A minimal partial realization algorithm for MIMO systems IV : A matrix Euclidean algorithm

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ABSTRACT

In this report we adapt algorithm A of TW79 and TW91 to get a possible generalization of the scalar algorithm of Euclides to matrix sequences or polynomials.
Given the matrix fraction description $T(z)^{-1}Q(z)$ this new algorithm can be used to compute a greatest common left divisor of the matrix polynomials $T(z)$ and $Q(z)$; moreover it leads to a matrix continued fraction representation of $T(z)^{-1}Q(z)$. Two examples are given to illustrate the algorithm.

1. Introduction

In report [10] we looked at an extension of the Berlekamp-Massey algorithm. The Berlekamp-Massey algorithm [3,4,9] solves the minimal partial realization problem for scalar sequences of Markov parameters. The matrix case is solved by algorithm A of section 12 in report [10]. This algorithm is in fact described by Dickinson, Morf and Kailath [5] and Anderson, Brasch and Lopresti [1]. In [2] Antoulas also gives an algorithm to solve the matrix case such that the solution is in the form of a matrix continued fraction. He calls his algorithm a matrix algorithm of Euclides while it is in fact a matrix Berlekamp-Massey algorithm.

The algorithm of Euclides is well known for finding the greatest common divisor of two (scalar) polynomials. But the algorithm has other useful applications such as the solution of the scalar minimal partial realization problem. The solution is given in the form of a continued fraction but in contrast with the matrix continued fraction of Antoulas each convergent has another structure i.e. the next convergent has more states than the previous one or the degree of the denominator polynomial of the convergent increases. To compute that part of the continued fraction that determines the next convergent the algorithm of Euclides uses more than one Markov parameter in contrast with the algorithm of Antoulas which uses just one matrix Markov parameter at each step. What we want to construct in the next section is an algorithm that constructs a matrix continued fraction such that the next convergent of the continued fraction has another structure than the previous one while maintaining as much detail as possible. In section 3 we give some examples.

2. A matrix Euclidean algorithm

Throughout this text we shall freely use the definitions and notations of [10].

We recall that, given the $p \times m$ Markov parameters $M_1, M_2, M_3, \cdots \in K^{p \times m}$ (K an arbitrary field), consider polynomial vectors $a(z) = \sum_{i=0}^{\alpha} a_i z^i$, $a_\alpha \neq 0$ which solve

$$\begin{bmatrix} M_{k+2} & M_{k+3} & \cdots & M_{k+\alpha+2} \\ M_{k+2} & M_{k+4} & \cdots & \vdots \\ \vdots & \vdots & & \\ M_{n+k+2-\alpha} & & \cdots & M_{n+k+2} \\ M_{n+k+3-\alpha} & & \cdots & M_{n+k+3} \\ \vdots & & & \vdots \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_\alpha \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ r_1 \\ r_2 \\ \vdots \end{bmatrix}, \quad \alpha \leqslant n$$

$r_1 \in K^p$ possibly nonzero is called the residual and $r_1, r_2, \cdots$ is called the residual sequence. n is the level and all solutions of this system are denoted as $S(n,\alpha)_k$. Introduce the operators R and RS by $Ra = r_1$ and $RSa = [r_1^T r_2^T \cdots]^T$. The residual series is denoted by $RSa(z) = \sum_{i=1}^{\infty} r_i z^{-n-k-1+\alpha-i}$. The notion of degree will be extended to series in z^{-1} to mean the largest exponent of z with a nonzero coefficient. Thus the degree of $RSa(z)$ above is $-n-k-2+\alpha$ if $r_1 \neq 0$. We shall use the notation $dRSa$ to denote the degree of $RSa(z)$ and we shall also refer to it as the degree of the corresponding residual sequence. We call it residual degree for short. Solution vectors $a_1^n(z), \ldots, a_m^n(z)$ of level n are solutions found on level n of lowest possible degree and with linear independent highest degree coefficients. Auxiliary vectors are solution vectors of level $< n$ with linear independent residuals that span the space of residuals. An auxiliary vector or a solution vector is said to have potential degree π (w.r.t. level n) if it is a solution vector of degree $\pi - j$ for level $n-j$ for some j, $j = 0,1, \ldots, \pi$.
If the solution vectors of level n are assembled as the columns of a $m \times m$ matrix $A(z)$, then $A(z)$ is the right denominator of a minimal partial realization of order $N = n+k+1$ for the given Markov sequence.

Because we want to construct a matrix continued fraction à la Antoulas, that is as simple as possible but whose convergents solve minimal partial realization problems of increasing orders, we construct solution vectors such that in table 2 of [10] they can be used at as much levels as possible.

