# A minimal partial realization algorithm for MIMO systems V : A canonical matrix continued fraction

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*ABSTRACT*

Based on the algorithms to generate minimal partial realizations which were given in previous parts of this series of reports, we give here a normalization which defines a unique (canonical) continued fraction whose convergents are uniquely defined minimal partial realizations and where each subsystem, represented by a term in the continued fraction is physically realizable.

## 1. Introduction

In report TW79 [7] we developed algorithm $A$ which is an extension of the Berlekamp-Massey algorithm much like the ones given by Dickinson, Morf and Kailath [5] and Anderson, Brasch and Lopresti [1]. In a subsequent report (TW91) [8] a connection was given with a continued fraction algorithm of Antoulas [2]. In TW94 [10], using a basis set put forward in TW93 [9] we modified the algorithm to generate a continued fraction whose convergents are minimal partial realizations having a different structure for each convergent. However at several steps of the algorithm, some freedom was left so that by making other choices, other convergents could be obtained. In this report, we want to generate a continued fraction which is unique, thus leaving no freedom to make a different choice at any of the steps of the algorithm. It will also turn out that the convergents have much nicer properties that can be expressed in terms of observability and controllability indices which are invariants.

In section 2 we shall give the extra conditions we shall impose on the minimal partial realizations which will make them unique together with a motivation for this choice. Section 3 gives some properties and equivalent uniqueness conditions. Section 4 describes the algorithm to generate the unique continued fraction and section 5 gives an example.

## 2. Uniqueness conditions

The main idea used implicitly or explicitly in our previous versions of the algorithm was to start from matrices

$$NM_0 = [NMA_0 \mid NMS_0] = [I_p \mid 0] \in K^{p\times(p+m)}[z]$$

(numerator matrix)

$$DM_0 = [DMA_0 \mid DMS_0] = [0 \mid I_m] \in K^{m\times(p+m)}[z]$$

(denominator matrix)

and

$$RSM_0 = [RSMA_0 \mid RSMS_0] = [-I_p \mid M(z)] \in K^{p\times(p+m)}[[z]]$$

(residual matrix)

The terminal letters $A$ and $S$ referred to auxiliary vectors and solution vectors repectively. $K$ is an arbitrary field and $M(z) = \sum_1^\infty M_k z^{-k}$ with $M_k \in K^{p\times m}$ the Markov parameters. Using essentially $RSM_k$ and some degree information, some unimodular matrices $V_k$ are generated such that

$$\begin{bmatrix} NM_k \\ DM_k \\ RSM_k \end{bmatrix} V_{k+1} = \begin{bmatrix} NM_{k+1} \\ DM_{k+1} \\ RSM_{k+1} \end{bmatrix} .$$

If we set as before

$$V_k = \begin{bmatrix} Y_k & C_k \\ X_k & A_k \end{bmatrix} , \quad k = 1,2, \cdots$$

and

$$V_{1,n} = V_1 \cdots V_n = \begin{bmatrix} Y_{1,n} & C_{1,n} \\ X_{1,n} & A_{1,n} \end{bmatrix} = \begin{bmatrix} NM_n \\ DM_n \end{bmatrix}$$

then $NMS_n\ (DMS_n)^{-1}$ is the $n$-th convergent of the matrix continued fraction

$$\cfrac{C_1 + Y_1 \cfrac{C_2+Y_2 \cfrac{\cdots}{\cdots}}{A_2+X_2 \cfrac{\cdots}{\cdots}}}{A_1 + X_1 \cfrac{C_2+Y_2 \cfrac{\cdots}{\cdots}}{A_2+X_2 \cfrac{\cdots}{\cdots}}}$$

where $\frac{A}{B}$ means $AB^{-1}$ and this convergent is a $n$-th order minimal partial realization of the given Markov parameters : i.e. for $n = 1,2, \cdots$

$$M(z) - C_{1,n}(z)A_{1,n}(z)^{-1} = O_-(z^{-n-1}). \tag{1}$$

There may however be several minimal partial realizations satisfying (1). Indeed in Antoulas [2, theorem 3.13] and TW93 [9] all minimal partial realizations of the sequence $M_1, M_2, \ldots, M_n$ are described in terms of $\rho$ parameters, where $\rho = \sum_{i,j} \max(0,\rho_{i,j})$ with $\rho_{i,j} = \kappa_j + \nu_i - n$ where $\nu_i$, $i = 1,2,\ldots,p$ are the observalibility indices and $\kappa_j$, $j = 1,2,\ldots,m$ are the controllability indices of the sequence. It is well known [4] that a unique minimal partial realization can be found among these for which the $ij$ entry of $M(z) - C_{1,n}(z)A_{1,n}(z)^{-1}$ is $O_-(z^{\kappa_j-\nu_i+1})$.
In our quest for uniqueness, this will be the first condition we shall impose :

**Condition C1 :** The minimal partial realizations have to satisfy

$$[M(z)-C_{1,n}(z)A_{1,n}(z)^{-1}]_{i,j} = O_-(z^{-\kappa_j(n)-\nu_i(n)-1})$$

where $\kappa_j(n)$, $j = 1,\ldots,m$; $\nu_i(n)$, $j = 1,\ldots,p$ are the controllabity and observability indices of the sequences $M_1, M_2, \ldots, M_n$.

Once we have selected this unique minimal partial realization, we shall chose a unique representation for it in the form $C_{1,n}(z)A_{1,n}(z)^{-1}$. This can be done by taking $A_{1,n}(z)$ column canonical by which we mean that [3]

- $A_{1,n}(z)$ is column and row reduced with the degree of column $j$ and row $j$ equal to $\kappa_j$
- the matrix of leading row coefficients is the unit matrix and the matrix of the leading column coefficients is unit upper triangular. I.e.

$$\begin{aligned} A_{1,n}(z) &= \operatorname{diag}(z^{\kappa_1}, z^{\kappa_2}, \ldots, z^{\kappa_m})[A_{1,n}]_{hr} + \text{lower degree terms} \\ &= [A_{1,n}]_{hc} \operatorname{diag}(z^{\kappa_1}, z^{\kappa_2}, \ldots, z^{\kappa_m}) + \text{lower degree terms} \end{aligned}$$

with $[A_{1,n}]_{hc}$ unit upper triangular and $[A_{1,n}]_{hr}$ the unit matrix.

