A minimal partial realization algorithm for MIMO systems II. A matrix continued fraction.

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ABSTRACT

We continue the investigation of the algorithm of report TW79 for the minimal partial realization of MIMO systems. In this part we give its relation to a method proposed by Antoulas who generates a matrix continued fraction. It is shown how the algorithm of TW79 can be modified to generate the matrix coefficients in this continued fraction.

A short note on the observability indices is added. It shows how the observability indices are related to the potential degrees of auxiliary vectors that appear in the algorithm.

1. Introduction

In [1] we studied rather extensively an algorithm for the minimal partial realization of MIMO systems given its Markov parameters $M_k \in K^{p\times m}$, $k \geqslant 1$ where K is some arbitrary field. As we mentioned in the introduction of [1], we became aware of some related work of Antoulas [2] when writing that paper. Antoulas describes a matrix continued fraction which has the form

$$Z = \cfrac{C_1 + Y_1 \cfrac{C_2+Y_2 \cfrac{\cdots}{\cdots}}{A_2+X_2 \cfrac{\cdots}{\cdots}}}{A_1 + X_1 \cfrac{C_2+Y_2 \cfrac{\cdots}{\cdots}}{A_2+X_2 \cfrac{\cdots}{\cdots}}} \quad ; \quad \begin{matrix} Y_k \in K^{p\times p} \\ C_k \in K^{p\times m} \\ X_k \in K^{m\times p} \\ A_k \in K^{m\times m} \end{matrix} \tag{1.1}$$

where $\frac{A}{B}$ means AB^{-1}. (Analogous results exist for the case of left multiplication with the inverse).

The N-th convergent of Z is a minimal partial realization of the partial sequence of Markov parameters $\{M_k\}_{k=1}^{N}$. The i-th additional term in the continued fraction is characterized by the matrix

$$V_i = \begin{bmatrix} Y_i & C_i \\ X_i & A_i \end{bmatrix} \in K^{(p+m)\times(p+m)} \tag{1.2}$$

which has the property that it is unimodular. The forward evaluation scheme for the continued fraction can be represented by matrix multiplications :

$$V = V_1 \cdot V_2 \cdots V_N \cdots$$

and if

$$V_{1,N} = V_1 \cdot V_2 \cdot \cdots V_N = \begin{bmatrix} Y_{1,N} & C_{1,N} \\ X_{1,N} & A_{1,N} \end{bmatrix} \tag{1.3}$$

then the N-th convergent is

$$Z_{1,N} = C_{1,N}\, A_{1,N}^{-1}$$

and this is a minimal partial realization of the sequence $\{M_k\}_{k=1}^{N}$.

In this note we shall extend algorithm A of [1] to contruct the unimodular matrices V_i so that in the end we can construct the continued fraction (1.1). We show that its convergents give minimal partial realizations of the given Markov parameters. Thus our extended algorithm can be used to construct Antoulas' continued fraction.

2. The extended algorithm

We shall extend algorithm A of [1] so that it also generates the matrices V_n.
We suppose the reader is familiar with the notation and terminology of [1].
We consider the classical case of algorithm A, i.e. with shift parameter $k = -1$ and we shall drop it from the notation for simplicity.

2.1 Updating of auxiliary vectors and solution vectors

The key observation to link (1.3) with algorithm A is that the polynomial matrix $A_{1,n} \in K^{m\times m}[z]$ has for its columns the solution vectors of level n, while $X_{1,n} \in K^{m\times p}[z]$ is a polynomial matrix which has the auxiliary vectors for its columns. This is the reason why we have altered the notation of Antoulas [2] so as to be consistent with the notation used in [1]. $V_{n+1} \in K^{(m+p)\times(m+p)}[z]$ will then be the matrix which represents the updating of $[X_{1,n} \quad A_{1,n}]$ in the n-th step of algorithm A.

$$[X_{1,n+1} \quad A_{1,n+1}] = [X_{1,n} \quad A_{1,n}]\, V_{n+1} \ .$$

Since at the start we have no auxiliary vectors in algorithm A, we set $X_0 = 0$, while the initialization of the solution vectors in algorithm A corresponds to setting $A_0 = I$.

