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p-refined Multilevel Quasi-Monte Carlo for Galerkin Finite Element Methods with applications in Geotechnical Engineering

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Outline

1 - Introduction - Setting and Scope

- We present **p-refined Multilevel Quasi-Monte Carlo** a novel algorithm which considerably speeds up the computation of statistics of a quantity of interest derived from the solution of a model described by a PDE with random coefficients

- This will be benchmarked against
  - Standard Multilevel Monte Carlo
  - Standard Multilevel Quasi-Monte Carlo

- Applied to a Non-linear Slope Stability Problem - (Geotechnical Engineering)

1 - Introduction - Case Presentation

- **Case**
  - Assess the stability of man made or natural slopes
  - Non-linear problem
  - Uncertainty located in the soil's cohesion
  - 2D Plane Strain

Source: Schijnbare cohesie van onverzadigde gronden - Geotechniek Januari 2011

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1 - Introduction - p-MLQMC

- p-MLQMC combines
  - a hierarchy of **higher order Finite Elements**
  - QMC sample points
- Because of the hierarchy of higher order Finite Elements
  - we cannot assign the randomness to the whole element because the number of elements remains the same on each level
  - we decouple the relation between the resolution of the random field and the resolution of the mesh
- Careful consideration needs to be given to the generation of random fields over successive levels

2 - p-MLQMC - Expected Value

- MLMC [Giles, 2008]

$$\mathbb{E}[P_L] = \frac{1}{N_0}\sum_{n=1}^{N_0} P_0(\mathbf{x}^{(n)}) + \sum_{\ell=1}^{L}\left\{\frac{1}{N_\ell}\sum_{n=1}^{N_\ell}\left(P_\ell(\mathbf{x}^{(n)}) - P_{\ell-1}(\mathbf{x}^{(n)})\right)\right\}$$

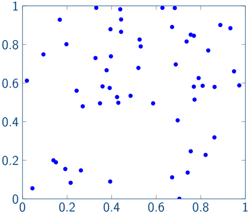
- MLQMC [Giles and Waterhouse, 2009]

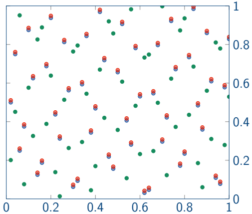
$$\mathbb{E}[P_L] = \frac{1}{R_0}\sum_{r=1}^{R_0}\frac{1}{N_0}\sum_{n=1}^{N_0} P_0(\mathbf{x}^{(r,n)}) + \sum_{\ell=1}^{L}\frac{1}{R_\ell}\sum_{r=1}^{R_\ell}\left\{\frac{1}{N_\ell}\sum_{n=1}^{N_\ell}\left(P_\ell(\mathbf{x}^{(r,n)}) - P_{\ell-1}(\mathbf{x}^{(r,n)})\right)\right\}$$

Take many computationally cheap samples on coarse meshes and few computationally expensive samples on fine meshes

2 - p-MLQMC - QMC Points

- For MLQMC, sample points are chosen according to a deterministic rule (rank-1 lattice rule) [Nuyens et al., 2016]





- Representation of the QMC points as open lattice rule,

$$\mathbf{x}^{(r,n)} = \mathrm{frac}\left(\phi_2(n)\mathbf{z} + \Xi_r\right), \text{for} \quad n \in \mathbb{N},$$

with the radical inverse function $\phi_2(\mathbf{x}_n)$ in base 2, the generating vector $\mathbf{z}$, and random shift Ξ_r

2 - p-MLQMC - Ritz-Galerkin

- By means of the variational formulation the PDE governing the displacement is discretized in the following form,

$$\mathbf{K}\mathbf{u} = \mathbf{f}$$

with **K** the global stiffness matrix of the problem resulting from the assembly of the element stiffness matrices,

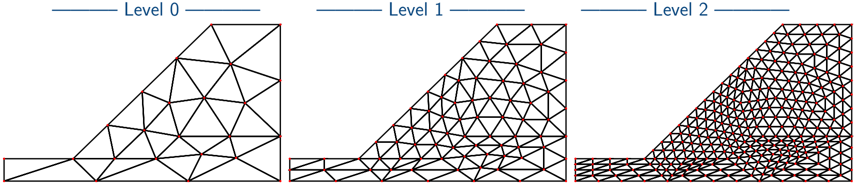
$$\mathbf{K^e} = \int_{\Omega} \mathbf{B}^T \mathbf{D}\mathbf{B} d\Omega$$

with **B** containing the derivatives of the element shape function and **D** the elastic/elastoplastic constitutive matrix.

- $\mathbf{K^e}$ is **numerically integrated** by means of Gauss Quadrature.