Up to a permutation, this is the same as the polynomial echelon form or Popov form as mentioned in Kailath [6,p. 481]. Hence our condition C2:

**Condition C2 :** For the minimal partial realizations satisfying condition C1 we choose a representation $C_{1,n}(z)A_{1,n}(z)^{-1}$ with $A_{1,n}(z)$ column canonical.

By combining conditions C1 and C2, we have fixed $C_{1,n}(z)$ and $A_{1,n}(z)$ uniquely. I.e. the second block column of $V_{1,n}$ is pinned down but $V_{1,n}$ isn't. If $V_{1,n}$ is unimodular, then this property will be kept if $V_{1,n}$ is multiplied by a unimodular matrix. So far we have applied the following unimodular transformations to $V_{1,n}$

$$V_{1,n} \cdot \begin{bmatrix} I & P \\ 0 & I \end{bmatrix} \begin{bmatrix} I & O \\ 0 & U_2 \end{bmatrix}$$

with $P$ polynomial and $U_2$ unimodular. But since $C_{1,n}(z)A_{1,n}(z)^{-1}$ has to be a minimal partial realization, the polynomial $P(z)$ is restricted viz. its $ij$ entry has to be 0 if $\rho_{i,j} \leqslant 0$ and is a

polynomial of degree $< \rho_{i,j}$ if $\rho_{i,j} > 0$, where $\rho_{i,j}$ is as defined before. The choice of $P(z)$ has been fixed by condition C1. The unimodular matrix $U_2$ has been fixed by condition C2. Since the second block column of $V_{1,n}$ is now completely defined, we can only apply unimodular transformations to $V_{1,n}$ which will not affect $C_{1,n}(z)$ and $A_{1,n}(z)$. This leaves us with a free unimodular matrix $U_1$ and a free polynomial matrix $Q(z)$ in the transformation

$$V_{1,n} \begin{bmatrix} I & P \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & U_2 \end{bmatrix} \begin{bmatrix} U_1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ Q & I \end{bmatrix} .$$

To fix these, we consider the matrix

$$W_{1,n} = V_{1,n}^{-1} = \begin{bmatrix} D_{1,n} & -B_{1,n} \\ -U_{1,n} & T_{1,n} \end{bmatrix} .$$

Because $W_{1,n} = V_{1,n}^{-1}$ it follows that

$$C_{1,n}(z)A_{1,n}(z)^{-1} = D_{1,n}(z)^{-1}B_{1,n}(z).$$

Thus $D_{1,n}(z)^{-1}B_{1,n}(z)$ is a left matrix fraction description of the same (unique) minimal partial realization for which we had a right matrix fraction description. Thus if $C_{1,n}(z)A_{1,n}(z)^{-1}$ satisfies condition C1, then $D_{1,n}(z)^{-1}B_{1,n}(z)$ will also satisfy C1 in the sense that

$$[M(z)-D_{1,n}(z)^{-1}B_{1,n}(z)]_{i,j} = O_-(z^{-\kappa_j-\nu_i-1}) \tag{2}$$

$U_1^{-1}$ (or what is the same, $U_1$) can now be used to get a unique left representation of this unique minimal partial realization. I.e. we require

**Condition C3 :** Choose $D_{1,n}(z)$ to be row canonical.

By symmetry considerations we say that $D_{1,n}(z)$ is row canonical if

- $D_{1,n}(z)$ is row reduced and column reduced with the degree of row $i$ and column $i$ equal to $\nu_i$
- the matrix of the leading column coefficients is the unit matrix and the matrix of leading row coefficients is unit lower triangular. I.e.

$$\begin{aligned} D_{1,n}(z) &= [D_{1,n}]_{hc}\ \mathrm{diag}(z^{\nu_1}, z^{\nu_2}, \ldots, z^{\nu^p}) + \text{lower degree terms} \\ &= \mathrm{diag}(z^{\nu_1}, z^{\nu_2}, \ldots, z^{\nu^p})[D_{1,n}]_{hr} + \text{lower degree terms} \end{aligned}$$

with $[D_{1,n}]_{hc}$ the unit matrix and $[D_{1,n}]_{hr}$ unit lower triangular.

Our final condition will fix the polynomial matrix $Q(z)$. This can be uniquely defined by requiring

**Condition C4 :** Choose $A_{1,n}(z)^{-1}X_{1,n}(z)$ strictly proper rational.

This can always be achieved without destroying the previous normalizations since it doesn't touch the second block column of $V_{1,n}$ nor the first block row of $W_{1,n}$.

By conditions C1-C4, we have defined a unique $V_{1,n}$ for $n = 1,2,\cdots$ hence also unique $V_k$, $k = 1,2,\cdots$. These $V$-matrices and all the quantities which are also uniquely related to it (the continued fraction, the minimal partial realizations, ...) will be called *canonical*.

## 3. Properties of the canonical continued fraction

Canonical minimal partial realizations have some remarkable properties.
We first introduce the following notations

$$Z_{k,l}(z) = C_{k,l}(z)A_{k,l}(z)^{-1} = D_{k,l}(z)^{-1}B_{k,l}(z)$$

and if $l = k$ we replace $k,l$ by $k$.
For the residual series we use

$$[S_{k,l}(z) \mid R_{k,l}(z)] = [-I \mid M(z)]V_{k,l}$$

and again if $l = k$ we use the single subscript $k$.
We set by definition

$$[S_0(z) \mid R_0(z)] = [-I \mid M(z)].$$

Furthermore :

$$M(z) - Z_{k,l}(z) = \Gamma_{k,l}(z) = R_{k,l}(z)A_{k,l}(z)^{-1}$$

and

$$\Delta_{k+1,l}(z) = D_{1,k}(z)^{-1}Z_{k+1,l}(z)A_{1,k}(z)^{-1}.$$

Then it can be shown that (see Antoulas [2,p. 1125])

$$Z_{1,l} = Z_{1,k} + (\Delta Z)_{k+1,l} \tag{3}$$

with

$$(\Delta Z)_{k+1,l} = \Delta_{k+1,l}\,(I + X_{1,k}D_{1,k}\Delta_{k+1,l})^{-1}. \tag{4}$$

