On the strong $\mathcal{H}_2$ norm of differential algebraic systems with multiple delays: finiteness criteria, regularization and computation

Marco A. Gomez, Raphaël M. Jungers, Wim Michiels

***Abstract*—The $\mathcal{H}_2$ norm of an exponentially stable system described by Delay Differential Algebraic Equations (DDAEs) might be infinite due to the existence of hidden feedthrough terms and, as shown in this paper, it might become infinite as a result of infinitesimal changes to the delay parameters. We first introduce the notion of strong $\mathcal{H}_2$ norm of semi-explicit DDAEs, a robustified measure that takes into account delay perturbations, and we analyze its properties. Next, we derive necessary and sufficient finiteness criteria for the strong $\mathcal{H}_2$ norm in terms of a frequency sweeping test over a hypercube, and in terms of a finite number of equalities involving multi-dimensional powers of a finite set of matrices. As the main contribution, we present a strengthened, sufficient, condition for finiteness of the strong $\mathcal{H}_2$ norm, along with an algorithm for checking it, which has significantly better scalability properties in terms of both the dimension of the system and the number of delays. We show that the satisfaction of the novel condition is equivalent to the existence of a simultaneous block triangularization of the matrices of the delay difference equation associated to the DDAE. The latter is instrumental to a novel regularization procedure that allows to transform the DDAE to a neutral type system with the same transfer matrix, without any need for differentiation of inputs or outputs. As we illustrate, this transformation enables for instance to compute the strong $\mathcal{H}_2$ norm using an established approach grounded in Lyapunov matrices. Finally, we investigate the conservatism of the sufficient finiteness condition. We show by a counterexample that the condition is in general not necessary, inducing open problems, but we also list several classes of DDAEs for which it is necessary and sufficient.**

***Index Terms*—$\mathcal{H}_2$ norm, time-delay systems, differential algebraic equations.**

I. Introduction

We consider systems described by linear delay-differential algebraic equations

$$\begin{aligned}\hat{E}\frac{d}{dt}x(t) &= \hat{A}_0x(t) + \sum_{j=1}^{m}\hat{A}_jx(t-h_j) + \hat{B}u(t)\\ y(t) &= \hat{C}x(t),\end{aligned} \tag{1}$$

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where $x(t) \in \mathbb{R}^{\hat{n}}$, $u(t) \in \mathbb{R}^{n_i}$, $y(t) \in \mathbb{R}^{n_o}$ are the state, input and output at time t, leading matrix $\hat{E} \in \mathbb{R}^{\hat{n}\times\hat{n}}$ is possibly singular with rank $\hat{E} = r$, the other matrices satisfy $\hat{A}_j \in \mathbb{R}^{\hat{n}\times\hat{n}}$, $j = 0,\dots,m$, $\hat{B} \in \mathbb{R}^{\hat{n}\times n_i}$, and $\hat{C} \in \mathbb{R}^{n_o\times\hat{n}}$, and $0 < h_1 \leq \cdots \leq h_m$ are the delay values. By considering the delays in this form we keep the possibility for delays to have the same value but represent different physical parameters. For instance, if two delays h_1 and h_2 are equal to one delay h but they correspond to independent physical parameters, then the system should be defined as $\hat{E}\frac{d}{dt}x(t) = \hat{A}_1x(t-h) + \hat{A}_2x(t-h)$. On the other hand, if they correspond to the same physical parameter then the system should be defined as $\hat{E}\frac{d}{dt}x(t) = (\hat{A}_1+\hat{A}_2)x(t-h)$. This distinction is important because, as we shall see, the $\mathcal{H}_2$ norm may be sensitive to infinitesimal delay perturbations.

Systems of the form (1) naturally appear in the modeling of electronic circuits, mechanical systems with algebraic constraints and interconnected systems with delays, just to mention a few [1]–[3]. It should be noted that linear time-invariant retarded and neutral equations, with discrete delays inputs, states and outputs, can be expressed in form (1), see [4], [5]. For the special case $r = 0$, system (1) corresponds to a difference equation in continuous time.

System (1) can be explicitly expressed as coupled delay differential equations and delay difference equations. Considering matrices $\begin{pmatrix}U_1 & U_2\end{pmatrix}$ and $\begin{pmatrix}V_1 & V_2\end{pmatrix}$, which are the left and right factor of the Singular Value Decomposition of matrix $\hat{E}$, respectively, where $U_1 \in \mathbb{R}^{\hat{n}\times r}$, $U_2 \in \mathbb{R}^{\hat{n}\times\hat{n}-r}$, $V_1 \in \mathbb{R}^{\hat{n}\times r}$, and $V_2 \in \mathbb{R}^{\hat{n}\times\hat{n}-r}$, we make the following assumption.

Assumption 1. *Matrix $U_2^T\hat{A}_0V_2$ is nonsingular.*

Assumption 1 implies that the differentiation index of (1) is one (a semi-explicit DDAE). If $r \geq 1$ then, premultiplying system (1) by $\begin{pmatrix}U_1 & U_2\end{pmatrix}^T$, and considering the change of coordinates

$$x(t) = \begin{pmatrix}V_1 & V_2\end{pmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix},\ x_1(t) \in \mathbb{R}^r,\ x_2(t) \in \mathbb{R}^n,$$

with $n := \hat{n} - r$, enable us to rewrite system (1) as

$$E\frac{d}{dt}x_1(t) = A_0^{(11)}x_1(t) + A_0^{(12)}x_2(t) + \sum_{j=1}^{m}A_j^{(11)}x_1(t-h_i)$$

$$
\begin{aligned}
&+\sum_{j=1}^{m} A_j^{(12)} x_2(t-h_j) + B_1 u(t)\\
x_2(t) =& A_0^{(21)} x_1(t) + \sum_{j=1}^{m} A_j^{(21)} x_1(t-h_j)\\
&+\sum_{j=1}^{m} A_j x_2(t-h_j) + Bu(t)\\
y(t) =& C_1 x_1(t) + C x_2(t),
\end{aligned} \tag{2}
$$

where we have assumed without loss of generality that $A_0^{(22)} = U_2^T \hat{A}_0 V_2 = -I$. Matrix $E = U_1^T \hat{E} V_1$ is invertible and the other matrices are given by

$$
\begin{aligned}
A_j^{(11)} =& U_1^T \hat{A}_j V_1,\ \ A_j^{(12)} = U_1^T \hat{A}_j V_2,\\
A_j^{(21)} =& U_2^T \hat{A}_j V_1, \quad\ A_j = U_2^T \hat{A}_j V_2,\ j = 0, \dots, m,
\end{aligned}
$$

and

$$
B_1 = U_1^T \hat{B},\ B = U_2^T \hat{B},\ C_1 = \hat{C} V_1,\ C = \hat{C} V_2.
$$

For $r = 0$, system (1) can be put in the form (2), where the differential equation is omitted and x_1 is set to zero in the other equations.

We consider initial functions φ for (1) that belong to the set of $\mathbb{R}^{\hat{n}}$-valued absolutely continuous functions $AC\left([-h_m,\ 0], \mathbb{R}^{\hat{n}}\right)$ and call them consistent if the corresponding initial value problem at $t = 0$ has at least one solution [6]. A function $x(t, \varphi)$ is called a (classical) solution of system (1) if it is absolutely continuous and satisfies (1) almost everywhere on $[0, \infty)$, and $x(\theta, \varphi) = \varphi(\theta)$ for $\theta \in [-h_m,\ 0]$, where φ is a consistent initial function. For a continuously differentiable input function, the space of consistent initial functions for (1) is given by

$$
\begin{aligned}
X := \Big\{ & \varphi \in AC\left([-h_m, 0], \mathbb{R}^{\hat{n}}\right) :\\
& U_2^T \hat{A}_0 \varphi(0) + \sum_{j=1}^{m} U_2^T \hat{A}_j \varphi(-h_j) + U_2^T \hat{B} u(0) = 0 \Big\},
\end{aligned}
$$

which corresponds to the set of initial conditions for which the second equation in (2) is satisfied at $t = 0$. Moreover, for every initial function belonging to X, a forward solution is uniquely defined [6]–[8].

Definition 1. *System (1), with zero input, is exponentially stable if there exist constant $\gamma > 0$ and $\sigma > 0$ such that for all initial condition $\varphi \in X$ we have*

$$
\|x(t)\| \leq \gamma e^{-\sigma t} \sup_{\theta \in [-h_m, 0]} \|\varphi\|,\ t \geq 0.
$$

The $\mathcal{H}_2$ norm is an important performance measure in the field of control theory [9]. For exponentially stable systems of the form (1), it is defined as

$$
\|G\|_{\mathcal{H}_2} := \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{Tr}(G^*(\imath\omega) G(\imath\omega)) d\omega}, \tag{3}
$$

where G is the transfer matrix of system (1), given by

$$
G(s) = \hat{C} \left(s\hat{E} - \hat{A}_0 - \sum_{j=1}^{m} \hat{A}_j e^{-s h_j} \right)^{-1} \hat{B}.
$$

In contrast to other classes of systems, the $\mathcal{H}_2$ norm of system (1) might be infinite even if the system is exponentially stable, as the DDAE formulation might hide a nontrivial feedthrough, and, besides the single delay case addressed in [10], there does not exist a Lyapunov matrix based procedure that can be directly applied for the computation. We illustrate the former with an example.