2 - p-MLQMC - Mesh Hierarchies

Standard ML(Q)MC, hence referred to as h-ML(Q)MC, makes use of mesh hierarchy based on nested **geometric refinement**



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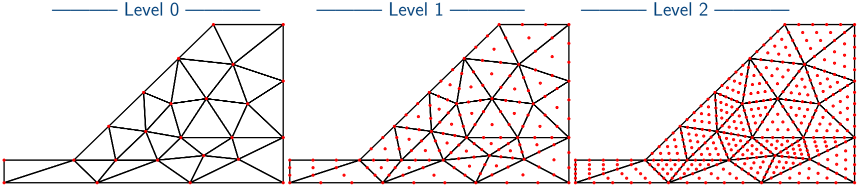
2 - p-MLQMC - Mesh Hierarchies

p-ML(Q)MC
makes use of mesh hierarchy based on increasing the
element's polynomial order



3 - Uncertainty Modeling - Random Fields

- The uncertainty in the material parameter
  - is chosen as the spatial variation of the soil's cohesion,
  - and is represented as a random field

- Ad hoc definition of a random field
  - Collection of random variables **at certain discrete locations**
  - Many different techniques possible
    - QR decomposition
    - Spectral decomposition
    - Circulant Embedding
    - Karhunen–Loève expansion
  - We will use and focus on the Karhunen–Loève expansion

3 - Uncertainty Modeling - Karhunen–Loève

Generation of the random field is a two-step process:

- **Construction** of a Gaussian random field by means of a Karhunen–Loève expansion,

$$Z(\mathbf{x},\omega) \approx \overline{Z}(\mathbf{x},.) + \sum_{n=1}^{s} \sqrt{\theta_n}\xi_n(\omega) b_n(\mathbf{x}),$$

  with a Matérn covariance Kernel,

$$C(\mathbf{x},\mathbf{y}) := \sigma^2 \frac{1}{2^{\nu-1}\Gamma(\nu)} \left(\sqrt{2\nu}\frac{\|\mathbf{x}-\mathbf{y}\|_2}{\lambda}\right)^{\nu} K_\nu \left(\sqrt{2\nu}\frac{\|\mathbf{x}-\mathbf{y}\|_2}{\lambda}\right).$$

- **Transformation** of the Gaussian random field to a Log-normal random field by applying the exponential

$$Z_{lognormal}(\mathbf{x},\omega) = \exp(Z(\mathbf{x},\omega))$$

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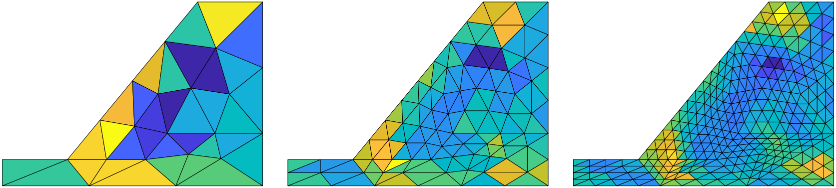
3 - Uncertainty Modeling - Stochastic Mapping

- Classically the **midpoint method** is used $\rightarrow$ each element is assigned one value of the random field

$$M(\mathcal{T}) = M(\text{Random Field})$$

- p-MLQMC uses the **integration point method** $\rightarrow$ each quadrature point is assigned one value of the random field

$$M(\mathcal{T}) < M(\text{Random Field})$$



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3 - Uncertainty Modeling - Stochastic Mapping

- How to generate the discrete values of the random field?
  - **Non-Nested approach**
  - **Global Nested approach**
  - **Local Nested approach**
- Why do we bother?

  We want to have a good correlation between successive levels

$$\rightarrow$$

good decrease of $\mathbb{V}[\Delta P_\ell]$

$$\rightarrow$$

lower number of samples per level

$$\rightarrow$$

lower computational cost

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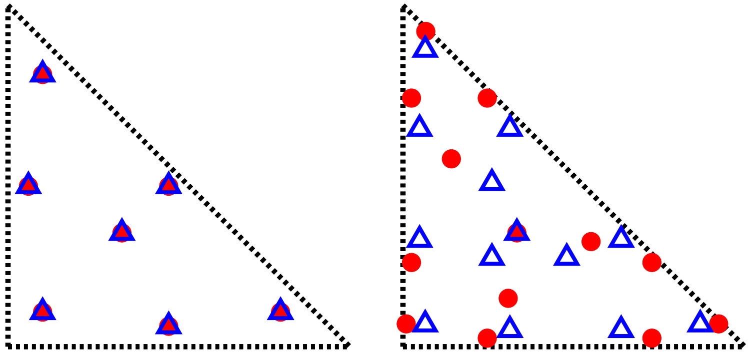
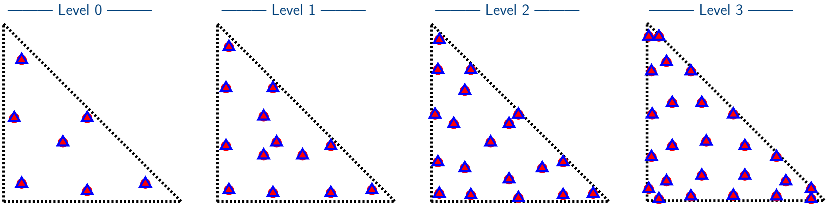
3 - Uncertainty Modeling - Stochastic Mapping