To find these solution vectors, we use our knowledge about all possible solution vectors at a certain point of the table. A basis for all these vectors is given by the solution vectors and auxiliary vectors (possibly shifted) [12]. There it was shown that $a(z) \in S(n,\alpha)_k$ has the form

$$a(z) = \sum_{l=1}^{m} \sum_{d=\alpha_l}^{\alpha} \nu_{l,d} z^{d-\alpha_l} a_l^n(z) + \sum_{i=1}^{s} \sum_{j=\pi_i}^{\min(n,\alpha)} \mu_{i,j} z^{j-\pi_i} x_i(z) \tag{1}$$

where $a_1^n(z), \ldots, a_m^n(z)$ are the solution vectors of degrees $\alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_m$ and $x_i(z)$, $i = 1,2,\ldots,s$ are the auxiliary vectors of potential degrees $\pi_1, \pi_2, \ldots, \pi_s$.
Using formula (1), we want to transform the solution vectors a_i^n, making them solution vectors of levels as deep as possible, i.e. with least possible residual degree. In doing so we can use a_i^n not only at level n but possibly also at deeper levels.
Let us search for solution vectors $\in S(0,0)_k$ with least possible residual degrees. We start with a basis for $S(0,0)_k$ consisting of m linearly independent constant vectors $a_1^0, \ldots, a_m^0 \in K^m$.

We only need to consider combinations of the form

$$a_i' = a_i^0 + \sum_{k=1}^{i-1} \gamma_{i,k} a_k^0.$$

Indeed, suppose $a_1, a_2, \ldots, a_m \in K^m$ are m linearly independent vectors such that their residual degrees are as low as possible and $dRSa_1 \leqslant dRSa_2 \leqslant \cdots \leqslant dRSa_m$, then they can be transformed into the form above without changing the $dRSa_i$ as can be seen from the following small procedure:
Initially let $A = \{a_1, a_2, \ldots, a_m\}$ and $A' = \emptyset$.

For $i = m, m-1, \ldots, 2$

1. Take the vector $a_i' = a_{k(i)} \in A$ with $k(i)$ as small as possible such that a_i' has a component a_i^0
2. Delete a_i' from A and add it to A'
3. Subtract suitable multiples of a_i', from the remaining vectors in A so that they have no component a_i^0 anymore.

This idea is used in section D.1.1 of the algorithm D to be described. We can extend this idea to solution vectors of arbitrary but equal degrees. Even more generally, it is easy to see that a similar reasoning shows that given m solution vectors $a_1^{n+1}, a_2^{n+2}, \ldots, a_m^{n+1}$ of degrees $\alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_m$ it is possible to find m solution vectors based on them, (hence with linear independent highest degree components) and with residual degrees as small as possible while we only need to consider combinations of the form

$$a_i^{n+1} + \sum_{l=1}^{i-1} \sum_{d=\alpha_l}^{\alpha_i} \nu_{l,d} z^{d-\alpha_l} a_l^{n+1} + \sum_{k=1}^{s} \sum_{j=\pi_k}^{\alpha_i} \mu_{k,j} z^{j-\pi_k} x_k \tag{2}$$

where x_k are the auxiliary vectors of potential degree π_k.
This idea is used in step D.2.4.1 of the algorithm.

Algorithm D we are going to describe now is an adaptation (along the lines given before) of algorithm A given in section 12 of [10] or its reformulation given in section 2.3 of [11]. The latter relates it to a matrix continued fraction as in [2].
It would take too long an exposition to give a complete rederivation of algorithm D. We suppose the reader is familiar with algorithm A of [11] and describe only the modifications and simplifications which are introduced.

1. Of course we shall need an extra matrix containing the residual series (or sequence) which will replace the simpler residual matrix of algorithm A. We denote it as

   $$RSM = [RSMA \mid RSMS] \in K^{p \times (p+m)}[[z^{-1}]].$$

   It contains the residual series *RSMA* of the auxiliary vectors and the residual series *RSMS* of the

solution vectors.

2. The ordering of the auxiliary vectors with respect to increasing potential degrees which was introduced in algorithm A in [10] was mainly for notational convenience but was never needed. Indeed, the residuals Rx_i of the auxiliary vectors x_i form a basis of the residual space, so, whenever Ra_i^n can be written as a linear combination of the Rx_j, this combination is unique. The reordering of the auxiliary vectors was maintained in [11] but it could easily be dropped there as well. Consequently, we don't need the permutations of algorithm A in steps A.2.1.3.1.6 and A.2.1.3.2.5. We pay for it with a slightly more involved formulation of step D.2.1.3.2.1.

3. Since in step D.2.1.3.2.1 α_i will always be less then π_r, step A.2.1.3.3 of algorithm A will never occur.

4. In algorithm A as formulated in [11], auxiliary vectors are introduced at the moment a new residual appeared which was not in the residual space obtained so far. This gave rise to initial terms of the continued fraction which could only be filled in when all auxiliary vectors were computed. A much easier solution of this problem is to start with a set of auxiliary vectors which have residuals that span K^p completely. This can be done by introducing p solution vectors for an artificial level -1 to be zero, so that their residuals (= residual series) are $M(z).0-I_p = -I_p$. They become auxiliary vectors on level 0 with potential degree 1. In that case, s, the number of auxiliary vectors will be p from the beginning, and no bookkeeping of s is required.
   This also implies that the residual Ra_i^n of the solution vector a_i^n can always be expressed as a linear combination of the residuals Rx_j, $j = 1, \ldots, p$ of the auxiliary vectors and therefore step A.2.1.3.1 of algorithm A will never occur.