Example 1. *We consider system (1) with $m = 1$ and matrices specified as $\hat{A}_0 = -I$,*

$$
\hat{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \hat{A}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\ \hat{B} = \hat{C}^T = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},
$$

whose transfer function is

$$
G(s) = (s+1)^{-1} + 2 - e^{-s h_1}.
$$

The term $2 - e^{-s h_1}$ corresponds to a (hidden) direct feedthrough from input to output, which induces an infinite $\mathcal{H}_2$ norm of the system [9]. Indeed, by elimination of variables, we have $\dot{x}_1(t) = -x_1(t) + u(t)$, $y(t) = x_1(t) + 2u(t) + u(t - h_1)$. In [10], [11], we have proved that the $\mathcal{H}_2$ norm is finite if and only if $2 - e^{-s h_1} \equiv 0$.

We recently showed in [11] that the finiteness of the $\mathcal{H}_2$ norm is determined by the condition $G_a \equiv 0$, where G_a is the *asymptotic transfer function*

$$
G_a(s) := C \left(I - \sum_{j=1}^{m} A_j e^{-s h_j} \right)^{-1} B,
$$

which describes the asymptotic behavior of $G(s)$ for $|s| \to \infty$ in the half plane $\Re(s) \geq 0$, see [5]. It can be interpreted as the transfer function of delay difference equation

$$
x_2(t) = \sum_{j=1}^{m} A_j x_2(t - h_j) + Bu(t), \quad y(t) = C x_2(t), \tag{4}
$$

obtained by setting $x_1 = 0$ in the second equation of (2). Obviously, $G_a = G$ if $r = 0$ in (1). In [11] we also presented algebraic necessary and sufficient conditions for determining whether $G_a \equiv 0$ is satisfied. In order to recall these results, we rely on matrix polynomials $P_{k_1,\dots,k_m}(A_1, \dots, A_m)$, with $k_j \in \mathbb{Z}_{\geq 0}, j = 1, \dots, m$, which are recursively defined through the following expressions:

$$
P_{0,\dots,0}(A_1, \dots, A_m) := I, \tag{5}
$$

$$
\begin{aligned}
&P_{k_1,\dots,k_m}(A_1, \dots, A_m) := A_1 P_{k_1-1,k_2,\dots,k_m}(A_1, \dots, A_m)\\
&+ A_2 P_{k_1,k_2-1,\dots,k_m}(A_1, \dots, A_m) + \dots +\\
&+ A_m P_{k_1,k_2,\dots,k_m-1}(A_1, \dots, A_m)
\end{aligned} \tag{6}
$$

and

$$
P_{k_1,\dots,k_m}(A_1, \dots, A_m) := 0 \text{ if any } k_j \in \mathbb{Z}_{<0}, j = 1, \dots, m. \tag{7}
$$

For instance, for $m = 2$ and $k_1 + k_2 \leq 3$, these matrix polynomials are

$$\begin{aligned}
&P_{0,0} = I, \\
&P_{1,0} = A_1, \quad P_{0,1} = A_2, \\
&P_{2,0} = A_1^2, \quad P_{1,1} = A_1A_2 + A_2A_1, \quad P_{0,2} = A_2^2, \\
&P_{3,0} = A_1^3, \quad P_{2,1} = A_1^2A_2 + A_1A_2A_1 + A_2A_1^2, \\
&P_{1,2} = A_1A_2^2 + A_2A_1A_2 + A_2^2A_1, \quad P_{0,3} = A_2^3.
\end{aligned}$$

The main results of [11] can now be stated.

Lemma 1. *[11] Let system* (1) *be exponentially stable. Assume that the delays* $(h_1, \ldots, h_m)$ *are rationally independent, i.e. the only solution* $(c_1, \ldots, c_m)$ *of*

$$\sum_{j=1}^{m} c_j h_j = 0, \quad c_i \in \mathbb{Z}, i = 1, \ldots, m,$$

is given by $(c_1, \ldots, c_m) = (0, \ldots, 0)$. *The following statements are equivalent.*

1) *The* $\mathcal{H}_2$ *norm of system* (1) *is finite.*
2) *Transfer matrix* $G_a(s)$ *is identically zero.*
3) *Equation*
$$CP_{k_1,\ldots,k_m}(A_1, \ldots, A_m)B = 0 \tag{8}$$
holds for all $(k_1, \ldots, k_m) \in \mathbb{Z}^m_{\geq 0}$.
4) *Equation* (8) *holds for all* $(k_1, \ldots, k_m) \in \mathbb{Z}^m$ *satisfying* $\sum_{j=1}^{m} k_j < mn$.

It is worthwhile observing the equivalence of Statement 3 and Statement 4, which enables checking the finiteness of the $\mathcal{H}_2$ norm in a *finite* number of mathematical operations. At the basis of the equivalence is the m-dimensional Cayley-Hamilton theorem, a generalization of the classic Cayley-Hamilton theorem in the sense of a recursion multiplication formula of a block matrix. We notice that for the one delay case condition (8) reduces to

$$CA_1^kB = 0, \ k = 0, 1, \ldots, n-1, \tag{9}$$

which is indeed the finiteness condition for system (1) with one delay obtained in [10]. Although the study of the $\mathcal{H}_2$ norm of system (1) has moved forward with the introduction of Lemma 1, there are still some problems of practical and theoretical interest, which are the main focus of this paper.

A *first* problem is that, as we shall see in Section II, the $\mathcal{H}_2$ norm may be a fragile measure in the sense of being sensitive with respect to infinitesimal delay perturbations. More specifically, the $\mathcal{H}_2$ norm might become infinite under infinitesimal perturbations of the delays. In order to address this problem, we define the *strong* $\mathcal{H}_2$ *norm*, a robustified measure, and present conditions for its finiteness. One of the necessary and sufficient conditions is found to be the same as condition (8), equivalent to the one in Statement 4. In the analysis we do not rely anymore on the assumption of rationally independent delays, which is restrictive from an application perspective. We also show that whenever the strong $\mathcal{H}_2$ norm is finite, it is equal to the $\mathcal{H}_2$ norm defined by (3).

A *second problem* is the computational cost of verifying the finiteness conditions of the $\mathcal{H}_2$ norm. Even though the expression in Statement 4 provides a finite test, the number of equations scales poorly with respect to the number of delays m and the dimension n, as we show at the end of Section II. We tackle this problem by introducing a novel strengthened, sufficient, condition for the finiteness of the (strong) $\mathcal{H}_2$ norm, along with an efficient computational test.

A *third problem* concerns the computation of the $\mathcal{H}_2$ norm, whenever it is finite. For the one delay case, the computation problem has been successfully addressed in [10] by constructing a neutral type system that has the same input-output map in the frequency domain whenever the $\mathcal{H}_2$ norm is finite. The construction of such a system is based on the existence of a transformation matrix that allows a block triangularization of matrices (A_1, B, C) as

$$(A_1, \ B, \ C) \to \left(\begin{pmatrix} A_{11} & 0 \\ A_{12} & A_{13} \end{pmatrix}, \begin{pmatrix} 0 \\ B_c \end{pmatrix}, \ \begin{pmatrix} C_u & 0 \end{pmatrix} \right),$$

whenever condition (9) holds. Following this idea, one may be tempted to conjecture that the set of matrices $(A_1, \ldots, A_m, B, C)$ in the multiple delay case admits a simultaneous block triangularization by a transformation matrix whenever condition (8) is satisfied. However, as we shall see, a necessary and sufficient condition for a simultaneous block-triangularization is the strengthened sufficient condition previously referred to, which is shown by a counterexample in Section IV not to be equivalent to (8), unless additional conditions are put on the coefficient matrices. The simultaneous block-triangularization of matrices $A_1, \ldots, A_m$ allows us to construct a neutral type system without derivatives of the input (which would appear when simply differentiating the second equation of (2)), enabling us to compute the $\mathcal{H}_2$ norm by state-of-the-art methods.

The remainder of the paper is organized as follows. In Section II we define and analyze the strong $\mathcal{H}_2$ norm and derive necessary and sufficient conditions for its finiteness. The strengthened finiteness criterion is introduced in Section III. There we also present an efficient computational algorithm for checking this criterion, we provide the construction of the equivalent neutral type system and we outline computation of the $\mathcal{H}_2$ norm via the so-called delay Lyapunov matrix. In Section IV we show that the strengthened condition is not equivalent to the original one, but that in a number of cases the strengthened condition is necessary and sufficient for the finiteness of the (strong) $\mathcal{H}_2$ norm. We also discuss an open problem regarding the construction of neutral systems. We illustrate the results by a numerical example in Section V, and conclude the paper with some final remarks.

We adopt the following notation. The set of non-negative integer, natural numbers, and non-negative real numbers is denoted by $\mathbb{Z}_{\geq 0}$, $\mathbb{Z}_{>0}$ and $\mathbb{R}_{\geq 0}$, respectively. The set of complex numbers is represented by $\mathbb{C}$, and the imaginary unit by $\imath$. We use $\vec{\theta} \in \mathbb{R}^m$ as short notation of $(\theta_1, \ldots, \theta_m)$. The Euclidian norm for vectors is denoted by $\|\cdot\|$, and the spectral norm for matrices by $\|\cdot\|_2$. The notations Im A, rank A and Tr A stand for the image, rank and trace of matrix A, respectively. Finally, in notations as $G(s;\vec{h})$ the semicolon is used to make distinction between variables of a function and parameters. The latter are only displayed when necessary for the understanding of the statement.

II. Sensitivity with respect to small delay perturbations and the strong $\mathcal{H}_2$ norm

System (1), with zero input, is exponentially stable if its spectral abscissa

$$\alpha(\vec{h}) := \sup_{s\in\mathbb{C}} \left\{ \Re(s) : \det\left(s\hat{E} - \hat{A}_0 - \sum_{j=1}^{m} \hat{A}_j e^{-sh_j} \right) = 0 \right\}$$

is strictly negative, where $\vec{h} = (h_1, \ldots, h_m)$ denotes the vector of delay values. It is well known that the exponential stability may be sensitive to infinitesimal perturbations of the delays. This has led to the introduction of the concept of strong stability in [4], which generalizes the corresponding notion for neutral functional differential equations in [12]. Letting

$$\mathcal{B}(\vec{h}, \epsilon) := \{\vec{\theta} \in \mathbb{R}^m_{\geq 0} : \|\vec{\theta} - \vec{h}\| < \epsilon\}, \tag{10}$$

strong stability can be defined as follows.