The matrix V_{n+1} will be constructed as a product of m unimodular matrices $V_{n+1}^{(i)}$ and (unimodular) permutation matrices $P_{n+1}^{(i)}$

$$V_{n+1} = V_{n+1}^{(1)} P_{n+1}^{(1)} V_{n+1}^{(2)} P_{n+1}^{(2)} \ \cdots \ V_{n+1}^{(m)} P_{n+1}^{(m)} \ .$$

The permutation matrices are used to represent the reorderings of solution vectors and auxiliary vectors in steps A.2.1.3.1.3 and A.2.1.3.2.3 of algorithm A.

The $V_{n+1}^{(i)}$ matrix corresponds to the computation of a_i^{n+1} A.2.1.3.
Suppose we set

$$[X_0|A_0]\, V_1 V_2 \cdots V_n V_{n+1}^{(1)} P_{n+1}^{(1)} \cdots V_{n+1}^{(i-1)} P_{n+1}^{(i-1)} =$$
$$= [x_1 \cdots x_s \ 0 \cdots 0 | a_1^{n+1} \cdots a_{i-1}^{n+1} a_i^n \cdots a_m^n].$$

Elementary column operations will be denoted as

$E_i(a)$ = unit matrix with column i multiplied by a
E_{ij} = unit matrix with column i and column j interchanged
$E_{ij}(a)$ = unit matrix with column j replaced by column j plus column i times a.

We now describe $V_{n+1}^{(i)}$ for the three possibilities of step A.2.1.3.

1. A.2.1.3.1
   Setting $a_i^{n+1} = z^{n+1-\alpha_i} a_i^n$ and adding a_i^n to the set of auxiliary vectors corresponds to a multiplication with

   $$V_{n+1}^{(i)} = E_{s+1,p+i} E_{s+1,p+i}(z^{n+1-\alpha_i}) \tag{2.1}$$

   since

   $$[x_1 \cdots x_s \ 0 \cdots 0 | a_1^{n+1} \cdots a_{i-1}^{n+1} \ a_i^n \ a_{i+1}^n \cdots a_m^n] V_{n+1}^{(i)} =$$
   $$\begin{matrix}[& x_1 & \cdots & x_s & a_i^n & 0 \cdots 0 | a_1^{n+1} \cdots a_{i-1}^{n+1} \ a_i^{n+1} \ a_{i+1}^n \cdots a_m^n]. \\ & (\pi_1) & \cdots & (\pi_s) & (\alpha_i) & \end{matrix}$$

   Reordering of the auxiliary vectors to keep increasing potential degree corresponds to a permutation of the first $s+1$ columns. this is $P_{n+1}^{(i)}$. s gets the value $s+1$.

2. A.2.1.3.2
   Setting $a_i^{n+1} = z^{\pi_r - \alpha_i} a_i^n + \sum_{j=1}^{r} \mu_j z^{\pi_r - \pi_j} x_j$ and replacing x_r by a_i^n corresponds to

$$V_{n+1}^{(i)} = E_{r,p+i}E_{p+i}(\mu_r)\prod_{j=1}^{r-1} E_{j,p+i}\,(\mu_j z^{\pi_r-\pi_j})E_{r,p+i}(z^{\pi_r-\alpha_i}) \tag{2.2}$$

because

$$[x_1 \cdots x_r \cdots x_s\ 0 \cdots 0 | a_1^{n+1} \cdots a_{i-1}^{n+1}\ a_i^n \cdots a_m^n]V_{n+1}^{(i)} =$$

$$\begin{matrix}[& x_1 & \cdots & a_i^n & \cdots & x_s & 0 \cdots 0 | a_1^{n+1} \cdots a_{i-1}^{n+1}\ a_i^{n+1}\ a_{i+1}^n \cdots a_m^n]. \\ & (\pi_1) & \cdots & (\alpha_i) & \cdots & (\pi_s) & \end{matrix}$$

Reordering the auxiliary vectors to keep increasing potential degree corresponds to a permutation of the first s columns. This gives $P_{n+1}^{(i)}$.

