AN ALGEBRAIC METHOD TO SOLVE THE MINIMAL PARTIAL REALIZATION PROBLEM FOR MATRIX SEQUENCES

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ABSTRACT

We give a new algebraic method to construct all minimal partial realizations of a finite sequence of constant matrices. We use the minimal number of parameters to characterize all these minimal partial realizations. The solution is obtained in the input-output canonical form of Beghelli and Guidorzi. A new interpretation of this form is given, relating it with linear dependencies of rows of the block Hankel matrix.
Three examples are worked out to illustrate the method.

1. Introduction

This paper is a generalization to matrix sequences of the method given in [7] to solve the minimal partial realization problem for scalar sequences. In part 2 we define the (minimal) (partial) realization problem for matrix sequences.
In part 3 the block Hankel matrix based on the matrix sequence is decomposed and from this decomposition the solution of the minimal realization problem can be constructed which is in the input-output canonical form of Beghelli and Guidorzi.
In part 4 we use the decomposition of part 3 to develop an algebraic algorithm to give all solutions of the minimal partial realization problem in input-output canonical form, i.e. with the minimal number of parameters.
Part 5 contains some examples.

2. The minimal partial realization problem

Consider an infinite sequence of $p \times m$ matrices $H_1, H_2, H_3, \ldots$.
The rational matrix function $F(z)^{-1}G(z)$ with $F(z)$ a regular $p \times p$ polynomial matrix and $G(z)$ a $p \times m$ polynomial matrix is a **realization** of the matrix series $H_1, H_2, H_3, \ldots$ iff

$$F(z)^{-1}G(z) = H_1 z^{-1} + H_2 z^{-2} + \cdots, \quad z \to \infty.$$

If N is finite and $F(z)^{-1}G(z) = H_1 z^{-1} + H_2 z^{-2} + \cdots H_N z^{-N} + O(z^{-N-1}), \ z \to \infty$ we call $F(z)^{-1}G(z)$ a **partial realization of order N.**

The degree n of the determinant of the denominator matrix polynomial $F(z)$ is called the **degree** of the (partial) realization.

If there is no other (partial) realization with smaller degree the (partial) realization is called minimal.

Note that $F(z)^{-1}G(z)$ is strictly proper, i.e. if we write $F(z)^{-1}G(z)$ as $F(z)^{-1}G(z) = H(z^{-1})$ then every entry $H_{i,j}(z^{-1})$ in the rational matrix function $H(z^{-1})$ is a strictly proper function i.e. the numerator polynomial has a smaller degree than the denominator polynomial.

The minimal partial realization problem for matrix sequences has applications for linear systems with multiple inputs and multiple outputs [6].

The polynomial matrix $F(z)$ can be considered as a set of p polynomial column vectors or as a set of p polynomial row vectors. By F_{hr} we shall denote the matrix consisting of the p highest degree coefficients of the polynomial row vectors and by F_{hc} we denote the corresponding matrix for the columns.

A (minimal) realization $F(z)^{-1}G(z)$ is said to be in **input-output canonical form** [1,2,3,4] if F_{hc} is the unit matrix and if F_{hr} is unit lower triangular.

Every realization can be brought into this canonical form with a unimodular transformation $U(z)$

$$(U(z)F(z))^{-1}(U(z)G(z)) = F(z)^{-1}G(z).$$

3. Decomposition of the block Hankel matrix

In this section a decomposition of the block Hankel matrix is given on which the algebraic algorithm in the next section will be based.

Consider the infinite sequence $\{H_1, H_2, H_3, \ldots\}$ of $p \times m$ matrices.

The block Hankel matrices connected with this series are denoted as :

$$H_{p,q} = \begin{bmatrix} H_1 & \cdots & H_q \\ \vdots & & \vdots \\ H_p & \cdots & H_{p+q-1} \end{bmatrix} \quad \text{for } p,q = 1,2,\cdots .$$

The l-th row of $H_{\infty,q}$ is denoted as $h_{l,q}$ (not the l-th block row).
The rows of $H_{\infty,\infty}$ are called Hankel rows and denoted as $h_{i,\infty} \equiv h_i$. If a Hankel row is linearly independent of the previous Hankel rows we shall indicate it with a $\times$ and if it is linearly dependent as $\bullet$. Now arrange the Hankel rows in a $p \times \infty$ table in groups of p rows : In column 0 of the table we place the rows $h_1, h_2, \ldots, h_p$ from top to bottom. In column 1, place the Hankel rows $h_{p+1}, h_{p+2}, \ldots, h_{2p}$ etc... . In general entry (i,j) will contain row h_{i+jp}.

Lets call this the row table. The natural ordering of the elements in this row table corresponds to $(i_1,j_1) < (i_2,j_2)$ iff $i_1+j_1p < i_2+j_2p$. The crate table is derived from the row table if each entry is replaced by a $\times$ if it is linearly independent on the naturally preceding entries and a $\bullet$ otherwise.