Definition 2. *[4] System (1), with zero input, is strongly stable if there exists a number $\epsilon > 0$ such that $\alpha(\vec{h}_\epsilon) < 0$ for all $\vec{h}_\epsilon \in \mathcal{B}(\vec{h}, \epsilon)$.*

A necessary and sufficient condition for strong stability of system (1) is that the nominal spectral abscissa $\alpha(\vec{h})$ is strictly negative, and

$$\max_{\vec{\theta}\in[0,\ 2\pi]^m} \rho\left(\sum_{j=1}^{m} A_j e^{\imath\theta_j} \right) < 1, \tag{11}$$

where $\rho(\cdot)$ is the spectral radius, see [4].

In this section we show that the sensitivity to small delay perturbations may also carry over to system norms. Accordingly, we introduce the strong $\mathcal{H}_2$ norm and provide necessary and sufficient finiteness conditions. Subsequently we discuss the computational complexity of verifying these conditions.

A. The strong $\mathcal{H}_2$ norm and finiteness conditions

With the following example we illustrate that the functions

$$\mathbb{R}^m_{\geq 0} \ni \vec{h} \mapsto \left\| G(\cdot;\vec{h}) \right\|_{\mathcal{H}_2}, \quad \text{and} \quad \mathbb{R}^m_{\geq 0} \ni \vec{h} \mapsto \left\| G(\cdot;\vec{h}) \right\|_{\mathcal{H}_\infty},$$

may not be continuous, even if the system is strongly stable. Recall that under the assumption of exponential stability, the $\mathcal{H}_\infty$ norm of the system (1) satisfies

$$\|G\|_{\mathcal{H}_\infty} := \sup_{\omega\in\mathbb{R}} \|G(\imath\omega)\|_2.$$

Example 2. *We consider system (1), already in the form (2) with $m = 2$ and $E = 1$, $A_0^{(11)} = -10$ and matrices $A_0^{(12)} = \begin{pmatrix} 1 & 1 \end{pmatrix}$, $A_0^{(21)} = \begin{pmatrix} 0 & 0 \end{pmatrix}^T$,*

$$\left(\begin{array}{c|c} A_1^{(11)} & A_1^{(12)} \\ \hline A_1^{(21)} & A_1 \end{array} \right) = \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & \frac{1}{4} & 0 \\ 0 & -1 & \frac{1}{4} \end{array} \right),$$

$$\left(\begin{array}{c|c} A_2^{(11)} & A_2^{(12)} \\ \hline A_2^{(21)} & A_2 \end{array} \right) = \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & \frac{1}{8} & \frac{1}{8} \\ 0 & 1 & \frac{1}{8} \end{array} \right),$$

$$\left(\begin{array}{c} B_1 \\ \hline B \end{array} \right) = \begin{pmatrix} 100 \\ 1 \\ 0 \end{pmatrix}, \ \left(\begin{array}{c|c} C_1 & C \end{array} \right) = \left(\begin{array}{c|cc} 1 & 0 & 1 \end{array} \right).$$

The system is exponentially stable for all delay values, and thus strongly stable. The left hand side of (11) is namely equal to 0.625. Furthermore, due to the tridiagonal structure, its spectrum consists of eigenvalue $s = -10$, supplemented with the spectrum of (4), which is confined to the open left half plane because (11) is satisfied. We now analyze its $\mathcal{H}_2$ norm from input u to output y. For nominal delays, i.e. for unperturbed delays, $\vec{h} = (1, 1)$, we regard the problem as a problem with a single delay. Since it can be easily verified that

$$C(A_1 + A_2)^i B = 0, \ i = 0, 1,$$

we conclude from Lemma 1, with $m = 1$ and A_1 replaced by $A_1 + A_2$, that the $\mathcal{H}_2$ norm from input u to output y is finite and, accordingly, that the asymptotic transfer function G_a is identically zero. On the contrary, if we take any *pair (h_1, h_2) of rationally independent delays and we apply Lemma 1 with $m = 2$, we conclude that the $\mathcal{H}_2$ norm is not finite, since*

$$CA_1B = -1, \quad CA_2B = 1.$$

In Figure 1 we show in the left the spectral norm of the transfer function G and in the right the the spectral norm of the asymptotic transfer function G_a evaluated on the imaginary axis, for $s = \imath\omega$. For $\vec{h} = (1, 1)$ there is clearly no feedthrough from input to output, inducing the finite $\mathcal{H}_2$ norm. Let us now consider rationally independent delays $\vec{h} = (1, 1 + \pi/\nu)$ with $\nu \in \mathbb{N}$. For $\nu = 50$ we see that functions G and G_a do not tend to zero as $\omega \to \infty$. If ν tends to infinity, the deviation from nominal delays $(1, 1)$ tends to zero. However, the $\mathcal{H}_2$ norms of G and G_a remain unbounded, while the significant mismatch of the transfer functions and the corresponding transfer functions for the limit $\vec{h} = (1, 1)$ only shifts towards higher frequencies. This is visualized in the figure by comparing the cases where $\nu = 50$ and $\nu = 200$.

The previous example may suggest that a discontinuity of the $\mathcal{H}_2$ norm can only occur if some of the delay values are equal to each other. This is, however, not the case, as demonstrated by the following example.

Example 3. *Consider a system of the form*

$$x_2(t) = A_1 x_2(t-h_1) + A_2 x_2(t-h_2) + A_3 x_2(t-h_3) + Bu(t),$$
$$y(t) = Cx_2(t)$$

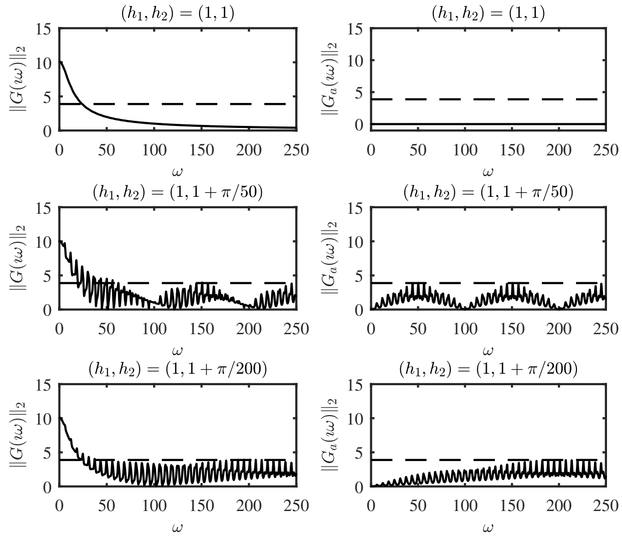

Fig. 1. Spectral norm of the transfer function (left) and of the asymptotic transfer function (right) of system in Example 2 as a function of $s=\imath\omega$, for three cases: $\vec{h}=(1,1)$ (top), $\vec{h}=(1,1+\pi/50)$ (middle) and $\vec{h}=(1,1+\pi/200)$ (bottom). The dashed line indicates the strong $\mathcal{H}_\infty$ norm of G_a.

with matrices

$$A_1=\begin{pmatrix}0&0&0\\ \frac{1}{4}&0&-\frac{1}{4}\\ 0&0&0\end{pmatrix},\ A_2=\begin{pmatrix}0&0&0\\0&0&0\\0&0&\frac{1}{8}\end{pmatrix},$$

$$A_3=\begin{pmatrix}0&0&0\\0&0&\frac{1}{32}\\0&0&0\end{pmatrix},B=\begin{pmatrix}1\\0\\1\end{pmatrix},\ C=\begin{pmatrix}0&1&0\end{pmatrix}.$$

The characteristic equation is given by

$$1-\frac{1}{8}e^{-sh_2}=0,$$

from which we conclude exponential stability for all delay values. The transfer function of the system is given by

$$G(s)=G_a(s)=-\frac{1}{4}\frac{\left(e^{-s(h_1+h_2)}-e^{-sh_3}\right)}{8-e^{-sh_2}}.$$

It is clear from this expression that $\|G\|_{\mathcal{H}_2}=0$ *if and only if* $h_3=h_1+h_2$, *while otherwise we have* $\|G\|_{\mathcal{H}_2}=+\infty$. *Thus,* $\|G\|_{\mathcal{H}_2}$ *has a discontinuity at each tuple* (h_1,h_2,h_3) *for which*

$$h_3=h_1+h_2. \tag{12}$$

These results are in agreement with Lemma 1, which states that for rationally independent delays the $\mathcal{H}_2$ *norm is infinite, following from* $CA_3B\neq 0$.

The possible discontinuity of the system norms brings us to the following robustified counter parts, which explicitly take into account infinitesimal delay perturbations. Note that the strong $\mathcal{H}_\infty$ norm was already introduced in [13].

Definition 3. *The strong* $\mathcal{H}_2$ *and strong* $\mathcal{H}_\infty$ *norm of* G *are defined as*

$$|||G(\cdot;\vec{h})|||_{\mathcal{H}_2}:=\lim_{\epsilon\to 0+}\sup\{\|G(\cdot;\vec{h}_\epsilon)\|_{\mathcal{H}_2}:\vec{h}_\epsilon\in\mathcal{B}(\vec{h},\epsilon)\},$$
$$|||G(\cdot;\vec{h})|||_{\mathcal{H}_\infty}:=\lim_{\epsilon\to 0+}\sup\{\|G(\cdot;\vec{h}_\epsilon)\|_{\mathcal{H}_\infty}:\vec{h}_\epsilon\in\mathcal{B}(\vec{h},\epsilon)\},$$

with $\mathcal{B}$ *given by (10).*

The strong $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of G_a are defined in a similar way. They satisfy the following properties.

