# A minimal partial realization algorithm for MIMO systems III. A parametrization of all solutions

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*Report TW 93, August 1987*

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*ABSTRACT*

In this report we derive all solutions of the minimal partial $k$-realization problem in the form of a polynomial matrix fraction.
All solutions are parametrized by the minimal number of parameters, i.e. a different choice for the parameter values gives a different rational matrix function. As an intermediate result we get an alternative basis for $S(n,\alpha)_k$ based on the (shifted) auxiliary and solution vectors at level $n$. An example is shown to clarify the theory.

## 1. Introduction

This note is a continuation of report TW79 [5] and TW91 [6] on the minimal partial realization of MIMO systems.
We shall give here a derivation of all possible solutions of the minimal partial $k$-realization problem in the form of a polynomial matrix fraction.
All solutions are parametrized by a minimal number of parameters, i.e. a different choice for the parameter values gives a different rational matrix function.

We shall use the terminilogy and the notations of TW79 [5].

When looking for the minimal partial $k$-realization of a $p \times m$ system, we had to solve certain block Hankel systems for a certain level $n$ and a certain degree $\alpha$.
Algorithm $A$ (section 12 of TW79 [5]) was based on a description of the solution space $S(n,\alpha)_k$ of these block Hankel systems using nested bases, described in section 6 of TW79 [5]. These bases were given in terms of solution vectors (for the definition see section 5 of TW79 [5]) in a triangle which contains the solution vectors for level $n$ and all previous levels. These bases were nested in two directions : with respect to $n$ and with respect to $\alpha$. To describe all the solutions for level $n$ we shall first construct a basis for $S(n,\alpha)_k$ which is described in terms of the solution vectors and the auxiliary vectors (for the definition see section 11 of TW79 [5]) of level $n$ only. This will be given in theorem 2.1. The main result is then given in theorem 2.2 which gives all possible solutions with a minimal number of parameters. This theorem is applied to an example.

## 2. Main results

The bases for $S(n,\alpha)_k$ $(n = 0,1,\cdots;\ \alpha = 0,1,2,\cdots)$ we shall give now are described in terms of the $m$ solution vectors $a_j^n(z)$ of degree $\alpha_j$ and the $s$ auxiliary vectors $x_j(z)$ of potential degree $\pi_j$ at level $n$. These bases will be nested with respect to the degree $\alpha$ only.
Note that $\alpha$ can take values $n, n+1, \cdots$, which was never used in TW79 [5], but by definition $S(n,\alpha)_k$ for $\alpha \geqslant n$ will be the space of all polynomial vectors of degree $\leqslant \alpha$ and its basis $B(n,\alpha)_k$ is a basis for this space.

**Theorem 2.1**

*A basis for* $S(n,\alpha)_k$, $n = 0,1,2,\cdots;\ \alpha = 0,1,2,\cdots$ *is given by*

$$B(n,\alpha)_k = \bigcup_{j=1}^{m} \{a_j^n(z)\}_{\alpha_j}^{\alpha} \cup \bigcup_{j=1}^{s} \{x_j(z)\}_{\pi_j}^{\min(\alpha,n)} \equiv A_\alpha \cup X_\alpha$$

*with*

$$\{a(z)\}_\alpha^\beta = \{a(z), za(z), \ldots, z^{\beta-\alpha}a(z)\} \quad \text{if} \quad \alpha \leqslant \beta$$
$$= \phi \quad \text{if} \quad \alpha > \beta$$

**Proof**

The proof is by induction on the degree $\alpha$.
It is clear that the theorem is true for $\alpha = 0$. I.e. a basis for $S(n,0)_k$ is given by the solution vectors of degree 0

$$B(n,0)_k = \{n,0\}_k.$$

Suppose now that the theorem is true for $\alpha - 1$. We prove that it is also valid for $\alpha$ $(\alpha = 1,2,\ldots,n-1)$. Therefore we shall decompose $a(z) \in S(n,\alpha)_k$ in a unique way as a sum

$$a(z) = a_1(z) + a_2(z) + a_3(z)$$

with $a_1(z) \in A_\alpha \backslash A_{\alpha-1}$, $a_2(z) \in X_\alpha \backslash X_{\alpha-1}$ and $a_3(z) \in S(n,\alpha-1)_k$. Since by induction

hypothesis $a_2(z)$ can be written uniquely in terms of the basis $A_{\alpha-1} \cup X_{\alpha-1}$, the induction step will be proved.