	0	1	2	$\cdots$
1	h_1	h_{p+1}	h_{2p+1}	$\cdots$
2	h_2	h_{p+2}	h_{2p+2}	$\cdots$
3	h_3	h_{p+3}	h_{3p+3}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
p	h_p	h_{2p}	h_{3p}	$\cdots$

row table

	0	1	2	3	4	5	$\cdots$
1	$\times$	$\times$	$\times$	$\times$	$\bullet$	$\bullet$	$\cdots$
2	$\times$	$\times$	$\times$	$\bullet$	$\bullet$	$\bullet$	$\cdots$
3	$\times$	$\times$	$\times$	$\times$	$\times$	$\bullet$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
p	$\times$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\cdots$

crate table

Let us call ν_i the column index of the first $\bullet$ appearing in row i of the crate table.
The numbers $\nu_1, \nu_2, \ldots, \nu_p$ are called the observability indices in system theory. Because of the block Hankel-structure, row i of the crate table will contain only dots from column ν_i on. Hence the crate table contains $n = \sum_i^p \nu_i$ crosses and this equals the number of linear independent rows of $H_{\infty,\infty}$, hence the rank of $H_{\infty,\infty}$. Note that controllability indices $\mu_1, \mu_2, \ldots, \mu_m$ can be similarly defined when working with columns of $H_{\infty,\infty}$. Because there must be n linearly independent columns we shall also have $n = \sum_{j=1}^m \mu_j$.

Place the Hankel rows corresponding to a $\times$ in their natural order and number them $j_1, j_2, \ldots, j_n$. Note that $h_{i+\nu_i p}$ is a $\bullet$ row and that $h_{i+(\nu_i-1)p}, h_{i+(\nu_i-2)p}, \ldots$ are all $\times$ rows.
Note also that there is a 1-1 relation between $\{\nu_1, \ldots, \nu_p\}$ and $\{j_1, j_2, \ldots, j_n\}$.

Every row of $H_{\infty,\infty}$ can be expressed as a linear combination of rows $h_{j_1}, h_{j_2}, \ldots, h_{j_n}$. Hence it follows that we can decompose the infinite block Hankel matrix as

$$H_{\infty,\infty} = A_{\infty,n}\, B_{n,\infty} \tag{1}$$

with

$$\begin{bmatrix} a_{j_1,n} \\ a_{j_2,n} \\ \vdots \\ a_{j_n,n} \end{bmatrix} = I_{n,n} \quad \text{and} \quad B_{n,\infty} = \begin{bmatrix} h_{j_1} \\ \vdots \\ h_{j_n} \end{bmatrix}$$

and with $a_{j,n}$ for $j_i < j < j_{i+1}$ $(j_0 = 0,\ j_{n+1} = \infty)$ of the form

$$a_{j,n} = [a_{j,n}^{(j_1)} a_{j,n}^{(j_2)} \cdots a_{j,n}^{(j_i)} \ 0 \cdots 0]$$

so that

$$h_j = \sum_{k=1}^{i} a_{j,n}^{(j_k)} h_{j_k}.$$

For $i = 1,2,\ldots,p$, the row $h_{i+\nu_i p}$ is a $\bullet$ row and

$$h_{i+\nu_i p} = a_{i+\nu_i p,n}\, B_{n,\infty} \quad , \quad i = 1,2,\ldots,p.$$

If S represents the shift operator over p Hankel rows i.e. $Sh_i = h_{i+p}$, then if h_{j_k} corresponds to the entry (m_k,l_k) in the Crate table, it is clear that

$$h_{j_k} = S^{l_k} h_{m_k} \quad , \quad 1 \leqslant m_k \leqslant p.$$

Therefore

$$\begin{aligned} h_{i+\nu_i p} &= \sum_{k=1}^{n} a_{i+\nu_i p,n}^{(j_k)} h_{j_k} \\ &= \sum_{j=1}^{p} a_{ij}(S) h_j \end{aligned}$$

with

$$a_{ij}(S) = a_{i+\nu_i p,n}^{(j)} + a_{i+\nu_i p,n}^{(j+p)} S + \cdots + a_{i+\nu_i p,n}^{(j+lp)} S^l$$

with

$$\begin{aligned} l &= \min\{\nu_i,\nu_j-1\} \quad \text{if } j < i \\ &= \min\{\nu_i-1,\nu_j-1\} \quad \text{if } j \geqslant i. \end{aligned} \tag{2}$$

When $l = -1$, we set $a_{ij}(S) = 0$.

Now introduce the z-transform :

$$h_i(z^{-1}) = h_i\, [z^{-1} z^{-2} \cdots]^T$$

then

$$S^k h_i(z^{-1}) = h_{i+k}(z^{-1}) + p_i(z)$$

with $p_i(z)$ a polynomial of degree at most $k-1$.