Proposition 1. *Assume that system* (1) *is strongly stable. Then its asymptotic transfer function* G_a *satisfies*

$$|||G_a|||_{\mathcal{H}_\infty}=\max_{\vec{\theta}\in[0,\ 2\pi]^m}\left\|C\left(I-\sum_{j=1}^{m}A_je^{-\imath\theta_j}\right)^{-1}B\right\|_2 \tag{13}$$

and

$$|||G_a|||_{\mathcal{H}_2}=\begin{cases}0, & \text{if (8) is satisfied,}\\ +\infty, & \text{otherwise.}\end{cases} \tag{14}$$

Proof. Property (13) is proven in [13], so we focus on (14). If condition (8) is satisfied, then it is proven in [11] that G_a is identically zero for *any* values of the delays, hence, its strong $\mathcal{H}_2$ norm is zero. If (8) is not satisfied, then by Lemma 1 the $\mathcal{H}_2$ norm is equal to infinity for any set of rationally independent delays, and since the ball $\mathcal{B}(\vec{h},\epsilon)$ contains rationally independent delay values for any $\epsilon>0$, the strong $\mathcal{H}_2$ norm of G_a must be equal to infinity as well. □

It is important to point out that the strong norms of the asymptotic transfer function do not depend on the delay values. We can now state the corresponding results for transfer function G.

Proposition 2. *If system* (1) *is strongly stable, then its transfer function* G *satisfies*

$$|||G(\cdot;\vec{h})|||_{\mathcal{H}_\infty}=\max\left\{\|G(\cdot;\vec{h})\|_{\mathcal{H}_\infty},|||G_a|||_{\mathcal{H}_\infty}\right\} \tag{15}$$

and

$$|||G(\cdot;\vec{h})|||_{\mathcal{H}_2}=\begin{cases}\|G(\cdot;\vec{h})\|_{\mathcal{H}_2}<+\infty, & \text{if (8) is satisfied,}\\ +\infty, & \text{otherwise.}\end{cases} \tag{16}$$

Furthermore, function $\mathbb{R}^m_{\geq 0}\ni\vec{h}\mapsto|||G(\cdot;\vec{h})|||_{\mathcal{H}_\infty}$ *is continuous whenever* (1) *is strongly stable. Function* $\mathbb{R}^m_{\geq 0}\ni\vec{h}\mapsto|||G(\cdot;\vec{h})|||_{\mathcal{H}_2}$ *is continuous whenever* (1) *is strongly stable and the strong* $\mathcal{H}_2$ *norm is finite.*

Proof. Property (15) and the continuity property of the strong $\mathcal{H}_\infty$ norm have again been proven in [13], so we restrict ourselves in what follows to the $\mathcal{H}_2$ norm. First, from the proof of Proposition 3.3 in [13], which describes the asymptotic matching of $G(\imath\omega)$ and $G_a(\imath\omega)$ for $\omega\to\infty$, we derive that there exists an $\epsilon>0$ such that the following holds: for all $\gamma>0$, there is a $\hat{\omega}>0$ such that $\|G(\imath\omega;\vec{\tau}_\epsilon)-G_a(\imath\omega;\vec{\tau}_\epsilon)\|_2<\gamma$ for all $\vec{\tau}_\epsilon\in\mathcal{B}(\vec{\tau},\epsilon)$ and $\omega\geq\hat{\omega}$. Second, function $\vec{\tau}\mapsto G(\imath\omega;\vec{\tau})$, is continuous for any ω. Third, condition (8) implies that the impulse response of system (1) does not contain impulses [11]. Combining these results with (14) directly leads to (16) and it induces the assertion on continuity of the strong $\mathcal{H}_2$ norm. □

Example 4. *We revisit Example 2 and consider nominal delay values* $(h_1, h_2) = (1, 1)$*. By evaluating* (13) - (14) *we arrive at* $|||G_a|||_{\mathcal{H}_\infty} = 3.88$*, whose corresponding level sets are the dashed horizontal lines in Figure 1, as well as* $|||G_a|||_{\mathcal{H}_2} = +\infty$*. The maximum in the right-hand side of* (15) *is attained by the first term, hence the strong* $\mathcal{H}_\infty$ *norm is reached at a finite frequency, here* $\omega = 0$ *(see Figure 1). Obviously, we have* $|||G|||_{\mathcal{H}_2} = +\infty$*.*

The developments so far were driven by the sensitivity of the $\mathcal{H}_2$ norm with respect to infinitesimal delay perturbations. One may argue that an infinitesimal perturbation to the system matrices may also induce a change from a finite to an infinite $\mathcal{H}_2$ norm (e.g. the first inequality in (8) reads as $CB = 0$, which is fragile with respect to perturbations to B and C). However, the situation is different. It is easy to see from (13) that function

$$(A_1, \ldots, A_m, B, C) \mapsto |||G_a(\cdot; \vec{h}, A_1, \ldots, A_m, B, C)|||_{\mathcal{H}_\infty} \tag{17}$$

is continuous whenever the system is strongly stable. Thus, if the strong $\mathcal{H}_2$ norm is finite (implying G_a is identically zero) and small perturbations to the coefficient matrices are applied, then by the continuity of (17), $\|G_a(j\omega)\|_2$ must remain small at *all* frequencies ω. From an application perspective this could indicate that a perturbation on the system matrices, rendering the $\mathcal{H}_2$ norm infinite, might correspond to a physically irrelevant perturbation that creates a non-existing direct link from input to output (note that in modeling systems in DDAE form, physical perturbations typically appear as highly structured perturbation to (1)). On the contrary, in the example visualized in Figure 1 an infinitesimal perturbation of a delay changes the $\mathcal{H}_\infty$ norm of G_a from zero to approximately 3.88, i.e. significant feedthrough terms are actually present, but for the nominal delays a kind of "catastrophic cancellation" takes place, resulting in a finite $\mathcal{H}_2$ norm.

The following theorem forms the starting point for the analysis in the remainder of the paper.

Theorem 1. *Assume that system* (1) *is strongly stable. The following statements are equivalent.*

1) *The strong* $\mathcal{H}_2$ *norm of* (1) *is finite.*
2) *Condition*

$$C\left(I - \sum_{j=1}^{m} A_j e^{-\imath\theta_j}\right)^{-1} B = 0, \tag{18}$$

$\forall(\theta_1, \ldots, \theta_m) \in [0,\ 2\pi]^m$, *is satisfied.*

3) *Conditions*

$$CP_{k_1,\ldots,k_m}(A_1, \ldots, A_m)B = 0, \tag{19}$$

$\forall(k_1, \ldots, k_m) \in \mathbb{Z}^m_{\geq 0}$ s.t. $\sum_{j=1}^m k_j < mn$ *are satisfied, with multi-powers* $P_{k_1,\ldots,k_m}(A_1, \ldots, A_m)$ *defined by* (5)-(7).

Furthermore, if the strong $\mathcal{H}_2$ *norm of* (1) *is finite, it equals its* $\mathcal{H}_2$ *norm.*

Proof. We first prove the equivalence between the first two statements. If the strong $\mathcal{H}_2$ norm for nominal delays $\vec{h}$ is finite, then there exist rationally independent delays $\vec{t} = (t_1, \ldots, t_m)$ for which the $\mathcal{H}_2$ norm is still finite. By Lemma 1 we also have $G_a(\imath\omega; \vec{t}) = 0$ for all $\omega \in \mathbb{R}$, or

$$C\left(I - \sum_{j=1}^{m} A_j e^{-\imath\omega t_j}\right)^{-1} B = 0,\ \forall\omega \in \mathbb{R}. \tag{20}$$

In addition, it follows from Kronecker's theorem that the set $\{(e^{-\imath\omega t_1}, \ldots, e^{-\imath\omega t_m}) :\ \omega \in \mathbb{R}\}$ is dense in the set $\left\{\left(e^{-\imath\theta_1}, \ldots, e^{-\imath\theta_m}\right) :\ (\theta_1, \ldots, \theta_m) \in [0,\ 2\pi]^m\right\}$. The latter result and (20) imply that (18) is satisfied. For the reverse implication, we consider a set $\vec{t}$ of rationally independent delays. From (18) we get $G_a(\imath\omega; \vec{t}) = 0$ for all $\omega \in \mathbb{R}$. From this result and the strong stability assumption it follows that $G_a(s; \vec{t})$ is identically zero. From Lemma 1 we conclude that (8) is satisfied, and from (16) we conclude in turn that the strong $\mathcal{H}_2$ norm is finite.

The equivalence between the first and the third statement directly follows from Proposition 2. □

B. Computational complexity

We conclude the section by discussing the numerical tractability of the finiteness conditions in Theorem 1. Criterion (18) is a semi-infinite equality, in the sense that the equality has to be satisfied for all parameters $\vec{\theta}$ in an m-dimensional hypercube. Due to the nonlinear dependence on $\vec{\theta}$, a direct evaluation requires a parameter sweep and a gridding procedure. Using a regular grid with N points in each direction, the number of equalities to check for the discretized condition is equal to N^m.

