Duality in Vector Padé-Hermite approximation problems

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Abstract

The article defines a class of dual vector Padé-Hermite problems. It describes dual basis matrices for the solution spaces of such problems and proves their properties. We show how degree restrictions can be imposed on vector Padé-Hermite problems and what the effect is on the dual basis matrices.

We also show that the classical duality between Padé-Hermite problems of type I and type II is a special case of vector Padé-Hermite duality. This is proved without the classical restrictive assumption of normality.

The article proposes an algorithm that solves two dual vector Padé-Hermite problems at the same time, returning dual basis matrices of the respective solution spaces.
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Introduction

In origin, this article is based on work of K. Mahler [6, 7], H. Jager [4] and J. Coates [2] on the Padé-Hermite problems of the *Latin* and *German* type. When they assumed normality of the data (i.e. when the corresponding linear system is non-singular), they proved some duality properties of these problems.

Both Padé-Hermite problems can be seen as special cases of vector Padé-Hermite problems. The aim of this article is to generalise the duality principle to the level of the vector Padé-Hermite approximants. We define a class of dual problems, in which the Latin and German problem again form a special case. In this way the duality of those problems is also enlarged to the non-normal case.

The paper is divided as follows. In Section 1 the vector Padé-Hermite approximation problem is defined and we formulate the conditions for a set of solutions to be a basis for its solution space. In section 2 we give a general definition for the duality of two vector Padé-Hermite problems. Section 3 describes how degree restrictions can be imposed on the solutions. Then we prove that the classical duality is only a special case of the generalised one. We show a similar duality between matrix Padé-Hermite approximants and matrix

simultaneous Padé systems and for matrix Padé approximants. Section 4 shows how the recursive solution algorithm for a vector Padé-Hermite problem also solves directly and recursively the dual problem.

1 Definition of the vector Padé-Hermite problem

Consider a field $\mathbf{F}$. Then we can denote the ring of polynomials by $\mathbf{F}[z]$, the ring of power series by $\mathbf{F}[[z]]$ and the $\mathbf{F}[z]$-module of polynomial vectors by $\mathbf{F}[z]^n$.

We generalise the notions 'degree' of a polynomial and 'order' of a power series in the following way for matrices and vectors :

Definition 1 : $O(z^\sigma)$, $O_s(z^\sigma)$, $O(z^{\vec{\sigma}})$, $\vec{\tau}$-$\deg(\vec{a}(z))$
Let $A(z) \in \mathbf{F}[[z]]^{m\times n}$, $\sigma \in \mathbb{N}$ *then*

$$A(z) = O(z^\sigma) \;\Leftrightarrow\; \exists A_p(z) \in \mathbf{F}[[z]]^{m\times n} : A(z) = z^\sigma A_p(z)$$
$$A(z) = O_s(z^\sigma) \;\Leftrightarrow\; \begin{cases} A(z) = z^\sigma \bar{A} + O(z^{\sigma+1}) \\ \bar{A} \in \mathbf{F}^{m\times n} \text{ has full rank.} \end{cases}$$

Let $\vec{f}(z) = \begin{bmatrix} f_1(z) & \dots & f_n(z) \end{bmatrix}^T \in \mathbf{F}[[z]]^n$ *and* $\vec{\sigma} \in \mathbb{N}^n$, *then*

$$\vec{f}(z) = O(z^{\vec{\sigma}}) \;\Leftrightarrow\; f_i(z) = O(z^{\sigma_i}),\ i = 1, \dots, n.$$

Let $\vec{\tau} \in \mathbb{Z}^n$, $\vec{a}(z) \in \mathbf{F}[z]^n$ *and say* $\deg(0) = -1$, *then*

$$\vec{\tau}\text{-}\deg(\vec{a}(z)) = \max\{\deg(a_i(z)) - \tau_i \mid 1 \le i \le n\}.$$

We can define the vector Padé-Hermite problem as an extension of the classical Padé-Hermite approximation problems of the Latin and the German type.

Definition 2 *Given a formal power series* $F_{VPH}(z) \in \mathbf{F}[[z]]^{k\times n}$ $(n > k)$, *with* $F_{VPH}(0)$ *of full rank, then a solution of the* **vector Padé-Hermite approximation problem of order** σ *(*$\sigma > 0$*) is a polynomial vector* $\vec{p}(z) \in \mathbf{F}[z]^n$ *such that*

$$F_{VPH}(z)\vec{p}(z) = O(z^\sigma).$$

For a given $F_{VPH}(z)$ *we denote the set of solutions* $\vec{p}(z)$ *– corresponding to a certain* σ *– by* S^σ.

The condition on the rank of $F_{VPH}(0)$ means that, associated with each $F_{VPH}(z)$, there exists at least one permutation matrix $P_{VPH} \in \mathbf{F}^{n\times n}$ and matrices $F_L(z) \in \mathbf{F}[[z]]^{k\times k}$, $F_R(z) \in \mathbf{F}[[z]]^{k\times(n-k)}$ such that

$$F_{VPH}(z) = \left[\begin{array}{cc} F_L(z) & F_R(z) \end{array} \right] P_{VPH},$$

where $F_L(0)$ is non-singular.

More generally, we can define the vector Padé-Hermite problem of order $\vec{\sigma} \in \mathbb{N}^k$ when we replace in the previous definition σ by $\vec{\sigma}$. This makes it possible to transform practically all vector Padé-Hermite problems into problems that have a full rank $F_{VPH}(0)$. We only have to recombine different rows and then divide them by z until their constant parts are non-zero and linearly independent.