By a notation like $O_-(z^{\gamma_{i,j}})$ we shall mean a matrix series in $z$ whose $(i,j)$ entry is $O_-(z^{\gamma_{i,j}})$. E.g. condition C1 says that

$$\Gamma_{1,n} = O_-(z^{-\kappa_j(n)-\nu_i(n)-1}).$$

In the proofs that follow we shall need the fact that the $\kappa_j(n)$ and the $\nu_i(n)$ are nondecreasing with $n$. This is a known fact. It was essentially proved in TW79 [7,section 7]. We give here a proof using a crate table mechanism.
Given the sequence $M_1, M_2, \ldots, M_n$, consider the infinite block Hankel matrix

$$H = \begin{bmatrix} M_1 & M_2 & M_3 & \cdots & M_n & x & \cdots \\ M_2 & M_3 & M_4 & \cdots & x & x & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \\ M_n & x & & \cdots & x & x & \cdots \\ x & x & & \cdots & x & x & \cdots \\ \vdots & \vdots & & & \vdots & \vdots & \end{bmatrix}$$

with the $x$ entries unspecified blocks.
Denote by $c_j$ the $j$-th column of $H$ and arrange these columns in a $\infty \times m$ table (the column table) with $c_{im+j}$ at position $(i,j)$.

| | 1 | 2 | $\cdots$ | $m$ |
|---|---|---|---|---|
| 0 | $c_1$ | $c_2$ | $\cdots$ | $c_m$ |
| 1 | $c_{m+1}$ | $c_{m+2}$ | $\cdots$ | $c_{2m}$ |
| 2 | $c_{2m+1}$ | $c_{2m+2}$ | $\cdots$ | $c_{3m}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ |
| n | $c_{nm+1}$ | $c_{nm+2}$ | $\cdots$ | $c_{(n+1)m}$ |
| $n+1$ | $x$ | $x$ | | $x$ |
| $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ |

column table

| | 1 | 2 | 3 | $\cdots$ | $m$ |
|---|---|---|---|---|---|
| 0 | × | × | × | $\cdots$ | × |
| 1 | × | ● | × | $\cdots$ | × |
| 2 | ● | ● | × | $\cdots$ | × |
| 3 | ● | ● | × | $\cdots$ | ● |
| 4 | ● | ● | ● | $\cdots$ | ● |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ |

crate table

Derive a crate table from the column table as follows. Place a × in position $(i,j)$ of the crate table if column $c_{im+j}$ at position $(i,j)$ of the column table is linearly independent of the preceding columns, otherwise place a ●. For the linear dependency, the unspecified entries $x$ should be taken such that they don't influence the dependencies. This means the following : consider $c_l$, $l = j+km$, then only its first $(n-k)p$ entries will be specified. Cut $c_l$ and all preceding columns $c_i$, $i < l$ after $(n-k)p$ entries and denote them by $\tilde{c}_i$, $i = 1,2,\ldots,l$. If $\tilde{c}_l$ is linearly independent of $\tilde{c}_i$, $i < l$ then $c_l$ will be linearly

independent of $c_i$, $i < l$, whatever the unspecified elements are. If however $\tilde{c}_l$ is linearly dependent of $\tilde{c}_i$, $i < l$, then the unspecified elements can be chosen such that also $c_l$ will be linearly dependent on $c_i$, $i < l$. The block Hankel structure of $H$ guarantees that if $c_j$ is linearly dependent on the previous columns, then the columns $c_{j+m}$, $c_{j+2m}$,... will be dependent too, i.e. if a dot appears in a column of the crate table, then all the following entries in that column will be dots too.

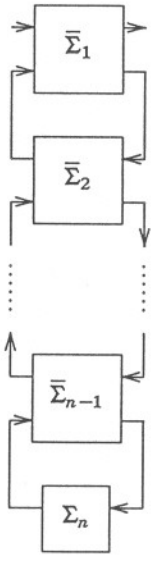
The controllability index $\kappa_j(n)$ is now defined as the row index of the first dot appearing in column $j$ of the crate table. It is clear that if $n$ is increased, then some previously unspecified elements become now specified and this can only delay the linear dependency of a certain column of the block Hankel matrix. Hence $\kappa_j(l) \geqslant \kappa_j(k)$ if $l \geqslant k$. A similar argumentation using the rows of the block Hankel matrix can be used to show that also $\nu_i(l) \geqslant \nu_i(k)$ if $l \geqslant k$.

Thus we have proved

**Lemma L1 :** *$\kappa_j(n)$ and $\nu_i(n)$ are nondecreasing with n for $j = 1,2,\ldots,m$ and $i = 1,2,\ldots,p$.*

□

It is described in Antoulas [2] how the $n$-th convergent of the continued fraction can be realized by a ladder network




where the transfer function of $\Sigma_n$ is $Z_n$ and the transfer function of the augmented system $\bar{\Sigma}_j$ is

$$\bar{Z}_j = \begin{bmatrix} Z_j & D_j^{-1} \\ A_j^{-1} & -A_j^{-1}X_j \end{bmatrix} = \begin{bmatrix} I & -C_j \\ 0 & A_j \end{bmatrix}^{-1} \begin{bmatrix} 0 & Y_j \\ I & -X_j \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 \\ -U_j & T_j \end{bmatrix} \begin{bmatrix} 0 & I \\ D_j & -B_j \end{bmatrix}^{-1} .$$

The overall transfer function being $Z_{1,n}$.
If each system $\bar{\Sigma}_j$ has to be physically realizable, then $\bar{Z}_j(z)$ should be proper rational. For a canonical continued fraction this will be true i.e. :

**Theorem T1 :**

*For a canonical continued fraction, each* $\bar{\Sigma}_j$ *will be causal. We shall prove the more general relations for* $l \geqslant k+1$

1. $C_{k+1,l}A_{k+1,l}^{-1} = Z_{k+1,l} = O_{-1}(z^{-1})$ *(strictly proper)*
2. $A_{k+1,l}^{-1} = O_{-}(z^{0})$ *(proper)*
3. $D_{k+1,l}^{-1} = O_{-}(z^{0})$ *(proper)*
4. $A_{k+1,l}^{-1}X_{k+1,l} = O_{-}(z^{-1})$ *(strictly proper)*

*which give for* $l = k+1$ *the causality of* $\bar{\Sigma}_{k+1}$.

