a PD controller with low pass filtering, strongly stabilizing controller parameters k_p and k_d can be understood as follows. The uncontrolled system ($k_p = 0$ and $k_d = 0$) is unstable and has an odd number of right half plane roots. First, the P part brings an additional root to the closed right half plane, i.e., the P part on its own is destabilizing. The D part subsequently moves these two right half plane roots to the left half plane via the imaginary axis. This is indicated with the arrows in Figure 3.

5. Design procedure for strongly stabilizing controllers. In this section, we describe a computational procedure to design a PID output feedback controller with a low pass filtering of the derivative signal which strongly stabilizes system (2.1). To synthesize such controllers, we will minimize the spectral abscissa of (2.3), i.e., the real part of its right-most characteristic root, in function of elements of the feedback matrices K_p, K_d and K_i, under the constraint that $BK_dC - I_n$ is Hurwitz. This leads to the following design procedure:

1. choose initial controller matrices K_p, K_d and K_i;
2. if $\alpha(BK_dC) = \max\{\Re(\lambda) : \det(I_n\lambda - BK_dC) = 0\} > 0.9$, then re-scale K_d by $0.9/\alpha(BK_dC)$
3. minimize the spectral abscissa of (2.3)

$$f(K_p, K_d, K_i) = \max_{\lambda\in\mathbb{C}} \Big\{ \Re(\lambda) : \det\Big(\lambda(I_n - BK_dC) - A_0 - \sum_{k=1}^{K} A_k e^{-\lambda\tau_k} - BK_pC - \tfrac{BK_iC}{\lambda}\Big) = 0 \Big\}$$

as function of the elements of K_p, K_d and K_i under the constraint that

$$\alpha(BK_dC) = \max\{\Re(\lambda) : \det(\lambda I_m - BK_dC) = 0\} < 1;$$

4. choose a sufficiently small T such that (2.1)-(4.1) is exponentially stable.

The constraint in step 3 ensures that the condition in Proposition 4.5 is fulfilled. This constraint is handled using a penalty method. More precisely, we minimize

$$f(K_p, K_d, K_i) + t\max\left(0, \alpha(BK_dC) - 1\right), \tag{5.1}$$

in the elements of the control matrices, where t is increased until the resulting K_d fulfills the constraint. Note, however, that this function is in general non-convex and, as a consequence, it might have multiple local optima. Therefore, to avoid ending up at a bad local minimum, we restart the optimization procedure from several initial points.

Next, we illustrate the effectiveness of this method on three example problems.

Example 5.1. We start with designing a controller for the system examined in Subsection 4.3. As initial parameters we use $(k_p, k_d) = (1.5, 1.2)$ and $t = 10^2$. This results in $(k_p, k_d) = (-1.08015, -1.04045)$. It thus follows from Figure 3 that the resulting controller with low-pass filtering is strongly stable for sufficiently small T. To indicate the importance of rescaling the initial K_d and the constraint in step 3, we redesign the controller starting from the same initial parameters without these

components. This results in the controller $(k_p, k_d) = (1.26832, 1.01777)$. The closed loop system without filter is now stable, but stability is lost when adding the low pass filter. ◇

Example 5.2. Secondly, we consider a system of form (2.1) with $n = 6$, $m = 3$, $p = 2$ and $K = 3$. The system matrices and delays are given in Appendix B. The open-loop system has 5 right half plane poles, the right-most of which are a complex conjugate pair located at $2.607 \pm \jmath 2.144$. After running the design procedure starting from a zero initial controller and $t = 10^5$ we obtained the feedback matrices given in Appendix B. The closed loop system with low-pass filter ($T = 10^{-4}$) is now exponentially stable with a decay rate of 0.10031. Furthermore, the real part of all eigenvalues of BK_dC is smaller than one, thus the closed loop with low pass filter is strongly stable. ◇

Example 5.3. Finally, as last example, we consider the stabilization of a quadcopter around its equilibrium point, i.e., hovering at a fixed position and a fixed orientation of the principal axes. We use the twelve dimensional linearized model given in [13, Equations (6.4-6.7)] for the parameters given in [13, Section 4]. By choosing an appropriate output matrix C we obtain the following system of form (2.1):