The solution set S^σ is a free $\mathbf{F}[z]$-module of dimension n (cfr. [3]). Hence, there is a set of basis vectors $\vec{a}_i(z)$ such that every solution $\vec{p}(z) \in S^\sigma$ can be written in a unique way as

$$\vec{p}(z) = \sum_{i=1}^{n} c_i(z)\vec{a}_i(z) \text{ for some } c_i(z) \in \mathbf{F}[z].$$

We represent a basis of S^σ, which is a submodule of $\mathbf{F}[z]^n$, as a square matrix, called a *basis matrix*

$$A(z) = \left[\begin{array}{ccc} \vec{a}_1(z) & \cdots & \vec{a}_n(z) \end{array} \right] \in \mathbf{F}[z]^{n\times n}.$$

Lemma 1 *Given a vector Padé-Hermite problem of order σ with $F_{VPH}(z) \in \mathbf{F}[[z]]^{k\times n}$ then a polynomial matrix $A(z) \in \mathbf{F}[z]^{n\times n}$ is a basis matrix of S^σ $\Leftrightarrow$*

$$\left\{ \begin{array}{l} F_{VPH}(z)A(z) = O(z^\sigma) \\ \deg(\det(A(z))) = k\sigma. \end{array} \right.$$

Proof : See [9]. □

The determinant of $A(z)$ is not only of strict degree $k\sigma$, we can prove a much stronger result.

Lemma 2 *Given a vector Padé-Hermite problem of order σ with $F_{VPH}(z) \in \mathbf{F}[[z]]^{k\times n}$ and a corresponding basis matrix $A(z)$ for the solution set S^σ.*
Then $\det(A(z)) = az^{k\sigma},\ a \neq 0$.

Proof : Because $\deg(\det(A(z))) = k\sigma$, we only must prove that $\det(A(z)) = O(z^{k\sigma})$.
Let $l = n - k$, then

$$\det\left(\left[\begin{array}{cc} F_L(z) & F_R(z) \\ 0 & I_l \end{array} \right] P_{VPH}A(z)\right) = \det\left(\left[\begin{array}{c} O(z^\sigma) \\ O(z^0) \end{array} \right]\right)$$

$$\text{i.e. } O_s(z^0)O_s(z^0)\det(A(z)) = O(z^{k\sigma}) \Rightarrow \det(A(z)) = O(z^{k\sigma}).$$

Because the columns of $A(z)$ form a basis, the determinant can not be 0 : $\det(A(z)) \neq 0 \Rightarrow a \neq 0$. $\square$

The proof of the lemma gives us necessary and sufficient conditions for a set of solutions to be a basis. We can say that for $A(z) \in \mathbf{F}[z]^{n\times n}$

$$A(z) \text{ is a basis matrix of } S^\sigma \Leftrightarrow \begin{cases} F_{VPH}(z)A(z) = O(z^\sigma) \\ \det(A(z)) = a(z) \neq 0 \\ \deg(a(z)) \leq k\sigma \end{cases}.$$

We do not use here the stronger condition $a(z) = az^{k\sigma}$ which is, by Lemma 2, an immediate consequence of $\deg(a(z)) \leq k\sigma$, because we will regard this equivalence mainly as a set of sufficient conditions for a set of polynomial vectors to be a basis matrix.

Definition 3 *We define an extension $\tilde{F}(z) \in \mathbf{F}[[z]]^{n\times n}$ of the matrix $F_{VPH}(z)$ as :*

$$\tilde{F}(z) = \begin{bmatrix} F_L(z) & F_R(z) \\ 0 & z^\sigma I_l \end{bmatrix} P_{VPH}.$$

We can then prove that the residual series $R(z) = F_{VPH}(z)A(z) = O_s(z^\sigma)$.
Indeed, we have

$$\begin{aligned} \det(\tilde{R}(z)) = \det(\tilde{F}(z)A(z)) &= \det(\tilde{F}(z))\det(A(z)) = O_s(z^{n\sigma}) \\ \Rightarrow \left[\det(z^{-\sigma}\tilde{R}(z))\right]_{z=0} &\neq 0 \\ \Rightarrow \tilde{R}(z) = O_s(z^\sigma) \Rightarrow R(z) &= O_s(z^\sigma). \end{aligned}$$

2 Duality

For a given matrix $F^{(1)}_{VPH}(z) = \begin{bmatrix} F_L(z) & F_R(z) \end{bmatrix} P_{VPH} \in \mathbf{F}[[z]]^{k\times n}$, consider the associated problem with

$$F^{(2)}_{VPH}(z) = \begin{bmatrix} -F_R(z)^T F_L(z)^{-T} & I_l \end{bmatrix} P_{VPH} \in \mathbf{F}[z]^{(n-k)\times n}. \quad (1)$$

It is clear that this new matrix defines a vector Padé-Hermite problem, because $rank F^{(2)}_{VPH}(0) = n - k = l$.

Both problems are complementary in the sense that

$$F^{(1)}_{VPH}(z)F^{(2)}_{VPH}(z)^T = 0.$$

For this reason and because their dimensions are also complementary, i.e. $k + l = n$, they are conventionally called *dual* problems. However, the set of dual problems is larger.