**Proof**

1. Because of condition C1

$$(\Delta Z)_{k+1,l} = (M - Z_{1,k}) - (M - Z_{1,l}) = \Gamma_{1,k} - \Gamma_{1,l}$$
$$= O_{-}(z^{-\kappa_j(k)-\nu_i(k)-1}) - O_{-}(z^{-\kappa_j(l)-\nu_i(l)-1})$$

and because the $\kappa_j(n)$ and $\nu_i(n)$ are not decreasing with $n$, we have

$$(\Delta Z)_{k+1,l} = O_{-}(z^{-\kappa_j(k)-\nu_i(k)-1}).$$

Now from (3,4) :

$$D_{1,k}^{-1}O_{-}(z^{-\kappa_j(k)-\nu_i(k)-1}) = Z_{k+1,l}A_{1,k}^{-1}(I + X_{1,k}Z_{k+1,l}A_{1,k}^{-1})^{-1}.$$

Because $D_{1,k}$ is column reduced with column degrees $\nu_i(k)$, the left hand side will be $O_{-}(z^{-\kappa_j(k)-1})$.

Condition C4 states that $A_{1,k}^{-1}X_{1,k}$ is strictly proper, while condition C2 includes that $A_{1,k}$ is row reduced with row degrees $\kappa_j(k)$. Therefore row $j$ of $X_{1,k}$ will be of degree $\leqslant \kappa_j(k)-1$. Thus we have

$$O_{-}(z^{-\kappa_j(k)-1})(I+X_{1,k}Z_{k+1,l}A_{1,k}^{-1}) = Z_{k+1,l}A_{1,k}^{-1}$$

or

$$O_{-}(z^{-\kappa_j(k)-1}) + O_{-}(z^{-2})Z_{k+1,l}A_{1,k}^{-1} = Z_{k+1,l}A_{1,k}^{-1}$$

or

$$O_{-}(z^{-\kappa_j(k)-1})A_{1,k} + O_{-}(z^{-2})Z_{k+1,l} = Z_{k+1,l}.$$

Again because $A_{1,k}$ is row reduced with row degrees $\kappa_j(k)$, we have

$$O_{-}(z^{-1}) + O_{-}(z^{-2})Z_{k+1,l} = Z_{k+1,l}$$

or

$$(I+O_{-}(z^{-2}))Z_{k+1,l} = O_{-}(z^{-1})$$

or

$$Z_{k+1,l} = O_{-}(z^{-1}).$$

2. Because

$$X_{1,k}C_{k+1,l} + A_{1,k}A_{k+1,l} = A_{1,l}$$

we have

$(A_{1,k}^{-1}X_{1,k})(C_{k+1,l}A_{k+1,l}^{-1}) + I = A_{1,k}^{-1}A_{1,l}A_{k+1,l}^{-1}$.

$A_{1,k}^{-1}X_{1,k} = O_-(z^{-1})$ by condition C4 and $Z_{k+1,l} = O_-(z^{-1})$ as we have proved in 1. Hence

$A_{1,l}^{-1}A_{1,k}(I+O_-(z^{-2})) = A_{k+1,l}^{-1}$

Because of the canonicity of the $A_{1,n}$ and because the $\kappa_j(n)$ are nondecreasing with $n$ we find that $A_{1,l}^{-1}A_{1,k} = O_-(z^0)$, so that also $A_{k+1,l}^{-1} = O_-(z^0)$.

3. This point can be shown as in 2 by duality.

4. By condition C4 we have that $A_{1,k}^{-1}X_{1,k}$ is strictly proper, because $A_{1,k}^{-1}X_{1,k} = U_{1,k}D_{1,k}^{-1}$, the latter will also be strictly proper. Now consider

   $U_{1,l}D_{1,l}^{-1} = [U_{k+1,l}D_{1,k}+T_{k+1,l}U_{1,k}]\,[D_{k+1,l}D_{1,k}+B_{k+1,l}U_{1,k}]^{-1}$.

   Use in the numerator of the right hand side

   $T_{k+1,l} = A_{k+1,l}^{-1}X_{k+1,l}B_{k+1,l} + A_{k+1,l}^{-1}$

   and

   $U_{k+1,l} = A_{k+1,l}^{-1}X_{k+1,l}D_{k+1,l}$

   to get

   $U_{1,l}D_{1,l}^{-1} = [A_{k+1,l}^{-1}X_{k+1,l}] + A_{k+1,l}^{-1}U_{1,k}\,[D_{k+1,l}D_{1,k}+B_{k+1,l}U_{1,k}]^{-1}$.

   The second term can be rewritten as

   $A_{k+1,l}^{-1}U_{1,k}D_{1,k}^{-1}\,[I+D_{k+1,l}^{-1}B_{k+1,l}U_{1,k}D_{1,k}^{-1}]^{-1}D_{k+1,l}$.

   Now is as shown in 1

   $D_{k+1,l}^{-1}B_{k+1,l} = C_{k+1,l}A_{k+1,l}^{-1} = O_-(z^{-1})$

   while

   $U_{1,k}D_{1,k}^{-1} = O_-(z^{-1})$

   as said in the beginning of this proof. Also

   $A_{k+1,l}^{-1} = O_-(z^0)$ and $D_{k+1,l}^{-1} = O_-(z^0)$

   as shown under 2 and 3. Hence we may conclude that the whole term is $O_-(z^0)$ and therefore

   $$\begin{aligned} A_{k+1,l}^{-1}X_{k+1,l} &= U_{1,l}D_{1,l}^{-1} - O_-(z^0) \\ &= O_-(z^{-1}) - O_-(z^0) = O_-(z^0) \end{aligned}$$

   which is 4.
   Note that by duality also $U_{k+1,l}D_{k+1,l}^{-1}$ will be $O_-(z^0)$.

□

Condition C3 requires $D_{1,n}$ to be row canonical. This matrix is a block in the $W$ matrix which doesn't appear explicitly in the algorithm of TW79 [7], TW91 [8]. It will be shown now that C3 can be replaced by the condition C5 which is a normalization of the residual series for the auxiliary vectors.

**Condition C5 :** $-S_{1,n}(z)$ is canonical in the following sense :

- $S_{1,n}(z)$ is column and row reduced with the degree of column $i$ and row $i$ equal to $-\nu_i(n)$.
- The matrix of leading row vectors is the unit matrix and the matrix of the leading column vectors is unit lower triangular. I.e.