In this context, a main contribution of [11] is that checking the semi-infinite equality can be reduced to checking a finite number of equalities (19), without introducing conservatism or an approximation. Let us now assess this number of equalities. The number of possible combinations of numbers $(k_1, \ldots, k_m) \in \mathbb{Z}^m_{\geq 0}$ such that $k_1 + \ldots + k_m = i$, with prescribed $i \in \mathbb{Z}_{\geq 0}$, can be interpreted as the number of multisets of cardinality i and underlying set $\{1, \ldots, m\}$, and it is given by the multiset coefficient

$$\left(\!\!\binom{m}{i}\!\!\right) = \binom{m+i-1}{i} = \frac{(m+i-1)!}{(m-1)!\ i!}.$$

Hence, the number of equalities in criterion (8) is equal to

$$\sum_{i=0}^{mn-1} \left(\!\!\binom{m}{i}\!\!\right). \tag{21}$$

It should also be noted that the number of distinct monomials in $P_{k_1,\ldots,k_m}$, for any given $(k_1, \ldots, k_m) \in \mathbb{Z}^m_{\geq 0}$, equals

$$\frac{(k_1 + \ldots + k_m)!}{k_1! \cdots k_m!}.$$

Table I illustrates how number (21) behaves with respect to the dimension n of the delay-difference equation (4) associated to (1), and the number of the delays m. Thus, even though we have a finite test for the finiteness of the

$\mathcal{H}_2$ norm, we notice that the number of equalities still exhibits a faster than exponential growth with respect to the number of delays m.

III. A strengthened finiteness condition

In this section we introduce and analyze a strengthened, sufficient, condition for the finiteness of the strong $\mathcal{H}_2$ norm of (1). We provide an algorithm to check this condition and analyze its computational complexity. We also show how the algorithm is instrumental in transforming system (1) to a neutral system while avoiding differentiation of the input or output, thereby enabling state-of-the-art methods for its $\mathcal{H}_2$ norm computation.

We consider a matrix multiplication of the form

$$A_{\sigma_1} \cdots A_{\sigma_k}$$

with $k \in \mathbb{Z}_{>0}$ and $\sigma_i \in \{1, \dots, m\}$, $i = 1, \dots, k$. We denote by $|\sigma|_i$ the number of occurrences of i in the finite sequence $\sigma := \{\sigma_1, \dots, \sigma_k\}$. In the matrix polynomial $P_{k_1,\dots,k_m}(A_1, \dots, A_m)$ every monomial is of order k_i in A_i, $i = 1, \dots, m$, thus, we can express

$$P_{k_1,\dots,k_m}(A_1, \dots, A_m) = \sum_{\sigma \in \Omega_{k_1,\dots,k_m}} A_{\sigma_1} \cdots A_{\sigma_{k_1+\dots+k_m}}$$

for $(k_1, \dots, k_m) \neq (0, \dots, 0)$, where

$$\Omega_{k_1,\dots,k_m} := \{\sigma \in \{1, \dots, m\}^{k_1+\dots+k_m} : \\ |\sigma|_i = k_i, \quad i = 1, \dots, m\}.$$

We observe that the strengthened conditions

$$\begin{aligned} CB &= 0, \\ CA_{\sigma_1} \cdots A_{\sigma_k} B &= 0, \forall k \in \mathbb{Z}_{>0}, \ \forall \sigma_i \in \{1, \dots, m\}, \end{aligned} \tag{22}$$

$i = 1, \dots, k$, imply that $CP_{k_1,\dots,k_m}(A_1, \dots, A_m)B = 0$ for any m-tuple $(k_1, \dots, k_m)$, and, hence, that finiteness criterion (19) of Theorem 1 is satisfied. Section IV will further elaborate on the connections between criteria (22) and (19).

A. An efficient computational test

Let us first define $\chi_0 := \text{Im } B$ and, for $k \geq 1$,

$$\chi_k := \text{span } \{B\} \cup \{A_{\sigma_1} \cdots A_{\sigma_{k'}} B : \\ 1 \leq k' \leq k, \ \sigma_i \in \{1, \dots, m\}, \ i = 1, \dots, k'\}, \tag{23}$$

where for the case of multiple inputs ($n_i \geq 1$) we take the convention that the span of a set of block vectors is simply the span of the columns of all the blocks combined. The finiteness condition (22) can now be rephrased as $\chi_k \subseteq \text{Ker } C$ for all $k \geq 0$, which implies

$$\dim \chi_k \leq n - \text{rank}(C). \tag{24}$$

At the same time we can express

$$\chi_{k+1} = \chi_k \cup \{A_j \chi_k : j = 1, \dots, m\} \tag{25}$$

and it holds that χ_k is a subspace of χ_{k+1}.

If (22) holds and we construct $\chi_0, \ \chi_1, \dots$, then we have $\chi_i \subseteq \text{Ker } C$, $i = 0, 1, \dots$, and because of the bound (24) and the inclusion $\chi_0 \subseteq \chi_1, \subseteq \chi_2, \dots$, we must have $\chi_{\hat{\imath}} = \chi_{\hat{\imath}-1}$ for some $\hat{\imath} \leq n - \text{rank } C$. The latter implies that $\chi_{\hat{\imath}-1}$ is an $(A_1, \dots, A_m)$-invariant subspace and consequently $\chi_k = \chi_{\hat{\imath}-1}$ for $k \geq \hat{\imath} - 1$. Note that the existence of such an invariant subspace in the kernel of C is equivalent to (22). On the contrary, if (22) does not hold and we construct again $\chi_0, \chi_1, \dots$ then condition $\chi_k \subseteq \text{Ker } C$ must be broken for some $k \leq n - \text{rank}(C)$. The argument is by contradiction: if the condition would only be broken for some bigger value of k or not be broken at all, then we must have $\chi_i = \chi_{i-1} \subseteq \text{Ker } C$ for some $i < k$, which would imply that (22) is satisfied. Algorithm 1 is based on these ideas.

Algorithm 1 Finiteness condition

Input: Matrices $(A_1, \dots, A_m, B, C)$, tolerance η
Output: Certificate cond for the satisfaction of condition (22)

Let $B = (U_0 \ \tilde{U}_0) \begin{pmatrix} \Sigma_{0,1} & \\ & \Sigma_{0,2} \end{pmatrix} \begin{pmatrix} V_0^T \\ \tilde{V}_0^T \end{pmatrix}$ be a reduced singular value decomposition
  with $\eta I < \Sigma_{0,1}$ and $\Sigma_{0,2} \leq \eta I$
Set $\mathcal{U}_0 = U_0$
for $i = 1$ to n **do**
  if $\|CU_{i-1}\|_2 > \eta$ **then** cond=FALSE **break**
  end if
  Set $Z = (A_1 U_{i-1} \ A_2 U_{i-1} \ \cdots \ A_m U_{i-1})$
  Set $Z = \left(I - \mathcal{U}_{i-1} \mathcal{U}_{i-1}^T\right) Z$
  if $\|Z\|_2 \leq \eta$ **then** cond=TRUE **break**
  end if
  Let $Z = (U_i \ \tilde{U}_i) \begin{pmatrix} \Sigma_{i,1} & \\ & \Sigma_{i,2} \end{pmatrix} \begin{pmatrix} V_i^T \\ \tilde{V}_i^T \end{pmatrix}$ be a reduced singular value decomposition
    with $\eta I < \Sigma_{i,1}$ and $\Sigma_{i,2} \leq \eta I$
  Set $\mathcal{U}_i = (\mathcal{U}_{i-1} \ U_i)$
end for

Let us analyze the algorithm at this point in the (hypothetical) case where all computations can be done in exact arithmetic, which allows us to set the tolerance η to zero. The following proposition relates matrices $\mathcal{U}_i$ with subspaces χ_i.

Proposition 3. *If all operations in Algorithm 1 are done in exact arithmetic and $\eta = 0$, then we have*

$$\text{Im } \mathcal{U}_i = \chi_i, \ \forall i \geq 0.$$

Proof. It is easy to see that it holds for $i = 0$ and $i = 1$. Furthermore we have by induction from $i - 2$ and $i - 1$ to i, with $i \geq 2$,

$$\begin{aligned} \chi_i &= \chi_{i-1} \cup \{A_j \chi_{i-1} : j = 1, \dots, m\} \\ &= \chi_{i-1} \cup \{\text{Im } A_j \mathcal{U}_{i-1} : j = 1, \dots, m\} \\ &= \chi_{i-1} \cup \{\text{Im } A_j [\mathcal{U}_{i-2} \ U_{i-1}] : j = 1, \dots, m\} \\ &= \chi_{i-1} \cup \{A_j \chi_{i-2} \cup \text{Im } A_j U_{i-1} : j = 1, \dots, m\} \\ &= \chi_{i-1} \cup \{\text{Im } A_j U_{i-1} : j = 1, \dots, m\} \\ &= \text{Im } \mathcal{U}_{i-1} \cup \{\text{Im } Z\} \\ &= \text{Im } [\mathcal{U}_{i-1} \ U_i] = \text{Im } \mathcal{U}_i, \end{aligned}$$

which completes the proof. □

	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$	$m=8$	$m=9$
$n=4$	36	364	3876	42504	475020	5379616	61523748	708930508
$n=5$	55	680	8855	118755	1623160	22481940	314457495	4431613550
$n=6$	78	1140	17550	278256	4496388	73629072	1217566350	20286591270
$n=7$	105	1771	31465	575757	10737573	202927725	3872894697	74473879480

TABLE I
NUMBER OF EQUALITIES (21) TO BE VERIFIED FOR ASSESSING THE FINITENESS OF THE $\mathcal{H}_2$ NORM OF (1) BY CRITERION (19), FOR DIFFERENT VALUES OF n AND m.