- Reference Triangular Finite Element with
  - ●, the location of the discrete values of the random field
  - △, the quadrature points



3 - Non-nested approach

- **Idea**: Use the locations of the quadrature points on each level as the location where the discrete values of the random field are to be generated, for $\ell = 0 \dots L$, $\bullet_\ell = \triangle_\ell$, RF$(\bullet_\ell)$



- **Advantage**: Extensible, an extra level can easily be added
- **Disadvantage**: Very high computational cost due to very slow decrease of $\mathbb{V}[\Delta P_\ell]$

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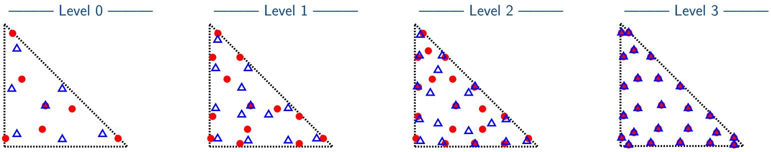
3 - Global Nested approach

- **Idea**: Starting from a user chosen maximum level L, use the quadrature points as location for values of the random field $\bullet_L = \triangle_L$, RF $(\bullet_L)$. On all coarser levels $\ell < L$, compute subsets of these points, $\bullet_0 \subseteq \bullet_\ell \subseteq \ldots \subseteq \bullet_L$, such that they are closest to the actual quadrature points of level ℓ.
- Example with $L = 3$

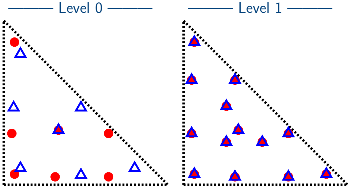


- **Advantage**: Good decrease of $\mathbb{V}[\Delta P_\ell]$
- **Disadvantage**: Maximum number of levels is fixed

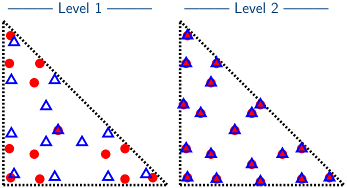
3 - Local Nested approach

- **Idea**: For each level ℓ from 1 to L, use the quadrature points as location for values of the random field $\bullet_\ell = \triangle_\ell$, RF $(\bullet_\ell)$. For the coarser level $\ell - 1$ compute a subset of these points, $\bullet_{\ell-1} \subseteq \bullet_\ell$, such that they are closest to the actual quadrature points of level ℓ.

- Example: Level 1 and Level 0



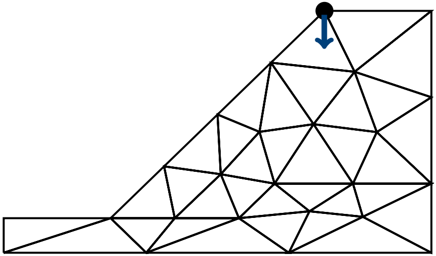
- Example: Level 2 and Level 1



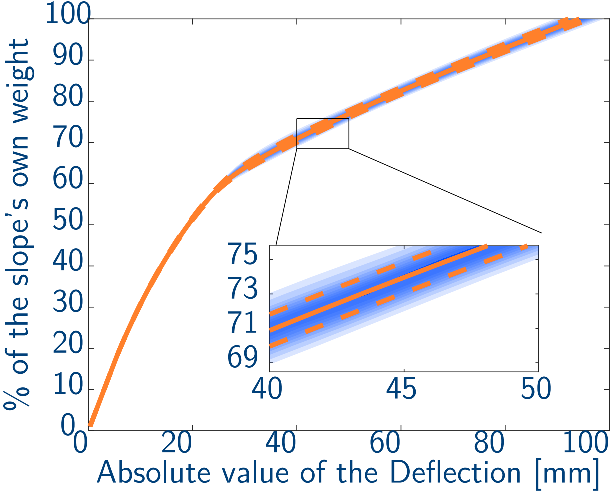
- **Advantage**: Level extensibility is easy
- **Disadvantage**: Complexer code

4 - Benchmarking and Results -QoI

- Quantity of Interest
  - Vertical displacement of node located at the upper left corner



4 - Results - Uncertainty on the Solution



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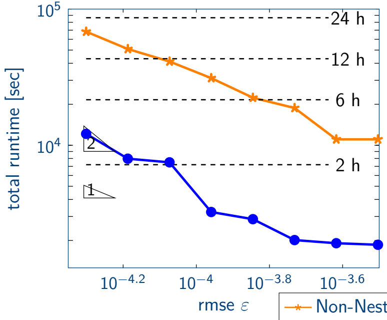
4 - Benchmarking - Comparison