a. Because $Ha_j^n$, $\alpha_j \leqslant \alpha$ form a basis for the highest degree coefficients,

$$Ha = a_\alpha = \sum_{\substack{j=1 \\ \alpha_j \leqslant \alpha}}^{m} \nu_j \, Ha_j^n$$

is a unique decomposition. So we have uniquely

$$a(z) = a_1(z) + a_4(z)$$

with

$$a_1(z) = \sum_{\substack{j=1 \\ \alpha_j \leqslant \alpha}}^{m} \nu_j z^{\alpha-\alpha_j} a_j^n(z) \quad \text{and} \quad a_4(z) \in S(n-1,\alpha-1)_k.$$

Note theat $a_4(z)$ has the same residual as $a(z)$ since $Ra_1 = 0$.

b. Let $r_4$ be the residual $Ra_4$ of $a_4(z)$. Then, because the $Rx_j$, $\pi_j \leqslant \alpha$ form a basis for the residual space, we have unique $\mu_j$ such that

$$r_4 = \sum_{\substack{j=1 \\ \pi_j \leqslant \alpha}}^{s} \mu_j Rx_j.$$

Hence, the decomposition

$$a_4(z) = a_2(z) + a_3(z)$$

with

$$a_2(z) = \sum_{\substack{j=1 \\ \pi_j \leqslant \alpha}}^{s} \mu_j z^{\alpha-\pi_j} x_j(z) \quad \text{and} \quad a_3(z) \in S(n,\alpha-1)_k$$

is unique.

For $\alpha > n$, the proof is similar, just note that then $S(n,\alpha)_k$ is the space of polynomials of degree $\leqslant \alpha$ so that in the summation for the highest degree coefficient we can drop the restriction $\alpha_j \leqslant \alpha$ and that $a_4(z)$ will be in $S(n,\alpha-1)_k$, so we don't need part b.

□

Theorem 2.1 gives a basis of $S(n,\alpha)_k$ based on the solution vectors and auxiliary vectors at level $n$. So we can construct all possible solution vectors of level $n$ by using this basis. Because every column reduced denominator polynomial matrix $A(z)$ of a minimal partial $k$-realization of order $N = n+k+1$ corresponds to a choice for the $m$ solution vectors, we can construct all column reduced denominator matrices $A(z)$ by using a specific choice for the solution and auxiliary vectors at level $n$. The corresponding numerator polynomial matrices $C(z)$ can be found by applying theorem 3.2 [5]. Different choices for the solution vectors give different denominator matrices $A_1(z)$ and $A_2(z)$. When we compute the corresponding numerator matrices $C_1(z)$ and $C_2(z)$ it is possible that the rational matrix functions $z^{-(k+1)}C_1(z) \, . \, A_1(z)^{-1}$ and $z^{-(k+1)}C_2(z) \, . \, A_2(z)^{-1}$ are the same. So the problem is : which of different choices will also give different realizations ? The next theorem solves this problem and gives all minimal partial $k$-realizations of order $N$ with the minimal number of parameters. $a_j^n(z)$, $j = 1,2,..,m$ denotes a specific choice for the $j$-th solution vector of degree $\alpha_j$ and $x_j(z)$, $j = 1, \ldots, s$ denotes a specific choice for the $j$-th auxiliary vector of potential degree $\pi_j$.

**Theorem 2.2 :**

*All different solutions of the N-th order partial k-realization problem are given by*

$$z^{-(k+1)}\, C^{\lambda}(z)\, .\, A^{\lambda}(z)^{-1}$$

*with the columns* $a_j^{\lambda}(z)$ *of* $A^{\lambda}(z)$ *given by :*

$$a_j^{\lambda}(z) = a_j^n(z) + \sum_{k=1}^{s} \sum_{i=\pi_k}^{\alpha_j} \lambda_{k,i}^j z^{i-\pi_k} X_k(z),\ j = 1,2,\ldots,m.$$