Hence we can translate the previous relations as

$$z^{\nu_i} h_i(z^{-1}) = \sum_{j=1}^{p} a_{i,j}(z) h_j(z^{-1}) + b_i(z) \quad , \quad i = 1,\ldots,p$$

with $b_i(z)$ a polynomial of degree at most ν_i-1.

Or

$$\operatorname{diag}(z^{\nu_1},z^{\nu_2},\ldots,z^{\nu_p}) \begin{bmatrix} h_1(z^{-1}) \\ \vdots \\ h_p(z^{-1}) \end{bmatrix} - \begin{bmatrix} a_{1,1}(z) & \cdots & a_{1,p}(z) \\ \vdots & & \vdots \\ a_{p,1}(z) & \cdots & a_{p,p}(z) \end{bmatrix} \begin{bmatrix} h_1(z^{-1}) \\ \vdots \\ h_p(z^{-1}) \end{bmatrix} = \begin{bmatrix} b_1(z) \\ \vdots \\ b_p(z) \end{bmatrix}. \tag{3}$$

Thus, if we set

$$A(z) = \operatorname{diag}(z^{\nu_1},z^{\nu_2},\ldots,z^{\nu_p}) - \begin{bmatrix} a_{1,1}(z) & \cdots & a_{1,p}(z) \\ \vdots & & \vdots \\ a_{p,1}(z) & \cdots & a_{p,p}(z) \end{bmatrix} \tag{4}$$

$H(z^{-1}) = \begin{bmatrix} h_1(z^{-1}) \\ \vdots \\ h_p(z^{-1}) \end{bmatrix} = \sum_{k=1}^{\infty} H_k z^{-k}$ and $B(z) = \begin{bmatrix} b_1(z) \\ \vdots \\ b_p(z) \end{bmatrix}$, then (3) gives

$$A(z)H(z^{-1}) = B(z) \tag{5}$$

with $A_{h,c}$ unit lower triangular with row degrees $\nu_1, \nu_2, \ldots, \nu_p$ and $A_{h,r} = I_{p,p}$ with column degrees $\nu_1, \ldots, \nu_p$. This implies that degree det $A(z) = \sum_{i=1}^{p} \nu_i = n =$ rank $H_{\infty,\infty}$. Because $A(z)$ is regular, we have

$$H(z^{-1}) = A(z)^{-1}B(z),$$

which means that $A(z)^{-1}B(z)$ is a partial realization of $\{H_1, H_2, \cdots\}$ and it has the input-output canonical form of Beghelli and Guidorzi [1,2,3,4].
The previous construction reveals the tight relation that exists between this canonical form and the linear dependencies of the Hankel rows. This interpretation is new.

The realization we constructed is minimal. If it were not, there would exist a realization of lower degree which can be brought into its input-output canonical form $A^*(z)^{-1}B^*(z)$ with deg det $A^*(z) <$ deg det $A(z) = n$ and with row degrees $\nu_1^*, \nu_2^*, \ldots, \nu_p^*$. None of the ν_i^* can be lower than ν_i because this would imply that $h_{i+\nu_i^* p}$ is linearly dependent on the previous Hankel rows with $\nu_i^* < \nu_i$ and this is impossible by construction of the ν_i. Hence deg det $A^*(z) = \sum_{i=1}^{p} \nu_i^* \geqslant n$ which contradicts our assumption.

In the scalar case $(p = m = 1)$ we have $j_k = k$, $k = 1,2,\ldots,n-1$ and $\nu_1 = n$, so that $A(z) = z^n - a_{1,1}(z)$ and $\beta_{n,\infty} = H_{n,\infty}$ and $A_{\infty,n}$ becomes the infinite companion matrix of $A(z)$ [7]. The decomposition (1) generalizes the scalar relation

$$H_{\infty,\infty} = A_{\infty,n}\, H_{n,\infty}$$

with $A_{\infty,n}$ the infinite companion of $A(z)$. For this reason we shall call $A_{\infty,n}$ in the non-scalar case the infinite companion matrix of the polynomial matrix $A(z)$. Given an arbitrary $p \times p$ polynomial matrix, its infinite companion is completely defined in terms of $A(z)$. First by left multiplication with a unimodular matrix we can make A_{hr} unit lower triangular and $A_{hc} = I_{p\times p}$. This brings $A(z)$ into the form (4) with restrictions (2). This form defines $\nu_1, \ldots, \nu_p$ and the rows $a_{i+\nu_i p,n}$ $i = 1,2,\ldots,p$ of $A_{\infty,n}$ (see (2)). The remaining rows can be found from this. Indeed, row $i+(j+1+\nu_i)p$ of $A_{\infty,n}$ is defined in terms of row $i+(j+\nu_i)p$ of $A_{\infty,n}$ and the rows $i+\nu_i p$ $i = 1,\ldots,p$. This can be seen as follows :

$$\begin{aligned} h_{i+(j+1+\nu_i)p,\infty} &= a_{i+(j+1+\nu_i)p,n}\, B_{n,\infty} \\ &= a_{i+(j+\nu_i)p,n} \begin{bmatrix} h_{j_1+p,\infty} \\ \vdots \\ h_{j_n+p,\infty} \end{bmatrix}. \end{aligned}$$

