# Rational bases for system identification

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# Systems

We consider discrete time systems




$$\left.\begin{array}{l} u \text{ is input} \rightarrow U(z) = \sum_k u_k z^{1-k} \\ y \text{ is output} \rightarrow Y(z) = \sum_k y_k z^{1-k} \\ G \text{ transfer function} \end{array}\right\} \quad Y(z) = G(z)U(z)$$

# Frequency Domain Identification

Estimate $G$ as $\hat{G}$ such that we minimize ($L^2(\mathbb{T})$-norm)

$$\|Y - \hat{Y}\| = \|GU - \hat{G}U\| = \|(G - \hat{G})U\| = \|G - \hat{G}\|_w$$

$$\min \frac{1}{2\pi} \int_{-\pi}^{+\pi} |G(z) - \hat{G}(z)|^2 w(z) d\omega, \quad z = e^{i\omega}, w(z) = |U(z)|^2$$

♣ ♣ ♣

Measurements $\{G(z_j)\}_{j=1}^N$, $z_j \in \mathbb{T}$, with variance $\{\sigma_j\}_{j=1}^N$.

$$\min \|G - \hat{G}\|_w = \min \left[ \sum_{j=1}^{N} \frac{|G(z_j) - \hat{G}(z_j)|^2}{\sigma_j^2} \right]^{1/2}, \quad w_j = \sigma_j^{-2},$$

# Frequency Domain Identification(2)

or

$$\begin{aligned}\|Y-\hat{Y}\| &= \|(G-\hat{G})U\| = \|G-\hat{G}\|_w \\ &= \left[\sum_{j=1}^{N}|G(z_j)-\hat{G}(z_j)|^2 w_j\right]^{1/2}, \quad w_j = |U(z_j)|^2/\sigma_j^2\end{aligned}$$

or

$$\begin{aligned}\|G-\hat{G}\| &= \left\|\frac{Y}{U}-\frac{B}{A}\right\| \\ &= \left\|\frac{YA-BU}{AU}\right\| = \|YA-BU\|_w, \quad w = 1/|AU|^2\end{aligned}$$

# Linear/Nonlinear problem

Since the approximant $\hat{G}$ is a rational function, we have a mixed linear/nonlinear approximation problem.

$$\hat{G}(z) = \frac{B(z)}{A(z)}, \quad A, B \in \Pi_n$$

$$\text{linear } B(z) = \sum_{k=0}^{n} b_k z^{-k}$$

$$\text{nonlinear } A(z) = \sum_{k=0}^{n} a_k z^{-k} = \prod_{k=1}^{n} (1 - \alpha_k / z).$$

Given an estimate for $A$,
finding $B$ is a weighted but linear least squares problem.

# Nonlinear problem

Given a stable and efficient method to solve the linear parameters $\lambda$ as a function of the nonlinear parameters $\nu$, we can consider the problem of minimizing a cost function $C(\nu) = K(\nu, \lambda(\nu))$.

Need system stability. If $\nu = \alpha$ (system poles), then need $|\alpha_k| < 1$. So have to solve a constrained nonlinear weighted least squares problem in $\mathbb{C}$.

Solve nonlinear problem with standard routine.
How to solve the linear problem?

# Orthogonal Rational Functions

$$\hat{G}(z) = \sum_{k=0}^{n} \lambda_k \phi_k(z),$$

with for given $\alpha_k$

$$\phi_k(z) \in \mathcal{L}_k = \left\{ \frac{p_k(z)}{\prod_{j=1}^{k}(1-\alpha_j/z)} : p_k \in \Pi_k \right\}$$

orthogonal basis functions with respect to an appropriate inner product. The inner product can be discrete or continuous, but is in general with respect to a weight (measure) on $\mathbb{T}$.

$$\langle f, g \rangle_\mu = \int_{\mathbb{T}} f(z)\overline{g(z)} d\mu(z) \quad \text{or} \quad \langle f, g \rangle_w = \sum_{j=1}^{N} f(z_j)\overline{g(z_j)} w_j.$$