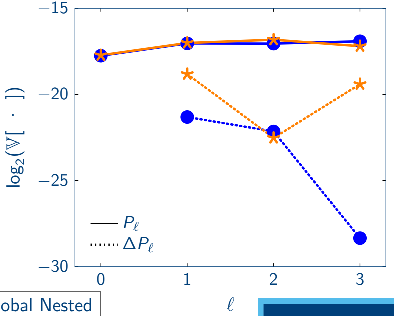
- Global Nested approach performs much better than Non-Nested approach
- Better decrease of $\mathbb{V}[\Delta P_\ell]$ for Global Nested approach

	p-ML(Q)MC			
Level	**Nel**	**DOF**	**Order**	**Nquad**
0	33	48	1	7
1	33	338	3	16
2	33	892	5	28
3	33	1720	7	37

———Runtime———

———Variance———





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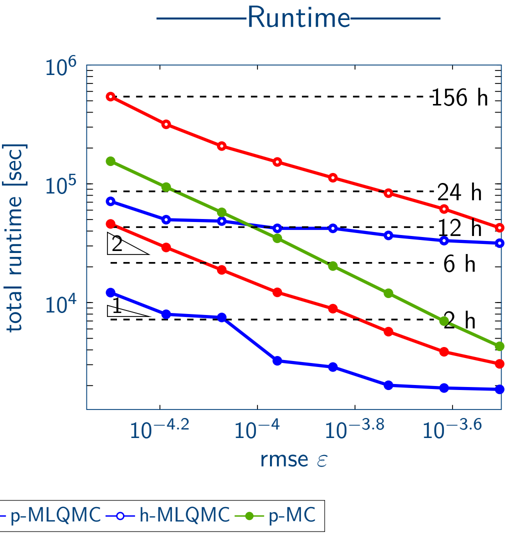
4 - Benchmarking - Global Nested Approach

	h-ML(Q)MC				p-ML(Q)MC			
Level	Nel	DOF	Order	Nquad	Nel	DOF	Order	Nquad
0	33	48	1	7	33	48	1	7
1	132	160	1	7	33	338	3	16
2	528	582	1	7	33	892	5	28
3	2112	2218	1	7	33	1720	7	37
4	8448	8658	1	7	/	/	/	/

- p-MLQMC $\sim$ 44 times faster than h-MLMC
- p-MLQMC $\sim$ 3 times faster than p-MLMC
- Cost of MLQMC $\sim \epsilon^{-1}$

[Blondeel et al., 2020]



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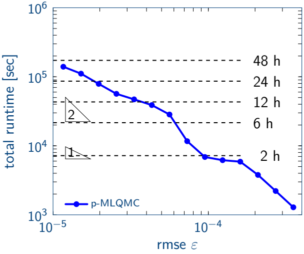
4 - Benchmarking - Global Nested Approach

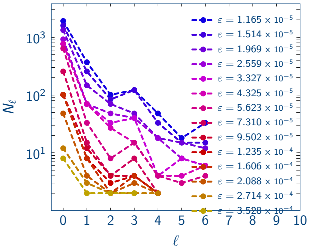
- Decrease of $\mathbb{V}[\Delta P_\ell]$ over the levels
- Cost of MLQMC $\sim \epsilon^{-1}$

p-ML(Q)MC				
Level	**Nel**	**DOF**	**Order**	**Nquad**
0	33	48	1	7
1	33	160	2	13
2	33	338	3	19
3	33	582	4	25
4	33	892	5	28
5	33	1268	6	33
6	33	1720	7	37
7	33	2218	8	61
8	33	2792	9	73

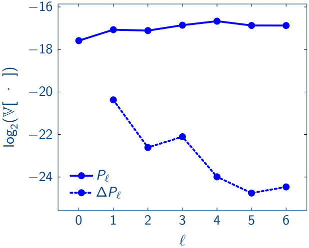
———Runtime———



———Samples———



———Variance———



6 - Conclusion and Outlook

- Conclusions
  - p-MLQMC
    - Speedup of a factor 44 with respect to h-MLMC
    - Global Nested and Local Nested approach for generating discrete values of the random field are the most promising
- Outlook
  - Extending p-MLQMC for 3D problems
    - Higher order Finite Elements based on Hierarchical Shape Functions
    - Nested Quadrature points over the levels based on Sparse Grids
    - Possibility of reusing Finite Element information over the levels
  - Multi-Index (Quasi) Monte Carlo for 2D/3D problems [Robbe et al., 2017]
  - Use of higher order Digital Nets instead of Rank-1 lattice rule for QMC points

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Thank you for your attention!