Now either $h_{j_k+p,\infty}$ is a $\times$ row and then j_k+p equals some j_t or $h_{j_k+p,\infty}$ is a $\bullet$ row and then j_k+p equals some $t+\nu_t p$. Thus there is some matrix $C_{n,n}$ such that

$$h_{i+(j+1+\nu_i)p,\infty} = a_{i+(j+\nu_i)p,n}\, C_{n,n} B_{n,\infty}$$

with the k-th row of $C_{n,n}$ equal to

$$\begin{cases} [0 \cdots 0 \overset{\overset{t}{\downarrow}}{1} 0 \cdots 0] & \text{if} \quad j_k+p = j_t \\ a_{t+\nu_t p,n} & \text{if} \quad j_k+p = t+\nu_t p \end{cases}$$

Thus $A_{\infty,n}$ is completely defined in terms of $A(z)$.

A new algebraic algorithm to solve the minimal realization problem is as follows :

Construct a constant matrix $Q_{\infty,\infty}$ such that

$$H_{\infty,\infty} \cdot Q_{\infty,\infty} = [A'_{\infty,n}\ 0_{\infty,\infty}] \tag{6}$$

with

$$\begin{bmatrix} a'_{j'_1,n} \\ \vdots \\ a'_{j'_n,n} \end{bmatrix} = I_{n,n}$$

and $j'_1 < j'_2 < \cdots < j'_n$ as small as possible.

Then we can write (6) as :

$$\begin{aligned} H_{\infty,\infty} &= [A'_{\infty,n}\ 0_{\infty,\infty}] \cdot Q^{-1}_{\infty,\infty} \\ &= A'_{\infty,n} \cdot B'_{n,\infty} \end{aligned}$$

with $B'_{n,\infty} = [Q^{-1}_{\infty,\infty}]_{n,\infty}$.

Because the decomposition $H_{\infty,\infty} = A_{\infty,n} \cdot B_{n,\infty}$ of (1) is unique we get that $A'_{\infty,n} = A_{\infty,n}$ and $B'_{n,\infty} = B_{n,\infty}$ and $j'_i = j_i$.
From the rows $a_{i+\nu_i p,n}$ of $A'_{\infty,n} = A_{\infty,n}$ we can construct the polynomial matrix $A(z)$. The numerator polynomial matrix $B(z)$ can be found as the polynomial part of

$$A(z)H(z^{-1}).$$

In the next section we are solving the problem when only the first N elements of the infinite sequence are known $\{H_1, \ldots, H_N\}$.

4. An algebraic algorithm to solve the minimal partial realization problem

When we only know a partial sequence $\{H_1, H_2, \ldots, H_N\}$ of N $p \times m$ matrices, it is clear from the block Hankel structure that there exists at least one extension sequence $\{H^*_{N+1}, H^*_{N+2}, \cdots\}$ such that the rank of the block Hankel matrix determined by the specified entries is equal to the rank of $H_{\infty,\infty}$ based on the infinite sequence $\{H_1, H_2, \ldots, H_N, H^*_{N+1}, H^*_{N+2}, \cdots\}$.

This implies that the knowledge of the linear dependencies of the block Hankel matrix $H_{N,N}$ is enough to solve the problem. When we use the ● and × notation to indicate the linear dependency, linear independency respectively of rows and columns of the block Hankel matrix $H_{N,N}$, only the unspecificied values of the extension sequence lying at the crossing of a ●-row and a ●-column have to get specific values to maintain the rank of the block Hankel matrix without increasing it. The elements of the extension sequence that are nowhere at the crossing of a ●-row and a ●-column can be chosen freely.

The algebraic algorithm consists of a sequence of elementary column operations performed on the block Hankel matrix $H_{N,N}$

$$H_{N,N} = \begin{bmatrix} H_1 & \cdots & H_{N-1} & H_N \\ H_2 & \cdots & H_N & H^*_{N+1} \\ \vdots & & \vdots & \vdots \\ H_N & \cdots & H^*_{2N-2} & H^*_{2N-1} \end{bmatrix}.$$

The algorithm consists of 3 parts.

Part 1

In the first part, the linear dependencies of the Hankel rows and columns are computed. None of the unspecified elements are needed. So, if we speak of Hankel rows or columns we mean only those parts which contain the specified elements. For these do the following operations :

1. For $j = 1,2,\ldots,N\,m$

   1.1 **If** there is a nonzero element in column j **then**

   — Take the first one and mark the corresponding row r_j and the column j by $\times$.

   — Mark also rows $r_j-p, r_j-2p, \cdots$ by $\times$.

   — Divide column j by this element.

