A minimal partial realization algorithm for MIMO systems VI : The time moment and mixed problems revisited

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ABSTRACT

We show how the results of previous reports about recursive computation of numerator and denominator by unimodular transformations, continued fraction expansion and the parametrization of all the solutions for the minimal partial realization problem can be extended for the case where mixed data (i.e. time moments and Markov parameters) and for the case where only time moments are given.

1. Introduction

This report is the next in the series [1-5] on the minimal partial realization problem. The problem of time moment matching and the mixed problem of matching both Markov parameters and time moments was only considered in the first of this series [1-5]. Subsequent reports were involved with the matching of Markov parameters only. We shall now come back to the mixed problem and see how [2,3,5] can be adapted for this setting.

All the different problems reduced to solving a block Hankel system of the form

$$\begin{bmatrix} H_1 & \cdots & H_{\alpha+1} \\ \vdots & & \vdots \\ H_{n-\alpha} & \cdots & H_n \end{bmatrix} \begin{bmatrix} b_0 \\ \vdots \\ b_\alpha \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \tag{1.1}$$

with $H_i \in K^{p\times m}$, K an arbitrary field. Setting $b(z) = \sum_{i=0}^{\alpha} b_i z^i$, being a solution of (1.1), we had to find m such solutions $b_j^n(z)$ of degree β_j with leading coefficients $H\, b_j^n(z)$ and constant terms $Lb_j^n(z)$ that had to satisfy

C1. The β_j are as low as possible
C2. The $Hb_j^n(z), j=1,...,m$ are linearly independent
C3. The $Lb_j^n(z), j=1,...,m$ are linearly independent

If no time moments are involved in the problem one can drop condition C3. Since the b_j^n are the columns of the denominator matrix $B(z)$, it should be invertible in the neighbourhood of $z=0$ if time moments have to be matched, which means that condition C3 should then be satisfied since it means that $B(0)$ is nonsingular.
Even if no Markov parameters are involved, condition C2 is still in order because it also relates to the minimality of the approximation. I.e. degree det $B(z)$ has to be minimal. Since the degree of det $B(z)$ equals the sum of the degrees β_j of the columns $b_j^n(z)$ if $B(z)$ is column reduced (hence if C2 is satisfied) and every solution can be brought into a column reduced form condition C2 is not only related to the causality of the solution but also to its minimality.
It is the presence of condition C3 which makes the problems involving time moments considerably more complicated.

We briefly recall the following notations from the previous reports : All solutions of system (1.1) form a solution space denoted as $S(n,\alpha)$. n is called the level, α is the degree. A set of m polynomial vectors satisfying conditions C1 and C2 for a certain level n are called solution vectors for level n and will be denoted by $a_j^n(z)$, $j=1,...,m$ and their degree will be α_j, $j=1,...,m$. The p-vector

$$Ra_j^n = \sum_{k=0}^{\alpha_j} H_{n+1-\alpha_j+k}\, a_{j,k}^n$$

is called the residual of $a_j^n(z)$ and auxiliary vectors are solution vectors of level $<n$ with linearly independent residuals (they span K^p). Auxiliary vectors will be denoted by x_j^n in general. An auxiliary or solution vector is said to have potential degree π (w.r.t. level n) if it is a solution vector of degree $\pi-j$ for level $n-j$ for some $j=0,1,...,\pi$.

As we have done before, we shall start with an artificial set of p auxiliary vectors equal to zero and which we give a potential degree $n+1$ with respect to level n.