3. A.2.1.3.3

Setting $a_i^{n+1} = a_i^n + \sum_{j=1}^{r} \mu_j z^{\alpha_i-\pi_j} x_j$ corresponds to

$$V_{n+1}^{(i)} = \prod_{j=1}^{r} E_{j,p+i}\,(\mu_j z^{\alpha_i-\pi_j}) \tag{2.3}$$

because

$$[x_1 \cdots x_s\ 0 \cdots 0 | a_1^{n+1} \cdots a_{i-1}^{n+1}\ a_i^n \cdots a_m^n]V_{n+1}^{(i)} =$$

$$\begin{matrix}[& x_1 & \cdots & x_s & 0 \cdots 0 | a_1^{n+1} \cdots a_{i-1}^{n+1}\ a_i^{n+1}\ a_{i+1}^n \cdots a_m^n]. \\ & (\pi_1) & \cdots & (\pi_s) & \end{matrix}$$

Since there is no reordering of the auxiliary vectors $P_{n+1}^{(i)} = I$ in this case.
Thus we have

$$V_{n+1} = V_{n+1}^{(1)}P_{n+1}^{(1)} \cdots V_{n+1}^{(m)}P_{n+1}^{(m)}$$

giving

$$[X_0 | A_0]V_1 \cdots V_n V_{n+1}.$$

Remarks

1. Note that all the transformations we have applied are unimodular and therefore $V_{1,n} = V_1 V_2 \cdots V_n$ are unimodular.
2. The matrix (2.1) has the form

$$V_{n+1}^{(i)} = \begin{bmatrix} 1 & & 0 & & 0 & \\ & \ddots & \vdots & & \vdots & \\ & & 0 & & 1 & \leftarrow s+1 \\ & & \vdots & \ddots & \vdots & \\ & & 0 & 1 & 0 & \\ \hdashline & & 0 & & 0 & \\ & & \vdots & & \vdots & \\ & & 1 & & z^{n+1-\alpha_i} & \leftarrow p+i \\ & & \vdots & & \vdots & \ddots \\ & & 0 & & 0 & 1 \end{bmatrix}$$

$$\uparrow \qquad\qquad \uparrow$$
$$s+1 \qquad\qquad p+i$$

The matrix (2.2) has the form

$$V_{n+1}^{(i)} = \begin{bmatrix} 1 & & 0 & & \mu_1 z^{\pi_r - \pi_1} \\ & \ddots & \vdots & & \vdots \\ & & 0 & & \mu_r \\ & & \vdots & \ddots & \vdots \\ & & 0 & 1 & 0 \\ \hline & & 0 & 1 & 0 \\ & & \vdots & \ddots & \vdots \\ & & 1 & & z^{\pi_r - \alpha_i} \\ & & \vdots & \ddots & \vdots \\ & & 0 & & 0 & \ddots & 1 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow r \\ \\ \\ \\ \leftarrow p+i \\ \\ \end{matrix}$$

$$\begin{matrix} & & \uparrow & & \uparrow \\ & & r & & p+i \end{matrix}$$

The matrix (2.3) has the form

$$V_{n+1}^{(i)} = \begin{bmatrix} 1 & & 0 & & \mu_1 z^{\alpha_i - \pi_1} \\ & \ddots & \vdots & & \vdots \\ & & 1 & & \mu_r z^{\alpha_i - \pi_r} \\ & & \vdots & \ddots & \vdots \\ & & 0 & 1 & 0 \\ \hline & & 0 & 1 & 0 \\ & & \vdots & \ddots & \vdots \\ & & 0 & & 1 \\ & & \vdots & \ddots & \vdots \\ & & 0 & & 0 & \ddots & 1 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow r \\ \\ \\ \\ \leftarrow p+i \\ \\ \end{matrix}$$

$$\begin{matrix} & & \uparrow & & \uparrow \\ & & r & & p+i \end{matrix}$$

2.2 Updating of the numerators

In the previous section it was shown how the denominators were updated by multiplication of $[X_0, A_0]$ with successsive V-matrices. This multiplication represented both

— the updating of the auxiliary vectors

— the updating of the solution vectors by taking linear combinations of the auxiliary vectors and the solution vector of the previous level.