  $S_{1,n}(z) = \mathrm{diag}(z^{-\nu_1(n)}, z^{-\nu_2(n)}, \ldots, z^{-\nu_p(n)})[S_{1,n}]_{hr}$ + lower order terms

$$= [S_{1,n}]_{hc} \operatorname{diag}(z^{-\nu_1(n)}, \ldots, z^{-\nu_p(n)}] + \text{lower order terms}$$

with $[-S_{1,n}]_{hr}$ the unit matrix and $[-S_{1,n}]_{hc}$ unit lower triangular.

This is important because the residual series can be easily computed along with the $V$ matrix as in the algorithm of TW94 [10] without using the $W$ matrix.

Theorem T2 :

*If for some minimal partial realization of $M_1, \ldots, M_n$ conditions C1, C2 , C4 are satisfied, then C3 is equivalent with C5.*

Proof

Using the definition of $S_{1,n}$ and C1 we get :

$$\begin{aligned} S_{1,n} &= MX_{1,n} - Y_{1,n} \\ &= (C_{1,n}A_{1,n}^{-1} + O_-(z^{-\kappa_j(n)-\nu_i(n)-1}))X_{1,n} - Y_{1,n} \\ &= (C_{1,n}A_{1,n}^{-1}X_{1,n} - Y_{1,n}) + O_-(z^{-\kappa_j(n)-\nu_i(n)-1})X_{1,n}. \end{aligned} \tag{5}$$

From C2 and C4 it follows that

$$X_{1,n} = O_-(z^{\kappa_i(n)-1}) \tag{6}$$

thus that the last term in (5) gives

$$O_-(z^{-\kappa_j(n)-\nu_i(n)-1})X_{1,n} = O_-(z^{-\nu_i(n)-2}).$$

From $V_{1,n}W_{1,n} = I$ we get

$$-D_{1,n}^{-1} = C_{1,n}A_{1,n}^{-1}X_{1,n} - Y_{1,n}$$

so that

$$S_{1,n} = -D_{1,n}^{-1} + O_-(z^{-\nu_i(n)-2}). \tag{7}$$

If C3 is satisfied, then

$$D_{1,n}^{-1} = \operatorname{diag}(z^{-\nu_1(n)}, \ldots, z^{-\nu_p(n)})[D_{1,n}]_{hc}^{-1} + \text{lower order terms}$$

$$\text{with } [D_{1,n}]_{hc}^{-1} = I_p \tag{8}$$

so that it follows from (7), that also (9)

$$S_{1,n} = \operatorname{diag}(z^{-\nu_1(n)}, \ldots, z^{-\nu_p(n)})[S_{1,n}]_{hr} + \text{lower order terms}$$

$$\text{with } [-S_{1,n}]_{hr} = I_p. \tag{10}$$

Conversely, if C5 is satisfied, then (9) holds and it then follows from (7) that also (8) holds, which implies $D_{1,n}$ column reduced with degrees $\nu_j(n)$ and $[D_{1,n}]_{hc} = I$.

To check the equivalence of the other parts, note that it follows from (6) that the column degree $d_j$ of the $j$-th column of $X_{1,n}$ will be bounded by $\kappa_i(n)-1$ if the leading coefficient has a nonzero $i$-th component and $l$-th components zero with $l = i+1, \ldots, m$. In other words, the $j$-th auxiliary vector has a degree lower than the $i$-th solution vector. Suppose the order of this auxiliary vector is $\sigma_j$, i.e.

$$MX_{1,n} - Y_{1,n} = S_{1,n} = O_-(z^{-\sigma_j-1+d_j})$$

with $\sigma_j$ as large as possible, so that the order is strict. Then $\sigma_j < n$, because $\sigma_j \geqslant n$ would imply that the $j$-th auxiliary vector had order $\geqslant n$, so that we can replace the $i$-th solution vector with this auxiliary vector and still get a realization of order $n$, but with $\deg \det A_{1,n} < \Sigma_1^m \kappa_j(n)$, which is impossible because the $\kappa_j(n)$ are the minimal values one can get for the degrees of the solution vectors. Thus we have

$$
\begin{aligned}
S_{1,n} &= O_-(z^{-\sigma_j-1+d_j}) \\
&= -D_{1,n}^{-1} + O_-(z^{-n-1})X_{1,n} \\
&= -D_{1,n}^{-1} + O_-(z^{-n-1+d_j})
\end{aligned}
\tag{11}
$$

with $\sigma_j < n$, and the order $-\sigma_j-1+d_j$ strict.
If C3 is satisfied, then

$$
D_{1,n}^{-1} = [D_{1,n}]_{hr}^{-1} \operatorname{diag}(z^{-\nu_1(n)}, \ldots, z^{-\nu_p(n)}) + \text{lower order terms}
$$

$$
\text{with } [D_{1,n}]_{hr}^{-1} \text{ unit lower triangular,} \tag{12}
$$

hence, it follows from (11) that $-\sigma_j-1+d_j = -\nu_j(n)$ and that

$$
S_{1,n} = [S_{1,n}]_{hc} \operatorname{diag}(z^{-\nu_1(n)}, \ldots, z^{-\nu_p(n)}) + \text{lower order terms}
$$

$$
\text{with } [-S_{1,n}]_{hc} \text{ unit lower triangular.} \tag{13}
$$

Conversely, if C5 is satisfied, then (13) holds with $-\nu_j(n) = -\sigma_j-1+d_j > -n-1+d_j$, so that it follows from (11) that (12) holds or $D_{1,n}$ row reduced with degrees $\nu_i(n)$ and $[D_{1,n}]_{hr}$ unit lower triangular.

Thus we have shown the equivalence of C3 and C5.

□

If we want to come to an algorithm to compute this canonical continued fraction, we can start from an algorithm like in TW79 [7] or TW91 [8] or TW94 [10] which gives a minimal partial realization and then perform after each step the normalizations defined by conditions C1-C4 or equivalently by conditions C1, C2, C4, C5. Satisfying condition C1 directly is not an easy task. Therefore we shall replace C1 by another condition.