$C^{\lambda}(z)$ *is the polynomial part of* $z^{k+1}\, M(z)A^{\lambda}(z)$ *and* $\lambda$ *is the vector containing all parameters* $\lambda_{k,i}^j$. *(This vector will contain*

$$\sum_{j=1}^{m} \sum_{k=1}^{s} \max\{0,\alpha_j - \pi_k + 1\}$$

*parameters* $\lambda_{k,i}^j$*).*

**Proof :**

It is well known in linear system theory, see [4,theorem 6.5-4,p.441], that the numerator and denominator polynomials of right coprime matrix fraction descriptions of the same rational matrix function are coupled by unimodular matrices : $C_1(z)\, .\, A_1(z)^{-1} = C_2(z)\, .\, A_2(z)^{-1}$ with $C_1(z)$, $A_1(z)$ and $C_2(z)$, $A_2(z)$ right coprime, if and only if there exists a unimodular matrix $U(z)$ such that :

$$C_1(z) = C_2(z)\, .\, U(z)$$

$$A_1(z) = A_2(z)\, .\, U(z).$$

This means that $z^{-(k+1)}\ (C(z).U(z)).(A(z).U(z))^{-1}$ with $U(z)$ any unimodular matrix represents all possible right coprime matrix fraction descriptions of the same rational function represented by the right coprime fraction $z^{-(k+1)}C(z)A(z)^{-1}$. The minimality of an $N$-th order minimal partial $k$-realization $z^{-(k+1)}C(z)A(z)^{-1}$ of $M(z)$ guarantees that $C(z)$ and $A(z)$ are right coprime. The denominator matrices of all right coprime matrix fraction representations of the same realization are given by $A(z)\, .\, U(z)$ with $U(z)$ any unimodular matrix. This means that if we have two minimal realizations

$$z^{-(k+1)}C_1(z)A_1(z)^{-1} \text{ and } z^{-(k+1)}C_2(z)A_2(z)^{-1}$$

they represent different rational matrix functions iff $A_1(z)$ can not be written as $A_2(z)U(z)$ with $U(z)$ unimodular.

The columns $a_j^{\mu,\nu}(z)$ of all possible column reduced denominator matrices $A^{\mu,\nu}(z)$ of the $N$-th order minimal partial $k$-realization problem are given by :

$$a_j^{\mu,\nu}(z) = \sum_{l=1}^{m} \sum_{d=\alpha_l}^{\alpha_j} \nu_{l,d}^j z^{d-\alpha_l} a_l^n(z) + \sum_{k=1}^{s} \sum_{i=\pi_k}^{\alpha_j} \mu_{k,i}^j z^{i-\pi_k} X_k(z),\ j = 1,2,\ldots,m.$$

with $\nu = \{\nu_{l,d}^j\}$ and $\mu = \{\mu_{k,i}^j\}$ parameters, the degree of $a_j^{\mu,\nu}(z)$ equal to $\alpha_j$ and $Ha_j^{\mu,\nu}(z)$, $j = 1,2,\ldots,m$ linearly independent.

Note that $A^{\mu,\nu}(z)$ is built up by these columns $a_j^{\mu,\nu}(z)$ in an arbitrary ordering but because right multiplication by a permutation, i.e. unimodular, matrix doesn't change the corresponding rational matrix function we get that all solutions can be represented with a denominator matrix

$$A^{\mu,\nu}(z) = [a_1^{\mu,\nu}(z)\ a_2^{\mu,\nu}(z)\ \cdots\ a_m^{\mu,\nu}(z)].$$

It is clear that this set can be further reduced by right multiplication by a unimodular matrix to the set $A^{\lambda}(z)$ given by the theorem.

When $a_j^{\mu,\nu}(z)$, $j = 1,\ldots,m$ are given, the $\nu_{l,d}^j$ and $\mu_{k,i}^j$ are unique because the (shifted) auxiliary and solution vectors form a basis for $S(n,\alpha_j)_k$. Because $Ha_j^{\mu,\nu}(z)$ and $Ha_j^n(z)$ are linearly independent there is a vector $a_{j_1}^{\mu,\nu}(z)$ of degree $\alpha_1$ such that $\nu_{1,\alpha_1}^{j_1} \neq 0$, i.e $a_1^n(z)$ is a component. From the other vectors $a_r^{\mu,\nu}(z)$ of degree equal to or greater than $\alpha_1$ we can subtract

$$\sum_{d=\alpha_1}^{\alpha_r} \nu_{1,d}^r \,.\, z^{d-\alpha_1} \,.\, a_{j_1}^{\mu,\nu}(z)$$