(5.2)
$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0_{3\times3} & I_3 & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & \begin{bmatrix} 0 & -g & 0 \\ g & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & I_3 \\ 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \end{bmatrix} x(t) + \begin{bmatrix} 0_{3\times4} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{-\theta}{m} & \frac{-\theta}{m} & \frac{-\theta}{m} & \frac{-\theta}{m} \end{bmatrix} \\ 0_{3\times4} \\ \begin{bmatrix} 0 & \frac{l\,\theta}{I_x} & 0 & \frac{-l\,\theta}{I_x} \\ \frac{l\,\theta}{I_y} & 0 & \frac{-l\,\theta}{I_y} & 0 \\ \frac{-2d\Omega_0}{I_z} & \frac{2d\Omega_0}{I_z} & \frac{-2d\Omega_0}{I_z} & \frac{2d\Omega_0}{I_z} \end{bmatrix} \end{bmatrix} u(t), \\ y(t) = \begin{bmatrix} I_3 & 0_{3\times1} & 0_{3\times1} & 0_{3\times1} & 0_{3\times3} & 0_{3\times1} & 0_{3\times1} & 0_{3\times1} \\ 0_{1\times3} & 1 & 1 & 1 & 0_{1\times3} & 0 & 0 & 0 \\ 0_{3\times3} & 0_{3\times1} & 0_{3\times1} & 0_{3\times1} & I_3 & 0_{3\times1} & 0_{3\times1} & 0_{3\times1} \\ 0_{1\times3} & 0 & 0 & 0 & 0_{1\times3} & 0 & 0 & 1 \end{bmatrix} x(t) \end{cases}$$

with $\theta = 2b\Omega_0$, $g = 9.8\,\mathrm{m\,s^{-2}}$, $m = 1.32\,\mathrm{kg}$, $b = 1.5108 \cdot 10^{-5}\,\mathrm{kg\,m}$, $l = 0.214\,\mathrm{m}$, $I_x = 9.3 \cdot 10^{-3}\,\mathrm{kg\,m^2}$, $I_y = 9.2 \cdot 10^{-3}\,\mathrm{kg\,m^2}$, $I_z = 151 \cdot 10^{-2}\,\mathrm{kg\,m^2}$, $d = 4.406 \cdot 10^{-7}\,\mathrm{kg\,m^2\,s^{-1}}$ and $\Omega_0 = \sqrt{(mg/(4b))}$. Starting from ten random initial parameter values and fixing t to 10^2, resulted in ten different PID controllers with low-pass filtering that each strongly stabilize (5.2) for sufficiently large cut-off frequencies. Next, we examine the controller with the best performance, i.e., the controller matrices that resulted in the smallest $f(K_p, K_d, K_i)$ and which is given in Appendix C. As cut-off frequency we choose $1/T = 10^6\ \mathrm{s^{-1}}$. By computing the characteristic roots of (4.2), we find that the closed loop system (with low pass filtering) is exponentially stable with a decay rate of 0.7526. Figure 4 shows these characteristic roots. As expected, four roots lie at approximately $\frac{\lambda_i - 1}{T}$ with λ_i the eigenvalues of K_dCB. The real part of almost all remaining roots is approximately equal to the spectral abscissa, which is a typical phenomenon for the direct optimization framework. ◇

6. Extensions. We present two extensions of the previous results. First, we consider systems which include an input delay in the nominal model. Second, we briefly comment on incorporating bounded (non-vanishing) perturbations on the sys-