2. A construction of the solution

In [1] we described an algorithm to construct the solutions for the time moment or mixed problem. This algorithm used so called generating vectors. We shall now describe a way of constructing the solution for a particular choice of these generating vectors.
It was shown in [3] that algorithm A [1,2] generates a nested basis for all solution spaces $S(n,\alpha)$, based on solution vectors $a_j^n(z)$ of degree α_j and the auxiliary vectors x_j^n of potential degree π_j. This basis was given by

$$BS(n,\alpha) = A_\alpha \bigcup B_\alpha \tag{2.1}$$

with

$$A_\alpha = \overset{m}{\underset{j=0}{U}} \{a_j^n(z)\}_{\alpha_j}^{\alpha} \quad \text{and} \quad X_\alpha = \overset{p}{\underset{j=1}{U}} \{x_j^n(z)\}_{\pi_j}^{\min(\alpha,n)}$$

where

$$\begin{aligned} \{a(z)\}_\alpha^\beta &= \{a(z), za(z), \ldots, z^{\beta-\alpha}a(z)\} && \text{if } \alpha \leqslant \beta \\ &= \varnothing && \text{if } \alpha > \beta \end{aligned}$$

The solution vectors $a_j^n(z)$ were constructed such that their leading coefficients $Ha_j^n(z)$ are linearly independent. Moreover the coefficient of z^α of a vector in $S(n,\alpha)$ will be generated by the leading coefficients $Ha_j^n(z)$ and is independent of the auxiliary vectors in the basis $BS(n,\alpha)$. Hence it directly follows that the $a_j^n(z), j=1,\ldots,m$ solve the Markov parameter problem (i.e. satisfy conditions C1 and C2).
The role of the auxiliary vectors was that they were vectors whose residuals generated the complete residual space $=K^p$. Similarly, generating vectors were vectors whose L-terms generated the complete space of constant terms $=K^m$.
We shall see now that these generating vectors can be selected from the current solution vectors and auxiliary vectors.

We start by investigating the spaces of constant terms of $S(n,\alpha)$ which we shall denote as $LS(n,\alpha)$.
Define

$$LS(n,\alpha{-}1{:}\alpha) = LS(n,\alpha) - LS(n,\alpha{-}1), \ \alpha = 0,1,\ldots,n$$

with $LS(n,-1)=\{0\}$. Suppose the dimension of $LS(n,\alpha{-}1{:}\alpha)$ is d_α, then

$$\dim LS(n,\alpha) = d_{0,\alpha} = \sum_{i=0}^{\alpha} d_i$$

and it follows from (2.1) that there are exactly d_α vectors in $BS(n,\alpha)$ whose L-coefficients are linearly independent among themselves and linearly independent of $LS(n,\alpha{-}1)$ and which have

- degree α, if they are solution vectors
- potential degree α, if they are auxiliary vectors

Since the potential degree of a solution vector is its degree, we can refrase the previous statement as : if $LS(n,\alpha{-}1{:}\alpha) = d_\alpha$, then there are exactly d_α vectors in the basis $BS(n,\alpha)$ of potential degree α whose L-coefficients can be added to a basis for $LS(n,\alpha{-}1)$ to form a basis for $LS(n,\alpha)$.
If we define the degrees

$$\gamma_j = \alpha \qquad d_{0,\alpha-1} < j \leqslant d_{0,\alpha}, \ \alpha = 0,1,\ldots,n$$

then we get exactly $m = d_{0,n} = d_0 + d_1 + \ldots + d_n$ numbers γ_j that are the minimal degrees for solutions of (1.1) with linearly independent constant terms, i.e. solutions of (1.1) satisfying conditions C1 and C3.

From this discussion, it follows that the solutions $b_j^n(z)$ of (1.1) satisfying all conditions C1-C3 will have degrees $\beta_j = \max(\alpha_j, \gamma_j)$, $j=1,\ldots,m$ if $\alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_m$.

If we start algorithm A of [2], [1] with solution vectors forming the unit matrix and auxiliary vectors equal to zero (as we did in [4], [5]), then it follows implicitly from the previous discussion that the L-coefficients of auxiliary and solution vectors for each level generate the whole space K^m. This can be seen from the fact that $S(n,n)$ is the space of all polynomial vectors of degree at most n. Thus its basis (2.1) must have L-coefficients that generate all constant polynomials, i.e. K^m. More generally, the L-coefficients of the joint set of auxiliary vectors and solution vectors form a matrix which at every stage of the algorithm has full rank because we only perform elementary operations of the form E_{ij}, $E_i(\mu)$ and $E_{ij}(q)$ (see [2], [5] for this notation) and these do not influence the rank.