It will be shown in this section that the updating of the numerators can be done in exactly the same way. I.e. by multiplying some

$[Y_0\ C_0]$ with successive V-matrices to give

$$[Y_0\ C_0]V_1V_2 \cdots V_nV_{n+1} = [Y_{1,n+1}\ C_{1,n+1}].$$

$C_{1,n+1}$ will have for its columns the numerator vectors corresponding to the solution vectors in $A_{1,n+1}$ and $Y_{1,n+1}$ will have for its columns the numerators for the auxiliary vectors in $X_{1,n+1}$. C_0 can be initialized as 0. Y_0 however will have to contain in column $s+1$ the residual Ra_i^n whenever a_i^n is introduced in step A.2.1.3.1 as an auxiliary vector in position $s+1$ as will become clear in a minute. Since we don't know these residuals at the start, we set $Y_0 = 0$ and fill in the residuals in the successive columns at the very moment we shall use this column in the updating.

It was shown in [1,Theorem 3.2] that the numerator $c(z)$ corresponding to a solution vector $a(z)$ for level n of degree α can be found as the polynomial part of

$$M(z)a(z)$$

with $M(z) = \sum_{k=1}^{\infty} M_k z^{-k}$.

If $a(z) = \sum_{k=0}^{\alpha} a_k z^k$ and $c(z) = \sum_{k=0}^{\alpha-1} c_k z^k$, then these are related by

$$\begin{bmatrix} & & M_1 \\ & M_1 & \vdots \\ & & M_\alpha \\ M_1 & \cdots & M_{\alpha+1} \\ \vdots & & \vdots \\ M_{n-\alpha} & \cdots & M_{n+1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_\alpha \end{bmatrix} = \begin{bmatrix} c_{\alpha-1} \\ \vdots \\ c_0 \\ 0 \\ \vdots \\ 0 \\ Ra \end{bmatrix}. \tag{2.4}$$

The numerators corresponding to a_i^n will be denoted as c_i^n and the numerators corresponding to x_j will be denoted as y_j.
The updating of the numerators in the three different cases can now be found as follows :
Denote :

$$[Y_0|C_0]\ V_1 \cdots V_n V_{n+1}^{(1)} P_{n+1}^{(1)} \cdots V_{n+1}^{(i-1)} P_{n+1}^{(i-1)} =$$
$$[y_1 \cdots y_s\ ?\ ? \cdots ?|c_1^{n+1} \cdots c_{i-1}^{n+1}\ c_i^n \cdots c_m^n].$$

(1) Case A.2.1.3.1
The denominator is given by $a^{n+1} = z^{n+1-\alpha_i} a_i^n$. As can be seen from (2.4). The polynomial part of $M(z)a^{n+1}(z)$ will be given by the relation (the level is $n+1$)

$$\begin{bmatrix} 0 & & & M_1 \\ & & & \vdots \\ & & & \\ 0 & M_1 & \cdots & M_{n+1} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_i^n \end{bmatrix} = \begin{bmatrix} c_i^n \\ 0 \\ \vdots \\ 0 \\ Ra_i^n \end{bmatrix} = c_i^{n+1}.$$

Hence

$$c_i^{n+1} = z^{n+1-\alpha_i} c_i^n + Ra_i^n.$$

Thus the updating of the numerator is described by two steps:

1. Replace column $s+1$ of Y_0 by Ra_i^n.
2. Multiply the current result

   $$[y_1 \cdots y_s \; Ra_i^n \; ? \cdots ? | c_1^{n+1} \cdots c_{i-1}^{n+1} \; c_i^n \cdots c_m^n]$$

   by $V_{n+1}^{(i)}$ of (2.1) to get

   $$[y_1 \cdots y_s \; c_i^n \; ? \cdots ? | c_1^{n+1} \cdots c_{i-1}^{n+1} \; c_i^{n+1} \; c_{i+1}^n \cdots c_m^n].$$

Because the auxiliary vectors are reordered after this according to $P_{n+1}^{(i)}$, their corresponding numerators should be reordered too. So to the previous result we also have to apply $P_{n+1}^{(i)}$.

(2) Case A.2.1.3.2.

Since $a_i^{n+1} = z^{\pi_r - \alpha_i} a_i^n + \sum_{j=1}^{r} \mu_j z^{\pi_r - \pi_j} x_j$, the polynomial part of $M(z) a_i^{n+1}(z)$ will be given by

$$c_i^{n+1}(z) = z^{\pi_r - \alpha_i} c_i^n + \sum_{j=1}^{r} \mu_j z^{\pi_r - \pi_j} y_j.$$

This can again be seen from (2.4), recalling that the x_j are old solution vectors. The number of zeros in the right hand side of (2.4) is always high enough to prevent the residual at the bottom to shift into the numerator as a consequence of the multiplications with powers of z.

Since in this case an auxiliary vector x_r is replaced by a_i^n, the corresponding numerator y_r is replaced by c_i^n. This is exactly what is done by the operation

$$[y_1 \cdots y_s \; ? \cdots ? | c_1^{n+1} \cdots c_{i-1}^{n+1} \; c_i^n \cdots c_m^n] V_{n+1}^{(i)} =$$

$$[y_1 \cdots c_i^n \cdots y_s \; ? \cdots ? | c_1^{n+1} \cdots c_{i-1}^{n+1} \; c_i^{n+1} \; c_{i+1}^n \cdots c_m^n]$$

when $V_{n+1}^{(i)}$ is as in (2.2).
$P_{n+1}^{(i)}$ then reorders the auxiliary vectors as well as the corresponding numerators.

(3) Case A.2.1.3.3.

The result is obtained with the same arguments as in the previous case except that no replacement of an auxiliary vector is required.

2.3 Reformulation of algorithm A

Using a compact representation of auxiliary vectors, solution vectors and corresponding numerators, we can reformulate algorithm A.

All the information can be stored in the following matrices:

$NMA = [y_1 \cdots y_s 0 \cdots 0] \in K^{p \times p}$
(numerator matrix of auxiliary vectors)

$DMA = [x_1 \cdots x_s 0 \cdots 0] \in K^{m \times p}$
(denominator matrix of auxiliary vectors)

$RMA = [Rx_1 \cdots Rx_s 0 \cdots 0] \in K^{p \times p}$
(residual matrix of auxiliary vectors)

$dMA = [\pi_1 \cdots \pi_s 0 \cdots 0] \in K^{1 \times p}$
(potential degree matrix of auxiliary vectors)

$NMS = [c_1^n \cdots c_m^n] \in K^{p \times m}$
(numerator matrix of solution vectors)

$DMS = [a_1^n \cdots a_m^n] \in K^{m \times m}$
(denominator matrix of solution vectors)

$dMS = [\alpha_1 \cdots \alpha_m] \in K^{1 \times m}$
(degree matrix of solution vectors)

and we use the short hand notation

$NM = [NMA \mid NMS]$

$DM = [DMA \mid DMS]$

$dM = [dMA \mid dMS]$

Algorithm A

A.1 {Initialisation}
Set
$NM = [0 \mid 0]$,
$DM = [0 \mid I]$,
$dM = [0 \mid 0]$,
$RMA = 0$.