   — Eliminate all the elements on row r_j in columns $j+1, j+2, \cdots$ by subtracting a suitable multiple of column j.

   1.2 **If** all the elements in column j are zero **then**

   — Mark column j by $\bullet$.

2. Mark all the rows which are not $\times$ rows by $\bullet$.

3. The rank of the original block Hankel matrix n equals the number of nonzero columns = the number of $\times$ rows = the number of $\times$ columns of the matrix which has been generated in step 1.

4. For $i = 1,2,\ldots,p$, the observability indices ν_i are defined as the smallest integers such that $i + \nu_i p$ is a $\bullet$ row.

5. For $i = 1,2,\ldots,p$, the controllability indices μ_i are defined as the smallest integers such that $i + \mu_i m$ is a $\bullet$ column.

6. The $\times$ rows have the numbers $j_1, j_2, \ldots, j_n$.

Part 2

1. Now the free elements are introduced. Because the transformations used in part 1 don't change the positions of the linear independent rows and columns, the free elements will be found at the same position as in the original block Hankel matrix. Now the unspecified elements in the original block Hankel matrix appear in several places (cfr. the block Hankel structure), but only those elements which are in none of their appearances at the crossing of a $\bullet$-row and a $\bullet$-column will be free, the free elements will be found at the same places in our transformed matrix. We introduce them in rows $i + \nu_i p$, which may require the introduction of extra rows at the bottom of the matrix. More precisely position $(\nu_i p + i,\ lm + j)$ will contain a free element if none of the positions $(\nu_i - k)p + i,\ (l+k)m + j),\ k = \cdots, -1, 0, 1, 2, \cdots$ is at the crossing of a $\bullet$ row and $\bullet$ column.

2. Consider the $\times$ rows r_j and eliminate the nonzero elements in this row which precede the 1 in column j by subtracting a suitable multiple of column j from the previous columns. These operations are also performed on the free elements just introduced in rows $Np+1, \ldots, (N+1)p$.

Part 3

Permute the columns, so that the n $\times$ columns are the first n columns and the number of initial zeros increases from left to right. In other words rows $j_1, j_2, \ldots, j_n$ are brought into the form $[I_n, 0_{n,Nm-n}[$.
The first n columns form now $A_{(N+1)p,n}$. From this matrix, $A(z)$ can be constructed as described before and $B(z)$ is the polynomial part of $A(z)H(z)$.

Note that the moment where the free elements are introduced is important. The proposed scheme ensures that $A(z)$ will be linear in the free elements.

5. Examples

(a) Consider example 14.1 of [6]. The finite sequence is

$$\left\{\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}\cdot\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}\cdot\begin{bmatrix}2 & 3\\ 0 & 0\end{bmatrix}\right\}.$$

So $N=3$ and $p=m=2$. The block Hankel matrix that gives enough information about the linear dependencies is $H_{3,3}$:

$$H_{3,3} = \begin{array}{c} \\ \times \\ \times \\ \\ \\ \\ \\ \end{array}\begin{array}{c}\begin{array}{cccccc}\times & & & & & \end{array}\\ \begin{bmatrix}1 & 1 & 1 & 1 & 2 & 3\\ 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 1 & 2 & 3 & & \\ 0 & 0 & 0 & 0 & & \\ 2 & 3 & & & & \\ 0 & 0 & & & & \end{bmatrix}\end{array}$$

The first element of the first column is different from zero, so we use this element as a pivot element to make the following entries of the first row zero.

$$H_{3,3}Q_1 = \begin{array}{c} \\ \times \\ \\ \times \\ \\ \times \\ \\ \end{array}\begin{array}{c}\begin{array}{cccccc} & \times & \times & & & \end{array}\\ \begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 1 & 2 & & \\ 0 & 0 & 0 & 0 & & \\ 2 & 1 & & & & \\ 0 & 0 & & & & \end{bmatrix}\end{array}$$

The 5-th element of the second column is the first element of the 5-th column different from zero. So we are going to use that element as pivot element.

Because the 5-th row is linearly independent of the previous rows also the thirth row has to be a X-row (because of the block Hankel structure of $H_{3,3}$).

$$H_{3,3}Q_1Q_2 = \begin{array}{c} \\ \times \\ \\ \times \\ \\ \times \\ \\ \end{array}\begin{array}{c}\begin{array}{cccccc} & \times & \times & \times & & \end{array}\\ \begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 1 & 2 & & \\ 0 & 0 & 0 & 0 & & \\ 2 & 1 & & & & \\ 0 & 0 & & & & \end{bmatrix}\end{array}$$

The 3-th element of the third column is its first element different from zero, so we are going to use that element as pivot element.

$$H_{3,3}Q_1Q_2Q_3 = \begin{array}{c} \\ \times \\ \bullet \\ \times \\ \bullet \\ \times \\ \bullet \end{array}\begin{array}{c}\begin{array}{cccccc} & \times & \times & \times & \bullet & \bullet & \bullet \end{array}\\ \begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 1 & 0 & & & \\ 0 & 0 & 0 & 0 & & & \\ 2 & 1 & & & & & \\ 0 & 0 & & & & & \end{bmatrix}\end{array}$$

The other rows and columns are $\bullet$-rows and $\bullet$-columns.