# ORF recurrence

Forward recursion

$$\begin{bmatrix} \phi_n(z) \\ \phi_n^*(z) \end{bmatrix} = e_n \frac{z-\alpha_{n-1}}{z-\alpha_n} \begin{bmatrix} 1 & \overline{L}_n \\ L_n & 1 \end{bmatrix} \begin{bmatrix} \zeta_{n-1}(z) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \phi_{n-1}(z) \\ \phi_{n-1}^*(z) \end{bmatrix}$$

$$L_n = -\frac{\left\langle \phi_k, \frac{1-\overline{\alpha}_{n-1}z}{z-\alpha_n}\phi_{n-1} \right\rangle}{\left\langle \phi_k, \frac{z-\overline{\alpha}_{n-1}}{z-\alpha_n}\phi_{n-1}^* \right\rangle}, \quad e_n = \left( \frac{1-|\alpha_n|^2}{1-|\alpha_{n-1}|^2} \frac{1}{1-|L_n|^2} \right)^{1/2}$$

$$\zeta_k(z) = \frac{1-\overline{\alpha}_k z}{z-\alpha_k}, \quad B_n = \zeta_1 \cdots \zeta_n, \quad \phi_n^*(z) = B_n(z)\overline{\phi_n(1/\overline{z})}.$$

Backward recursion = Nevanlinna-Pick algorithm

# Inner product evaluation

If data are available in $\mathbf{z} = \{z_j\}_{j=1}^N \subset \mathbb{T}$ with corresponding weights $\mathbf{w} = \{w_j\}_{j=1}^N$ then choose the discrete inner product.

For a continuous inner product, choose points on the circle, e.g. equidistant $z_j = \exp\{2ij\pi/N\}$, $j = 1, \ldots, N$ and evaluate weight $w_j = w(z_j)$ and use discrete inner product.

For example $w(z) = |U(z)|^2$, and $z_j$ equidistant on $\mathbb{T}$, then given time domain data $u_k$, this can be computed very efficiently by FFT.

We can evaluate the moments $c_l$ in $|U(z)|^2 = \sum_{l\in\mathbb{Z}} c_l z^l$ by convolution and use a Nevanlinna-Pick type algorithm on the PR function $\Omega(z) = c_0/2 + \sum_{l=1}^{\infty} c_l z^l$. (approximately)

# The linear least squares problem

So we can compute

$$\begin{aligned} \boldsymbol{\Phi}(\boldsymbol{\alpha}) = \boldsymbol{\Phi}(\mathbf{z};\boldsymbol{\alpha}) &= [\phi_0(\mathbf{z}) \mid \phi_1(\mathbf{z}) \mid \cdots \mid \phi_n(\mathbf{z})] \in \mathbb{C}^{N\times n+1} \\ \mathbf{W}^{1/2} &= \operatorname{diag}(\mathbf{w})^{1/2}, \quad \boldsymbol{\lambda} = \{\lambda_k\}_{k=0}^{n}, \quad \mathbf{G} = G(\mathbf{z}) \end{aligned}$$

Setting $\hat{G}_n = \sum_{k=0}^{n} \lambda_k \phi_k$, solve in least squares sense

$$\mathbf{W}^{1/2}\boldsymbol{\Phi}(\boldsymbol{\alpha})\boldsymbol{\lambda}(\boldsymbol{\alpha}) = \mathbf{W}^{1/2}\mathbf{G}$$

by orthogonality however, $\boldsymbol{\lambda}(\boldsymbol{\alpha}) = \boldsymbol{\Phi}(\boldsymbol{\alpha})^{H}\mathbf{W}\mathbf{G}$.

# Related work

Work by Ninnes, Van den Hof, Hueberger, Bokor, and others:

Use ORF with respect to Lebesgue measure. These have explicit expressions

$$\phi_n(z) = \frac{\sqrt{1-|\alpha_n|^2}\, z}{z-\alpha_n} B_{n-1}(z)$$

but that does not help for the condition number of $\mathbf{\Phi}^H \mathbf{W} \mathbf{\Phi}$ in the linear least squares problem.

And/or use a finite number of $\alpha_k$ that are cyclically repeated.