In steps 7-10 we test whether the space spanned by the columns of $\mathcal{U}_{i-1}$ is $(A_1,\ldots,A_m)$-invariant. Because of property (25) it is sufficient, and more efficient, to apply $A_1,\ldots,A_m$ only to the vectors added in the previous iteration, which is done in line 7. Then we have to check whether or not $\text{Im } Z \subseteq \text{Im } \mathcal{U}_{i-1}$. For this we replace Z in line 8 by its projection on the orthogonal complement of $\mathcal{U}_{i-1}$. If the space $\mathcal{U}_{i-1}$is invariant then we necessarily have $Z=0$. If it is not, then we only add the directions to $\mathcal{U}_{i-1}$, which are not yet present in $\text{Im } \mathcal{U}_{i-1}$, when we expand $\mathcal{U}_{i-1}$ to $\mathcal{U}_i$ in line 13. With the singular value decompositions in line 1 and line 11 we remove redundant information: the columns of U_0 in line 1 form a minimal orthogonal basis of the column space of B, while in line 13 the columns of U_i form a minimal orthogonal basis for the column space of Z.

If there is no break in iterations $1,\ldots,i$, then we have

$$\text{rank } \mathcal{U}_i > \text{rank } \mathcal{U}_{i-1} > \cdots > \text{rank } \mathcal{U}_0 = \text{rank } B,$$

while $\text{rank } \mathcal{U}_{i-1} \le n - \text{rank } C$. This means that the number of iterations before termination is bounded by $n - \text{rank } B - \text{rank}(C) + 1$. Let us now assess the two termination conditions.

- If the algorithm terminates in line 9 for iteration $\hat{\imath}$, then we recall that $\chi_{\hat{\imath}-1}$ is an $(A_1,\ldots,A_m)$-invariant space. At the same time, as the condition in line 5 is not satisfied for $i=1,\ldots,\hat{\imath}$, we have $\chi_{\hat{\imath}-1} \subseteq \ker C$. These two properties correspond to condition (22).
- If the algorithm terminates in line 5 for iteration $\hat{\imath}$, then there exists a vector in $\chi_{i-1} = \text{Im } \mathcal{U}_{i-1}$ which does not belong to the kernel of C, violating condition (22).

Finally, if the algorithm terminates in iteration $\hat{\imath}$ with cond=TRUE or in iteration $\hat{\imath}+1$ with cond=FALSE, then we must have $k_0 + k_1 + \ldots + k_{\hat{\imath}-1} \le n - \text{rank } C$ with $k_i = \text{rank } U_i$. Consequently, the number of matrix-vector products involving $A_1,\ldots,A_m$ is bounded by

$$m(n - \text{rank } C). \tag{26}$$

When computing in finite precision arithmetic and accounting errors on the system data, the check $CU_{i-1}=0$ in line 5 is done up to some prescribed tolerance $\eta > 0$. It should also be taken into account that the rank of a matrix is fragile, in the sense of being sensitive to infinitesimal perturbations to it. Therefore, tolerance η is also used to determine numerically the rank of matrices B and Z in lines 1 and 11, and their corresponding truncations to U_0 and U_i. It is important to point out that the bound on the number of matrix-vector products (26) remains valid (an over-estimation of the rank would affect numbers k_i but not the bound, (26) itself). It scales linearly with both n and m, which is a major improvement with respect to (21), which merely concerns the number of (complicated) equalities (8).

B. Simultaneous block triangularization and computation of the $\mathcal{H}_2$ norm

We provide a Lyapunov matrix based formula for computing the $\mathcal{H}_2$ norm (3). Inspired by the one delay case [10], we introduce a neutral type system whose transfer matrix is equivalent to transfer matrix G of system (2) (in the sense of forthcoming Theorem 2), whenever the strengthened finiteness condition (22) holds. This allows us to directly apply previous results on the computation of the $\mathcal{H}_2$ norm for neutral systems [14].

The following key lemma shows that a necessary and sufficient condition for the existence of a transformation that *simultaneously* brings matrices $A_1,\ldots,A_m$, B and C to their corresponding controllable/observable canonical form is given by condition (22).

Lemma 2. *Let system* (1) *with $r<n$ be strongly stable, and $B \neq 0$ and $C \neq 0$. There exists a matrix $T_c \in \mathbb{R}^{n\times n}$ such that*

$$\begin{gathered} T_c^{-1} A_j T_c = \left(\begin{array}{c|c} A_{j1} & 0 \\ \hline A_{j2} & A_{j3} \end{array}\right),\ j=1,\ldots,m, \\ T_c^{-1} B = \begin{pmatrix} 0 \\ B_c \end{pmatrix},\ CT_c = \left(\begin{array}{c|c} C_u & 0 \end{array}\right) \end{gathered} \tag{27}$$

if and only if condition (22) *holds. Here, the location of separation is uniform over the matrices, $B_c \in \mathbb{R}^{r_2 \times n_i}$, and $C_u \in \mathbb{R}^{n_o \times r_1}$ with $r_1 + r_2 = n$.*

Proof. The forward implication is trivial. Indeed, it follows from the structure of matrices in (27) that condition (22) holds. Therefore we focus on the reverse implication in what follows.

Let χ_k be defined as in (23). Condition (22) implies that $\chi_k \subseteq \text{Ker } C$ for all $k \in \mathbb{Z}_{\ge 0}$. Recall that if $\chi_k = \chi_{k+1}$, then $\chi_k = \chi_{k'}$ for all $k' \ge k$. Thus, there exists $\bar{k} \le n - \text{rank } C$ such that $\chi_{\bar{k}}$ is an $(A_1,\ldots,A_m)$-invariant subspace of $\text{Ker } C$. Define matrix

$$T_c := \begin{pmatrix} T_{c1} & T_{c2} \end{pmatrix},$$

such that the columns of T_{c2} form an orthogonal basis of $\chi_{\bar{k}}$, and the columns of T_{c1} an orthogonal basis of its orthogonal complement $\chi_{\bar{k}}^{\perp}$. By the $(A_1, \ldots, A_m)$- invariance of $\chi_{\bar{k}}$ we have

$$T_c^{-1} A_j T_c = T_c^T A_j T_c = \begin{pmatrix} A_{j1} & 0 \\ A_{j2} & A_{j3} \end{pmatrix},\ j = 1, \ldots, m,$$

and we also have, since Im $B \subseteq \chi_{\bar{k}} \subseteq$ Ker C,

$$T_c^{-1} B = \begin{pmatrix} 0 \\ B_c \end{pmatrix}, \text{ and } CT_c = \begin{pmatrix} C_u & 0 \end{pmatrix}.$$

which completes the proof. □

Under assumption that condition (22) holds and with the rank of E satisfying $r \geq 1$, we construct the following neutral type system:

$$\begin{aligned} \frac{d}{dt}\left(\sum_{j=0}^{m} \mathcal{D}_j z(t-h_j)\right) &= \sum_{j=0}^{m} \mathcal{A}_j z(t-h_j) + \mathcal{B}u(t) \\ y(t) &= \mathcal{C}z(t), \end{aligned} \tag{28}$$

where $h_0 = 0$, $z^T(t) = \begin{pmatrix} x_1^T(t) & \xi^T(t) \end{pmatrix}$, $\xi(t) = T^{-1} x_2(t)$,

$$\mathcal{D}_0 = \begin{pmatrix} E & A_0^{(12)} T Q_2 \\ Q_1 T^{-1} A_0^{(21)} & I \end{pmatrix},$$

$$\mathcal{D}_j = \begin{pmatrix} 0 & A_j^{(12)} T Q_2 \\ Q_1 T^{-1} A_j^{(21)} & Q_1 T^{-1} A_j T + T^{-1} A_j T Q_2 \end{pmatrix},$$

$$\mathcal{A}_0 = \begin{pmatrix} A_0^{(11)} & A_0^{(12)} T \\ T^{-1} A_0^{(21)} & -I \end{pmatrix},\ \mathcal{A}_j = \begin{pmatrix} A_j^{(11)} & A_j^{(12)} T \\ T^{-1} A_j^{(21)} & T^{-1} A_j T \end{pmatrix},$$

where $j = 1, \ldots, m$,

$$\mathcal{B}^T = \begin{pmatrix} B_1^T & 0 & B_c^T \end{pmatrix},\ \mathcal{C} = \begin{pmatrix} C_1 & C_u & 0 \end{pmatrix},$$

with

$$Q_1 = -\begin{pmatrix} I_{r_1} & 0 \\ 0 & 0 \end{pmatrix}, \text{ and } Q_2 = -\begin{pmatrix} 0 & 0 \\ 0 & I_{r_2} \end{pmatrix},$$

and r_1 and r_2 as in Lemma 2. Matrix $T = T_c$, with T_c inducing (27) and specified in the proof of Lemma 2, if $B \neq 0$ and $C \neq 0$, otherwise T is considered as an arbitrary nonsingular matrix. If $B = 0$, then $r_2 = 0$ and $Q_2 = 0$, and if $C = 0$, then $r_1 = 0$ and $Q_1 = 0$.

We observe that, since $Q_2 Q_1 = 0$, we have $A_0^{(12)} T Q_2 Q_1 T^{-1} A_0^{(21)} = 0$, which implies that

$$\det \mathcal{D}_0 = \det E \neq 0,$$

and system (28) is regular. For the special case $r = 0$, where (1) reduces to a delay-difference equation, all blocks in the matrices of (28) corresponding to partial state x_1 should be removed.

The transfer matrix of the neutral type system is given by

$$\mathcal{G}(s) := \mathcal{C}\mathcal{H}^{-1}(s)\mathcal{B},$$

where

$$\mathcal{H}(s) = s\sum_{j=0}^{m} \mathcal{D}_j e^{-sh_j} - \sum_{j=0}^{m} \mathcal{A}_j e^{-sh_j},$$

and its spectrum by

$$\Lambda_n := \{s \in \mathbb{C} : \det \mathcal{H}(s) = 0\}.$$

In the following theorem, we correlate the transfer matrices and spectra of system (28) and system (1).