From this result we get the observability indices which are: $\nu_1 = 3$, $\nu_2 = 0$ and the controllability indices $\mu_1 = 2$, $\mu_2 = 1$.

In the second part of the algorithm we extend $H_{3,3}$ and introduce the three free parameters u, v, w.

$$H_{4,3}Q_1Q_2Q_3 = \begin{array}{c} \\ \times \\ \bullet \\ \times \\ \bullet \\ \times \\ \bullet \\ \bullet \\ \bullet \end{array} \begin{array}{c} \begin{array}{cccccc} \times & \times & \times & \bullet & \bullet & \bullet \end{array} \\ \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & & \\ 0 & 0 & 0 & 0 & & \\ 2 & 1 & & & & \\ 0 & 0 & & & & \\ u & v & w & & & \\ & & & & & \end{array}\right] \end{array}$$

We have to make the (3,1) and (5,1) elements equal to zero.

$$H_{4,3}Q_1Q_2Q_3Q_4 = \begin{array}{c} \\ \times \\ \bullet \\ \times \\ \bullet \\ \times \\ \bullet \\ \bullet \\ \bullet \end{array} \begin{array}{c} \begin{array}{cccccc} \times & & \times & \times & \bullet & \bullet & \bullet \end{array} \\ \left[\begin{array}{cccccc} 1 & & 0 & 0 & 0 & 0 & 0 \\ 0 & & 0 & 0 & 0 & 0 & 0 \\ 0 & & 0 & 1 & 0 & & \\ 0 & & 0 & 0 & 0 & & \\ 0 & & 1 & & & & \\ 0 & & 0 & & & & \\ u-w-2v & & v & w & & & \\ & & & & & & \end{array}\right] \end{array}$$

The third part of the algorithm permutes the first three columns such that the first three elements of the resulting first, third and fifth row form the unit 3×3 matrix :

$$H_{4,3}Q_1Q_2Q_3Q_4P = \begin{array}{c} \\ \times \\ \bullet \\ \times \\ \bullet \\ \times \\ \bullet \\ \bullet \\ \bullet \end{array} \begin{array}{c} \begin{array}{cccccc} & & & & & \end{array} \\ \left[\begin{array}{cccccc} 1 & & 0 & 0 & 0 & 0 & 0 \\ 0 & & 0 & 0 & 0 & 0 & 0 \\ 0 & & 1 & 0 & 0 & & \\ 0 & & 0 & 0 & 0 & & \\ 0 & & & 1 & & & \\ 0 & & & 0 & & & \\ u-w-2v & & w & v & & & \\ & & & & & & \end{array}\right] \end{array}$$

The denominator polynomial matrix $A(z)$ has three free parameters u, v, w :

$$A(z) = \begin{bmatrix} z^3-vz^2-wz-(u-2v-w) & 0 \\ 0 & 1 \end{bmatrix}.$$

The numerator polynomial matrix can be found as : $B(z) =$ the polynomial part of $A(z) . H(z^{-1})$ with

$$H(z^{-1}) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} z^{-1} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} z^{-2} + \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix} z^{-3} + \cdots .$$

this results into

$$B(z) = \begin{bmatrix} (2-v-w)+(1-v)z+z^2 & (3-v-w)+(1-v)z+z^2 \\ 0 & 0 \end{bmatrix}.$$

So $A(z)^{-1}B(z)$ represents all minimal partial realizations with three free parameters. This means that for a different choice for u, v, w we get another rational matrix function $A(z^{-1})B(z)$.

The minimal number of states is three = rank of the block Hankel matrix $H_{3,3} = \sum_{i=1}^{2} \nu_i = \sum_{j=1}^{2} \mu_j$.

(b) Consider example 14.3 of [6].

The finite sequence is equal to :

$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right\}$ so $N=6$, $p=2$, $m=1$.