**Condition C6 :** $S_{1,n}^{-1}R_{1,n} = O_-(z^{-1})$ i.e. strictly proper rational.

We shall prove now

**Theorem T3 :**

*For a minimal partial realization satisfying C2, C3, C4 and C6 (14) or C2, C5, C4 and C6 (15), condition C1 will be satisfied automatically.*

**Proof**

Suppose $S_{1,n}^{-1}R_{1,n} = O_-(z^{-1})$ or $R_{1,n} = S_{1,n}O_-(z^{-1})$, then

$$
\begin{aligned}
R_{1,n} &= (MX_{1,n}-Y_{1,n})O_-(z^{-1}) \\
&= [(C_{1,n}A_{1,n}^{-1}+\Gamma_{1,n})X_{1,n}-Y_{1,n}]O_-(z^{-1}) \\
&= (C_{1,n}A_{1,n}^{-1}X_{1,n}-Y_{1,n})O_-(z^{-1}) + \Gamma_{1,n}A_{1,n}A_{1,n}^{-1}X_{1,n}O_-(z^{-1}).
\end{aligned}
$$

Because

$$
-D_{1,n}^{-1} = C_{1,n}A_{1,n}^{-1}X_{1,n}-Y_{1,n}
$$

is row reduced with row degrees $-\nu_i(n)$ by C3, we find that the first term gives

$$
-D_{1,n}^{-1}O_-(z^{-1}) = O_-(z^{-\nu_i(n)-1}).
$$

By C4 we find that

$$
A_{1,n}^{-1}X_{1,n} = O_-(z^{-1})
$$

so that

$$R_{1,n} = O_-(z^{-\nu_i(n)-1}) + \Gamma_{1,n}A_{1,n}O_-(z^{-2})$$
$$= O_-(z^{-\nu_i(n)-1}) + R_{1,n}O_-(z^{-2})$$

or

$$R_{1,n}(I+O_-(z^{-2})) = O_-(z^{-\nu_i(n)-1})$$

or

$$R_{1,n} = MA_{1,n} - C_{1,n} = O_-(z^{-\nu_i(n)-1}). \tag{16}$$

To get formula (16) we have used conditions (14). Equivalently using conditions (15) we can get (16) as follows : From C5 we get that $S_{1,n}$ is row reduced with row degrees $-\nu_i(n)$. From C6 formula (16) is obtained.
Bacause $A_{1,n}$ is row reduced with row degrees $\kappa_j(n)$ by C2, if follows from (16) that

$$M-C_{1,n}A_{1,n}^{-1} = O_-(z^{\nu_i(n)-1})A_{1,n}^{-1} = O_-(z^{-\kappa_j(n)-\nu_i(n)-1})$$

which is in C1.

□

**Corollary C1 :**

*C1, C2, C3, C4 ⇔ C1, C2, C4, C5 ⇔ C2, C4, C5, C6 ⇔ C2, C3, C4, C6*

**Proof**

All implications are proved if we can prove e.g. C1, C2, C3, C4 ⇒ C6. This is easy to prove because C1-C4 ⇒ C5 which implies that

$$S_{1,n} = [S_{1,n}]_{hc}\ \mathrm{diag}(z^{-\nu_1(n)}, \ldots, z^{-\nu_p(n)}) + \text{lower order terms}$$

while

$$R_{1,n} = \Gamma_{1,n}A_{1,n}.$$

C1 and C2 give

$$R_{1,n} = O_-(z^{-\kappa_j(n)-\nu_i(n)-1})O_-(z^{\kappa_i(n)})$$
$$= O_-(z^{-\nu_i(n)-1}).$$

Consequently

$$S_{1,n}^{-1}R_{1,n} = \mathrm{diag}(z^{\nu_1(n)}, \ldots, z^{\nu_p(n)})[S_{1,n}]_{hc}^{-1}O_-(z^{-\nu_i(n)-1}) + \text{lower order terms}$$
$$= O_-(z^{-1})$$

and this is C6.

□

## 4. The algorithm

The normalization will be obtained by satisfying conditions C2, C4, C5 and C6 for a minimal partial realization. Suppose we have an algorithm (like the algorithms in TW79 [7], TW91 [8], TW94 [10]) that starts from

$$\begin{bmatrix} X_0 & A_0 \\ S_0 & R_0 \end{bmatrix} = \begin{bmatrix} O & I_m \\ -I_p & M(z) \end{bmatrix}$$

and generates a minimal partial realization $A_{1,n}C_{1,n}^{-1}$ of onder $n$ by

$$\begin{bmatrix} X_{1,n} & A_{1,n} \\ S_{1,n} & R_{1,n} \end{bmatrix} = \begin{bmatrix} X_0 & A_0 \\ S_0 & R_0 \end{bmatrix} V_{1,n}$$

with unimodular

$$V_{1,n} = \begin{bmatrix} Y_{1,n} & C_{1,n} \\ X_{1,n} & A_{1,n} \end{bmatrix} .$$

Suppose also that $S_{1,n}$ is column reduced with column degrees $-\sigma_j + d_j - 1$ where $d_j$ is the degree of column $j$ of $X_{1,n}$ and $\sigma_j < n$. For the inital zero auxiliary vectors we can take $\sigma_j = d_j = -\infty$. I.e. the orders of the auxiliary vectors $X_{1,n}$ are less than $n$. This will be the case for the algorithms in TW79 [7], TW91 [8], TW94 [10] since the $S_{1,n}$ are the residual series for the auxiliary vectors $X_{1,n}$ whose residuals span the residual space.

Note that these suppositions on $S_{1,n}$ imply that the column degrees of $S_{1,n}$ are given by a permutation of the $-\nu_j(n)$. This can be seen from a relation we have given before viz.

$$\begin{aligned} D_{1,n}^{-1} &= -S_{1,n} + O_-(z^{-n-1})X_{1,n} \\ &= -S_{1,n} + O_-(z^{-n-1+d_j}). \end{aligned}$$

Since $S_{1,n} = O_-(z^{-\sigma_j-1+d_j})$ with $\sigma_j < n$, it follows that $D_{1,n}$ is row reduced with row degrees $\sigma_j + 1 - d_j$. But $C_{1,n}A_{1,n}^{-1} = D_{1,n}^{-1}B_{1,n}$ is a minimal partial realization of which $D_{1,n}$ is a left denominator. Hence its row degrees must be a permutation of the $\nu_i(n)$, since these are invariants. Thus $\sigma_j + 1 - d_j$ is equal to some $\nu_i(n)$ or $S_{1,n}$ has column degrees which are given by a permutation of the $-\nu_j(n)$.

The normalization is now obtained by the following steps which correspond to multiplications to the right with unimodular matrices $N_i$.