A.2 **For** $n = 0,1,2,\ldots$
Suppose we know
$NM = [y_1 \cdots y_s 0 \cdots 0 | c_1^n \cdots c_m^n]$
$DM = [x_1 \cdots x_s 0 \cdots 0 | a_1^n \cdots a_m^n]$
$dM = [\pi_1 \cdots \pi_s 0 \cdots 0 | \alpha_1 \cdots \alpha_m]$
with $\pi_1 \leqslant \pi_2 \leqslant \cdots \leqslant \pi_s$ and
$RMA = [Rx_1 \cdots Rx_s 0 \cdots 0]$

A.2.1 **For** $i = 1,2,\ldots,m$

A.2.1.1. Compute
$$Ra_i^n = [M_{n-\alpha_i+1} \cdots M_{n+1}]a_i^n$$

A.2.1.2. Try to write Ra_i^n as a linear combination of $Rx_1, \ldots, Rx_r$:
$$Ra_i^n = -\sum_{j=1}^{r} \mu_j Rx_j$$

with r as low as possible ($\mu_r \neq 0$).
If $Ra_i^n = 0$ set $r = 0$ and $\pi_0 = 0$.

A.2.1.3 {Updating for solution vector a_i^{n+1} based on a_i^n}

A.2.1.3.1. **If** $r \leqslant s$ does not exist **then**

A.2.1.3.1.1. $V^{(i)} = E_{s+1,p+i}E_{s+1,p+i}(z^{n+1-\alpha_i})$

A.2.1.3.1.2. Store $\alpha_i^{n+1} = n+1$ at position $p+i$ of dM

A.2.1.3.1.3. Replace s by $s+1$

A.2.1.3.1.4. Store $\begin{bmatrix} Ra_i^n \\ Ra_i^n \\ \alpha_i \end{bmatrix}$ at position s of $\begin{bmatrix} NMA \\ RMA \\ dMA \end{bmatrix}$

A.2.1.3.1.5. Replace $\begin{bmatrix} NM \\ DM \end{bmatrix}$ by $\begin{bmatrix} NM \\ DM \end{bmatrix} V^{(i)}$

A.2.1.3.1.6. Compute the permutation matrix $PA^{(i)}$ which sets the first s elements of dM in increasing order and replace

$$\begin{bmatrix} NMA \\ DMA \\ RMA \\ dMA \end{bmatrix} \text{ by } \begin{bmatrix} NMA \\ DMA \\ RMA \\ dMA \end{bmatrix} PA^{(i)}$$

A.2.1.3.2. **If** $r \leqslant s$ does exist and $\alpha_i < \pi_r$ **then**

A.2.1.3.2.1. $V^{(i)} = E_{r,p+i}E_{p+i}(\mu_r)\prod_{j=1}^{r-1} E_{j,p+i}(\mu_j z^{\pi_r - \pi_j})E_{r,p+i}(z^{\pi_r - \alpha_i})$

A.2.1.3.2.2. Store $\alpha_i^{n+1} = \pi_r$ at position $p+i$ of dM

A.2.1.3.2.3. Store $\begin{bmatrix} Ra_i^n \\ \alpha_i \end{bmatrix}$ at position r of $\begin{bmatrix} RMA \\ dMA \end{bmatrix}$

A.2.1.3.2.4. Replace $\begin{bmatrix} MN \\ DN \end{bmatrix}$ by $\begin{bmatrix} MN \\ DN \end{bmatrix} V^{(i)}$

A.2.1.3.2.5. Compute the permutation matrix $PA^{(i)}$ which sets the first s elements of dM in increasing order and replace

$$\begin{bmatrix} NMA \\ DMA \\ RMA \\ dMA \end{bmatrix} \text{ by } \begin{bmatrix} NMA \\ DMA \\ RMA \\ dMA \end{bmatrix} PA^{(i)}$$

A.2.1.3.3. **If** $r \leqslant s$ does exist and $\alpha_i \geqslant \pi_r$ **then**