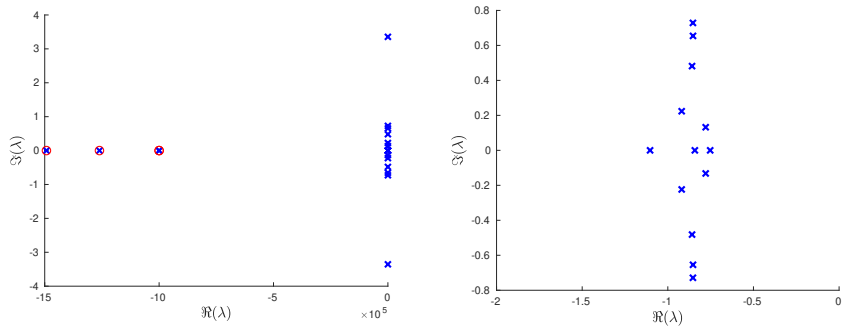


Fig. 4: Blue crosses: roots of the instance of characteristic function (4.2) associated with the closed loop quadcopter system for the controller matrices that minimized $f(K_p, K_d, K_i)$ and $\frac{1}{T} = 10^6$ s^{-1}. In the right-hand panel we zoomed in on the right-most characteristic roots.
Red circles: eigenvalues of $\frac{K_d C B - I_4}{T}$.

tem matrices and nominal delays, as so far we dealt with analyzing and resolving the sensitivity of stability with respect to perturbations that can be made sufficiently small (see also Remark 4.9).

6.1. Systems with input delay. This subsection considers systems with a fixed input delay $\tau_u > 0$,

$$\left\{ \begin{array}{l} \dot{x}(t) = A_0\, x(t) + \sum_{k=1}^{K} A_k\, x(t-\tau_k) + B\, u(t-\tau_u), \\ y(t) = C\, x(t). \end{array} \right. \tag{6.1}$$

The stability of the nominal closed loop system is now characterized by the roots of (3.2), where r is fixed to τ_u. It thus follows from a similar argument as in the proof of Proposition 3.1, that $\rho(BK_dC) < 1$ is a necessary condition for the exponential stability of the nominal system. In the following proposition, we examine strong stability for systems with input delay. In contrast to systems without input delay, there is now no additional constraint (besides stabilizing the system) on K_d to achieve strong stability using a low pass filter with a sufficiently large cut-off frequency.

PROPOSITION 6.1. *Assume that the gain matrices K_p, K_d and K_i are such that the closed loop system (6.1)-(2.2) is exponentially stable. Then the closed loop of (6.1) and (4.1) is strongly stable.*

Proof. By incorporating the low-pass filter and the perturbations $\{R_i(\lambda; r_i)\}_{i=1}^3$, the characteristic function becomes:

$$H_6(\lambda; T, r_1, r_2, r_3) := \det \left(\lambda \begin{bmatrix} I_n & 0 \\ 0 & I_q \end{bmatrix} - \begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix} - \sum_{k=1}^{K} \begin{bmatrix} A_k & 0 \\ 0 & 0 \end{bmatrix} e^{-\lambda \tau_k} \right.$$
$$\left. \begin{bmatrix} BR_1(\lambda; r_1) e^{-\lambda \tau_u} & 0 \\ 0 & I_q \end{bmatrix} \begin{bmatrix} K_p + \frac{\lambda}{\lambda T + 1} K_d R_2(\lambda; r_2) & U_i \\ V_i & 0 \end{bmatrix} \begin{bmatrix} R_3(\lambda; r_3) C & 0 \\ 0 & I_q \end{bmatrix} \right).$$

We have already established that the exponential stability of the closed loop system of (6.1) and (2.2) requires $\rho(BK_dC) < 1$. Next, we will show that this condition implies

that the low-pass filter is not destabilizing for $R_1(\lambda; r_1) = R_2(\lambda; r_2) = R_3(\lambda; r_3) = I$. If $\rho(BK_dC) < 1$, there exist $c > 0$ and $\widehat{T} > 0$ such that $\rho\left(BK_dCe^{-\lambda\tau_u}\frac{1}{\lambda T+1}\right) < 1$ for all $\lambda \in V := \{\lambda \in \mathbb{C} : \Re(\lambda) > -c\}$ and all $T \in (0, \widehat{T})$. This implies that $I - BK_dCe^{-\lambda\tau_u}\frac{1}{\lambda T+1}$ is invertible on V for $T \in (0, \widehat{T})$. It now follows from a similar argument as in Proposition 4.1 that the low-pass filter does not destabilize the system for sufficiently small T. Strong stability follows from a similar argument as in Proposition 4.5. ∎

The proposition above shows that if a PID controller stabilizes the system (6.1), then including a low-pass filter with a sufficiently large cut-off frequency makes the closed loop system strongly stable. It thus suffices to design controller matrices K_p, K_d and K_i such that the closed loop system of (6.1) and (2.2) is stable. Such controller matrices can, for instance, be obtained using the method presented in [19]. More precisely, suitable controller matrices are derived by minimizing the spectral abscissa of the closed loop system under the constraint that $\rho(BK_dC) < 1$. As in Section 5, this constraint is incorporated in the cost function. However, instead of including a penalty term, a logbarier method is used to enforce that the constraint is fulfilled in each iteration step.