1. $N_1 = I_p \oplus U_1$ with $U_1$ unimodular such that $A_{1,n}$ satisfies condition C2, i.e. such that $A_{1,n}$ is column canonical in the sense of C2.
   Such a type of algorithm can be found e.g. in Kailath [6]. It uses elementary column operations represented by
   $E_i(a)$ : multiply column $i$ by a constant $a$
   $E_{ij}$ : interchange column $i$ and column $j$
   $E_{ij}(q)$ : replace column $j$ by column $j$ plus $q$ times column $i$ with $q$ polynomial.
   Note that this normalizes the minimal partial realization without changing it.

2. $N_2 = \begin{bmatrix} I_p & O \\ Q_2(z) & I_m \end{bmatrix}$ with $Q_2(z)$ polynomial such that C4 is satisfied, i.e. such that

   $X_{1,n} = O_-(z^{-\kappa_i(n)-1})$.

   This means subtract polynomial multiples of the solution vectors such that the required condition is satisfied. Note that this doesn't change $[S_{1,n}]_{hc}$ for the following reason : If a polynomial multiple $(q)$ of the $i$-th solution vector (degree $\kappa_i(n)$) is added to the $j$-th auxiliary vector (degree $d_j$) then clearly $\kappa_i \leqslant d_j$. This operation corresponds to a multiplication with an elementary matrix $E_{i+p,j}(q)$. The effect on the corresponding residual $s_j$ ($r_i$ is the residual of the solution vector) is

   $$\begin{aligned} s_j(z) + q(z)r_i(z) &= O_-(z^{-\sigma_j-1+d_j}) + O_-(z^{d_j-\kappa_i})O_-(z^{-n-1+\kappa_i}) \\ &= O_-(z^{-\sigma_j-1+d_j}) + O_-(z^{-n-1+d_j}). \end{aligned}$$

   Since we supposed that $\sigma_j < n$, the leading column coefficient is not affected. Hence $S_{1,n}$ remains column reduced with column degrees a permutation of the $-\nu_j(n)$.

3. $N_3 = \begin{bmatrix} U_3 & 0 \\ Q_3 & I_m \end{bmatrix}$ with $U_3$ unimodular and $Q_3$ polynomial such that C5 is satisfied. $U_3$ consists of

elementary operations of the form

- $E_i(a)$ with $a$ constant (to make the leading diagonal elements of $S_{1,n}$ equal to -1).
- $E_{ij}$. These are simply row interchanges that don't change the row degrees of $X_{1,n}$ so that C4 is not influenced by this operation.
- $E_{ij}(q)$ with $q$ polynomial. This however may undo the achievements of step 2 and violate C4. Fortunately, it can be repaired by the $Q_3$ polynomial. Since this is a complex operation we explain this with a typical example. Suppose $x_k$ is the $k$-th column of $X_{1,n}$ of degree $d_k$ with order $\sigma_k$, i.e. with the $k$-th column of $S_{1,n}$ e.g. of the form ($p = 2$)

  $$s_k = \begin{bmatrix}1\\0\end{bmatrix} z^{-\omega_k} + O_-(z^{-\omega_k-1})$$

  with $-\omega_k = -\sigma_k-1+d_k$. Let the $l$-th auxiliary vector be $x_l$ of degree $d_l$ and order $\sigma_l$ with residual e.g.

  $$s_l = \begin{bmatrix}1\\1\end{bmatrix} z^{-\omega_l} + O_-(z^{-\omega_l-1})$$

  with $-\omega_l = -\sigma_l-1+d_l$ and suppose $\omega_k < \omega_l$. The algorithm to make $S_{1,n}$ canonical then requires an operation which subtracts $z^{\omega_l-\omega_k}s_l$ from $s_k$. This causes also $x_k$ to be replaced by $x_k' = x_k-x_l z^{\omega_l-\omega_k}$. Since the $i$-th element of $x_l$ and $x_k$ had a degree less than $\kappa_i$ by step 2, this may not be true anymore for $x_k'$. Now we can redo operations of step 2, i.e. add polynomial multiples of the solution vectors to $x_k'$ to obtain some $x_k''$ which has again on row $i$ a polynomial of degree less than $\kappa_i$. The latter operations don't change the leading term of $s_k'$ so that $s_k''$ has still the required form. The same type of operations can be done in each step of the process to make $S_{1,n}$ canonical. Each time an auxiliary vector is shifted and corrected with solution vectors so that the required operations can be done on $S_{1,n}$ without violating C4.

  To check that the leading coefficients of the residuals of the auxiliary vectors are not influenced by this operation, note that $[S_{1,n}]_{hc}$ is regular, which implies e.g. that a combination of the form

  $$x_k + z^{\rho_l-\rho_k}x_l$$

  where $-\rho_i = -\sigma_i+d_i-1$, i.e. $S_{1,n} = O_-(z^{-\rho_j})$ will give a residual which is strictly $O_-(z^{-\rho_k})$ if $\rho_k \geqslant \rho_l$. The degree of the combination of the auxiliary vectors is $d \leqslant \max(d_k, d_j-\rho_k+\rho_l)$. So that the order of the combination is $\sigma = \rho_k+d-1$ which is always less than $n$ because $\sigma_k$ and $\sigma_l$ are. We also have to add combinations of solution vectors, but this doesn't influence the leading coefficient of the residual, hence $\rho_k$ is the same. On the other hand, this operation can only decrease $d$, which means that the order is not increased, hence less than $n$. Note that this also follows from C2 and C4 as proved in T2.

After step 3 $S_{1,n} = O_-(z^{-\nu_j} = O_-(z^{-\nu_l})$ and $X_{1,n}$ is still $O_-(z^{-\kappa_l-1})$ (i.e. C2, C4, C5 are satisfied).

4. $N_4 = \begin{bmatrix}I_p & Q_4\\ 0 & U_4\end{bmatrix}$ with $U_4$ unimodular and $Q_4$ polynomial to satisfy C6.