5. As in [11] we shall take the shift parameter k to be -1, which corresponds to the classical problem.

We are now ready to formulate algorithm D where the E matrices represent elementary operations as in [11] viz.:
$E_i(a)$ = unit matrix with column i multiplied by a
E_{ij} = unit matrix with column i and column j interchanged
$E_{ij}(a)$ = unit matrix with column j replaced by column j plus a times column i.

Algorithm D

D.1 {Initialisation}
Define (A refers to auxiliary vectors, S refers to solution vectors)
$NM = [NMA \mid NMS] = [I_p \mid 0] \in K^{p\times(p+m)}[z]$
(numerator matrix)
$DM = [DMA \mid DMS] = [0 \mid I_m] \in K^{m\times(p+m)}[z]$
(denominator matrix)
$RSM = [RSMA \mid RSMS] = [-I_p \mid M(z)] \in K^{p\times(p+m)}[[z^{-1}]]$
(residual matrix)
$dM = [dMA \mid dMS] = [1 \cdots 1 \mid 0 \cdots 0] \in K^{1\times(p+m)}$
(degree matrix)

D.1.1 For $i = 2,3, \ldots, m$

D.1.1.1 Search for

$$a = a_i^0 + \sum_{l=1}^{i-1} \gamma_{i,l} a_l^0$$

with $dRSa(z)$ as low as possible

D.1.1.2 Replace $\begin{bmatrix} a_i^0 \\ RSa_i^0 \end{bmatrix}$ by $\begin{bmatrix} a \\ RSa \end{bmatrix}$ in $\begin{bmatrix} DMS \\ RSMS \end{bmatrix}$

D.2 **For** $n = 0,1,2, \cdots$

Suppose we know

$NM = [y_1 \cdots y_p \mid c_1^n \cdots c_m^n]$

$DM = [x_1 \cdots x_p \mid a_1^n \cdots a_m^n]$

$RSM = [RSx_1 \cdots RSx_p \mid RSa_1^n \cdots RSa_m^n]$

$dM = [\pi_1 \cdots \pi_p \mid \alpha_1 \cdots \alpha_m]$

D.2.1 **For** $i = 1,2, \ldots, m$

D.2.1.1 Take Ra_i^n from $RSMS$ (the coefficient of $z^{-n+\alpha_i-1}$)

D.2.1.2 **If** $Ra_i^n = 0$ **then** a_i^{n+1} can be chosen as a_i^n and

$$V^{(i)} = I_{p+m}$$

D.2.1.3 **If** $Ra_i^n \neq 0$ **then**

D.2.1.3.1 Write Ra_i^n as a linear combination of $Rx_1, \ldots, Rx_p$:

$$Ra_i^n = -\sum_{j=1}^{p} \mu_j \, Rx_j$$

D.2.1.3.2 {Update for solution vector a_i^{n+1} based on a_i^n}

D.2.1.3.2.1 Take r to be the index of the auxiliary vector with highest potential degree which is used actively in this linear combination, i.e. with $\mu_r \neq 0$ (note that $\alpha_i < \pi_r$)

D.2.1.3.2.2 Set

$$V^{(i)} = E_{r,p+i} E_{p+i}(\mu_r) \prod_{j \neq r} E_{j,p+i}(\mu_j z^{\pi_r - \pi_j}) E_{r,p+i}(z^{\pi_r - \alpha_i})$$

D.2.1.3.2.3 Replace

$$\begin{bmatrix} NM \\ DM \\ RSM \end{bmatrix} \text{ by } \begin{bmatrix} NM \\ DM \\ RSM \end{bmatrix} V^{(i)}$$

D.2.1.3.2.4 {The corresponding interchange of degrees}
Replace dM by $dM \, . \, E_{r,p+i}$

D.2.2 {Reordering w.r.t. dMS}
Define the permutation matrix PS such that $dMS.PS$ has its entries in non-decreasing order from left to right and reorder all other entries correspondingly i.e. replace

$$\begin{bmatrix} NMS \\ DMS \\ RSMS \\ dMS \end{bmatrix} \text{ by } \begin{bmatrix} NMS \\ DMS \\ RSMS \\ dMS \end{bmatrix} . \; PS.$$

D.2.3 {Prepare for the next level}
Increase the entries of dMA by 1

D.2.4 {Using formula (1), we compute now the final solution vectors for level $n+1$}

D.2.4.1 For $i = 1,2, \ldots, m$

D.2.4.1.1 **If** a_i^{n+1} is not a solution vector of level n **then**

D.2.4.1.1.1 Find a linear combination

$$a = a_i^{n+1} + \sum_{l=1}^{i-1} \sum_{d=\alpha_l}^{\alpha_i} \nu_{l,d} z^{d-\alpha_l} a_l^{n+1} + \sum_{k=1}^{p} \sum_{j=\pi_k}^{\alpha_i} \mu_{k,j} z^{j-\pi_k} x_k$$

such that $dRSa$ is as small as possible.
(Note that there may be more than 1 solution).