Definition 4 *Consider two vector Padé-Hermite problems of order σ defined by $F_1(z) \in \mathbf{F}[[z]]^{k\times n}$ and $F_2(z) \in \mathbf{F}[[z]]^{l\times n}$ with $k+l=n$. Then these two problems,*

$$F_1(z)\vec{p}(z) = O(z^\sigma)$$
$$F_2(z)\vec{p}(z) = O(z^\sigma),$$

are called dual if

$$F_1(z)F_2(z)^T = O(z^\sigma).$$

If an extension $\tilde{F}_1(z)$ of the matrix $F_1(z)$ is given as in Definition 3, then we can define a dual extension for the matrix $F_2(z)$ as

$$\tilde{F}_2(z) = \begin{bmatrix} F_2(z)P_1^{-1} \\ z^\sigma I_k \quad 0 \end{bmatrix} P_1 = \begin{bmatrix} F_{2,R}(z) & F_{2,L}(z) \\ z^\sigma I_k & 0 \end{bmatrix} P_1.$$

So both extensions have the same permutation matrix $P_1 = P_2$. This is a correct extension, i.e. $rank F_{2,L}(0) = l$; the rows of $F_2(0)$ cover the complete null space of the rows of $F_1(0)$, because $n = k + l$ and $F_1(0)F_2(0)^T = 0$ (for $\sigma > 0$).
For the basis matrices of dual problems, we can prove the following theorem.

Theorem 1 *Consider two dual vector Padé-Hermite problems of order σ with $F_1(z)$ and $F_2(z)$. Let $A_1(z)$ be a basis matrix for the solution space of the first problem. Then the matrix $A_2(z) = z^\sigma A_1(z)^{-T}$ is a basis matrix for the solution space of the second vector Padé-Hermite problem. $A_2(z)$ is called the dual basis matrix for $A_1(z)$.*

Proof : First, we can see that

$$\tilde{F}_1(z)\tilde{F}_2(z)^T = \begin{bmatrix} F_{1,L}(z) & F_{1,R}(z) \\ 0 & z^\sigma I_l \end{bmatrix} P_1 P_1^T \begin{bmatrix} F_{2,R}(z)^T & z^\sigma I_k \\ F_{2,L}(z)^T & 0 \end{bmatrix} = \begin{bmatrix} O(z^\sigma) & O_s(z^\sigma) \\ O_s(z^\sigma) & 0 \end{bmatrix} = O_s(z^\sigma)$$

and not only $O(z^\sigma)$.
Notice that $A_2(z) = z^\sigma A_1(z)^{-T}$ is a polynomial matrix because

$$\tilde{F}_1(z)A_1(z) = O_s(z^\sigma)$$

$$\Rightarrow O_s(z^{-\sigma})\tilde{F}_1(z) = A_1(z)^{-1} \Rightarrow z^\sigma A_1(z)^{-1} = O_s(z^0)\tilde{F}_1(z) = O(z^0).$$

Because $A_1(z)A_2(z)^T = z^\sigma I_n = A_1(z)^T A_2(z)$,

$$\begin{aligned} \tilde{F}_2(z)\left[A_2(z)A_1(z)^T\right]\tilde{F}_1(z)^T &= z^\sigma \tilde{F}_2(z)\tilde{F}_1(z)^T = O_s(z^{2\sigma}) \\ = \left[\tilde{F}_2(z)A_2(z)\right]\left[\tilde{F}_1(z)A_1(z)\right]^T &= O(z^\alpha)O_s(z^\sigma) \end{aligned}$$

So $\alpha = \sigma$, $A_2(z)$ is a matrix of solutions of the second problem. Because $\deg(\det(A_2(z))) = n\sigma - k\sigma = l\sigma$, it is also a basis matrix. $\Box$

Note : It is important to remember that for non-square residual series, $R_1(z) = O_s(z^\sigma)$ and $R_2(z) = O_s(z^\sigma)$ does not imply that the product of both should be of strict order : $R_1(z)R_2(z)^T \neq O_s(z^{2\sigma})$. Columns of both coefficient matrices can always fit in each others null space. Examples are both previous complementary problems (1), for which (for residuals of dual basis matrices) $R_1(z)R_2(z)^T = 0$.

3 Solutions with degree restrictions

Most of the time we want to find all solutions having a specific degree structure, and this will be reflected by the degree structure of the basis matrix. If the intended degrees are represented by a vector $\vec{\tau}$, then we look for a basis matrix which is $\vec{\tau}$*-reduced* [9].

Let $\vec{\tau} \in \mathbb{Z}^n$. Any polynomial matrix $A(z) \in \mathbf{F}[z]^{n \times m}$ can be written as

$$T_{-\vec{\tau}}(z)A(z) = (\bar{A} + O(\tfrac{1}{z}))T_{\vec{\alpha}}(z),$$

where

$$\begin{cases} T_{\vec{c}}(z) = diag(z^{c_1}, \ldots, z^{c_n}) \\ \vec{\alpha} \in \mathbb{Z}^m, \ (\alpha_i = \vec{\tau}\text{- }\deg(\vec{a}_i(z))). \end{cases}$$

In this equation, $\bar{A}$ consists of the *highest* $\vec{\tau}$*-degree coefficients* of the basis vectors. If these coefficients are linearly independent, we call the matrix $\vec{\tau}$-reduced. If $A(z)$ has zero-columns, then for each zero-column in $A(z)$ a corresponding zero-column in $\bar{A}$ is allowed.