such that all other vectors $a_r^{\mu,\nu}(z)$, $r \neq j$ have no component $z^{d-\alpha_1}a_1^n(z)$ anymore, $d = \alpha_1,\alpha_1+1,\ldots,\alpha_r$. Note that this transformation is unimodular.
We can do the same thing for $a_2^n(z)$.
Because the $Ha_j^{\mu,\nu}(z)$ and $Ha_j^n((z)$ are linearly independent there is a vector $a_{j_2}^{\mu,\nu}(z)$, with $j_2 \neq j_1$ such that it has degree $\alpha_2$ and has $a_2^n(z)$ as a component.
Using this vector $a_{j_2}^{\mu,\nu}(z)$ we can kill all other occurrences of $a_2^n(z)$ in the other vectors $a_r^{\mu,\nu}(z)$, $r \neq j_2$.
Finally we get after reordering :

$$a_j^{\mu,\nu}(z) = a_j^n(z) + b(z) \quad \text{with} \quad b(z) \in X_{\alpha_j}.$$

Suppose $\lambda_1$ and $\lambda_2$ are different choices for the parameters $\lambda_{k,i}^j$ then it is impossible to find a unimodular matrix $U(z) \neq I$ such that $A^{\lambda_1}(z) = A^{\lambda_2}(z) \,.\, U(z)$ because

$$A_{\alpha_j} \cup B_{\alpha_j} = \bigcup_{l=1}^{m} \{a_l^n(z)\}_{\alpha_l}^{\alpha_j} \cup \bigcup_{k=1}^{s} \{x_k(z)\}_{\pi_k}^{\alpha_j}$$

is a basis for $S(n,\alpha_j)_k$.

Indeed, in the $j$-th column of $U(z)$ we can't have nonzero elements except at position $j$ because knowing that $A_{\alpha_j} \cup B_{\alpha_j}$ is a basis for $S(n,\alpha_j)_k$ this would make some $\nu_{l,d}^j$ with $l \neq j$ different from zero. So $U(z) = I$

The remaining part of the theorem is easy to see.

□

We get the same result as described in theorem 2.2 by using the connection with the work of A.C. Antoulas. From now on we assume that $k$ is equal to -1.

In report TW91 we made the link between the algorithm of Dickinson, Morf, Kailath [3], Anderson, Brasch, Lopresti [1] and the matrix continued fraction algorithm of A.C Antoulas [2]. We derived that the observability indices $\nu_i$, $i = 1,2,\ldots,p$ are connected with the potential degrees $\pi_i$ of the auxiliary vectors $x_i$, $i = 1,2,\ldots,s$ :

$$\begin{cases} \nu_i = N - \pi_i + 1 & i = 1,2,\ldots,s \\ \nu_i = 0 & i = s+1,\ldots,p \end{cases}$$

with $N$ the order of the minimal partial realization and $\pi_i$ the potential degree of the auxiliary vector $x_i$ with respect to level $N$.

Theorem 3.13 of A.C. Antoulas [2] gives all solutions of the $N$-th order minimal partial realization problem with the minimal number of free parameters :

$$C_\alpha(z) \,.\, A_\alpha(z)^{-1}$$

with

$$C_\alpha(z) = C(z) + Y(z)P_\alpha(z)$$

and

$$A_\alpha(z) = A(z) + X(z)P_\alpha(z)$$

with $P_\alpha(z)$ a $p \times m$ polynomial matrix defined as follows :

Set $\rho_{ij} = \kappa_j + \nu_i - N$, $j = 1,2,\ldots,m$, $i = 1,2,\ldots,p$ ($\kappa_j$ = the controllability index) then

$(P_\alpha(z))_{ij} = 0$ iff $\rho_{ij} \leqslant 0$

$(P_\alpha(z))_{ij}$ = a polynomial of degree less than $\rho_{ij}$ with arbitrary coefficients, iff $\rho_{ij} > 0$.

When we write this theorem in the language of report TW79 [5], we get

$\rho_{ij} = \alpha_j + N - \pi_i + 1 - N \quad i = 1,2,\ldots,s$

$= \alpha_j + 0 - N, \quad i = s+1,\ldots,p.$

Because $\alpha_j \leqslant N$, $j = 1,2,\ldots,m$ we get :

$\rho_{ij} \leqslant 0$ or $(P_\alpha(z))_{ij} = 0$ , $i = s+1,\ldots,p$

i.e. the last $p-s$ rows of $P_\alpha(z)$ are zero.