The $b_j^n(z)$ polynomials can be found as follows : suppose we have for a certain level n all solution vectors $a_j^n(z)$ of (potential) degree α_j and suppose for simplicity that they are ordered such that $\alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_m$. Extend this set with the auxiliary vectors $x_j^n(z)$ of potential degree π_j, $j=1,\ldots,p$ and call the resulting set C with elements c_j of potential degree ν_j, ordered such that $\nu_1 \leqslant \nu_2 \leqslant \cdots \leqslant \nu_{p+m}$. We can now construct the set B of vectors $b_j^n(z)$ by the following algorithm.

1. Set $k_0=1$ and $B=\varnothing$
2. **For** $j=1,2,\ldots,m$
   - 2.1 **If** $La_j^n \notin$ span $\{Lb \mid b \in B\}$ **then** place a_j^n in B
   - 2.2 **If** $La_j^n \in$ span $\{Lb \mid b \in B\}$ **then**
     - 2.2.1 Choose $l = \min_{k_0 \leqslant k \leqslant m+p} \{k \mid Lc_k \notin \text{span}\{Lb \mid b \in B\}\}$
     - 2.2.2 Set $s = \max(\nu_l - \alpha_j, 0)$
     - 2.2.3 Place $b = z^s a_j^n + c_l$ in B
     - 2.2.4 Set $k_0 = l+1$

3. V-matrices, numerators and continued fractions

It was shown in [2] how all the operations of algorithm A of [1] can be expressed as unimodular transformations on the columns of a matrix which is initialized as

$$\mathbf{M}_0 = \begin{bmatrix} I_p & 0_{p\times m} \\ 0_{m\times p} & I_m \\ -I_p & H(z) \end{bmatrix} \quad \text{with } H(z) = \sum_{k=1}^{\infty} H_k z^{-k}$$

The transformations are characterized by a unimodular V-matrix. If V_n represents the transformations performed by algorithm A of [1], then generating the $b_j^n(z)$ vectors adds to this V_n matrix some extra steps which are defined by (we use the notation of the algorithm of the previous section)

For $j=1,2,\ldots,m$

2.1 $V^{(j)}=I_{p+m}$

2.2 **If** c_l is a solution vector $a_k^n(z)$ **then**

$$V^{(j)}=E_{p+j,p+k}(z^{\alpha_k-\alpha_j})E_{p+j,p+k}$$

If c_l is an auxiliary vector $x_k^n(z)$ **then**

$$V^{(j)}=E_{p+j,k}(z^s)E_{p+j,k} \text{ with } s=max\ (\pi_k-\alpha_j,0)$$

We denote by $E_{i,j}$ the unit matrix with column i and column j interchanged and by $E_{i,j}(q)$ the unit matrix with column j replaced by column j plus q times column i.
The complete transformation for level n is then given by

$$\hat{V}_n=V_n\ V_n' \quad \text{with} \quad V_n'=V^{(1)}V^{(2)}\ \ldots\ V^{(m)}$$

Note that we have slightly changed the algorithm of the previous section to make all the $V^{(j)}$ unimodular. The difference is that here the solution vectors $a_j^n(z)$ are replaced by the vectors $b_j^n(z)$. This makes it necessary to replace in step 2.2 when c_l is some $a_k^n(z)$ the solution vector $a_k^n(z)$ by $a_j^n(z)$. In this way we may consider $b_j^n(z)$ to be based on $a_k^n(z)$ while the b-vector based on $a_j^n(z)$ will be found later.
To construct the numerator for the mixed problem, we introduce the following notations.