D.2.4.1.1.2 The previous linear combination is assembled in the unimodular matrix $W^{(i)}$:

$$W^{(i)} = \prod_{l=1}^{i-1} E_{p+l,p+i} \left(\sum_{d=\alpha_l}^{\alpha_i} \nu_{l,d} z^{d-\alpha_l} \right) \cdot \prod_{k=1}^{p} E_{k,p+i} \left(\sum_{j=\pi_k}^{\alpha_i} \mu_{k,j} z^{j-\pi_k} \right)$$

D.2.4.1.1.3 Replace

$$\begin{bmatrix} NM \\ DM \\ RSM \end{bmatrix} \text{ by } \begin{bmatrix} NM \\ DM \\ RSM \end{bmatrix} W^{(i)}$$

D.2.4.1.2 **If** a_i^{n+1} is a solution vector of level n **then** set $W^{(i)} = I_{p+m}$

D.2.5 Define

$$V_{n+1} = \begin{bmatrix} Y_{n+1} & C_{n+1} \\ X_{n+1} & A_{n+1} \end{bmatrix} = V^{(1)} \cdots V^{(m)} \cdot P \cdot W^{(1)} \cdots W^{(m)}$$

with $P = I \oplus PS$.

Notes

1. The reordering in step D.2.2 is not necessary but the formulation of formula (2) used in D.2.4.1.1.1 demands that the solution vectors are ordered.
2. In step D.2.4 we could find the linear combinations giving the least possible residual degree based on the old solution vectors (i.e. without replacing them immediately by the new solution vector). The replacement is then done just before we go to the next level. This would give a global unit upper triangular unimodular matrix WS.
3. It is possible that in step D.2.4.1.1.1 there is more than one solution $a(z)$. It is sufficient to take one of them as the new solution vector $a_i(z)$. See also example 1.
4. Instead of the initializations $NMA = I_p$ and $DMS = I_m$, we could have taken $NMA = Y_0 \in K^{p \times p}$ and $DMS = A_0' \in K^{m \times m}$ to be any nonsingular matrices. If A_0 is the transformation of A_0' as performed in step D.1.1, then we could define

$$V_0 = \begin{bmatrix} Y_0 & C_0 \\ X_0 & A_0 \end{bmatrix} \text{ with}$$

$X_0 = 0_{m,p}$ and $C_0 = 0_{p,m}$, to complete the set of unimodular matrices

$$V_n = \begin{bmatrix} Y_n & C_n \\ X_n & A_n \end{bmatrix}, \quad n = 1,2,\ldots \text{ of step D.2.5.}$$

If you now set

$$V_{0,n} = V_0 V_1 \cdots V_n = V_{0,n-1} V_n = \begin{bmatrix} Y_{0,n} & C_{0,n} \\ X_{0,n} & A_{0,n} \end{bmatrix}$$

$$= \begin{bmatrix} NMA_n & NMS_n \\ DMA_n & DMS_n \end{bmatrix}$$

then

$$Y_0 \cfrac{C_1+Y_1 \cfrac{\cdots}{\cdots}}{A_1+X_1 \cfrac{\cdots}{\cdots}}$$
$$\overline{\qquad\qquad A_0}$$

has convergents $C_{0,n}A_{0,n}^{-1}$ giving solutions of the minimal partial realization problem at level n, $n = 0,1,2,\cdots$, i.e. of order $N = n$ (because $k = -1$). This was shown in [11].

5. If we only know the first N Markov parameters M_k, $k = 1,\ldots,N$, we cannot compute the complete residual sequence but only some of its elements. Besides the initial zeros, you will only be able to compute $r_1, r_2, \ldots, r_{N-n-k-1}$. However the subsequent elements never enter the computations explicitly, so minimal partial realizations of order N can be computed.

6. It can be seen that the matrices NM and DM are never explicitly needed in algorithm D. The only information needed is RSM and the degrees dM. Hence these matrices can be dropped from the algorithm. The algorithm computes the (matrix) continued fraction expansion of the series $M(z)$. If the transformations V_i are performed on the NM and DM matrices, the numerators and denominators of the successive convergents are obtained by the forward evaluation method as new terms in the continued fraction are computed. If these transformations are not explicitly performed, the numerator and denominator can always be recovered by multiplying the V_i matrices.

7. When algorithm D is applied to a sequence of scalar Markov parameters ($p = m = 1$) the algorithm reduces to THE scalar algorithm of Euclides. So algorithm D can be seen as a generalization of the scalar algorithm of Euclides. The unimodular matrices V_n are in this case :

$$V_n = \begin{bmatrix} y_n(z) & c_n(z) \\ x_n(z) & a_n(z) \end{bmatrix} = \begin{bmatrix} 0 & c_n \\ 1 & a_n(z) \end{bmatrix}$$

where c_n is a constant different from zero such that $a_n(z)$ is a monic polynomial of degree $\geqslant 1$. Because $y_n(z) \equiv 0$ we obtain a scalar continued fraction not going upwards but only going downwards :

$$\cfrac{c_1}{|a_1(z)} + \cfrac{c_2}{|a_2(z)} + \cfrac{c_3}{|a_3(z)} + \cdots$$

For further details see [6,8].

