BLUES function method for partial differential equations

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Introduction

Solving nonlinear differential equations analytically without resorting to perturbation, linearization or discretization is no trivial task. Some methods (VIM, ADM, HAM) have been developed to tackle these problems in an iterative manner. In this work we propose a new method which is based on extending the theory of Green functions to the nonlinear domain. Earlier work on this topic (1; 2) has resulted in a powerful theory for boundary value problems in ordinary differential equations (ODEs). Here we extend the theory to partial differential equations (PDEs) where the initial condition assumes the role of a source. Three examples are studied.

BLUES function methods for PDEs

Consider a (nonlinear) PDE with an initial condition ϕ, which can be written as an external source containing a delta function at e.g. $t = 0$ i.e.,

$$\mathcal{N}u = \delta(t)\phi = \psi \tag{1}$$

and an associated linear operator with the same initial condition

$$\mathcal{L}u = \delta(t)\phi = \psi \tag{2}$$

Assume that the solution to (1) can be written in the form of a convolution of a Green function G with an associated source ξ

$$u = (G * \xi) \tag{3}$$

where G is the solution of the linear DE with a Dirac delta source at $t = s$

$$\mathcal{L}G = \delta(t - s) \tag{4}$$

Now define a residual operator as the difference between the linear (2) and nonlinear (1) operators, i.e., $\mathcal{R}u = \mathcal{L}u - \mathcal{N}u$. Apply to the solution u and rearrange to find the associated source ξ, i.e.,

$$\begin{aligned} \mathcal{R}(G*\xi) &= \mathcal{L}(G*\xi) - \mathcal{N}(G*\xi) \\ &= \xi - \psi \\ \xi &= \psi + \mathcal{R}(G*\xi)\,, \end{aligned} \tag{5}$$

which can be iterated in the Picard sense

$$\begin{aligned} \xi^{(0)} &= \psi \\ \xi^{(n+1)} &= \psi + \mathcal{R}(G*\xi^{(n)}) \qquad n \geq 0 \end{aligned} \tag{6}$$

The n—th order solution u of the nonlinear PDE (1) is consequently the convolution of the Green function with the n—th order associated source (6), i.e.,

$$\begin{aligned} u^{(0)} &= G*\xi^{(0)} = G*\psi \\ u^{(n+1)} &= G*\xi^{(n+1)} = u^{(0)} + G*\mathcal{R}(u^{(n)}) \qquad n \geq 0 \end{aligned} \tag{7}$$

Examples

1. Reaction-Diffusion-Convection equation (RDCE):

$$\mathcal{N}u = u_t - u_{xx} + uu_x + u(u+2) = 0 \tag{8}$$

with initial condition $u(x,0) = \phi(x) = \mathrm{e}^{-x}$.

Green function:

$$G(t,s) = \mathrm{e}^{-2(t-s)} \tag{9}$$

Exact solution:

$$u_{ex}(x,t) = \mathrm{e}^{-(x+t)} \tag{10}$$

2. Black-Scholes equation (BSE)

$$v_\tau - \frac{\hat{\sigma}^2}{\sigma^2}(v_{xx} - v_x) - D(v_x - v) = 0 \tag{11}$$

with $D > 0$, $\sigma > 0$ and initial condition $v(x,0) = \phi(x) = \mathrm{e}^x - 1$. $\hat{\sigma}^2$ is a nonlinear function of v, v_x, v_{xx}.

Green function:

$$G(t,s) = \mathrm{e}^{-D(\tau - s)} \tag{12}$$

Exact solution:

$$v_{ex}(x,t) = \mathrm{e}^x - \mathrm{e}^{-D\tau} \tag{13}$$

3. Porous medium equation with growth/decay (PME):

$$\mathcal{N}w = w_t - \Delta(w^m) - \beta w = 0 \tag{14}$$

with $m > 1$, $\beta \in \mathbb{R}$ and initial condition $w(x,0) = \phi(x) = x$.