Note : To avoid the confusion that is inherent in the notation '$O(z^{-1})$', we explicitly use '$O(\frac{1}{z}) = z^{-1}A_{-1} + z^{-2}A_{-2} + \ldots$'.

Definition 5 *A matrix $A(z) \in \mathbf{F}[z]^{n \times m}$ is called $\vec{\tau}$-reduced for a $\vec{\tau} \in \mathbb{Z}^n$ if the matrix of the highest $\vec{\tau}$-degree coefficients $\bar{A}$ has full rank.*

The notion of $\vec{\tau}$-reduced basis matrix is strongly related to the definition of σ-bases in [1], where an almost complementary approach is used, using a 'defect' instead of a $\vec{\tau}$-degree.

In practice one tries to compute a $\vec{\tau}$-reduced basis matrix for the set $S^\sigma$of all solutions of a vector Padé-Hermite problem. $\vec{\tau}$-reducedness makes it possible to retrieve solutions with a special degree structure and allows us to map specific linear systems onto the vector Padé-Hermite framework.

As a small example of this mapping, we can retrieve the solution of the Hankel system

$$H\mathbf{x} = \mathbf{b},$$

$$\begin{cases} H \in \mathbf{F}^{(n+1)\times(n+1)},\ H_{ij} = h_{i+j},\ 0 \le i,j \le n \\ \mathbf{b} = \left[\begin{array}{ccc} b_0 & \cdots & b_n \end{array}\right]^T \end{cases}$$

as of the solution with $\vec{\tau}$-degree $= 0$ of the vector Padé-Hermite problem with

$$\begin{cases} F_{VPH}(z) = \left[\begin{array}{ccc} \sum_{i=0}^{2n} h_i z^i & -1 & -\sum_{i=0}^{n} b_i z^{n+i} \end{array}\right] = \left[\begin{array}{ccc} h(z) & -1 & -b(z) \end{array}\right] \\ \vec{\tau} = \left[\begin{array}{ccc} n & n-1 & 0 \end{array}\right] \\ \sigma = 2n \end{cases}$$

The generalisation towards block-Hankel systems is straightforward.

Definition 6 *The restricted solution set $S_\delta^\sigma(\vec{\tau})$ is defined as*

$$S_\delta^\sigma(\vec{\tau}) = \{\vec{a}(z) \mid \vec{a}(z) \in S^\sigma,\ \vec{\tau}\text{-}\deg(\vec{a}(z)) \le \delta\},\ (\delta \in \mathbb{Z})$$

$S_\delta^\sigma(\vec{\tau})$ is no longer an $\mathbf{F}[z]$-module, but it still is a vector space over $\mathbf{F}$.
If $A_\sigma(z)$ is a $\vec{\tau}$-reduced basis matrix for the problem of order σ then

$$\{z^{\alpha_i}\vec{a}_i(z) \mid 0 \le \alpha_i \le \delta - \vec{\tau}\text{-}\deg(\vec{a}_i(z)),\ \vec{a}_i(z) \in A_\sigma(z),\ 1 \le i \le n\}$$

is a basis for $S_\delta^\sigma(\vec{\tau})$.

Theorem 2 *If $A(z)$ is a basis matrix of S^σ, then*

$$A(z) \text{ is } \vec{\tau}\text{-reduced} \ \Leftrightarrow\ \sum_{i=1}^{n} \vec{\tau}\text{-}\deg(\vec{a}_i(z)) \text{ is minimal (over the set of all possible bases).}$$

Proof : [$\Rightarrow$] Suppose that the sum of $\vec{\tau}$-degrees is not minimal, then there is a basis matrix $B(z)$ with a lower sum and $B(z) = A(z)M(z)$ for a unimodular $M(z)$, i.e., $M(z) \in \mathbf{F}[z]^{n\times n}$ and $\det(M(z)) = \mu \in \mathbf{F}\backslash\{0\}$. If we call $\vec{\alpha}$ and $\vec{\beta}$ the vector of $\vec{\tau}$-degrees of $A(z)$ and $B(z)$ respectively, then

$$\begin{aligned} B(z) = T_{\vec{\tau}}(z)(\bar{B} + O(\tfrac{1}{z}))T_{\vec{\beta}}(z) &= T_{\vec{\tau}}(z)(\bar{A} + O(\tfrac{1}{z}))T_{\vec{\alpha}}(z)M(z) \\ \Rightarrow \det(\bar{B} + O(\tfrac{1}{z})) &= \det(\bar{A} + O(\tfrac{1}{z}))\det\left[T_{\vec{\alpha}}(z)M(z)T_{-\vec{\beta}}(z)\right] \\ \Rightarrow \det(\bar{B}) + O(\tfrac{1}{z}) &= (\det(\bar{A}) + O(\tfrac{1}{z}))\mu z^\delta,\ \ \delta = \textstyle\sum_{i=1}^{n}(\alpha_i - \beta_i). \end{aligned}$$

But $\det(\bar{A}) \neq 0 \ \Rightarrow\ \delta \le 0 \Rightarrow \sum_{i=1}^{n}\alpha_i \le \sum_{i=1}^{n}\beta_i$.
[$\Leftarrow$] Suppose $\exists \vec{q} \in \mathbf{F}^n : \bar{A}\vec{q} = 0$.
Let $\vec{\tau}\text{-}\deg(\vec{a}_k(z)) = \max\{\vec{\tau}\text{-}\deg(\vec{a}_i(z)) \mid q_i \neq 0, i = 1\ldots n\}$ and replace $\vec{a}_k(z)$ in $A(z)$ by