# Nonlinear problem

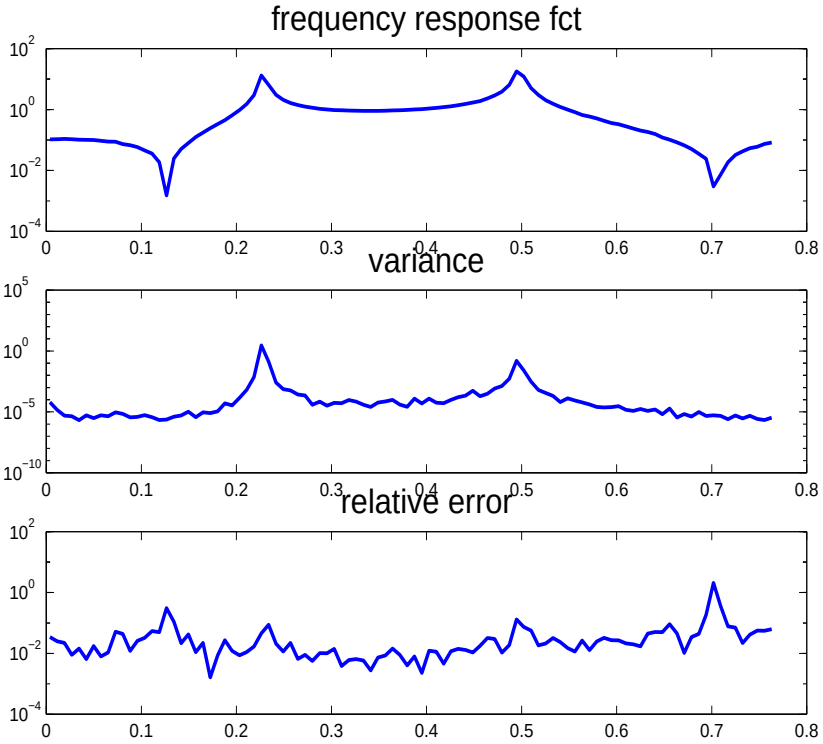
$$
\begin{aligned}
K(\boldsymbol{\alpha}) &= \sum_j |G(z_j) - \sum_{k=0}^{n} \lambda_k(\boldsymbol{\alpha}) \phi_k(z_j; \boldsymbol{\alpha})|^2 w_j \Rightarrow \min_{\boldsymbol{\alpha} \subset \mathbb{D}} K(\boldsymbol{\alpha}) \\
&= \|(\mathbf{I} - \boldsymbol{\Phi}(\boldsymbol{\alpha}) \boldsymbol{\Phi}(\boldsymbol{\alpha})^H \mathbf{W}) \mathbf{G}\|_{\mathbf{W}}^2
\end{aligned}
$$

constraint: set $\alpha_j = r_j e^{i\omega_j}$ with $-1 \leq r_j \leq 1$, $j = 1, \ldots, n$.

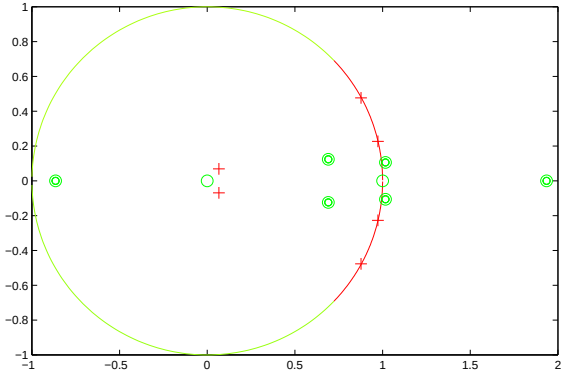
$\alpha_j$ can be forced to be real or complex conjugate.

# Robot arm

100 data points, 200 Hz, condensed in the beginning, order [6/6]