Green function:

$$G(t,s) = \mathrm{e}^{\beta(t-s)} \tag{15}$$

Exact solution ($m = 2$):

$$w_{ex}(x,t) = x\mathrm{e}^{\beta t} + \frac{2}{\beta}\mathrm{e}^{2\beta t} - \frac{2}{\beta}\mathrm{e}^{\beta t}\,, \tag{16}$$

Other iterative methods:

- Variational Iteration Method (VIM)
- Variational Iteration Method + Green function (GVIM)
- Adomian Decomposition Method (ADM)

Results

Reaction-Diffusion-Convection PDE (8):

$$\begin{aligned} u_0(x,t) &= \mathrm{e}^{-2t-x} \\ u_1(x,t) &= \mathrm{e}^{-2t-x} + \mathrm{e}^{-2t-x}t \\ u_2(x,t) &= \mathrm{e}^{-2t-x} + \mathrm{e}^{-2t-x}t + \mathrm{e}^{-2t-x}\frac{t^2}{2!} \\ &\vdots \\ u_n(x,t) &= \mathrm{e}^{-2t-x}\sum_{i=0}^{n}\frac{t^i}{i!} \\ &\Downarrow \\ u(x,t) &= \lim_{n\to\infty} u_n(x,t) = \mathrm{e}^{-(x+t)} \end{aligned}$$

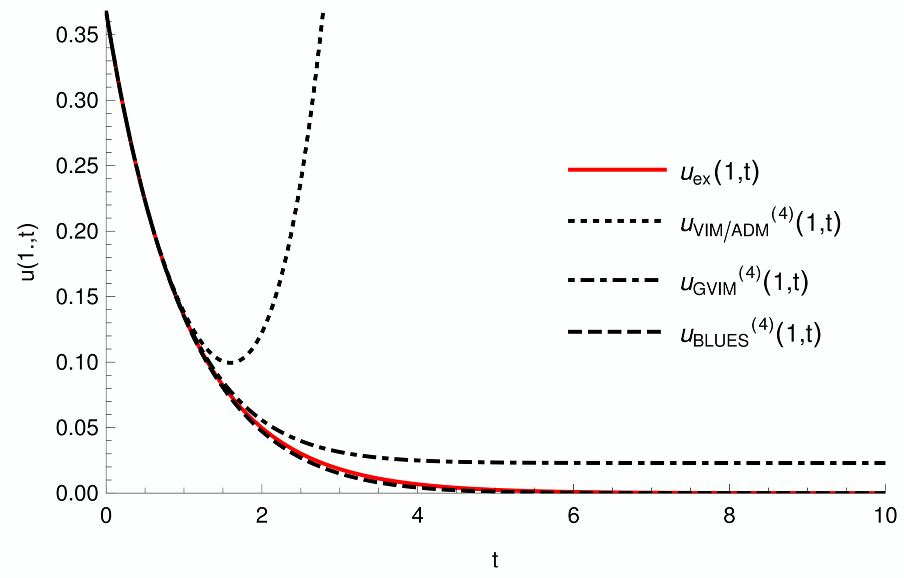

Figure 1: Exact solution (10) of the RDCE (8) (red line) and the approximants of order $n = 10$ for the four methods, at $x = 1$.

Black-Scholes PDE (11):

$$\begin{aligned} v_0(x,\tau) &= -\mathrm{e}^{-D\tau} + \mathrm{e}^{x-D\tau} \\ v_1(x,\tau) &= -\mathrm{e}^{-D\tau} + \mathrm{e}^{x-D\tau} + \mathrm{e}^{x-D\tau}D\tau \\ v_2(x,\tau) &= -\mathrm{e}^{-D\tau} + \mathrm{e}^{x-D\tau} + \mathrm{e}^{x-D\tau}D\tau \\ &\quad + \mathrm{e}^{x-D\tau}\frac{D^2\tau^2}{2!} \\ &\vdots \\ v_n(x,\tau) &= -\mathrm{e}^{-D\tau} + \mathrm{e}^{x-D\tau}\sum_{i=0}^{n}\frac{(D\tau)^i}{i!} \\ &\Downarrow \\ v(x,t) &= \lim_{n\to\infty} v_n(x,t) = \mathrm{e}^x - \mathrm{e}^{-D\tau} \end{aligned}$$