$\vec{a}_*(z) = \sum_{i=1}^{n} q_i \vec{a}_i(z) z^{\alpha_k - \alpha_i}$. Because $\vec{\tau}\text{-}\deg(\vec{a}_*(z)) < \vec{\tau}\text{-}\deg(\vec{a}_k(z))$, we would have found a new basis with a smaller sum of $\vec{\tau}$-degrees. □

The above theorem gives an alternative condition for $\vec{\tau}$-reducedness of a basis :

$$A(z) \text{ is a } \vec{\tau}\text{-reduced basis of } S^\sigma$$

$$\Leftrightarrow$$

$A(z)$ is an independent set of n solution vectors and has a minimal $\vec{\tau}$-degree sum.

Another consequence is that if any basis exists for the solution space, then there always will be at least one $\vec{\tau}$-reduced basis, because the set of integers

$$\{\sum_{i=1}^{n} \vec{\tau}\text{-}\deg(\vec{a}_i(z)) \mid A(z) = \left[\begin{array}{ccc} \vec{a}_1(z) & \cdots & \vec{a}_n(z) \end{array} \right] \text{ is a basis for } S^\sigma\}$$

has a lower bound, and so must have a minimum.

Theorem 3 *Given two dual vector Padé-Hermite problems of order σ, with two respective dual basis matrices $A_1(z)$ and $A_2(z)$ then*

$$A_1(z) \textit{ is } \vec{\tau}\textit{-reduced} \Leftrightarrow A_2(z) \textit{ is } (-\vec{\tau})\textit{-reduced},$$

$$\vec{\tau}\text{-}\deg(\vec{a}_{1,i}(z)) = \alpha_i \Leftrightarrow -\vec{\tau}\text{-}\deg(\vec{a}_{2,i}(z)) = \sigma - \alpha_i,\ i = 1, \ldots, n.$$

Proof : Set $\vec{\sigma} = (\sigma, \ldots, \sigma)$ then

$$\begin{aligned} A_1(z) &= T_{\vec{\tau}}(\bar{A}_1 + O(\tfrac{1}{z}))T_{\vec{\alpha}} = A_2(z)^{-T} T_{\vec{\sigma}} \\ \Rightarrow A_2(z) &= T_{-\vec{\tau}}(\bar{A}_1 + O(\tfrac{1}{z}))^{-T} T_{\vec{\sigma}-\vec{\alpha}} = T_{-\vec{\tau}}(\bar{A}_1{}^{-T} + O(\tfrac{1}{z}))T_{\vec{\sigma}-\vec{\alpha}}. \end{aligned}$$

Thus $\bar{A}_2 = \bar{A}_1{}^{-T}$ has full rank. □

Classical Padé-Hermite problems

Based on Hermite's results – on the function set of exponential functions $e^{\omega_i z}$ – Mahler [7] developed a theory on the (dual) Padé-Hermite problems. These problems can be seen as special cases of vector Padé-Hermite problems with strict degree conditions imposed. Also *normality* is always assumed in Mahler's work, i.e. the associated linear systems are never singular. He worked with a set of $(n+1)$ formal power series $(f_0(z), \ldots, f_n(z))$.

Definition 7 *[4]*
Given $\vec{\rho} = (\rho_0, \ldots, \rho_n) \in \mathbb{N}_0^{n+1}$, $\sigma = \sum_{i=0}^{n} \rho_i$, *define the following problems* ($\mathbb{N}_0 = \mathbb{N} \setminus \{0\}$).

Padé-Hermite-problem of type I (Latin type) :
Find a non-trivial set of polynomials $(a_0(z), \ldots, a_n(z))$ *such that*

$$\begin{cases} \deg(a_\nu(z)) \leq \rho_\nu - 1, \ 0 \leq \nu \leq n \\ r(z) = \sum_{\nu=0}^{n} f_\nu(z) a_\nu(z) = O(z^{\sigma-1}). \end{cases}$$

Padé-Hermite-problem of type II (German type) :
Find a non-trivial set of polynomials $(\mathfrak{a}_0(z), \ldots, \mathfrak{a}_n(z))$ *such that*

$$\begin{cases} \deg(\mathfrak{a}_\nu(z)) \leq \sigma - \rho_\nu, \ 0 \leq \nu \leq n \\ \mathfrak{r}_{k,l}(z) = \mathfrak{a}_k(z) f_l(z) - \mathfrak{a}_l(z) f_k(z) = O(z^{\sigma+1}), \ 0 \leq k, l \leq n. \end{cases}$$

If we set

$$\begin{cases} a_\nu(z) = a_\nu(z \mid \rho_0, \ldots, \rho_n) \\ \mathfrak{a}_\nu(z) = \mathfrak{a}_\nu(z \mid \rho_0, \ldots, \rho_n) \end{cases}$$

we can define

$$\begin{cases} a_{\mu,\nu}(z \mid \rho_0, \ldots, \rho_n) = a_\nu(z \mid \rho_0 + \delta_{\mu,0}, \ldots, \rho_n + \delta_{\mu,n}) \\ \mathfrak{a}_{\mu,\nu}(z \mid \rho_0, \ldots, \rho_n) = \mathfrak{a}_\nu(z \mid \rho_0 - \delta_{\mu,0}, \ldots, \rho_n - \delta_{\mu,n}). \end{cases}$$