Example 6.2. We revisit Example 5.3 with a (fixed) feedback delay of $\tau_u = 0.1$s. Although $\rho(BK_dC) = 0.4925 < 1$ for the controller obtained in Section 5, the closed loop system is unstable for $\tau_u = 0.1$s, both with and without filtering (recall that stability is only preserved for a sufficiently small feedback delay). By applying the procedure of [19] starting from this controller, we arrive at a stable closed loop system with an exponentially decay rate (with $T = 10^{-6}$s) of less than 0.0100 and $\rho(BK_dC) = 0.9990 < 1$. However, when starting from another initial controller, created by swapping the initial K_i and K_d, we arrived at an exponentially decay rate (with $T = 10^{-6}$s) of 1.1797, illustrating the non-convexity of the optimization problem at hand and the influence of the chosen starting values. The corresponding controller matrices are given in Appendix C.

6.2. Bounded uncertainties on system matrices and state delays. In this subsection, we generalize the algorithm presented in Section 5 to systems with bounded uncertainties on the system matrices and the state delays. More specifically, we consider L real-valued matrix uncertainties

$$\delta_1 \in \mathbb{R}^{\varrho_1 \times \varsigma_1}, \ldots, \delta_L \in \mathbb{R}^{\varrho_L \times \varsigma_L}$$

that are bounded in Frobenius norm by 1. For notational convenience we denote the combination of uncertainties as $\delta = (\delta_1, \ldots, \delta_L)$, and the set of all admissible uncertainty values as $\mathcal{D}$:

$$\mathcal{D} = \left\{(\delta_1, \ldots, \delta_L) : \delta_l \in \mathbb{R}^{\varrho_l \times \varsigma_l} \text{ and } \|\delta_l\|_F \leq 1 \text{ for } l = 1, \ldots, L\right\}$$

These matrix uncertainties affect the system matrices in an affine way:

$$\widetilde{R}(\delta) = R + \sum_{l=1}^{L} G_{R,l}\delta_l H_{R,l}$$

with R the nominal value of the system matrix. The matrices $G_{R,l}$ and $H_{R,l}$ are real-valued and allow to target specific blocks or parameters in the system matrices.

The uncertainties on the state delays, $\delta\tau_1, \ldots, \delta\tau_K$, are real-valued and bounded in absolute value by $\overline{\delta\tau_k} < \tau_k$. The open loop system now becomes

$$\begin{cases} \dot{x}(t) = \widetilde{A}_0(\delta)\, x(t) + \sum_{k=1}^{K} \widetilde{A}_k(\delta)\, x\big(t - (\tau_k + \delta\tau_k)\big) + \widetilde{B}(\delta)\, u(t), \\ y(t) = \widetilde{C}(\delta)\, x(t). \end{cases}$$

The goal is to design a PID controller with low-pass filtering that strongly stabilizes the system for all admissible uncertainty values. To this end, the the design procedure of Section 5 has to be modified in the following way. Firstly, the constraint $\alpha(\widetilde{B}(\delta)K_d\widetilde{C}(\delta)) < 1$ now has to hold for all admissible uncertainty values or in other words,

(6.2)
$$\alpha^{\mathrm{ps}}(\widetilde{B}K_d\widetilde{C}) = \max\Big\{\Re(\lambda) : \exists\delta \in \mathcal{D} \text{ such that } \det\big(\lambda I_m - \widetilde{B}(\delta)K_d\widetilde{C}(\delta)\big) = 0\Big\} < 1.$$