I.e. make the row degrees of $R_{1,n}$ less than $-\nu_i$, the column degrees of $S_{1,n}$. Thus $R_{1,n} = O_-(z^{-\nu_l-1})$. The operations we have to perform now are much like in the previous step. Suppose the $(i,j)$ entry of $R_{1,n}$ is of the form

$$O_-(z^{-n-\mu_{i,j}+\kappa_j-1})$$

with $-n-\mu_{i,j}+\kappa_j-1 \geqslant -\nu_i$ and $\mu_{i,j} \geqslant 0$ then this degree should be decreased. This can be obtained by adding a multiple of the $i$-th column of $S_{1,n}$ (because $[S_{1,n}]_{hc} = -I_p$). This causes the $j$-th solution vector $a_j$ to be replaced by

$$a_j' = a_j + \alpha x_i z^{\omega}$$

with $\omega = -n-\mu_{i,j}+\kappa_j-1+\nu_i$. The degree $d_i$ of $x_i$ is now bounded by $n-\nu_i$ because its order is less than $n$. Hence the degree of $x_i z^{\omega}$ is bounded by

$$n-\nu_i-n-\mu_{i,j}+\kappa_j-1+\nu_i = \kappa_j-1-\mu_{i,j} < \kappa_j.$$

Hence the leading column coefficient of $a_j'$ is the same as for $a_j$. However the row degrees could be influenced. Therefore we do as in step 3, i.e. subtract polynomial multiples of solution vectors from $x_i z^{\omega}$ to make its $k$-th entry of degree less than $\kappa_k$, while keeping the corresponding leading coefficient of its residual. These operations give us a transformation of the form $N_4$. It is still to show that $U_4$ (caused by the polynomial multiples of the solution vectors) is not only polynomial but also unimodular. Decreasing the row degrees in $x_i z^{\omega}$ corresponds to a contribution to $U_4$ which is of the form

$$\begin{bmatrix} 1 & & p_1 & & \\ & & \vdots & & \\ & 1 & p_{j-1} & & \\ & & p_j & & \\ & & p_{j+1} & 1 & \\ & & \vdots & & \\ & & p_m & & 1 \end{bmatrix}$$

with polynomials $p_l$ in column $j$. We shall show that $p_j$ is a constant. Then $U_4$ is built up of a product of unimodular transformations, hence it is unimodular. $p_j$ will be a constant if all the entries of $x_i z^{\omega}$ have degree $< \kappa_j$. But we have just seen that the degree of $x_i z^{\omega}$ is less than $\kappa_j$. This proves that $p_j$ is a constant.

## 5. Example

Consider the Markov parameters :

$$M(z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} z^{-1} + \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} z^{-2} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} z^{-3}$$
$$+ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} z^{-4} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} (z^{-5} + z^{-6} + z^{-7} + \cdots).$$

For level 0 we have that

$$[Y_0 \mid C_0] = \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

$$[X_0 \mid A_0] = \left[\begin{array}{cc|ccc} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

with residuals

$$[S_0 \mid R_0] = \left[\begin{array}{cc|ccc} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{array}\right] z^0 +$$
$$\left[\begin{array}{cc|ccc} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] z^{-1} +$$
$$\left[\begin{array}{cc|ccc} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] z^{-2} +$$
$$\left[\begin{array}{cc|ccc} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] z^{-3} +$$

$$\left[\begin{array}{cc|ccc} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] z^{-4} +$$

$$\left[\begin{array}{cc|ccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] (z^{-5} + z^{-6} + z^{-7} + \cdots).$$

With algorithm $A$ of (TW79) [7] we find for level 1

$$[Y_{1,1} \mid C_{1,1}] = \left[\begin{array}{cc|ccc} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$[X_{1,1} \mid A_{1,1}] = \left[\begin{array}{cc|ccc} 1 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & z \end{array}\right]$$

$$[S_{1,1} \mid R_{1,1}] = \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{array}\right] z^{-1} +$$

$$\left[\begin{array}{cc|ccc} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right] z^{-2} +$$

$$\left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] z^{-3} +$$

$$\left[\begin{array}{cc|ccc} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] z^{-4} + \cdots$$

To get the canonical solution we continue as follows

1. $A_{1,1}$ is canonical
2. $A_{1,1}^{-1} X_{1,1} = O_{-}(z^{-1})$
3. To make $S_{1,1}$ canonical multiply the first 2 columns by -1.
4. To make $S_{1,1}^{-1} R_{1,1} = O_{-}(z^{-1})$ we have to add the first column (with the sign change) to the last column and add the second column (with sign change) to the third column.

We then get

$$V_1 = \left[\begin{array}{cc|ccc} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ - & - & - & - & - \\ -1 & 0 & z & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & z \end{array}\right] \quad \text{and} \quad W_1 = \left[\begin{array}{cc|ccc} z & -1 & -1 & 0 & 0 \\ -1 & z & 0 & 0 & -1 \\ - & - & - & - & - \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

while

$$[S_{1,1} \mid R_{1,1}] = \left[\begin{array}{cc|ccc} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{array}\right] z^{-1} +$$

$$\left[\begin{array}{cc|ccc} 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 \end{array}\right] z^{-2} +$$

$$\left[\begin{array}{cc|ccc} -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] z^{-3} +$$

$$\left[\begin{array}{cc|ccc} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] z^{-4} + \cdots$$

For level 2, algorithm $A$ will have nothing to do, so that

$$V_2 = W_2 = I_{p+m}.$$

For level 3, it finds

$$[Y_{1,3} \mid C_{1,3}] = \left[\begin{array}{cc|ccc} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$[X_{1,3} \mid A_{1,3}] = \left[\begin{array}{cc|ccc} -1 & 0 & z & 0 & -1 \\ 0 & 1 & 0 & z^2 & z \\ 0 & 0 & -1 & -1 & z \end{array}\right]$$

and

$$[S_{1,3} \mid R_{1,3}] = \left[\begin{array}{cc|ccc} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] z^{-1} + \left[\begin{array}{cc|ccc} 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{array}\right] z^{-2} + \left[\begin{array}{cc|ccc} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right] z^{-3} + \left[\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] z^{-4} + \cdots$$

To make this canonical, only $S_{1,3}$ has to be made canonical. This is obtained by multiplying the second column by -1 and by substracting $z$ times this sign changed column from the first one. This results in

$$V_3 = \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ - & - & - & - & - \\ 0 & 0 & 1 & 0 & 0 \\ z & -1 & 0 & z^2 & z \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \quad \text{and} \quad W_3 = \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ z & z^2 & 0 & -1 & z \\ - & - & - & - & - \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

and this is also a solution for level 4, which gives

$$V_4 = W_4 = I_{p+m}.$$

It can easily be checked that each subsystem that is connected with a $V_i$ or $W_i$, $i = 1,2,3,4$ is physically realizable.

## 6. Conclusion

We have extended algorithm $A$ of TW79 [7] and TW91 [8]. The new algorithm computes the elements of a canonical matrix continued fraction and each of these elements is physically realizable.

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