8. The scalar algorithm of Euclides can also be used to compute a greatest common divisor of two polynomials. This property can also be extended to algorithm D. See [2]. Suppose we have a left matrix fraction description : $T(z)^{-1}Q(z)$ and we want to compute a greatest common left divisor of $T(z)$ and $Q(z)$. Starting from the Markov parameters of $T(z)^{-1}Q(z)$ we compute the unimodular matrices $V_0(z), V_1(z), V_2(z), \ldots, V_N(z)$ using algorithm D until we get a minimal realization of these Markov parameters. This means that

$$[T(z)|-Q(z)]V_0(z)V_1(z)V_2(z)\cdots V_N(z) = [T(z)|-Q(z)] \begin{bmatrix} Y_{0,N}(z) & C_{0,N}(z) \\ X_{0,N}(z) & A_{0,N}(z) \end{bmatrix} = [D(z)|0]$$

or $D(z)$ is a greatest common left divisor of $T(z)$ and $Q(z)$ unique upto a right multiplication with a unimodular matrix (See [7,p.376-378]). Note that $T(z)Y_{0,N}(z) - Q(z)X_{0,N}(z) = D(z)$, so the auxiliary vectors and the corresponding numerator vectors form the solutions of this Bezout equation.

3. Examples

Example 1

In the first example we consider 3 Markov parameters :

$$M(z) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} z^{-1} + \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} z^{-2} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} z^{-3} + \cdots .$$

The initialization of algorithm D gives :

$$NM = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \quad , \quad dM = [1 \;\; 1 \mid 0 \;\; 0]$$

$$DM = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

with residual sequences

$$RSM = \left[\begin{array}{cc|cc} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] z^0 +$$

$$\left[\begin{array}{cc|cc} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] z^{-1} +$$

$$\left[\begin{array}{cc|cc} 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right] z^{-2} +$$

$$\left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] z^{-3} + \cdots$$

Step D.1.1 replaces the second solution vector $[0 \;\; 1]^T$ by $[-1 \;\; 1]^T$ and we get

$$DMA = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = A_0$$

with

$$RSMA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} z^0 +$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} z^{-1} +$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} z^{-2} +$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} z^{-3} + \cdots$$

So

$$V_0 = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right] .$$

For $n = 0$ we get $Ra_1^0 = [1 \;\; 0]^T$.
The linear combination of D.2.1.3.1 is

$$Ra_1^0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = - \left(1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right)$$

so $r = 1$ and $\pi_r = 1$ and

$$V^{(1)} = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 1 & 0 & z & 0 \\ 0 & 0 & 0 & 1 \end{array}\right], \quad dM = [0\ \ 1 \mid 1\ \ 0]$$

so that after multiplication with $V^{(1)}$ we get

$$NM = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

$$DM = \left[\begin{array}{cc|cc} 1 & 0 & z & -1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$RSM = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] z^0 +$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] z^{-1} +$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \end{array}\right] z^{-2} +$$

$$\left[\begin{array}{cc|cc} 0 & 0 & * & 0 \\ 0 & 0 & * & 0 \end{array}\right] z^{-3} + \cdots$$

The residual $Ra_2^0 = [0\ \ 0]^T$ so that $V^{(2)} = I$ and nothing changes.
By section D.2.2 we have to set $PS = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ which reorders the last two columns to give

$$NM = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

$$DM = \left[\begin{array}{cc|cc} 1 & 0 & -1 & z \\ 0 & 0 & 1 & 0 \end{array}\right]$$

and

$$RSM = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] z^0 +$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right] z^{-1} +$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] z^{-2} +$$

$$\left[\begin{array}{cc|cc} 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \end{array}\right] z^{-3} + \cdots$$

After section D.2.3 and D.2.4, dM looks as follows

$$dM = [1\ \ 2 \mid 0\ \ 1].$$

Section D.2.4 replaces only the second solution vector a_2' by

$$a_2^1 + za_1^1 - 2x_1 = \begin{bmatrix} z \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ z \end{bmatrix}$$

with residual sequence

$$\begin{bmatrix}0\\0\\1\\1\\0\\0**\end{bmatrix} + \begin{bmatrix}0\\0\\1\\-1\\0\\0**\end{bmatrix} - 2\begin{bmatrix}0\\0\\1\\0\\1\\1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\\-2\\-2**\end{bmatrix}$$

Note that we could also take

$$a_2^1 + \alpha a_1^1 + z a_1^1 - 2x_1 = \begin{bmatrix}-2-\alpha\\ z+\alpha\end{bmatrix}$$

with $\alpha \in \mathbb{R}$.
Thus $W^{(1)} = I_4$ and

$$W^{(2)} = \left[\begin{array}{cc|cc} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Now applying $W^{(2)}$ gives

$$NM = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

In DM the last column is replaced by $[-2 \;\; z]^T$ and in RSM the last column is replaced by the residual sequence given above.