Secondly, instead of minimizing the spectral abscissa, we now have to minimize the pseudo-spectral abscissa, which is defined as the worst-case value of the spectral abscissa over all admissible uncertainty values,

$$\begin{aligned} (6.3)\quad f(K_p, K_d, K_i) = \max_{\lambda\in\mathbb{C}} \Bigg\{ & \Re(\lambda) : \exists\delta \in \mathcal{D} \text{ and } |\delta\tau_k| \le \overline{\delta\tau_k} \text{ for } k = 1, \ldots K \\ & \text{such that } \det\Bigg(\lambda\big(I_n - \widetilde{B}(\delta)K_d\widetilde{C}(\delta)\big) - \widetilde{A}_0(\delta) - \sum_{k=1}^{K} \widetilde{A}_k(\delta)e^{-\lambda(\tau_k+\delta\tau_k)} \\ & \qquad - \widetilde{B}(\delta)K_p\widetilde{C}(\delta) - \frac{\widetilde{B}(\delta)K_i\widetilde{C}(\delta)}{\lambda}\Bigg) = 0\Bigg\} \end{aligned}$$

By noting that the characteristic functions in (6.2) and (6.3) can be rewritten as

$$\det\left(\lambda \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & \widetilde{B}(\delta) & 0 \\ 0 & -I_m & K_d \\ \widetilde{C}(\delta) & 0 & -I_p \end{bmatrix}\right)$$

and

$$\det\left(\lambda \begin{bmatrix} I_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \tilde{A}_0(\delta) & \tilde{B}(\delta) & \tilde{B}(\delta) & 0 & 0 & 0 & 0 \\ 0 & -I_m & 0 & K_p & U_i & 0 & 0 \\ 0 & 0 & -I_m & 0 & 0 & K_d & 0 \\ \tilde{C}(\delta) & 0 & 0 & -I_p & 0 & 0 & 0 \\ 0 & 0 & 0 & V_i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -I_p & \tilde{C}(\delta) \\ 0 & 0 & 0 & 0 & 0 & 0 & I_n \end{bmatrix} - \sum_{k=1}^{K} \begin{bmatrix} \tilde{A}_k(\delta) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} e^{-\lambda(\tau_k+\delta\tau_k)}\right),$$

respectively, both (6.2) and (6.3) can be computed using a slight modification of the method presented in [3, 2]. Furthermore, explicit expressions for the derivative of these quantities with respect to the elements of K_p, K_d and K_i, which are required in the optimization process, can be obtained using [2, Theorem 4.2].

7. Concluding remarks. This work analyzed the potential sensitivity of stability for PID control of dynamical systems with multiple state delays with respect to arbitrarily small modelling errors. This led to the introduction of the new notion of *strong stability*, which requires the preservation of stability under sufficiently small perturbations. We showed that, under a certain algebraic constraint on the derivative gain matrix K_d, strong stability can be achieved by adding a low-pass filter to the control loop. A computational procedure to design strongly stabilizing PID controllers was provided. A MATLAB^{TM} implementation of this algorithm is available from http://twr.cs.kuleuven.be/research/software/delay-control/strong-stability. The code used to generate Examples 5.1 to 5.3 and Example 6.2 is available from the same location.

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Appendix A. Proof that system (4.3) is not stabilizable with P or PI control.

Case I: P control ($k_i = 0$) In this case, the characteristic polynomial becomes:

$$\lambda^3 + (1 - k_p)\lambda^2 - \lambda/3 - 1 - k_p.$$

The result follows immediately from the Routh-Hurwitz conditions for polynomials of degree three.

Case II: PI control ($k_i \neq 0$) In this case, the characteristic polynomial becomes:

$$\lambda^4 + (1 - k_p)\lambda^3 + (-1/3 - k_i)\lambda^2 + (-1 - k_p)\lambda - k_i.$$

The corresponding Routh-Hurwitz conditions for polynomials of degree four are:

$$k_p < 1 \tag{A.1}$$

$$k_i < -1/3$$

$$k_p < -1 \tag{A.2}$$

$$k_i < 0 \tag{A.3}$$

$$2/3 - k_i + (4/3)k_p + k_p k_i > 0$$

$$-2/3 - 2k_p + 2k_i - 2k_i k_p - 4/3k_p^2 > 0 \tag{A.4}$$

Condition (A.4) implies that

$$-(2/3) - 2k_p - 4/3k_p^2 > 2k_i(k_p - 1).$$

Furthermore, because $k_p - 1 < 0$ due to (A.1), this is equivalent with

$$\frac{-(2/3)(k_p + 1/2)(k_p + 1)}{k_p - 1} < k_i.$$

Because the right hand side of this inequality is strictly positive for $k_p < -1$, it follows from (A.2) that k_i must be strictly larger then 0. This contradicts however (A.3).