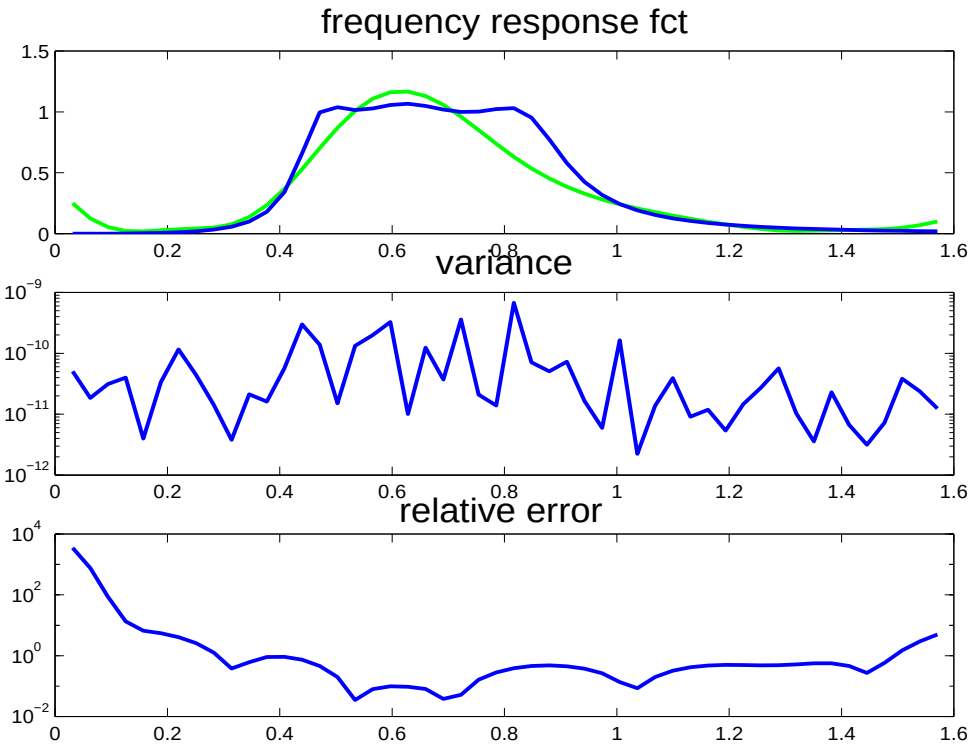



| poles |
|---|
| $0.0665 \pm 0.0690i$ |
| $0.8770 \pm 0.4765i$ |
| $0.9733 \pm 0.2266i$ |
| **zeros** |
| $1.9350, \quad -0.8649$ |
| $1.0164 \pm 0.1056i$ |
| $0.6905 \pm 0.1238i$ |

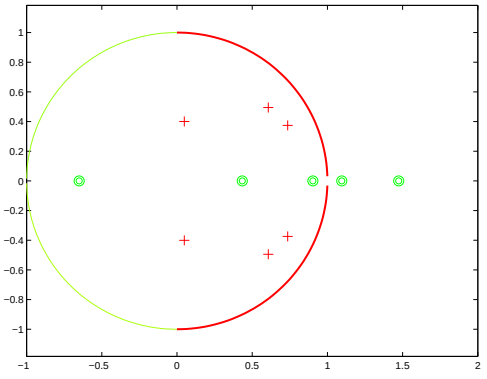


# Band pass filter

50 data points, 20 kHz, in lower half, order [6/6]