For $n = 0$

$$V_1 = V^{(1)}V^{(2)}PW^{(1)}W^{(2)} = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & z-2 \\ 0 & 0 & 1 & z \end{array}\right]$$

For $n = 1$ we find the following transformations summarized in

$$\begin{aligned} V_2 &= V^{(1)}V^{(2)}PW^{(1)}W^{(2)} \\ &= \left[\begin{array}{cc|cc} 1 & 0 & -z & 0 \\ 0 & 0 & -1 & 0 \\ \hline 0 & 1 & z^2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \cdot I \cdot \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \cdot I \cdot \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & -z/2 \\ 0 & 0 & 0 & 1 \end{array}\right] \\ &= \left[\begin{array}{cc|cc} 1 & 0 & 0 & -z \\ 0 & 0 & 0 & -1 \\ \hline 0 & 1 & 0 & z^2 \\ 0 & 0 & 1 & -z/2 \end{array}\right] \end{aligned}$$

So that we get

$$dM = [2 \;\; 1 \mid 1 \;\; 2]$$

$$NM = \left[\begin{array}{cc|cc} 0 & 0 & 1 & -z/2 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

$$DM = \left[\begin{array}{cc|cc} 1 & -1 & -2 & -z^2 \\ 0 & 1 & z & z^2/2 \end{array}\right]$$

$$RSM = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] z^0 +$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] z^{-1} +$$

$$\left[\begin{array}{cc|cc} 1 & 1 & -2 & * \\ 1 & -1 & -2 & * \end{array}\right] z^{-2} +$$

$$\left[\begin{array}{cc|cc} 0 & 0 & * & * \\ 0 & 0 & * & * \end{array}\right] z^{-3} + \cdots$$

The results for $n = 2$ are

$$V_3 = V^{(1)}V^{(2)}PW^{(1)}W^{(2)} = V^{(1)}$$

$$= \left[\begin{array}{cc|cc} 0 & 0 & 4 & 0 \\ 0 & 1 & -2z & 0 \\ \hline 1 & 0 & z & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

so that

$$NM = \left[\begin{array}{cc|cc} 1 & 0 & z & -z/2 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

$$DM = \left[\begin{array}{cc|cc} -2 & -1 & +4 & -z^2 \\ z & 1 & -2z+z^2 & z^2/2 \end{array}\right]$$

$$RSM = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] z^0 +$$

$$\left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] z^{-1} +$$

$$\left[\begin{array}{cc|cc} -2 & -1 & * & * \\ -2 & -1 & * & * \end{array}\right] z^{-2} +$$

$$\left[\begin{array}{cc|cc} * & 0 & * & * \\ * & 0 & * & * \end{array}\right] z^{-3} + \cdots$$

$$V_{0,1} = V_0V_1 = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & z-2 \\ 0 & 0 & 1 & z \end{array}\right] = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \hline 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & z \end{array}\right]$$

and

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & z \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} z^{-1} + \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} z^{-2} + \begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix} z^{-3} + \begin{bmatrix} 8 & 8 \\ 0 & 0 \end{bmatrix} z^{-4} + \cdots$$

$$V_{0,2} = V_{0,1}V_2 = \left[\begin{array}{cc|cc} 0 & 0 & 1 & -z/2 \\ 0 & 0 & 0 & -1 \\ \hline 1 & -1 & -2 & -z^2 \\ 0 & 1 & z & z^2/z \end{array}\right]$$

and

$$\begin{bmatrix} 1 & -z/2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -2 & -z^2 \\ z & z^2/2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} z^{-1} + \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} z^{-2} + \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} z^{-3} + \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} z^{-4} + \cdots$$

$$V_{0,3} = V_{0,2}V_3 = \left[\begin{array}{cc|cc} 1 & 0 & z & -z/2 \\ 0 & 0 & 0 & -1 \\ \hline -2 & -1 & -\frac{}{4} & -z^2 \\ z & 1 & -2z+z^2 & z^2/2 \end{array}\right]$$

and

$$\begin{bmatrix} z & -z/2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 4 & -z^2 \\ z^2-2z & z^2/2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} z^{-1} + \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} z^{-2} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} z^{-3} + \begin{bmatrix} -2 & -4 \\ -2 & -4 \end{bmatrix} z^{-4} + \cdots$$

Example 2

Consider the 5 Markov parameters given by Antoulas [2].

$$M(z) = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 0 \end{bmatrix} z^{-1} + \begin{bmatrix} 2 & 2 & 1 \\ 4 & 4 & 0 \end{bmatrix} z^{-2} + \begin{bmatrix} 4 & 6 & 3 \\ 8 & 10 & 2 \end{bmatrix} z^{-3} +$$

$$\begin{bmatrix} 8 & 11 & 6 \\ 16 & 21 & 7 \end{bmatrix} z^{-4} + \begin{bmatrix} 17 & 25.5 & 15.5 \\ 34 & 46 & 21 \end{bmatrix} z^{-5} + \cdots$$