For the remaining rows of $P_\alpha(z)$ we derive :

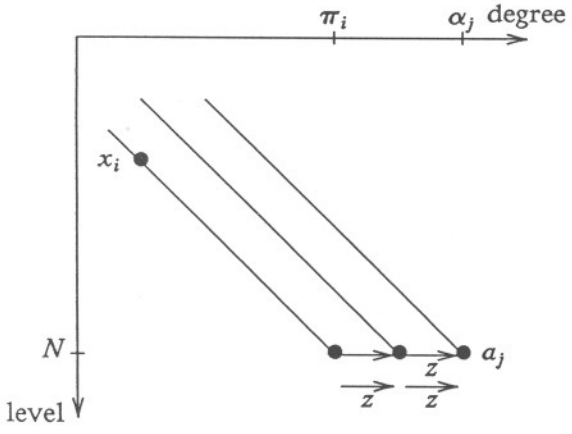
$\rho_{ij} = \alpha_j - \pi_i + 1 \quad i = 1,2,\ldots,s$

i.e. we also have a minimal partial realization of order $N$ if we add to a solution vector $a_j(z)$ a linear polynomial combination of auxiliary vectors such that the potential degree of this linear combination is smaller than or equal to the degree $\alpha_j$ of $a_j(z)$. So :

$$a'_j(z) = a_j + \sum_{i=1}^{s} (P_\alpha(z))_{ij} \cdot x_i(z) \qquad j = 1,2,\ldots,m$$

with degree $(P_\alpha(z))_{ij} \leqslant \rho_{ij} - 1 = \alpha_i - \pi_i$.

We can interpret this result in the context of table 2 ([5]) :



## 3. Example

We take the example of A.C. Antoulas ($k = -1$) [2]. There are given 5 Markov parameters, $N = 5$ :

$$M(z) = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 0 \end{bmatrix} z^{-1} + \begin{bmatrix} 2 & 2 & 1 \\ 4 & 4 & 0 \end{bmatrix} z^{-2} + \begin{bmatrix} 4 & 6 & 3 \\ 8 & 10 & 2 \end{bmatrix} z^{-3} + \begin{bmatrix} 8 & 11 & 6 \\ 16 & 21 & 7 \end{bmatrix} z^{-4} + \begin{bmatrix} 17 & 25.5 & 15.5 \\ 34 & 46 & 21 \end{bmatrix} z^{-5} + \cdots .$$

We use algorithm $A$ of section 12 to get a specific choice for the solution vectors $a_j^n(z)$ and auxiliary vectors $x_j(z)$ of level $n = 0,1,\ldots,N$.

$$\mathbf{M}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{matrix} \\ \\ \\ (0)\ (0)\ (0) \end{matrix}$$

$$\mathbf{RM}_0 = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 0 \end{bmatrix}$$

$$\rightarrow A_1 = \begin{bmatrix} z & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z \end{bmatrix} .$$

$$\mathbf{M}_1 = \left[\begin{array}{cc|ccc} 1 & 0 & z & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & z \end{array}\right] \quad \begin{matrix} \\ \\ (1)\ (1) \quad (1)\ (0)\ (1) \end{matrix}$$

$$\mathbf{RM}_1 = \left[\begin{array}{cc|ccc} 1 & 1 & 2 & 0 & 1 \\ 2 & 0 & 4 & 0 & 0 \end{array}\right]$$

$$\rightarrow A_2 = \begin{bmatrix} z-2 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z-1 \end{bmatrix} .$$

$$\mathbf{M}_2 = \left[\begin{array}{cc|ccc} 1 & 0 & z-2 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & z-1 \end{array}\right] \quad \begin{matrix} \\ \\ (2)\ (2) \quad (1)\ (0)\ (1) \end{matrix}$$

$$\mathbf{RM}_2 = \left[\begin{array}{cc|ccc} 1 & 1 & 0 & 2 & 2 \\ 2 & 0 & 0 & 2 & 2 \end{array}\right]$$

$$\rightarrow A_3 = \begin{bmatrix} z-2 & -z^2-1 & +z \\ 0 & z^2 & -z \\ 0 & -1 & z-1 \end{bmatrix} .$$

$$M_3 = \left[\begin{array}{cc|ccc} -1 & 0 & z-2 & -z^2-1 & z \\ 1 & 0 & 0 & z^2 & -z \\ 0 & 1 & 0 & -1 & z-1 \\ (1) & (3) & (1) & (2) & (1) \end{array}\right]$$

$$RM_3 = \left[\begin{array}{cc|ccc} 2 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\rightarrow A_4 = \begin{bmatrix} z-2 & -z^3+1/2z^2-z & z \\ 0 & z^3-1/2z^2 & -z \\ 0 & -z+1 & z-1 \end{bmatrix}.$$