Because $\sum_{i=0}^{n} \rho_i \pm \delta_{\mu,i} = \sigma \pm 1$ for these problems, both the $\mathfrak{r}_{k,l}(z)$ and the $r(z)$ will be of order σ.
Construct the $(n+1) \times (n+1)$ matrices

$$\begin{cases} A(z \mid \rho_0, \ldots, \rho_n) = [a_{\mu,\nu}(z \mid \rho_0, \ldots, \rho_n)]_{(\mu=0,\ldots,n;\ \nu=0,\ldots,n)} \\ \mathfrak{U}(z \mid \rho_0, \ldots, \rho_n) = [\mathfrak{a}_{\mu,\nu}(z \mid \rho_0, \ldots, \rho_n)]_{(\mu=0,\ldots,n;\ \nu=0,\ldots,n)}. \end{cases}$$

If the solution space of each of these problems is one dimensional, these matrices are well defined. Then we will call $\vec{\rho}$ a *normal point*. In such a normal point the following special relationships where retrieved [2, 4].

$$A(z \mid \rho_0, \ldots, \rho_n) \mathfrak{U}(z \mid \rho_0, \ldots, \rho_n)^T = \begin{bmatrix} c_0 z^\sigma & & 0 \\ & \ddots & \\ 0 & & c_n z^\sigma \end{bmatrix}, \ c_i \neq 0,$$

and the result for the determinants

$$\begin{cases} \det(A(z \mid \rho_0, \ldots, \rho_n)) = b z^\sigma, \ b \neq 0 \\ \det(\mathfrak{U}(z \mid \rho_0, \ldots, \rho_n)) = a z^{n\sigma}, \ a \neq 0 \end{cases}$$

Both the type I and type II problems are vector Padé-Hermite problems. We are able to solve them and then fulfill the restrictions on the degrees. This can be easily seen. Assume $f_i(0) \neq 0$, for some value of i.

1. For the type I problem, we set $F^I_{VPH}(z) = \left[\begin{array}{ccc} f_0(z) & \cdots & f_n(z) \end{array} \right]$.

2. To get the type II problem we have to put

$$F^{II}_{VPH}(z) = \begin{bmatrix} -f_i(z) & & & f_0(z) & & & \\ & \ddots & & \vdots & & & \\ & & -f_i(z) & f_{i-1}(z) & & & \\ & & & f_{i+1}(z) & -f_i(z) & & \\ & & & \vdots & & \ddots & \\ & & & f_n(z) & & & -f_i(z) \end{bmatrix}.$$

Theorem 4 *Consider a system $(f_0(z), \ldots, f_n(z))$ which is normal for the point $\vec{\rho} = (\rho_0, \ldots, \rho_n)$.*

1. *$A(z \mid \rho_0, \ldots, \rho_n)^T$ is a $\vec{\rho}$-reduced basis matrix for the vector Padé-Hermite problem of type I and order σ.*

2. *$\mathfrak{U}(z \mid \rho_0, \ldots, \rho_n)^T$ is a $-\vec{\rho}$-reduced basis matrix for the vector Padé-Hermite problem of type II and order σ.*

Proof :

$$\begin{cases} F^I_{VPH} A(z)^T = O(z^\sigma) \\ \det(A(z)^T) = bz^\sigma, b \neq 0 \end{cases} \Rightarrow A(z)^T \text{ is a basis of } S^\sigma_I.$$

$$T_{-\vec{\rho}} A(z)^T = D + O(\tfrac{1}{z})$$

with D a non-singular diagonal matrix because it is a normal point, so $\det(D + O(\frac{1}{z})) = bz^{\sigma-\sigma} = b \neq 0$. Hence, $A(z)^T$ is a $\vec{\tau}$-reduced basis.
The proof for $\mathfrak{U}(z)$ is analogous. □

Matrix Padé-Hermite and matrix simultaneous Padé approximants

We can show that the duality properties also hold between matrix Padé-Hermite and matrix simultaneous Padé approximants. The definition of these problems is as in definition 7, where the power series $f_\nu(z)$ are replaced by square matrices $F_\nu(z) \in \mathbf{F}[[z]]^{p \times p}$.

In [6, section 5], Labahn constructs the square $(n+1)p \times (n+1)p$ polynomial matrix

$$\Gamma(z \mid \vec{\rho}) = \begin{bmatrix} P(z) & U(z) \\ Q(z) & V(z) \end{bmatrix}.$$

To be consistent with our notation, we have replaced n of [6] by $\vec{\rho}$ and k by n.

Let us consider the matrix

$$A_1(z) = \Gamma(z \mid \vec{\rho}) \begin{bmatrix} z^2 I_p & 0 \\ 0 & I_{pn} \end{bmatrix} = \begin{bmatrix} z^2 P(z) & U(z) \\ z^2 Q(z) & V(z) \end{bmatrix}.$$

When a block Hankel striped matrix [6 (1.1)] is nonsingular then $\det(A_1(z)) \neq 0$. Note that the latter is a normality condition. Moreover, with $\vec{\tau} = (\rho_0, \ldots, \rho_1, \ldots, \rho_1, \ldots, \rho_k, \ldots, \rho_k)$, where each ρ_i repeated p times, the $\vec{\tau}$-degree of the first n columns of $A_1(z)$ is ≤ 1, while the $\vec{\tau}$-degree of the remaining pn columns is ≤ 0. Hence, $\deg(\det(A_1(z))) \leq p\sigma + p$.