We take $Y_0 \neq I$

$$V_0 = \left[\begin{array}{cc|ccc} 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$RSM = \left[\begin{array}{cc|ccc} -1 & -1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 \\ \hline 0 & 0 & 4 & 2 & 3 \\ 0 & 0 & 8 & 3 & 6 \\ \hline 0 & 0 & 16 & 5 & 7 \\ 0 & 0 & 17 & 8.5 & 15.5 \\ \hline 0 & 0 & 34 & 12 & 21 \end{array}\right]$$

The computations for level $n = 0$ are summarized as

$$V_1 = V^{(1)}V^{(2)}V^{(3)}PW^{(1)}W^{(2)}W^{(3)}$$

$$= \left[\begin{array}{cc|ccc} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] . I . \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & z \end{array}\right] \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] . I . \left[\begin{array}{cc|ccc} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \cdot \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ \hline 0 & 0 & 1 & 0 & -z \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{cc|ccc} 0 & 0 & 0 & 1 & -1 \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{1} \\ 1 & 0 & 0 & z-2 & -z+2 \\ 0 & 0 & 1 & 0 & -z \\ 0 & 1 & 0 & 0 & z-1 \end{array}\right]$$

with residual series :

$$RSM = \left[\begin{array}{cc|ccc} 0 & 0 & 0 & 0 & 0 \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\ 1 & 1 & 0 & 0 & 0 \\ \underline{2} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\ 2 & 1 & 0 & 0 & 0 \\ \underline{4} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\ 4 & 3 & 2 & 0 & 0 \\ \underline{8} & \underline{2} & \underline{2} & \underline{0} & \underline{0} \\ 8 & 6 & 3 & 1 & 0 \\ \underline{16} & \underline{7} & \underline{5} & \underline{2} & \underline{0} \\ 17 & 15.5 & 8.5 & * & * \\ 34 & 21 & 12 & * & * \end{array}\right]$$

In step 2, no computations are done, i.e. $V_2 = I$. In step 3 we have

$$V_3 = V^{(1)}V^{(2)}V^{(3)}PW^{(1)}W^{(2)}W^{(3)}$$

$$= \left[\begin{array}{cc|ccc} 0 & 0 & -1 & 0 & 0 \\ \underline{0} & \underline{1} & \underline{-1} & \underline{0} & \underline{0} \\ 1 & 0 & z^2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] . I . \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ \underline{0} & \underline{1} & \underline{0} & \underline{0} & \underline{0} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] I . I . I$$

$$= \left[\begin{array}{cc|ccc} 0 & 0 & 0 & 0 & -1 \\ \underline{0} & \underline{1} & \underline{0} & \underline{0} & \underline{-1} \\ 1 & 0 & 0 & 0 & z^2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

With residuals :

$$RSM = \left[\begin{array}{cc|ccc} 0 & 0 & 0 & 0 & 0 \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\ 0 & 1 & 0 & 0 & 0 \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\ 0 & 1 & 0 & 0 & 0 \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{1} \\ 2 & 3 & 0 & 0 & 1.5 \\ \underline{2} & \underline{2} & \underline{0} & \underline{0} & \underline{2} \\ 3 & 6 & 1 & 0 & * \\ \underline{5} & \underline{7} & \underline{2} & \underline{0} & \underline{*} \\ 8.5 & 15.5 & * & * & * \\ 12 & 21 & * & * & * \end{array}\right]$$

The computations for $n = 3$ give

$$V_4 = V^{(1)}V^{(2)}V^{(3)}PW^{(1)}W^{(2)}W^{(3)}$$

$$= I\,.\,I\,.\left[\begin{array}{cc|cc|c} 1 & 0 & 0 & 0 & -z^2/2 \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{\quad 1 \quad} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & z \end{array}\right] I\,.\,I\,.\,I\,.\left[\begin{array}{cc|cc|c} 1 & 0 & 0 & 0 & -z/2 \\ \underline{0} & \underline{1} & \underline{0} & \underline{0} & \underline{1.5} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{cc|cc|c} 1 & 0 & 0 & 0 & -z^2/2-z/2 \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{\quad 1 \quad} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & z+1.5 \end{array}\right]$$

with residuals

$$RSM = \left[\begin{array}{cc|ccc} 0 & 0 & 0 & 0 & 0 \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\ 0 & 0 & 0 & 0 & 0 \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\ 0 & 0 & 0 & 0 & 0 \\ \underline{0} & \underline{1} & \underline{0} & \underline{0} & \underline{0} \\ 2 & 1.5 & 0 & 0 & * \\ \underline{2} & \underline{2} & \underline{0} & \underline{0} & \underline{*} \\ 3 & * & 1 & 0 & * \\ \underline{5} & \underline{*} & \underline{2} & \underline{0} & \underline{*} \\ 8.5 & * & * & * & * \\ 12 & * & * & * & * \end{array}\right]$$