Appendix B. System definition and resulting feedback matrices for Example 5.2. The system is defined by

$$A_0 = \begin{bmatrix} -0.3430 & 0.6843 & -0.1278 & 0.4078 & -0.7793 & -1.4286 \\ 1.6663 & 0.0558 & -1.4103 & 0.2430 & -1.7622 & -1.1146 \\ -0.7667 & -1.4018 & 0.6029 & 0.3975 & -1.9355 & 0.9132 \\ 2.6355 & -1.3601 & -0.4569 & -0.1757 & 1.5269 & 0.9764 \\ -0.0168 & -1.5217 & -0.1397 & -0.3175 & 0.6787 & -1.5769 \\ 0.3042 & 1.0547 & -0.9833 & -1.1016 & -2.2772 & 0.2041 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -1.1636 & -0.0632 & -0.0153 & 0.1706 & 1.0161 & 0.3321 \\ 0.4537 & 0.3837 & -1.5704 & 1.0775 & 1.0633 & -1.2500 \\ 0.6882 & -0.1188 & -0.6172 & 0.1081 & -0.9434 & -0.8816 \\ 0.4581 & 0.4896 & -0.7158 & 0.2237 & -0.2411 & -0.2983 \\ 0.9957 & 0.0992 & -0.1938 & 0.4602 & -0.9461 & -0.2692 \\ -1.0270 & 0.3322 & 0.6574 & 0.5190 & 0.0591 & 0.3468 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -0.1620 & -0.7600 & 0.2185 & 0.6680 & -0.0869 & 0.2811 \\ 0.3548 & 1.1226 & -0.1035 & 0.0191 & 0.8869 & -0.1554 \\ -0.0062 & -0.1772 & -0.5372 & -0.1446 & 0.4768 & -0.2087 \\ 0.1849 & -0.5384 & 0.2619 & 0.4180 & -0.8080 & 0.7966 \\ 1.0530 & -0.2093 & -0.4522 & -0.6498 & -0.9190 & 0.9116 \\ 0.2829 & 0.0122 & -0.0370 & 0.0390 & -0.0724 & -0.0892 \end{bmatrix},$$

$$A3 = \begin{bmatrix} 2.4775 & 1.0257 & 1.5891 & 0.8326 & -1.3239 & 1.7045 \\ -0.3964 & 1.2449 & -0.4890 & -1.5695 & -0.8312 & 0.5800 \\ -2.2922 & 2.2344 & -2.0149 & -0.2128 & 0.6352 & 2.5780 \\ 0.1915 & 1.4213 & -0.8818 & 1.5595 & -1.1228 & -1.1024 \\ 0.8404 & 0.6028 & -1.7115 & -0.7325 & -1.8020 & 2.0994 \\ 0.8771 & 1.2225 & -0.6453 & -2.3762 & 2.2157 & 0.2430 \end{bmatrix},$$

$$B = \begin{bmatrix} -0.7928 & -0.1048 & 1.7634 \\ 0.3175 & 0.8241 & -0.0609 \\ 0.3013 & -1.0316 & -0.1341 \\ 0.8311 & -1.2573 & 0.5240 \\ -2.4712 & 0.2176 & 0.0554 \\ 0.4338 & -0.4529 & 1.1995 \end{bmatrix},$$

$$C = \begin{bmatrix} 1.5959 & -0.8172 & 0.7905 & 1.9030 & -0.3477 & -0.9110 \\ -0.0133 & 0.1929 & 0.4005 & -1.1900 & -0.2924 & -0.5558 \end{bmatrix},$$

$\tau_1 = 0.11$, $\tau_2 = 0.21$ and $\tau_3 = 1$.