Also, with $F^I_{VPH}(z) = \begin{bmatrix} F_0(z) & \cdots & F_n(z) \end{bmatrix}, F_i(z) \in \mathbf{F}[[z]]^{p \times p}$,

$$F^I_{VPH}(z) A_1(z) = O_s(z^{\sigma+1}).$$

Thus, $A_1(z)$ is a basis matrix of $S^{\sigma+1}$ for $F^I_{VPH}(z)$.

If we then consider the dual problem with

$$F^{II}_{VPH}(z) = \begin{bmatrix} -F_i(z)^T & & & F_0(z)^T & & & \\ & \ddots & & \vdots & & & \\ & & -F_i(z)^T & F_{i-1}(z)^T & & & \\ & & & F_{i+1}(z)^T & -F_i(z)^T & & \\ & & & \vdots & & \ddots & \\ & & & F_n(z)^T & & & -F_i(z)^T \end{bmatrix},$$

we get $A_2(z) = z^{\sigma+1} A_1(z)^{-T}$ as a basis matrix of the solution space $S^{\sigma+1}$ for $F^{II}_{VPH}(z)$.

Moreover, because $A_1(z)$ is $\vec{\tau}$-reduced with $\vec{\tau}$-degrees $(\underbrace{1, \ldots, 1}_{p}, \underbrace{0, \ldots, 0}_{pn})$, $A_2(z)$ is $-\vec{\tau}$-reduced with $-\vec{\tau}$-degrees $(\underbrace{\sigma, \ldots, \sigma}_{p}, \underbrace{\sigma+1, \ldots, \sigma+1}_{pn})$ (or $(\sigma - \vec{\tau})$-reduced with $\vec{\tau}$-degrees $(0, \ldots, 0, 1, \ldots, \ldots, 1)$).

Comparing with the results in [5], it turns out that

$$A_2(z)^T = \begin{bmatrix} V^*(z) & U^*(z) \\ -z^2 Q^*(z) & -z^2 P^*(z) \end{bmatrix}.$$

Matrix Padé approximants

In [5 (3.17)], Labahn constructs a matrix polynomial

$$\Gamma_{m,n}(z) = \begin{bmatrix} P(z) & U(z) \\ Q(z) & V(z) \end{bmatrix} \in \mathbf{F}[z]^{(p+q)\times(p+q)}.$$

If we consider now the matrix

$$A_1(z) = \Gamma_{m,n}(z) \begin{bmatrix} z^2 I_p & 0 \\ 0 & I_q \end{bmatrix},$$

it turns out that if the square block Hankel matrix $H_{m,n}$ ($mq = np$) in [5 (1.1)] is nonsingular (again a normality condition) then $\det(A_1(z)) \neq 0$.

Moreover, with $\vec{\tau} = (m, \ldots, m)$, the $\vec{\tau}$-degree of the first p columns of $A_1(z)$ is 1, while the remaining q columns have $\vec{0}$-degree 0. Hence, $\deg(\det(A_1(z))) \leq (p+q)m + p$.

On the other hand,

$$F^I_{VPH}(z)A_1(z) = \begin{bmatrix} I_p & -F(z) \end{bmatrix} A_1(z) = O(z^{m+n+1}) = O(z^\sigma).$$

Therefore, $A_1(z)$ is a basis matrix of S^σ for $F^I_{VPH}(z)$. From Theorem 1, we know that $A_2(z) = z^\sigma A_1(z)^{-T}$ is a basis matrix of S^σ for

$$F^{II}_{VPH}(z) = \begin{bmatrix} F(z)^T & I_q \end{bmatrix}.$$

Again, Theorem 3 states that the $-\vec{\tau}$-degree of the columns of $A_2(z)$ is $(\underbrace{\sigma-1, \ldots, \sigma-1}_{p}, \underbrace{\sigma, \ldots, \sigma}_{q})$ (or, with $\vec{\tau}_n = (n, \ldots, n)$, the $\vec{\tau}_n$-degree is $(\underbrace{0, \ldots, 0}_{p}, \underbrace{1, \ldots, 1}_{q})$).

Comparing with the results of [5 (3.12)], it is easy to see that

$$A_2(z)^T = \begin{bmatrix} -V^*(z) & U^*(z) \\ z^2 Q^*(z) & -z^2 P^*(z) \end{bmatrix}.$$

As a consequence, we get that the right matrix Padé approximant $U(z)V(z)^{-1} = F(z) + O(z^{m+n+1})$ is equal to the left matrix Padé approximant $V^*(z)^{-1}U^*(z)$.

4 An algorithm to solve dual problems

Given two dual problems, the algorithm that we propose recursively builds a $\vec{\tau}$-reduced basis matrix for the first problem and at the same time a $-\vec{\tau}$-reduced basis matrix for

the second problem. In each step it increases the order σ of approximation by one, introducing in each problem respectively k and l new degrees of freedom. The algorithm is a generalisation of the algorithm in [8]. An algorithm that solves dual problems could be used to compute the elements needed for the inversion components of structured matrices.

Given $F_1(z) \in \mathbf{F}[z]^{k \times n}$ and $F_2(z) \in \mathbf{F}[z]^{l \times n}$ with $F_1(z)F_2(z)^T = O(z^\sigma)$ and $F_1(0), F_2(0)$ of full rank.