Finally for $n = 4$ we get

$$V_5 = V^{(1)}V^{(2)}V^{(3)}PW^{(1)}W^{(2)}W^{3}$$

$$= \left[\begin{array}{cc|ccc} 1 & 0 & -0.5z & 0 & 0 \\ \underline{0} & \underline{0} & \underline{\quad 1 \quad} & \underline{0} & \underline{0} \\ 0 & 1 & z^2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] I\,.\,I\,.\left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ \underline{0} & \underline{1} & \underline{0} & \underline{0} & \underline{0} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] .I\,.I\,.I$$

$$= \left[\begin{array}{cc|ccc} 1 & 0 & 0 & -0.5z & 0 \\ \underline{0} & \underline{0} & \underline{0} & \underline{-1} & \underline{\quad 0 \quad} \\ 0 & 1 & 0 & z^2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$V_{0,1} = V_{0,2} = V_0V_1V_2 = \left[\begin{array}{cc|ccc} 0 & 0 & 0 & 1 & 0 \\ \underline{0} & \underline{0} & \underline{0} & \underline{2} & \underline{-2} \\ 1 & 0 & -1 & z-2 & 2 \\ 0 & 0 & 1 & 0 & -z \\ 0 & 1 & 0 & 0 & z-1 \end{array}\right]$$

gives the convergent

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} -1 & z-2 & 2 \\ 1 & 0 & -z \\ 0 & 0 & z-1 \end{bmatrix}^{-1}$$

$$V_{0,3} = V_{0,2}V_3 = \left[\begin{array}{ccc|ccc} 0 & 0 & & 1 & 0 & 0 \\ 0 & 0 & & 2 & -2 & 0 \\ \hline -1 & 0 & & z-2 & 2 & -1-z^2 \\ 1 & 0 & & 0 & -z & z^2 \\ 0 & 1 & & 0 & z-1 & -1 \end{array}\right]$$

gives the convergent

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -2 & 0 \end{bmatrix} \begin{bmatrix} z-2 & 2 & -1-z^2 \\ 0 & -z & z^2 \\ 0 & z-1 & -1 \end{bmatrix}^{-1}$$

and

$$V_{0,4} = V_{0,3}V_4 = \left[\begin{array}{cc|ccc} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 \\ \hline -1 & -1-z^2 & z-2 & 2 & -z^3-z^2-0.5z-1.5 \\ 1 & z^2 & 0 & -z & z^3+z^2-0.5z \\ 0 & -1 & 0 & z-1 & -z-0.5 \end{array}\right]$$

gives the convergent

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -2 & 0 \end{bmatrix} \begin{bmatrix} z-2 & 2 & -z^3-z^2-0.5z-1.5 \\ 0 & -z & z^3+z^2-0.5z \\ 0 & z-1 & -z-0.5 \end{bmatrix}^{-1}$$

and finally

$$V_{0,5} = V_{0,4}V_5 = \left[\begin{array}{cc|ccc} 0 & 1 & 0 & z^2 & 0 \\ 0 & 2 & -2 & 2z^2 & 0 \\ \hline -1 & z-2 & 2 & z^3-z^2+0.5z+1 & -z^3-z^2-0.5z-1.5 \\ 1 & 0 & -z & -z^2-0.5z & z^3+z^2-0.5z \\ 0 & 0 & z-1 & 1 & -z-0.5 \end{array}\right]$$

gives a minimal partial realization of all the 5 Markov parameters :

$$\begin{bmatrix} 0 & z^2 & 0 \\ -2 & 2z^2 & 0 \end{bmatrix} \begin{bmatrix} 2 & z^3-z^2+0.5z+1 & -z^3-z^2-0.5z-1.5 \\ -z & -z^2-0.5z & z^3+z^2-0.5z \\ z-1 & 1 & -z-0.5 \end{bmatrix}^{-1} =$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 0 \end{bmatrix} z^{-1} + \begin{bmatrix} 2 & 2 & 1 \\ 4 & 4 & 0 \end{bmatrix} z^{-2} + \begin{bmatrix} 4 & 6 & 3 \\ 8 & 10 & 2 \end{bmatrix} z^{-3} + \begin{bmatrix} 8 & 11 & 6 \\ 16 & 21 & 7 \end{bmatrix} z^{-4} +$$

$$\begin{bmatrix} 17 & 25.5 & 15.5 \\ 34 & 46 & 21 \end{bmatrix} z^{-5} + \begin{bmatrix} 37 & 51.5 & 33 \\ 72.5 & 99 & 52 \end{bmatrix} z^{-6} + \cdots .$$

4. Conclusion

We have adapted algorithm A of [10] and [11], which gives a solution of the minimal partial realization problem for matrix sequences, to get algorithm D, which can be seen as A generalization of THE scalar algorithm of Euclides (See notes 7 and 8). Two examples illustrate the algorithm.

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