If $M+1$ time moments T_i are given and N Markov parameters M_i then,

$$\begin{aligned} H_{i+1} &= -T_{M-i} && i=0,1,\ldots,M \\ &= M_{i-M} && i=M+1,M+2,\ldots,M+N \end{aligned}$$

and if we set

$$T(z) = \sum_{i=0}^{\infty} T_i\, z^i\,, \qquad z\to 0$$

$$M(z) = \sum_{i=1}^{\infty} M_i\, z^{-i}\,, \qquad z\to\infty$$

$$T_M(z) = \sum_{i=0}^{M} T_i\, z^i$$

then

$$H(z) = z^{-(M+1)}\,(-T_M(z)+M(z)). \tag{3.1}$$

The unimodular transformation for level n is represented by

$$\hat{V}_n = V_n\,V_n' = \begin{bmatrix} Y_n & C_n \\ X_n & A_n \end{bmatrix}\begin{bmatrix} Y_n' & C_n' \\ X_n' & A_n' \end{bmatrix} \quad \text{while} \quad V_k = \begin{bmatrix} Y_k & C_k \\ X_k & A_k \end{bmatrix} \quad \text{for} \quad k < n$$

and we set

$$V_{1,n}=V_1\,V_2\,\cdots\,V_n = \begin{bmatrix} Y_{1,n} & C_{1,n} \\ X_{1,n} & A_{1,n} \end{bmatrix} \text{ and } \hat{V}_{1,n}=V_{1,n}\,V_n' = \begin{bmatrix} \hat{Y}_{1,n} & \hat{C}_{1,n} \\ \hat{X}_{1,n} & \hat{A}_{1,n} \end{bmatrix}.$$

Since we had the relation [1]

$$H(z) - \hat{C}_{1,n}\,\hat{A}_{1,n}^{-1} = O\,(z^{-n-1}),\ z \to \infty$$

it follows that $\hat{C}_{1,n}$ is the numerator and $\hat{A}_{1,n}$ the denominator for the minimal partial realization for $H(z)$ satisfying conditions C1-C3.

The problem is to find the common numerator $\hat{T}_{0,n}(z)$, such that

$$T(z) - \hat{T}_{0,n}\,\hat{A}_{1,n}^{-1} = O\,(z^{M+1}), \qquad z \to 0 \tag{3.2}$$

and

$$M(z) - \hat{T}_{0,n}\,\hat{A}_{1,n}^{-1} = O\,(z^{M-n}), \qquad z \to \infty \tag{3.3}$$

(this is only meaningful if $n \geqslant M+1$).

Therefore we take the initialization

$$\mathbf{T}_0 = [S_0\ T_0]\,,\quad S_0 = z^{M+1}I_p\,,\quad T_0 = T_M(z)$$

and set

$$\hat{\mathbf{T}}_{0,n} = T_0\,\hat{V}_{1,n} = [\hat{S}_{0,n}\ \hat{T}_{0,n}]$$

We claim that $\hat{T}_{0,n}$ is the required numerator.

This can be seen as follows :

$$[-I_p \mid T(z)]\begin{bmatrix} S_0 & T_0 \\ X_0 & A_0 \end{bmatrix} = [O\,(z^{M+1}) \quad O\,(z^{M+1})],\ z \to 0.$$

Hence

$$[-I_p \mid T(z)]\begin{bmatrix} \hat{S}_{0,n} & \hat{T}_{0,n} \\ \hat{X}_{0,n} & \hat{A}_{0,n} \end{bmatrix} = [O\,(z^{M+1}) \quad O\,(z^{M+1})]\,\hat{V}_{1,n},\ z \to 0$$

or, since $\hat{V}_{1,n}$ is polynomial.

$$T(z)\,\hat{A}_{0,n} - \hat{T}_{0,n} = O\,(z^{M+1}),\ z \to 0$$

and because $\hat{A}_{0,n}=\hat{A}_{1,n}$ has a nonsingular constant term, it follows that

$$T(z) - \hat{T}_{0,n}\,\hat{A}_{1,n}^{-1} = O\,(z^{M+1}),\ z \to 0$$

which is (3.2).