The block Hankel matrix that gives all the essential information is given by :

$$H_{6,6} = \begin{array}{c} \times \\ \times \end{array} \begin{bmatrix} 1 & 1 & 1 & 2 & 1 & 3 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & 2 & 1 & 3 & \\ 2 & 3 & 4 & 5 & 6 & \\ 1 & 2 & 1 & 3 & & \\ 3 & 4 & 5 & 6 & & \\ 2 & 1 & 3 & & & \\ 4 & 5 & 6 & & & \\ 1 & 3 & & & & \\ 5 & 6 & & & & \\ 3 & & & & & \\ 6 & & & & & \end{bmatrix}$$

Element (1,1) is used as pivot element.

$$H_{6,6}Q_1 = \begin{array}{c} \\ \times \\ \times \end{array} \begin{array}{c} \begin{array}{cccccc} \times & \times & & & & \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 2 & 4 & 3 \\ 1 & 0 & 1 & -1 & 2 & \\ 2 & 1 & 2 & 1 & 4 & \\ 1 & 1 & 0 & 1 & & \\ 3 & 1 & 2 & 0 & & \\ 2 & -1 & 1 & & & \\ 4 & 1 & 2 & & & \\ 1 & 2 & & & & \\ 5 & 1 & & & & \\ 3 & & & & & \\ 6 & & & & & \end{bmatrix} \end{array}$$

Element (2,2) is used as pivot element.

$$H_{6,6}Q_1Q_2 = \begin{array}{c} \\ \times \\ \times \\ \times \end{array} \begin{array}{c} \begin{array}{cccccc} \times & \times & \times & & & \end{array} \\ \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & -2 & \\ 2 & 1 & 0 & -1 & 0 & \\ 1 & 1 & -2 & -1 & & \\ 3 & 1 & 0 & -2 & & \\ 2 & -1 & 3 & & & \\ 4 & 1 & 0 & & & \\ 1 & 2 & & & & \\ 5 & 1 & & & & \\ 3 & & & & & \\ 6 & & & & & \end{array}\right] \end{array}$$

Element (3,3) is used as pivot element.

$$H_{6,6}Q_1Q_2Q_3 = \begin{array}{c} \\ \times \\ \times \\ \times \\ \times \end{array} \begin{array}{c} \begin{array}{cccccc} \times & \times & \times & \times & & \end{array} \\ \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & \\ 2 & 1 & 0 & -1 & 0 & \\ 1 & 1 & -2 & -3 & & \\ 3 & 1 & 0 & -2 & & \\ 2 & -1 & 3 & & & \\ 4 & 1 & 0 & & & \\ 1 & 2 & & & & \\ 5 & 1 & & & & \\ 3 & & & & & \\ 6 & & & & & \end{array}\right] \end{array}$$

Element (4,4) is used as pivot element.

$$H_{6,6}Q_1Q_2Q_3Q_4 = \begin{array}{c} \\ \times \\ \times \\ \times \\ \times \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \begin{array}{c} \begin{array}{cccccc} \times & \times & \times & \times & \bullet & \bullet \end{array} \\ \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & \\ 2 & 1 & 0 & 1 & 0 & \\ 1 & 1 & -2 & 3 & & \\ 3 & 1 & 0 & 2 & & \\ 2 & -1 & 3 & & & \\ 4 & 1 & 0 & & & \\ 1 & 2 & & & & \\ 5 & 1 & & & & \\ 3 & & & & & \\ 6 & & & & & \end{array}\right] \end{array}$$

From this matrix we get that the observability indices are $\nu_1 = \nu_2 = 2$ and the controllability index $\mu_1 = 4$.
So the degree of the minimal partial realization will be 4.

We don't need to introduce free parameters in the second part of the algorithm. We have to make the 4×4 upper left square matrix equal to the unit 4×4 matrix :

$$H_{6,6}Q_1Q_2Q_3Q_4Q_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -2 & -2 & 3 \\ 0 & -1 & -1 & 2 \end{bmatrix}.$$

The denominator polynomial matrix $A(z)$ of the minimal partial realization is unique and equal to :

$$A(z) = \begin{bmatrix} z^z+2z+1 & -3z+2 \\ 0 & z^2-2z+1 \end{bmatrix}.$$

the numerator polynomial vector $B(z)$ is equal to :

$$B(z) = \begin{bmatrix} z \\ z \end{bmatrix}.$$

So the unique minimal partial realization $A(z)^{-1}B(z)$ has degree 4, so it can be realized with 4 states, and is equal to

$$A(z)^{-1}B(z) = \begin{bmatrix} z^3+z^2-z \\ z^3+2z^2+z \end{bmatrix} \quad / \quad z^4-2z^2+1.$$

(c) Consider the scalar finite sequence $\{1,2,3\}$. So $N=3$ and $p=m=1$.

The 3×3 Hankel matrix contains all the necessary information.

$$H_{3,3} = \begin{matrix} \times \end{matrix} \begin{matrix} \\ \times \\ \\ \end{matrix} \begin{matrix} \times \\ \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & \\ 3 & & \end{bmatrix} \end{matrix}$$

(1,1) is the pivot element.

$$H_{3,3}Q_1 = \begin{matrix} \\ \times \\ \times \\ \\ \end{matrix} \begin{matrix} \times & \times \\ \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & \\ 3 & & \end{bmatrix} \end{matrix}$$

(2,2) is the pivot element.

$$H_{3,3}Q_1Q_2 = \begin{matrix} \\ \times \\ \times \\ \bullet \\ \end{matrix} \begin{matrix} \times & \times & \bullet \\ \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & \\ 3 & & \end{bmatrix} \end{matrix}$$

So the controllability and observability index is equal to $\mu_1 = \nu_1 = 2$.