A.2.1.3.3.1. $V^{(i)} = \prod_{j=1}^{r} E_{j,p+i}(\mu_j z^{\alpha_i - \pi_j})$

A.2.1.3.3.2. Store $\alpha_i^{n+1} = \alpha_i$ at position $p+i$ of dM

A.2.1.3.3.3. Replace $\begin{bmatrix} MN \\ DN \end{bmatrix}$ by $\begin{bmatrix} MN \\ DN \end{bmatrix} V^{(i)}$

A.2.1.3.3.4. Set $PA^{(i)} = I$

A.2.2. Define

$$V_n = \begin{bmatrix} Y_{n+1} & C_{n+1} \\ X_{n+1} & A_{n+1} \end{bmatrix} = V^{(1)}P^{(1)} \ldots V^{(m)}P^{(m)}$$

with $P^{(i)} = PA^{(i)} \oplus I$

A.2.3 {Prepare for the next level}

A.2.3.1 Increase the first s elements of dM with 1.

3. The matrix continued fraction

By our extended algorithm of the previous section we have generated some matrices V_1, V_2,... which are unimodular and it follows from the construction that if

$$\begin{bmatrix} Y_0 & C_0 \\ X_0 & A_0 \end{bmatrix} V_1 \cdots V_n = \begin{bmatrix} Y_{0,n} & C_{0,n} \\ X_{0,n} & A_{0,n} \end{bmatrix}$$

with

$C_0 = 0$, $X_0 = 0$, $A_0 = I$ and
the i-th column of Y_0 is Ra_i^n whenever a_i^n is a solution vector which becomes an auxiliary vector in step A.2.1.3.1. of the algorithm, and zero otherwise,

then

$C_{0,n}A_{0,n}^{-1}$ will be a minimal partial realization of the sequence $M_1, M_2, \ldots, M_n$.
Furthermore, the successive convergents of the continued fraction

$$Y_0 \cfrac{}{A_1 + X_1 \cfrac{C_2+Y_2 \cfrac{\cdots}{\cdots}}{A_2+X_2 \cfrac{\cdots}{\cdots}}} \qquad C_1 + Y_1 \cfrac{C_2+Y_2 \cfrac{\cdots}{\cdots}}{A_2+X_2 \cfrac{\cdots}{\cdots}}$$

are given by $C_{0,n}A_{0,n}^{-1}$, $n = 1,2,\ldots$. Of course, if you want to have the form (1.1), you have to replace $[Y_1\ C_1]$ by $[Y_1'\ C_1'] = Y_0[Y_1\ C_1]$, or equivalently, by replacing V_1 by $V_1' = V_0V_1$ where

$$V_0 = \begin{bmatrix} Y_0 & C_0 \\ X_0 & A_0 \end{bmatrix}.$$

To be consistent with the Antoulas notation, this V_1' (or equivalently V_0) should be unimodular. By our choice which takes the redundant columns with number $s+1, \ldots, p$ of Y_0 as zero, this will not be the case. But since these columns are not essential, we can always choose them to be linearly independent of the residuals which constitute the first s columns of Y_0.
Let us denote Y_0 by $Y_0 = [Y_0' \mid Y_0'']$ with $Y_0' \in K^{p\times s}$ and $Y_0'' \in K^{p\times(p-s)}$. It is by definition of the auxiliary vectors that the columns of Y_0' generate the residual space of the final level n. During the updating Y_0'' will be changed according to the change in auxiliary vectors. Indeed, $Y_{0,n}'$ will represent the numerators for the auxiliary vectors of level n. However, the replacement of auxiliary vectors will never change the residual space. This implies that $[RX_{0,n} \mid Y_0'']$ will still be a regular constant matrix.
Thus we have constructed V-matrices that have essentially the same properties as the V-matrices of Antoulas:

- $V_{0,n}$ is unimodular
- $A_{0,n}$ is column reduced
- $C_{0,n}A_{0,n}^{-1}$ is a minimal partial realization of the sequence $M_1, M_2, \ldots, M_n$.

Antoulas has one extra condition viz.

- $A_{0,n}^{-1}X_{0,n}$ is strictly proper rational.