On the other hand,

$$[\hat{S}_{0,n} \mid \hat{T}_{0,n}] = [S_0 \mid T_0]\, \hat{V}_{1,n}$$
$$= [z^{M+1} \hat{Y}_{1,n} + T_M(z)\, \hat{X}_{1,n} \mid z^{M+1} \hat{C}_{1,n} + T_M(z)\, \hat{A}_{1,n}]$$

or

$$\hat{C}_{1,n}\, \hat{A}_{1,n}^{-1} = z^{-(M+1)}\, \hat{T}_{0,n}\, \hat{A}_{1,n}^{-1} - z^{-(M+1)}\, T_M(z).$$

Thus for $z \to \infty$ we have

$$H(z) - \hat{C}_{1,n}\, \hat{A}_{1,n}^{-1} = O\,(z^{-n-1})$$
$$= z^{-(M+1)}\, [-T_M(z) + M(z)] - z^{-(M+1)}\, [-T_M(z) + \hat{T}_{0,n}\, \hat{A}_{1,n}^{-1}]$$
$$= z^{-(M+1)}\, [M(z) - \hat{T}_{0,n}\, \hat{A}_{1,n}^{-1}]$$

which implies (3.3).

It is then also simple to give the continued fraction expansion. Indeed,

$$\hat{T}_{0,n}\, \hat{A}_{1,n}^{-1} = T_M(z) + z^{M+1}\, \hat{C}_{1,n}\, \hat{A}_{1,n}^{-1}$$

and since we know from [2] the continued fraction expansion $C_{1,n}\, A_{1,n}^{-1}$, we find that the $(n+1)st$ convergent of the continued fraction

$$T_M(z) + z^{M+1}\, I_p \cfrac{}{A_1 + X_1 \cfrac{C_2 + Y_2 \cfrac{\cdots}{\cdots}}{A_2 + X_2 \cfrac{\cdots}{\cdots}}}$$

is given by $\hat{T}_{0,n}\, \hat{A}_{1,n}^{-1}$ (see also [6]). The $(n+1)st$ convergent is obtained by setting

$$\begin{bmatrix} Y_{n+1} & C_{n+1} \\ X_{n+1} & A_{n+1} \end{bmatrix} = \begin{bmatrix} O & C_n' \\ O & A_n' \end{bmatrix}$$

in this continued fraction.

This expansion is possible for every level n since the A_k for levels $k \leqslant n$ are generated by algorithm A and are therefore nonsingular while A_n' is the A-part of V_n' and this is also nonsingular. One can show that there exists a permutation matrix P such that $P\, A_n'$ has the form

$$P\, A_n' = \text{block diag}\, (\mathbf{a}_1, \ldots, \mathbf{a}_r)$$

with either

$$\mathbf{a}_i = [z^{s_i^{(1)}}] \quad \text{or} \quad \mathbf{a}_i = \begin{bmatrix} z^{s_i^{(1)}} & \cdots & \cdot & z^{s_i^{(d_i)}} \\ 1 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}$$

so that $\det P\, A_n' = z^s$ with $s = \sum_{i=1}^{r} s_i^{(d_i)}$ with d_i the order of $\mathbf{a}_i$ and $s_i^{(1)} \leqslant \cdots \leqslant s_i^{(d_i)}$

If C'_n is conformally subdivided into block columns grouping d_i columns :

$$C'_n = [\mathbf{c}_1 | \mathbf{c}_2 | \cdots | \mathbf{c}_r]$$

then $\mathbf{c}_i$ will have zeros everywhere except in exactly one row of the last column

$$\mathbf{c}_i = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ & & & 1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix} .$$

If the canonical version of algorithm A is used [5] then it can also be shown that in

$$\hat{V}_n = V_n \, V'_n = \begin{bmatrix} \hat{Y}_n & \hat{C}_n \\ \hat{X}_n & \hat{A}_n \end{bmatrix}$$

$\hat{A}_n$ is nonsingular. Indeed,

$$\hat{A}_n = X_n \, C'_n + A_n \, A'_n$$

and suppose that the previously mentioned permutation matrix P is part of A'_n. Then

$$\hat{A}_n = A_n \, (A_n^{-1} \, X_n \, C'_n \, A'^{-1}_n + I) \, A'_n \quad .$$