Theorem 2. *If system* (1) *is strongly stable and condition* (22) *is satisfied, then its spectrum* $\Lambda := \{s \in \mathbb{C} :\ \det(s\hat{E} - \hat{A}_0 - \sum_{j=1}^{m} \hat{A}_j e^{-sh_j}) = 0\}$ *satisfies* $\Lambda = \Lambda_n \setminus \{-1\}$ *or* $\Lambda = \Lambda_n$. *Furthermore, we have*

$$G(s) = \mathcal{G}(s),\ s \in \mathbb{C} \setminus \Lambda_n.$$

Proof. We restrict ourselves to a proof for $r \geq 1$, from which the proof for $r = 0$ can be deduced in a straightforward way. We first note that the transfer function G of (1), via the transformation to (2), can equivalently be expressed as

$$G(s) := \begin{pmatrix} C_1 & C \end{pmatrix} H^{-1}(s) \begin{pmatrix} B_1 \\ B \end{pmatrix}$$

where

$$H(s) := \begin{pmatrix} sE - \sum_{j=0}^{m} A_j^{(11)} e^{-sh_j} & -\sum_{j=0}^{m} A_j^{(12)} e^{-sh_j} \\ -\sum_{j=0}^{m} A_j^{(21)} e^{-sh_j} & I - \sum_{j=1}^{m} A_j e^{-sh_j} \end{pmatrix}.$$

Now we can express

$$\sum_{j=0}^{m} \mathcal{D}_j e^{-sh_j} = \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} + \widehat{Q}_1 \widehat{T}^{-1} H(s) \widehat{T} + \widehat{T}^{-1} H(s) \widehat{T} \widehat{Q}_2,$$

and

$$-\sum_{j=0}^{m} \mathcal{A}_j e^{-sh_j} = \widehat{T}^{-1} H(s) \widehat{T} - \begin{pmatrix} sE & 0 \\ 0 & 0 \end{pmatrix},$$

where

$$\widehat{T} = \begin{pmatrix} I_r & 0 \\ 0 & T \end{pmatrix},\ \widehat{Q}_1 = \begin{pmatrix} 0_{r,r} & 0 \\ 0 & -Q_1 \end{pmatrix},\ \widehat{Q}_2 = \begin{pmatrix} 0_{r,r} & 0 \\ 0 & -Q_2 \end{pmatrix}.$$

Hence, we have

$$\begin{aligned} \mathcal{H}(s) &= s\widehat{Q}_1 \widehat{T}^{-1} H(s) \widehat{T} + s\widehat{T}^{-1} H(s) \widehat{T} \widehat{Q}_2 + \widehat{T}^{-1} H(s) \widehat{T} \\ &= \left(I + s\widehat{Q}_1\right) \widehat{T}^{-1} H(s) \widehat{T} \left(I + s\widehat{Q}_2\right), \end{aligned} \tag{29}$$

where the last equality follows from the fact that, by Lemma 2,

$$\begin{aligned} &\widehat{Q}_1 \widehat{T}^{-1} H(s) \widehat{T} \widehat{Q}_2 \\ &\quad = \begin{pmatrix} 0 & 0 \\ 0 & Q_1 \left(I - \sum_{j=1}^{m} T^{-1} A_j e^{-sh_j} T\right) Q_2 \end{pmatrix} = 0. \end{aligned}$$

From equation (29) we get

$$\det \mathcal{H}(s) = (s+1)^{r_1 + r_2} \det H(s),$$

implying that

$$\Lambda_n = \Lambda \cup \{-1\}.$$

Thus,

$$\mathcal{G}(s) = \mathcal{C}\left(I - s\widehat{Q}_2\right)^{-1} \widehat{T}^{-1} H^{-1}(s) \widehat{T} \left(I - s\widehat{Q}_1\right)^{-1} \mathcal{B},$$

where $s \in \mathbb{C} \setminus \Lambda_n$. Finally, it follows from

$$I + s\widehat{Q}_2 = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & (1+s)I \end{pmatrix},$$

$$I + s\widehat{Q}_1 = \begin{pmatrix} I & 0 & 0 \\ 0 & (1+s)I & 0 \\ 0 & 0 & I \end{pmatrix},$$

and Lemma 2 that

$$\mathcal{C} = \begin{pmatrix} C_1 & C \end{pmatrix} \widehat{T} \left(I + s\widehat{Q}_2 \right)$$

and

$$\mathcal{B} = \left(I + s\widehat{Q}_1 \right) \widehat{T}^{-1} \begin{pmatrix} B_1 \\ B \end{pmatrix}.$$

Therefore,

$$\mathcal{G}(s) = \begin{pmatrix} C_1 & C \end{pmatrix} H^{-1}(s) \begin{pmatrix} B_1 \\ B \end{pmatrix} = G(s),\ s \in \mathbb{C} \setminus \Lambda_n.$$

□

The procedure for constructing neutral type system (28) also has an interpretation in the time domain. It extends the one used in [10] for systems with one delay. We summarize it in three steps as follows:

1) Set $x_2(t) = T\xi(t)$ and apply the operator $I - Q_1 \dfrac{d}{dt}$ to the delay-difference part of system (2). From this, one obtains a neutral type system with matrices $(\hat{\mathcal{D}}_j, \hat{\mathcal{A}}_j, \hat{\mathcal{B}}, \hat{\mathcal{C}})$, $j = 1, \dots, m$.
2) Consider the dual of the neutral system obtained in Step 1, i.e. a neutral system with matrices $(\hat{\mathcal{D}}_j^T, \hat{\mathcal{A}}_j^T, \hat{\mathcal{C}}^T, \hat{\mathcal{B}}^T)$, $j = 1, \dots, m$.
3) Apply now the operator $I - Q_2 \dfrac{d}{dt}$ to the corresponding delay-difference part of the dual system. The dual of the obtained system is system (28).

It is well known that DDAEs can be transformed to neutral type equations by applying differentiation and shifting operations in the difference part (see, e.g. [3], [6], [7]). As already noticed, direct differentiation of $x_2(t)$ in system (2) results in a system with derivative in the input. With the above procedure, we show that it is possible to construct a neutral type system without derivatives of the input signal.

This result is particularly relevant for providing a formula for the computation of the $\mathcal{H}_2$ norm, as shown in the next corollary, which directly follows from Theorem 2 and Theorem 1 in [14]. Note that if the strong $\mathcal{H}_2$ norm is finite, it equals the $\mathcal{H}_2$ norm for the nominal delay values; see Theorem 1.

Corollary 1. *If system (1) is strongly stable and condition (22) is satisfied, then we have*

$$|||G|||_{\mathcal{H}_2} = \|G\|_{\mathcal{H}_2} = \|\mathcal{G}\|_{\mathcal{H}_2} = \sqrt{\mathrm{Tr}\left(\mathcal{B}^T U(0) \mathcal{B}\right)},$$

where $U : [-h_m, h_m] \mapsto \mathbb{R}^{n \times n}$ is the so-called delay Lyapunov matrix associated with $\mathcal{C}^T\mathcal{C}$ of neutral type system (28).

The computation of the delay Lyapunov matrix U for systems of the form (28) has been widely studied in the literature, [14], [15]. Thus, Corollary 1 provides an effective basis for computing the $\mathcal{H}_2$ norm of system (2) whenever condition (22) holds.

IV. Connections between the finiteness conditions and an open problem

In this section we establish relations between the necessary and sufficient condition (19) for the finiteness of the strong $\mathcal{H}_2$ norm, and sufficient condition (22), which is more tractable from a computational point of view and is at the basis of the transformation of system (1) to neutral system (28). The following counterexample shows that both conditions are not equivalent, and in turn that condition (22) is, in general, only a sufficient condition for the finiteness of the strong $\mathcal{H}_2$ norm.

Example 5. *Consider the matrices*

$$A_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix},\ C^T = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

One can verify by direct calculation that

$$CP_{k_1,k_2}(A_1, A_2)B = 0$$

for every (k_1, k_2) such that $k_1 + k_2 < 8$, but

$$CA_1A_2B = -1, \quad CA_2A_1B = 1,$$

showing that condition (19) does not imply in general condition (22).

Remark 1. *From Lemma 2 and Example 5 we conclude that condition (19) does not imply in general the existence of a transformation resulting in matrices of the form (27).*

The previous example, with a very special structure of the coefficient matrices, leads us to the question whether there exist classes of systems for which the equivalence of both finiteness conditions holds. The next proposition provides an answer.

Proposition 4. *Criterion (19) of Theorem 1 is equivalent to criterion (22) if any of the following conditions hold:*

1) *Matrices $A_1, \dots, A_m$, mutually commute.*
2) *Matrices $A_1, \dots, A_m$, B and C belong to $\mathbb{R}^{n\times n}_{\geq 0}$, $\mathbb{R}^{n\times n_i}_{\geq 0}$ and $\mathbb{R}^{n_o\times n}_{\geq 0}$, respectively.*
3) *The number n satisfies $n = \mathrm{rank}\, B + \mathrm{rank}\, C$.*
4) *The number n satisfies $n \leq 3$.*

Proof. It is obvious that if condition (22) holds, then condition (19) does so. Hence, we focus on the other direction in each of the items.

Statement 1. If matrices A_j mutually commute, observe that for any $(k_1, \dots, k_m) \in \mathbb{Z}^m_{\geq 0} \setminus (0, \dots, 0)$, there exist an integer $r > 0$ and a sequence $\{\sigma_1, \sigma_2, \ldots, \sigma_{k_1+\ldots+k_m}\}$ such that

$$P_{k_1,\ldots,k_m}(A_1,\ldots,A_m) = rA_{\sigma_1}\cdots A_{\sigma_{k_1+\ldots+k_m}}.$$

Thus, if equality (19) is satisfied, then (22) is also satisfied.