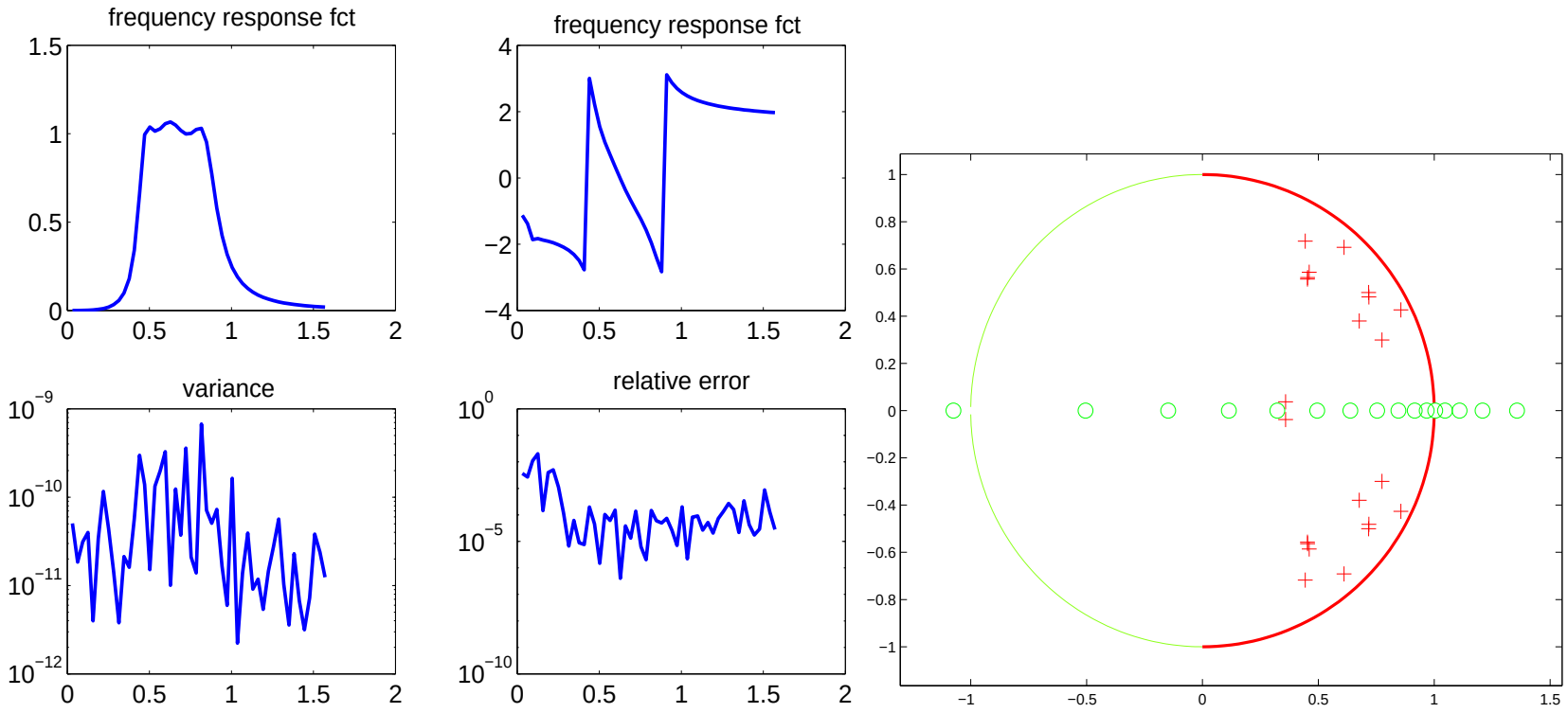



| poles | zeros |
|---|---|
| 0.0487 - 0.4006i | 9.6766 |
| 0.0487 + 0.4006i | 1.4742 |
| 0.6070 - 0.4944i | 1.0955 |
| 0.6070 + 0.4944i | 0.9032 |
| 0.7356 - 0.3739i | -0.6507 |
| 0.7356 + 0.3739i | 0.4337 |



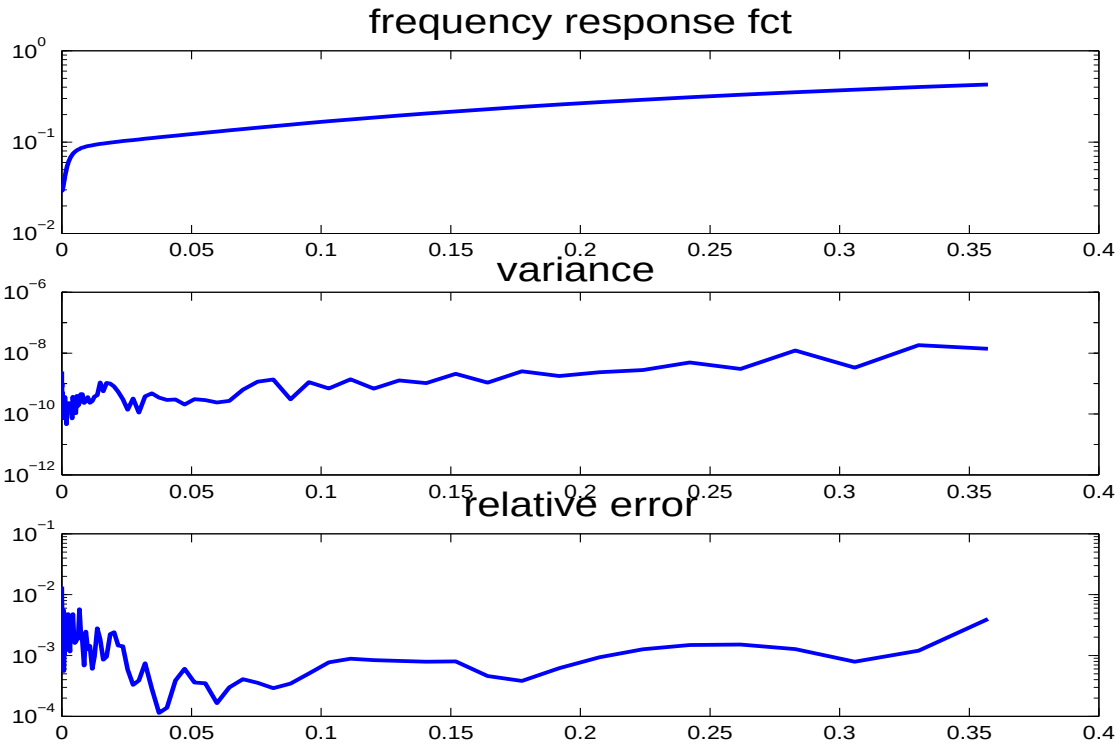
# Band pass filter

50 data points, 20 kHz, in lower half, order [22/22]