In the canonical case we know that $A_n^{-1} \, X_n$ is strictly proper and it follows from the form of C'_n and A'_n as explained above that $C'_n \, A'^{-1}_n$ is proper. Thus the central factor of $\hat{A}_n$ is I + strictly proper part, hence has full rank. Moreover, since A_n and A'_n have full rank, also $\hat{A}_n$ has full rank.
It also follows that in this case

$$\deg \det \hat{A}_n = \deg \det A_n + \deg \det A'_n \quad .$$

Note :
It is shown how the $b_j^n(z)$ vectors are computed from the $a_j^n(z)$ vectors as generated by algorithm A. We could generate the $a_j^{n+1}(z)$ vectors for the next level from the final result $b_j^n(z)$ of level n. E.g. by the transformation

$$(V'_n)^{-1} \, V_{n+1}$$

by which we go back to the $a_j^n(z)$ vectors first and then do as in algorithm A.

Thus the transformation to go from the $b_j^n(z)$ vectors to the $b_j^{n+1}(z)$ vectors is represented by

$$(V'_n)^{-1} \, V_{n+1} \, V'_{n+1} = \begin{bmatrix} \tilde{Y}_{n+1} & \tilde{C}_{n+1} \\ \tilde{X}_{n+1} & \tilde{A}_{n+1} \end{bmatrix} \quad .$$

This is a unimodular transformation, but it does not have the property that

$$\deg \det \hat{A}_{1,n+1} - \deg \det \hat{A}_{n,1} = \deg \det \tilde{A}_{n+1}$$

This can be seen from the following example :

Take

$$H_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} , \quad H_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} , \quad H_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Then

$$V_2' = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 1 & 0 & z & 0 \\ 0 & 0 & 0 & 1 \end{array}\right], \quad (V_2')^{-1} = \left[\begin{array}{cc|cc} -z & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

and

$$V_3 = \left[\begin{array}{cc|cc} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & z^2 \end{array}\right], \quad V_3' = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

so that

$$(V_2')^{-1} V_3 V_3' = \left[\begin{array}{cc|cc} 1 & 0 & 1 & z \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & z^2 \end{array}\right]$$

with deg det $\tilde{A}_3 = 0$, while it transforms the

$$[b_j^2(z)]_{j=1,2} = \begin{bmatrix} z^2+1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{into} \quad [b_j^3(z)]_{j=1,2} = \begin{bmatrix} z & -1 \\ 1 & z^2 \end{bmatrix}$$

which shows an increase in determinant degree equal to 1.

4. A parametrization of all solutions

As in theorem 2.2 of [3] we can give all solutions of the time moment or mixed problem. If $b_j^n(z)$, of degree β_j, $j=1,2,\ldots,m$ is a set of solutions and the $x_j(z)$ have potential degree π_j, then all different solutions are given by

$$T^\lambda(z)\, B^\lambda(z)^{-1}.$$

The columns $b_j^\lambda(z)$ of $B^\lambda(z)$ are given by

$$b_j^\lambda(z) = b_j^n(z) + \sum_{k=1}^{p} \sum_{i=\pi_k}^{\beta_j} \lambda_{k,i}^j\, z^{i-\pi_k}\, x_k(z), \quad j=1,\ldots,m.$$

or

$$B^\lambda(z) = B(z) + X(z)\,\Lambda(z)$$

with $\Lambda(z) = [\lambda_k^j(z)]_{\substack{k=1,\ldots,p \\ j=1,\ldots,m}}$

where $\lambda_k^j(z) = \sum_{i=\pi_k}^{\beta_j} \lambda_{k,i}^j\, z^{i-\pi_k}$

with the additional condition that the $b_j^\lambda(0)$ are linearly independent. I.e. the matrix

$$B(0) + X(0)\,\Lambda(0)$$

should be nonsingular.

The set

$$\lambda = \{\, \lambda_{k,i}^j, k=1,\ldots,p, j=1,\ldots,m, i=\pi_k,\ldots,\beta_j \mid B(0) + X(0)\,\Lambda(0) \text{ is nonsingular} \,\}$$

is a minimal set of free parameters which gives all the different solutions.