The designed gain matrices are

$$K_p = \begin{bmatrix} 2.3919 & 1.2543 \\ 2.3057 & -1.0215 \\ -0.5508 & 3.3244 \end{bmatrix}, \; K_d = \begin{bmatrix} 0.6934 & -0.7823 \\ 3.8363 & 1.7142 \\ 1.3728 & 1.5151 \end{bmatrix} \text{ and } K_i = \begin{bmatrix} 0.8078 & -0.1195 \\ 2.2107 & -1.1656 \\ -0.2982 & 1.7709 \end{bmatrix}.$$

Appendix C. Resulting gain matrices for Examples 5.3 and 6.2. For Example 5.3, the resulting gain matrices are

$$K_p = \begin{bmatrix} -5.4621 & -37.839 & 50.320 & 2.2475 & -81.876 & -81.384 & 20.740 & 20.948 \\ -43.189 & -19.153 & 28.186 & -49.341 & -5.5520 & -3.2277 & -23.318 & -28.907 \\ -13.891 & -20.990 & 43.097 & 3.3196 & 66.371 & 64.368 & 32.764 & 44.411 \\ 17.057 & 7.0390 & 9.8189 & 73.132 & 21.407 & 16.281 & -22.868 & -12.137 \end{bmatrix},$$

$$K_d = \begin{bmatrix} 2.6020 & -35.292 & 30.733 & 4.6261 & -26.267 & -44.933 & 19.415 & 11.927 \\ -46.694 & -28.348 & 24.584 & -47.105 & -52.664 & 0.5357 & -26.764 & -12.026 \\ -25.728 & 5.1204 & 21.746 & 16.797 & 19.493 & 48.842 & 43.154 & -1.4350 \\ 33.934 & 13.093 & 26.771 & 62.267 & 45.204 & -3.5705 & -10.754 & 8.9672 \end{bmatrix}$$

and

$$K_i = \begin{bmatrix} 20.372 & -4.9418 & 23.075 & 9.2952 & -5.5020 & -0.1435 & 10.171 & 20.704 \\ -15.792 & -15.709 & -12.478 & -32.234 & -5.9862 & 5.4981 & -16.945 & -23.147 \\ -35.206 & 0.9211 & 1.7405 & 6.6778 & 0.9162 & 1.9864 & 11.170 & 32.211 \\ -6.5286 & 10.778 & -21.887 & 36.470 & 0.9330 & -2.7628 & -24.332 & -23.208 \end{bmatrix}.$$

For Example 6.2, the resulting gain matrices are

$$K_p = \begin{bmatrix} 16.643 & -63.313 & 62.181 & 4.4335 & -66.888 & -68.060 & 16.321 & 16.515 \\ -50.853 & -19.330 & 30.241 & -36.155 & -9.8565 & -12.996 & -31.706 & -26.147 \\ -50.085 & 5.6587 & 42.992 & 13.233 & 58.886 & 52.449 & 55.428 & 42.728 \\ 10.317 & 15.647 & 8.9039 & 75.015 & 19.351 & 16.477 & -32.157 & -8.2343 \end{bmatrix},$$

$$K_d = \begin{bmatrix} 24.157 & -13.275 & 29.809 & 17.304 & 4.6970 & -33.340 & 5.7379 & -1.0075 \\ -25.779 & 4.0052 & -9.0192 & -1.2406 & -19.611 & -13.106 & -14.185 & -4.0257 \\ -44.969 & 8.0431 & 14.295 & 8.2878 & 6.5073 & 0.3612 & 9.4864 & -1.4291 \\ -16.865 & -2.2785 & 3.3891 & 26.649 & 11.830 & -24.990 & -20.429 & -1.1029 \end{bmatrix}$$

and

$$K_i = \begin{bmatrix} 3.7628 & -33.633 & 37.426 & 13.118 & -27.117 & -45.319 & 15.887 & 7.5073 \\ -45.325 & -37.653 & 26.177 & -52.892 & -50.652 & 1.5547 & -25.018 & -20.414 \\ -18.774 & -0.9240 & 14.856 & 7.1468 & 20.219 & 49.838 & 41.447 & 21.228 \\ 28.956 & 27.573 & 27.615 & 63.219 & 43.872 & -2.5157 & -6.3298 & -0.3217 \end{bmatrix}.$$