$$M_4 = \left[\begin{array}{cc|ccc} -1 & -z^2-1 & z-2 & -z^3+1/2z^2-z & z \\ 1 & z^2 & 0 & z^3-1/2z^2 & -z \\ 0 & -1 & 0 & -z+1 & z-1 \\ (2) & (3) & (1) & (3) & (1) \end{array}\right]$$

$$RM_4 = \left[\begin{array}{cc|ccc} 2 & 0 & 1 & 1 & 1 \\ 2 & 1 & 2 & -0.5 & 2 \end{array}\right]$$

$$\rightarrow A_5 = \begin{bmatrix} z^3-z^2+1/2z+1 & 1/2z^3-5/2z^2+1/4z & 2 \\ -z^2-1/2z & z^3-1/2z^2-5/4z & -z \\ 1 & -z+1 & z-1 \end{bmatrix}.$$

$$M_5 = \left[\begin{array}{cc|ccc} z-2 & -1 & z^3-z^2+1/2z+1 & 1/2z^3-5/2z^2+1/4z & 2 \\ 0 & 1 & -z^2-1/2z & z^3-1/2z^2-5/4z & -z \\ 0 & 0 & 1 & -z+1 & z-1 \\ (2) & (3) & (3) & (3) & (1) \end{array}\right].$$

The numerator vectors corresponding to the columns of $M_5$ are :

$$NM_5 = \left[\begin{array}{cc|ccc} 1 & 0 & z^2 & 3/2z^2 & 0 \\ 2 & 0 & 2z^2 & 3z^2 & -2 \end{array}\right].$$

Using theorem 2.2 we can write all solutions of the 5-th order minimal partial -1 realization problem as :

$$C^{\lambda}(z) \, . \, A^{\lambda}(z)^{-1}$$

with

$$C^{\lambda}(z) = [c_1^{\lambda}(z) \; c_2^{\lambda}(z) \; c_3^{\lambda}(z)]$$

and

$$A^{\lambda}(z) = [a_1^{\lambda}(z) \; a_2^{\lambda}(z) \; a_3^{\lambda}(z)]$$

and

$$c_1^{\lambda}(z) = \begin{bmatrix} z^2 \\ 2z^2 \end{bmatrix} + \lambda_{1,2}^1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \lambda_{1,3}^1 \begin{bmatrix} z \\ 2z \end{bmatrix} + \lambda_{2,3}^1 \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$c_2^{\lambda}(z) = \begin{bmatrix} 3/2z^2 \\ 3z^2 \end{bmatrix} + \lambda_{1,2}^2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \lambda_{1,3}^2 \begin{bmatrix} z \\ 2z \end{bmatrix} + \lambda_{2,3}^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$c_3^{\lambda} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$a_1^{\lambda}(z) = \begin{bmatrix} z^3 - z^2 + 1/2z + 1 \\ -z^2 - 1/2z \\ 1 \end{bmatrix} + \lambda_{1,2}^1 \begin{bmatrix} z-2 \\ 0 \\ 0 \end{bmatrix} + \lambda_{1,3}^1 \begin{bmatrix} z^2-2z \\ 0 \\ 0 \end{bmatrix} + \lambda_{2,3}^1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$a_2^{\lambda}(z) = \begin{bmatrix} 1/2z^3 - 5/2z^2 + 1/4z \\ z^3 - 1/2z^2 - 5/4z \\ -z+1 \end{bmatrix} + \lambda_{1,2}^2 \begin{bmatrix} z-2 \\ 0 \\ 0 \end{bmatrix} + \lambda_{1,3}^2 \begin{bmatrix} z^2-2z \\ 0 \\ 0 \end{bmatrix} + \lambda_{2,3}^2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$a_3^{\lambda}(z) = \begin{bmatrix} 2 \\ -z \\ z-1 \end{bmatrix} .$$

Note that we get the same number of free parameters as A.C. Antoulas [2] and that it is also possible to write the solution as a matrix continued fraction.

## 4. Conclusion

In this paper all solutions of the mimimal partial $k$-realization problem are constructed using solution vectors and auxiliary vectors with the minimal number of parameters. Algorithm $A$ of section 12 TW79 [5] gives a special choice for these solution and auxiliary vectors.
This parametrization of all solutions is derived in two ways using the concepts of TW79 [5] and using the connection with the work of A.C. Antoulas [2]. An example shows the application of the obtained results.

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