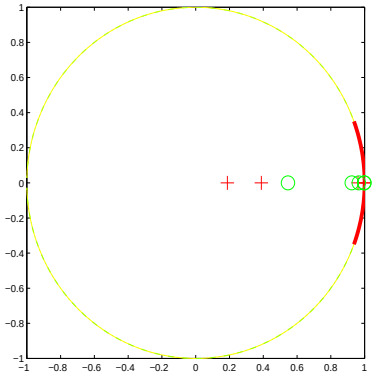


# Electrical Machine

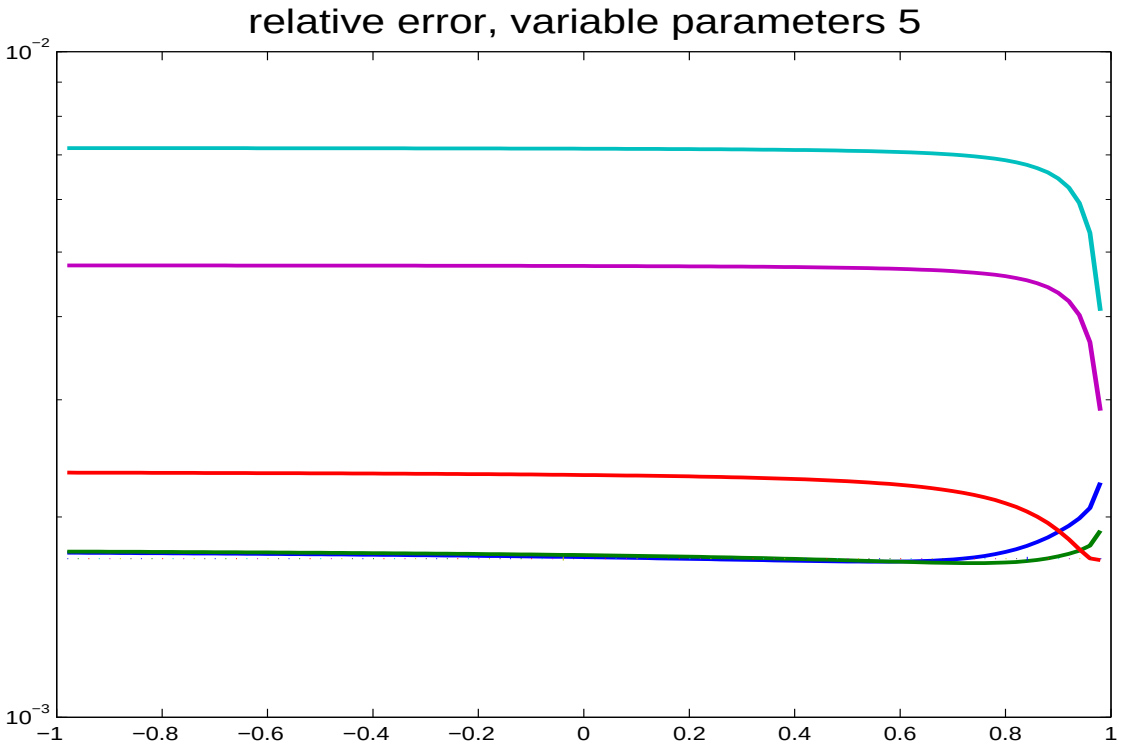
110 data points, 4 kHz, condensed in the beginning, order [5/5]




| poles | zeros |
|---|---|
| 0.1875 | 0.5461 |
| 0.3886 | 0.9239 |
| 0.9609 | 0.9656 |
| 0.9966 | 0.9969 |
| 0.9970 | 0.9990 |



# Sensitivity, Electrical Machine



# Polynomial method

$$\min \left\| \frac{Y}{U} - \frac{B}{A} \right\|^2 = \min \|YA - UB\|^2_{|AU|^{-2}}$$

The weight depends on the solution.