To compute the corresponding numerator columns $c_j^\lambda(z)$, we should use the following procedure which depends on $M+1$, the number of time moments and the degree β_j of $b_j^\lambda(z)$.

If $\beta_j \leqslant M+1$ then

$$\begin{aligned} c_j^\lambda(z) &= T_M(z)\, b_j^\lambda(z) \mod z^{M+1} \\ &= T_M(z)\, b_j^\lambda(z) \mod z^{\beta_j} \end{aligned}$$

If $\beta_j > M+1$ then

$$c_j^\lambda(z) = z^{M+1}\, c_{j,1}^\lambda(z) + c_{j,2}^\lambda(z)$$

with

$$c_{j,1}^\lambda(z) = \text{the polynomial part of } z^{-(M+1)}\, M(z)\, b_j^\lambda(z)$$

$$c_{j,2}^\lambda(z) = T_M(z)\, b_j^\lambda(z) \mod z^{M+1}$$

The proof that $B^\lambda(z)$ gives all solutions for the denominator is as in [3] and will not be repeated here.
The proof that the corresponding numerator is given by the relations above follows from the observations below :

If $c_j^\lambda(z) = \sum_{k=0}^{\beta_j-1} c_{j,k}^\lambda \, z^k$ and $b_j^\lambda(z) = \sum_{k=0}^{\beta_j} b_{j,k}^\lambda \, z^k$, then

$$c_{j,k}^\lambda = \sum_{l=0}^{\beta_j} T_l \, b_{j,l}^\lambda \tag{4.1}$$

If $\beta_j \leqslant M+1$, then all $c_{j,k}^\lambda$ can be computed from (4.1).
If $\beta_j > M+1$, then only $c_{j,k}^\lambda$ for $k=0,1,\ldots,M$ can be computed from (4.1), since for larger k some unknown $T_{M+1}, T_{M+2}, \ldots$ will enter. However as we have shown in [1], it are precisely the coefficients $c_{j,M+1}^\lambda, \ldots, c_{j,\beta_j-1}^\lambda$ that can then be computed from

$$c_{j,k}^\lambda = \sum_{l=k+1}^{\beta_j} M_{l-k} \, b_{j,l}^\lambda \tag{4.2}$$

5. Conclusion

In this report we have considered the mixed problem of constructing the minimal rational approximation problem if some time moments as well as some Markov parameters are given, and the time moment problem where only some time moments are given. The matrix denominator should then be column reduced and have a nonsingular constant term as well.
We described the unimodular transformations involved, the computation of the numerator in a recursive way and a continued fraction expansion. Also a parametrization of all solutions is given.

6. References

[1] M.VAN BAREL, A.BULTHEEL : A Minimal partial realization algorithm for MIMO systems. Report TW79, July 1986, (Revised: April 1987), Department of Computer Science, K.U.Leuven.

[2] M.VAN BAREL, A.BULTHEEL : A Minimal partial realization algorithm for MIMO systems, II:A matrix continued fraction. Report TW91, June 1987, Department of Computer Science, K.U.Leuven.

[3] M.VAN BAREL, A.BULTHEEL : A Minimal partial realization algorithm for MIMO systems, III:A parametrization of all solutions. Report TW93, August 1987, Department of Computer Science, K.U.Leuven.

[4] M.VAN BAREL, A.BULTHEEL : A Minimal partial realization algorithm for MIMO systems, IV:A matrix Euclidean algorithm. Report TW94, December 1987, Department of Computer Science, K.U.Leuven.

[5] M.VAN BAREL, A.BULTHEEL : A Minimal partial realization algorithm for MIMO systems, V:A canonical matrix continued fraction. Report TW100, January 1988, Department of Computer Science, K.U.Leuven.

[6] M.VAN BAREL, A.BULTHEEL : A Matrix Euclidean algorithm and matrix Padé approximation. Report TW104, January 1988, Department of Computer Science, K.U.Leuven.