In the second part of the algorithm we have to introduce a free parameter u :

$$H_{3,3}Q_1Q_2 = \begin{array}{c} \\ \times \\ \times \\ \bullet \end{array} \begin{array}{c} \begin{array}{ccc} \times & \times & \bullet \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & \\ 3 & u & \end{bmatrix} \end{array}$$

We make the (2,1) element zero.

$$H_{3,3}Q_1Q_2Q_3 = \begin{array}{c} \\ \times \\ \times \\ \bullet \end{array} \begin{array}{c} \begin{array}{ccc} \times & \times & \bullet \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & \\ 3-2u & u & \end{bmatrix} \end{array}$$

All minimal partial realizations of $\{1,2,3\}$ are given by :

$$\frac{z+(2-u)}{z^2-uz+(2u-3)} = 1z^{-1}+2z^{-2}+3z^{-3}+(-u+6)z^{-4}+(-u^2+9)z^{-5}+\cdots .$$

ADDENDUM : ALL MINIMAL PARTIAL REALIZATIONS IN STATE SPACE FORM

For those who are familiar with linear system theory, we give here a state space representation of the input-output canonical form of [1,2,3,4]. This can be done in several similar ways. We can use methods constructed for a general row-reduced denominator, numerator pair $A(z)^{-1}B(z)$ (see [5] p. 403-439]).
another algorithm was given by Beghelli and Guidorzi themselves (see [1,2,3,4]). We are not going to repeat the method here but we shall directly apply it to the two examples of part 5.

(a) All minimal partial realizations can be presented as :

$$\begin{bmatrix} z^3-vz^2-wz-(u-2v-w) & 0 \\ 0 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} z^2+(1-v)z+(2-v-w) & z^2+(1-v)z+(3-v-w) \\ 0 & 0 \end{bmatrix}.$$

The matrices A and C of the canonical state-space representation of Beghelli and Guidorzi can be written immediately :

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ u-2v-w & w & v \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The matrix B can be found as the solution of :

$$\begin{bmatrix} -w & -v & 1 \\ -v & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} B = \begin{bmatrix} -v-w+2 & -v-w+3 \\ -v+1 & -v+1 \\ 1 & 1 \end{bmatrix}.$$

This gives :

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}.$$

(b) The unique solution of example (b) is equal to :

$$\begin{bmatrix} z^2+2z+1 & -3z+2 \\ 0 & z^2-2z+1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} z \\ z \end{bmatrix}.$$

This leads to the following A and C matrices :

$$A = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -1 & -2 & -2 & 3 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 2 \end{array}\right] \quad C = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

The vector B can be solved from :

$$\left[\begin{array}{cc|cc} 2 & 1 & -3 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

This leads to :

$$B = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}.$$

For further properties of these canonical realizations, see [1,2,3,4].

(c) All minimal partial realizations of example (c) are equal to $\frac{z+(2-u)}{z^2-uz+(2u-3)}$ and can be represented in state space form as :

The matrices A and C of the canonical state-space representation of Beghelli and Guidorzi can be constructed immediately.

$$A = \begin{bmatrix} 0 & 1 \\ -2u+3 & u \end{bmatrix} \qquad C = [1 \quad 0].$$

The matrix B can be found as the solution of :

$$\begin{bmatrix} -u & 1 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 2-u \\ 1 \end{bmatrix}.$$

This results in : $B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

REFERENCES :

[1] S. Beghelli, R. Guidorzi :
Computational and theoretical aspects in the transformations to a canonical input-output form for multivariable systems.
Report nr. 30 - October 1975 (Istituto di automatica, Facoltà di Ingegneria - Università di Bologna).

[2] S. Beghelli, R. Guidorzi :
Computational and theoretical aspects in the realization of linear multivariable systems.
Report nr. 31 - November 1975 (Istituto di automatica, Facoltà di Ingegneria - Università di Bologna).

[3] S. Beghelli, R. Guidorzi :
A new realization algorithm for linear multivariable systems.
Ricerche di Automatica 7, 164-181 (1976 no. 2-3).

[4] S. Beghelli, R. Guidorzi :
A new input-output canonical form for multivariable systems.
IEEE Transactions on automatic control AC-21, 692-696 (1976).
IEEE Transactions on automatic control AC-23, 93 (1978) (Comments).

[5] Thomas Kailath :
Linear Systems.
Prentice Hall, Englewood Cliffs, N.J. 1980.

[6] Marc Van Barel, Adhemar Bultheel :
A minimal partial realization for MIMO systems.
TW report nr. 79, Dept. of Computer Science, K.U.Leuven.

[7] Marc Van Barel, Adhemar Bultheel :
An algebraic method to solve the minimal partial realization problem for scalar sequences.
Submitted to Linear Algebra and Its Applications.