Estimate $A$ so that we have an estimate for the weight $1/|AU|$,

Set $w = [Y/AU \;\; -U/AU], P = [A \; B]^T \Rightarrow$

$$\min \sum_{j=1}^{N} P(z_j)^H W_j P(z_j), \quad W_j = w(z_j)^H w(z_j)$$

need orthogonal block polynomials $\phi_k$ and write $P = \sum_{k=0}^{n} \phi_k \lambda_k$, $\phi_k \in \Pi_k^{2\times 2}, \lambda_k \in \mathbb{C}^{2\times 1}$. Constraint: $P$ "monic".

# Monic solution

$$\mathbf{\Phi} = [\phi_0(\mathbf{z}) \ \ldots \ \phi_n(\mathbf{z})] \in \mathbb{C}^{2N \times 2(n+1)}, \quad \mathbf{W} = \mathrm{diag}(W_j) \in \mathbb{C}^{2N \times 2N}$$

$$\min \boldsymbol{\lambda}^H \mathbf{\Phi}^H \mathbf{W} \mathbf{\Phi} \boldsymbol{\lambda}, \quad \mathbf{\Phi}\boldsymbol{\lambda} \text{ monic}$$

orthogonal polynomial $\mathbf{\Phi}^H \mathbf{W} \mathbf{\Phi} = \mathbf{I} \Rightarrow P = \phi_n \lambda_n$.

# Szegő-type recurrence

$$\begin{aligned} \phi_n \sigma_n &= z\phi_{n-1} + \phi_{n-1}^* \gamma_n \\ \phi_n^* \sigma_n^H &= z\phi_{n-1} \gamma_n^H + \phi_{n-1}^* \end{aligned}$$

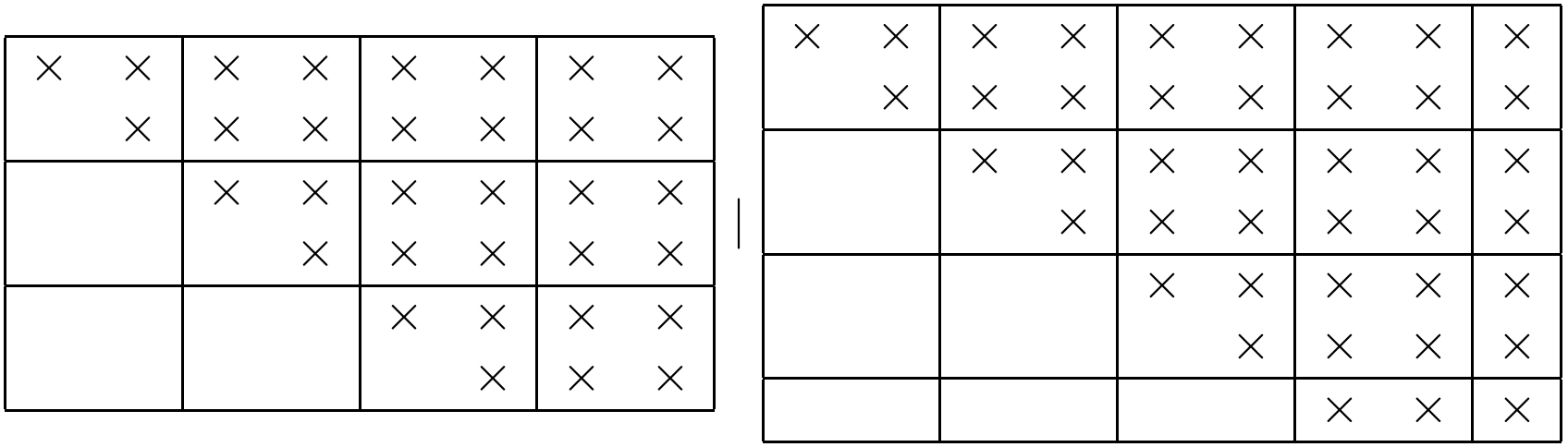
$\sigma_n, \gamma_n \in \mathbb{C}^{2\times 2}$ (block Schur parameters)

# Hessenberg matrix

$$\left[\begin{array}{c|ccc} w_1 & z_1 & & \\ w_2 & & z_2 & \\ & & & \ddots \\ w_N & & & z_N \end{array}\right]$$

unitary similarity $\Rightarrow$ transf.