Statement 2. We observe that, since the elements of all the matrices are nonnegative, the expression

$$CP_{k_1,\ldots,k_m}(A_1,\ldots,A_m)B = \sum_{\sigma\in\Omega_{k_1,\ldots,k_m}} CA_{\sigma_1}\cdots A_{\sigma_k}B$$

is made of only nonnegative terms; hence it is zero if and only if all the terms are zero, which proves the assertion.

Statement 3. Under the rank assumption, equality $CB = 0$ implies Im B = ker C. As a consequence, equations $CA_jB = 0$ imply that Im $A_jB \subseteq$ Im B, $j = 1,\ldots,m$, that is, Im B is an $(A_1,\ldots,A_m)$-invariant subspace equal to ker C. The latter implies that (22) is satisfied.

Statement 4. For the cases where $n = 1$ and $n = 2$, as well as the cases where $B = 0$ or $C = 0$, the proof is trivial. Therefore, we consider $n = 3$ and nonzero B and C in what follows. Because of Statement 3. we can further restrict ourselves to the single input, single output case, $n_i = n_o = 1$.

From $CB = 0$ we can assume without loss of generality that

$$C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

as this structure can be induced by a state transformation. Due to condition $CA_jB = 0$, $j = 1,\ldots,m$, matrices A_j must have the structure

$$A_j = \begin{pmatrix} a_j & b_j & 0 \\ c_j & d_j & e_j \\ f_j & g_j & h_j \end{pmatrix}, \quad j = 1,\ldots,m.$$

Condition $CA_i^2B = 0$ can be rephrased as

$$b_ie_i = 0, \; i = 1,\ldots,m, \tag{30}$$

while condition $C(A_iA_j + A_jA_i)B = 0$ leads to

$$b_ie_j + b_je_i = 0, \quad \forall i,j \in \{1,\ldots,m\}. \tag{31}$$

Considering (30) for $i = 1$ and $i = 2$, along with (31) for $i = 1, j = 2$, lead us to either $b_1 = b_2 = 0$ or $e_1 = e_2 = 0$. For $m > 2$ we distinguish between three cases. If $e_1 \neq 0$ then from (31) with $i = 1$ and $j > 2$, along with $b_je_j = 0$ we get $b_j = 0$ for $j > 2$. Similarly, if $b_1 \neq 0$ we get $e_j = 0$ for $j > 2$. Finally, if $b_1 = c_1 = 0$, we go through the same process, starting from matrix A_2 and by considering the cases $e_2 \neq 0$, $d_2 \neq 0$ and $d_2 = e_2 = 0$. Hence, with a recursive argument we always arrive at

$$b_i = 0, \; i = 1,\ldots,m, \text{ or } e_i = 0, \; i = 1,\ldots,m.$$

This property implies that the matrices A_j, $j = 1,\ldots,m$, B and C have the block structure as in (27), which implies that (22) is satisfied. □

As shown in Subsection III-B, the construction of a neutral system that is equivalent to system (1) in the sense stated in Theorem 2 is possible whenever (22) is fulfilled. However, as shown by Example 5 conditions (19) and (22) are not equivalent in general. In addition to getting more insight in the relation between the two conditions, this leads us to the following open question:

If (19) *is satisfied but* (22) *is not, is it still possible to find a transformation that allows constructing an equivalent neutral type system without differentiation of input and output?*

V. Illustrative example

We consider a strongly exponentially stable system (1) already in the form (2) with two (nominal) delays $h_1 = 1$ and $h_2 = 2$, $E = 1$, $A_0^{(11)} = -2$ and matrices $A_0^{(12)} = A_0^{(21)^T} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$,

$$\left(\begin{array}{c|c} A_1^{(11)} & A_1^{(12)} \\ \hline A_1^{(21)} & A_1 \end{array}\right) = \left(\begin{array}{c|ccc} 0 & -1 & -2 & 1 \\ \hline 0 & 0.1 & -0.5 & 0.4 \\ 0 & 0 & 0.3 & 0 \\ 0.75 & 0 & -0.5 & 0.5 \end{array}\right),$$

$$\left(\begin{array}{c|c} A_2^{(11)} & A_2^{(12)} \\ \hline A_2^{(21)} & A_2 \end{array}\right) = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline -0.5 & -0.05 & 0.15 & 0.05 \\ -0.25 & 0 & -0.05 & 0 \\ -0.75 & 0 & 0 & -0.1 \end{array}\right),$$

and

$$\left(\begin{array}{c} B_1 \\ \hline B \end{array}\right) = \left(\begin{array}{cc} 1 & 1 \\ \hline 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{array}\right), \; \left(\begin{array}{c|c} C_1 & C \end{array}\right) = \left(\begin{array}{c|ccc} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right).$$

We observe that rank(C)+rank$(B) = 3$, hence it follows from either Statement 3 or Statement 4 of Proposition 4 that condition (19) and (22) are equivalent for this system. One can check either that (19) holds for all (k_1, k_2) such that $k_1 + k_2 < 6$ by direct computation, or that (22) is satisfied by using Algorithm 1, hence the strong $\mathcal{H}_2$ norm is finite. Notice however that condition (19) requires to check 21 equations of the form $CP_{k_1,k_2}(A_1, A_2)B = 0$, whereas Algorithm 1 stops at iteration $i = 1$ at line 9 since Im B is an (A_1, A_2)-invariant space.

Figure 2 illustrates the spectral norm of transfer matrices G and G_a of the system as a function of $s = \imath\omega$ for both nominal and perturbed delays $(h_1, h_2) = (1, 2 + 0.05\sqrt{2})$. As expected from the previous discussion, $\|G_a(\imath\omega)\|_2$ is zero in both cases for all $\omega \in \mathbb{R}_{\geq 0}$, showing the finiteness of the strong $\mathcal{H}_2$ norm.

From Lemma 2 it follows that there exists a matrix T_c that allows us to bring matrices (A_1, A_2, B, C) to controllable/observable canonical form. Thus, by considering

$$T = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

and

$$Q_1 = -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Q_2 = -\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

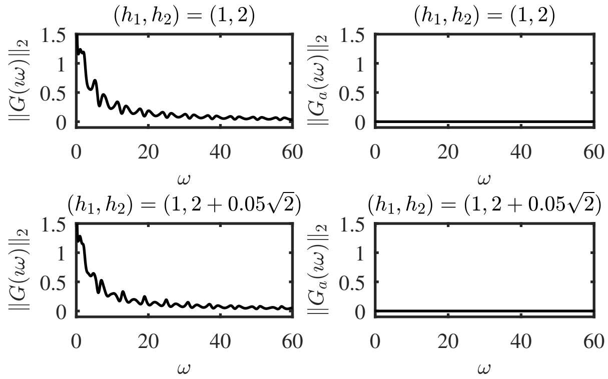

Fig. 2. Spectral norm of the transfer function (left) and of the asymptotic transfer function (right) of the system of Section V for two cases: $(h_1, h_2) = (1, 2)$ (top), and $(h_1, h_2) = (1, 2 + 0.05\sqrt{2})$ (bottom).

we can construct a neutral type system of the form (28) with matrices $\mathcal{D}_0 = I$,

$$\mathcal{D}_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -0.3 & 0 & 0 \\ 0 & 0 & -0.5 & 0 \\ 0 & 0 & 0 & -0.1 \end{pmatrix},$$

$$\mathcal{D}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0.25 & 0.05 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \\ 0 & 0 & -0.1 & 0.05 \end{pmatrix},$$

$$\mathcal{A}_0 = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\mathcal{A}_1 = \begin{pmatrix} 0 & -2 & 0 & -1 \\ 0 & 0.3 & 0 & 0 \\ 0.75 & -0.5 & 0.5 & 0 \\ -0.75 & 0 & 0 & 0.1 \end{pmatrix},$$

$$\mathcal{A}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -0.25 & -0.05 & 0 & 0 \\ -0.75 & 0 & -0.1 & 0 \\ 0.25 & 0.15 & 0.1 & -0.05 \end{pmatrix},$$

and

$$\mathcal{B} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \\ 1 & -1 \end{pmatrix}, \mathcal{C} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

According to Corollary 1, we have that for the nominal case $(h_1, h_2) = (1, 2)$

$$|||G|||_{\mathcal{H}_2} = \|\mathcal{G}\|_{\mathcal{H}_2} \approx 3.01,$$

and for the perturbed delay case $(h_1, h_2) = (1, 2+0.05\sqrt{2})$

$$|||G|||_{\mathcal{H}_2} = \|\mathcal{G}\|_{\mathcal{H}_2} \approx 3.02,$$

where $\|\mathcal{G}\|_{\mathcal{H}_2}$ is computed via the delay Lyapunov matrix based formula introduced in [14].

VI. Conclusions

We introduced the strong $\mathcal{H}_2$ norm for systems of the form (1), which is a robustified measure taking into account infinitesimal perturbations of the delays. We provided necessary and sufficient conditions for its finiteness, presented in Theorem 1, and sufficient conditions given by (22). We showed that the strengthened condition (22) is numerically more tractable by introducing an efficient algorithm in order to verify it. Some classes of systems for which conditions (22) and (19) are equivalent were also presented.

We also introduced a methodology for constructing a neutral type system whose transfer matrix is the same as the transfer matrix of system (1) whenever the sufficient condition (22) is satisfied. A notable characteristic of constructed neutral system (28) is that differentiation of the input was avoided. The latter was particularly useful for providing a formula based on the delay Lyapunov matrix for the $\mathcal{H}_2$ norm computation. Future research work includes addressing the open problem stated in Section IV, and exploring other applications of the regularization procedure to a neutral system.

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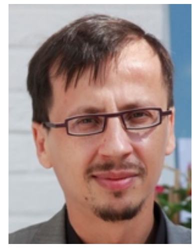

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