This condition however doesn't seem to be essential except that it is used by Antoulas to make the V-matrix unique. It is always possible to attach to each of our V-matrices some extra unimodular factor to meet this condition.

All this implies that all properties Antoulas has for V-matrices are also true for the V-matrices we have constructed.

4. A note on the observability indices

In [1] we have shown that the controllability indices are equal to the degrees of the solution vectors for the right minimal partial realization with column reduced denominator. By duality, it should be clear that the observability indices are equal to the degrees of the solution vectors of a left minimal partial realization with row reduced denominator. As we shall show below, these observability indices are also related to the potential degrees of our auxiliary vectors for the right minimal partial realization. It follows from Antoulas [2] that we get the left minimal partial realization by inverting the V-matrix. This gives, in the terminology of Antoulas, the W-matrix. If

1. W is a unimodular matrix
$$W = \begin{bmatrix} T & -Q \\ -U & R \end{bmatrix}$$
with T row reduced and if
2. the V-matrix
$$V = \begin{bmatrix} Y & C \\ X & A \end{bmatrix}$$
with A column reduced, is as before the V-matrix of the right minimal partial realization $C(z)A(z)^{-1}$, and if
3. moreover $VW = I$,

then

$T(z)^{-1}Q(z)$ will be a left minimal partial realization.
Using the inversion formula

$$\begin{bmatrix} Y & C \\ X & A \end{bmatrix} = \begin{bmatrix} \Delta^{-1} & -\Delta^{-1}F \\ -E\Delta^{-1} & A^{-1} + E\Delta^{-1}F \end{bmatrix} = \begin{bmatrix} T & -Q \\ -U & R \end{bmatrix}$$

with

$$\Delta = Y - CA^{-1}X$$
$$E = A^{-1}X$$
$$F = CA^{-1}.$$

We have after splitting up Y and X as

$$\begin{bmatrix} Y \\ X \end{bmatrix} = \begin{bmatrix} Y' & Y'' \\ X' & X'' \end{bmatrix} \qquad \begin{matrix} Y' \in K^{p\times s} & ,Y'' \in K^{p\times(p-s)} \\ X' \in K^{m\times s} & ,X'' \in K^{m\times(p-s)} \end{matrix}$$

with $X'' = X_0'' = 0$ and with $Y'' = Y_0''$ such that Y_0 is regular

that

$$\Delta = [Y' - CA^{-1}X' \mid Y'']$$
$$= [-Rx_1 z^{-n_1+d_1-1} + O(z^{-n_1+d_1-2}) \cdots -Rx_s z^{-n_s+d_s-1} + O(z^{-n_s+d_s-2}) \mid Y''] \quad , \quad z \to \infty$$

with Rx_i the residuals of the auxiliary vectors x_i and d_i the degrees of x_i.

Suppose the i-th auxiliary vector was equal to the solution vector $a_j^{n_i}$, then $n_i - d_i \geqslant 0$. As we have noticed in the previous section, $[RX' \mid Y'']$ will be regular and this means that Δ is column reduced. Therefore $T = \Delta^{-1}$ will be row reduced with row degrees

$$\begin{cases} n_i - d_i + 1 & i = 1,2,\ldots,s \\ 0 & i = s+1,\ldots,p. \end{cases}$$

Note that $n_i - d_i + 1 = N - \pi_i$ if π_i is the potential degree of x_i with respect to level $N-1$. Moreover, since V is unimodular, V^{-1} will be polynomial, hence also T. T is the row reduced denominator of a left minimal partial realization and therefore its row degrees are the observability indices ν_i. In other words, we have shown that these observability indices satisfy

$$\begin{cases} \nu_i = N - \pi_i & i = 1,2,\ldots,s \\ \nu_i = 0 & i = s+1,\ldots,p \end{cases}$$

with

π_i the potential degree of the auxiliary vector x_i with respect to level $N-1$ and

s = the number of nonzero observability indices = the dimension of the residual space.

5. References

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