Block Upper Hessenberg



# Efficient algorithm

Store the block upper Hessenberg matrix in factored form $H = G_1 G_2 \cdots G_m$

$$G_k = \begin{bmatrix} I_{2(k-1)} & & & \\ & -\gamma_k & \hat{\sigma}_k & \\ & \sigma_k & \hat{\gamma}_k & \\ & & & I_{\ldots} \end{bmatrix}$$

Chasing the elements to block upper Hessenberg form by similarity transformations percolating through the product requires operations on $3 \times 3$ or $5 \times 5$ blocks $\Rightarrow$ fast algorithm.

# Nonlinear problem

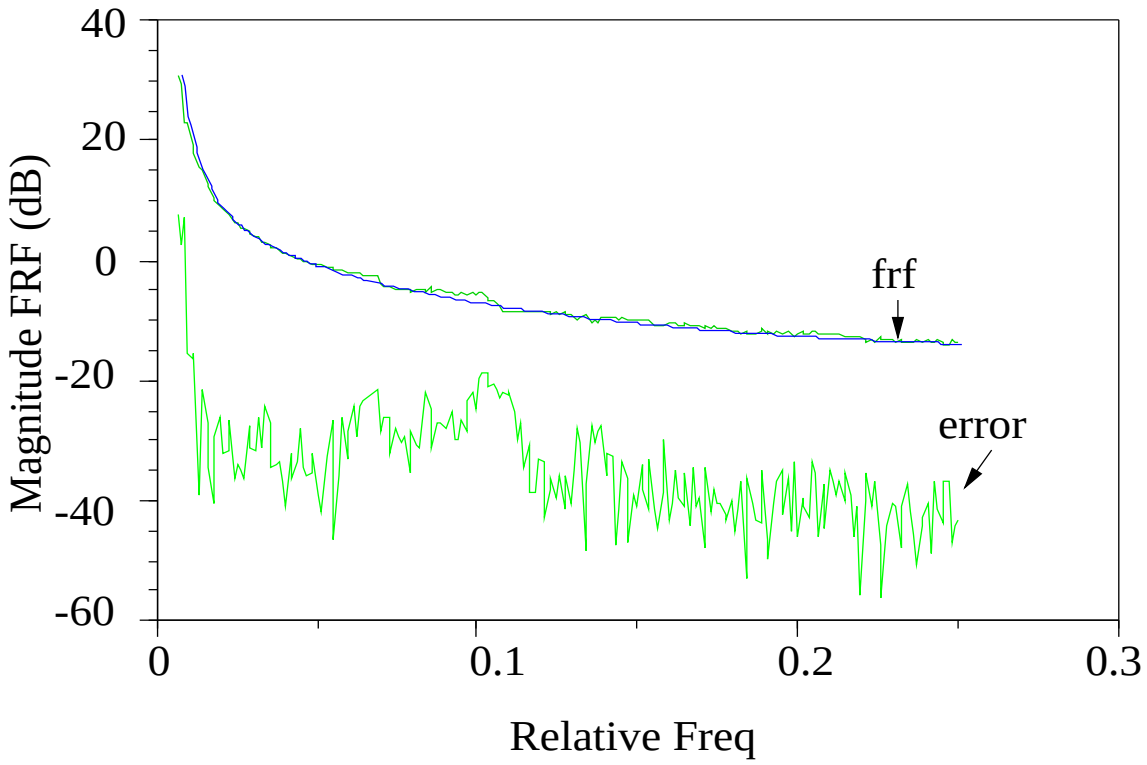
Once $P = [A\ B]^T$ has been found, use A to modify the weight $w = 1/|AU|$ and reiterate.

or

Use nonlinear method to improve the estimate $\sum_{k=0}^{n} \phi_k \lambda_k$. Weight depends on $A \Rightarrow$ loss of orthogonality, but condition number of Jacobian increases only slightly.

# CD's radial servo system

sampling frequency 9.7 kHz, order [5/5]



# Conclusion

- Rational approximation in weighted discrete least squares sense.
- Linear and nonlinear parameters. Solve for linear in terms of nonlinear. Use orthogonal basis to minimize the condition number.
- Either direct orthogonal rational basis or linearized vector polynomial as a combination of orthogonal block polynomials.

# References ORF

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[4] M. Van Barel, A. Bultheel, Orthogonal polynomial vectors and least squares approximation for a discrete inner product, *Electron. Trans. Numer. Anal.* 3 (1995) 1-23.