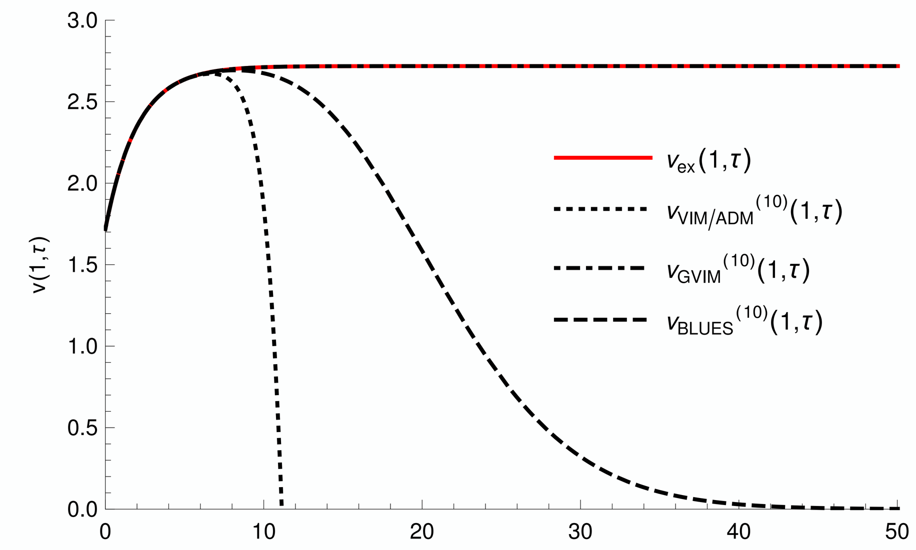

Figure 2: Exact solution (13) of the BSE (11) (red line) and the approximants of order $n = 10$ for the four methods, at $x = 1$. Parameter values are $\sigma = 1$ and $D = 1/2$.

Porous Medium PDE (14):

$$\begin{aligned} w_0(x,t) &= \mathrm{e}^{\beta t}x \\ w_1(x,t) &= x\mathrm{e}^{\beta t} + \frac{2}{\beta}\mathrm{e}^{2\beta t} - \frac{2}{\beta}\mathrm{e}^{\beta t} \\ w_2(x,t) &= x\mathrm{e}^{\beta t} + \frac{2}{\beta}\mathrm{e}^{2\beta t} - \frac{2}{\beta}\mathrm{e}^{\beta t} \\ &\vdots \\ w_n(x,t) &= x\mathrm{e}^{\beta t} + \frac{2}{\beta}\mathrm{e}^{2\beta t} - \frac{2}{\beta}\mathrm{e}^{\beta t} \\ &\Downarrow \\ w(x,t) &= x\mathrm{e}^{\beta t} + \frac{2}{\beta}\mathrm{e}^{2\beta t} - \frac{2}{\beta}\mathrm{e}^{\beta t} \end{aligned}$$