Algorithm :

$A_1^0(z) = I_n = A_2^0(z)$
for $\sigma = 1, 2, 3, \ldots$
Compute $R_1^\sigma \in \mathbf{F}^{k \times n}$ as
$F_{VPH}(z) A_1^{\sigma-1}(z) = z^{\sigma-1} R_1^\sigma + O(z^\sigma)$
Compute the permutation matrix $P^\sigma \in \mathbf{F}^{n \times n}$ such that
(i) the first k columns of $R_1^\sigma P^\sigma$ are linearly independent,
(ii) the k corresponding vectors $\vec{a}_i^{\sigma-1}(z)$
in the matrix $A_1^{\sigma-1}(z)$ have minimal $\vec{\tau}$-degree (within (i)).
Compute $C \in \mathbf{F}^{k \times l}$ such that

$$R_1^\sigma P^\sigma \begin{bmatrix} C \\ I_l \end{bmatrix} = 0$$

Compute
$A_1^\sigma(z) = A_1^{\sigma-1}(z) V_1^\sigma(z)$,
$A_2^\sigma(z) = A_2^{\sigma-1}(z) V_2^\sigma(z)$,
where

$$V_1^\sigma(z) = P^\sigma \begin{bmatrix} zI_k & C \\ 0 & I_l \end{bmatrix},$$

$$V_2^\sigma(z) = z V_1^\sigma(z)^{-T} = P^\sigma \begin{bmatrix} I_k & 0 \\ -C^T & zI_l \end{bmatrix}.$$

Theorem 5 *The algorithm constructs $\vec{\tau}$- and $-\vec{\tau}$-reduced dual basis matrices for the solution spaces S_1^σ and S_2^σ.*

Proof : The algorithm constructs for both problems independent sets of n solutions with $\deg(\det(A_1^\sigma(z))) = k\sigma$ and $\deg(\det(A_2^\sigma(z))) = l\sigma$.
In each step k new degrees of freedom are introduced in $A_1^\sigma(z)$: k basis vectors are multiplied by z (their $\vec{\tau}$-degrees are raised with 1). The l other vectors are recombined (to solve the problem) in such a way that their $\vec{\tau}$-degree does not grow. So the sum of $\vec{\tau}$-degrees remains minimal and the basis matrix is $\vec{\tau}$-reduced. The case of $A_2^\sigma(z)$ is analogous. □

The duality properties can be seen in each step of the recursion. Indeed, $A_1^\sigma(z)A_2^\sigma(z)^T = z^\sigma I_n = A_1^\sigma(z)^T A_2^\sigma(z)$, because $(P^{\sigma T}P^\sigma = I_n)$

$$V_1^\sigma(z)^T V_2^\sigma(z) = \begin{bmatrix} zI_k & 0 \\ C^T & I_l \end{bmatrix} P^{\sigma T} P^\sigma \begin{bmatrix} I_k & 0 \\ -C^T & zI_l \end{bmatrix} = zI_n.$$

Notes :

1. The row space of lowest degree coefficients of the residual series R_1^σ and R_2^σ are complementary $(k + l = n)$.

$$\begin{aligned} \left[F_1(z)A_1^{\sigma-1}(z)\right]\left[A_2^{\sigma-1}(z)^T F_2(z)^T\right] &= F_1(z)\left[A_1^{\sigma-1}(z)A_2^{\sigma-1}(z)^T\right]F_2(z)^T \\ O(z^{\sigma-1})O(z^{\sigma-1}) &= F_1(z)z^{\sigma-1}I_n F_2(z)^T \\ R_1^\sigma R_2^{\sigma T} z^{2\sigma-2} + O(z^{2\sigma-1}) &= O(z^{2\sigma-1}) \\ \Rightarrow R_1^\sigma R_2^{\sigma T} &= 0. \end{aligned}$$

2. We know that $V_1^\sigma(z)$ and $V_2^\sigma(z)$ can be found in the way we described :

$$\begin{cases} R_2^\sigma R_1^{\sigma T} = 0, \\ rank R_1^\sigma = k,\ rank R_2^\sigma = l \end{cases} \Rightarrow \begin{bmatrix} R_1^\sigma \\ R_2^\sigma \end{bmatrix} \text{ has full rank.}$$

   This means that the complementary part in R_2 of a square full rank submatrix of R_1 is never singular. So if P^σ fulfills the condition (i) for R_1^σ, then P^σ automatically fits the same condition for R_2^σ. It also means that the rows of $\begin{bmatrix} C^T & I_l \end{bmatrix}$ cover the same space as the rows of $R_2^\sigma P^\sigma$, which is equal to the null space of the rows of $R_1^\sigma P^\sigma$. So in the algorithm R_2^σ, which must not be computed, may be used instead of $\begin{bmatrix} C^T & I_l \end{bmatrix}$. But then the generated basis matrices are not dual any more.

3. The basis matrices that are constructed by the algorithm have some additional properties. For $A_1^\sigma(z)$ holds

$$\begin{cases} A_L(0) = \begin{bmatrix} \vec{a}_{1,1}(0) & \cdots & \vec{a}_{1,k}(0) \end{bmatrix} = 0 \\ F_1(z)A_L(z) = O_s(z^\sigma) \end{cases}$$
$$A_R(0) = \begin{bmatrix} \vec{a}_{1,k+1}(0) & \cdots & \vec{a}_{1,n}(0) \end{bmatrix} \text{ has full rank.}$$

   For $A_2(z)$ similar properties are true, but shifted as in (1).

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