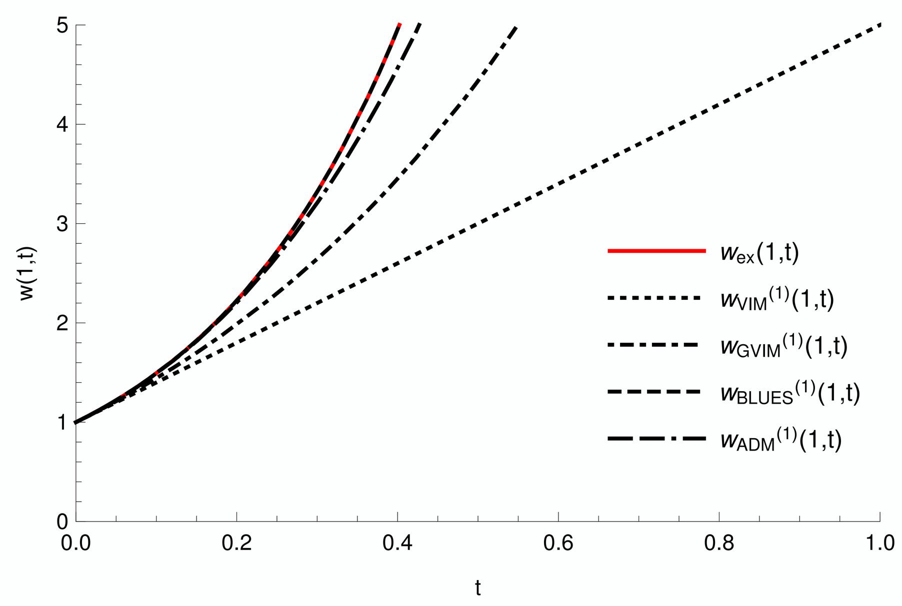

Figure 3: Exact solution (16) of the PME (14) (red line) and the approximants of order $n = 1$ for the four methods, at $x = 1$. The BLUES method generates the exact solution.

Errors

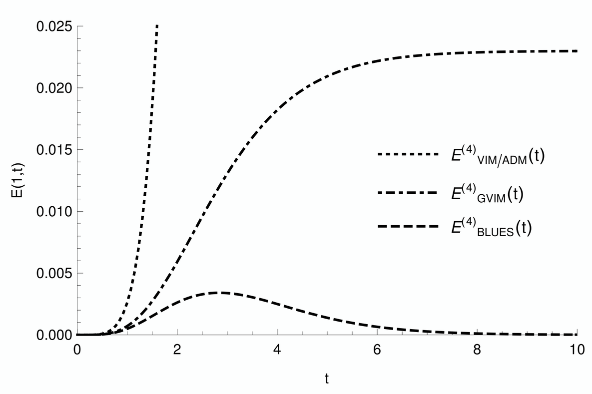

Figure 4: Error between the exact solution (10) of the RDCE (8) and the approximant of order $n = 10$ for the four methods, at $x = 1$.

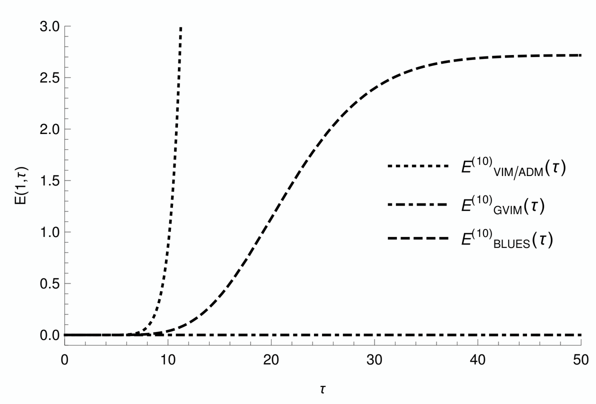

Figure 5: Error between the exact solution (13) of the BSE (11) and the approximant of order $n = 10$ for the four methods, at $x = 1$.

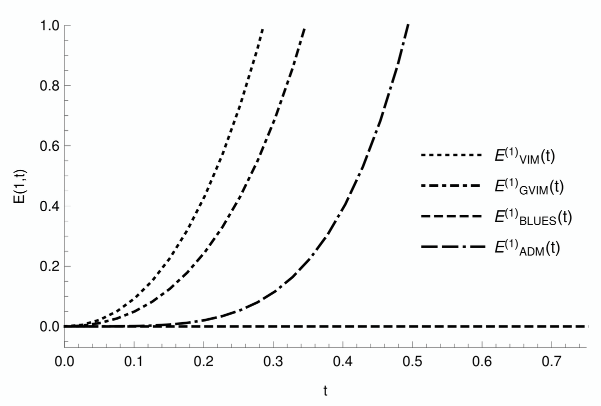

Figure 6: Error between the exact solution (16) of the PME (14) and the approximant of order $n = 1$ for the four methods, at $x = 1$.

Conclusions

The BLUES function method can efficiently generate iterative solutions for highly nonlinear PDEs without the need for linearization or smallness arguments. Comparison with other well-established methods shows that the BLUES method is more efficient in some cases.

References

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